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Identification of linear systems with nonlinear Luenberger observers

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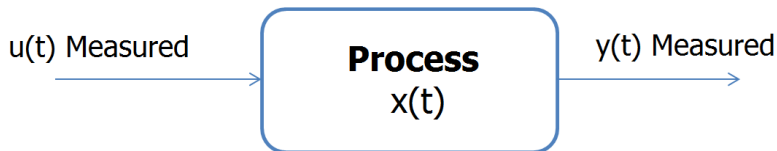
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Outline

- 1 Problem description
- 2 Solution by Luenberger observers approach
- 3 Numerical illustration
- 4 Perspectives

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We are looking for a mathematical model that describes the dynamic behaviour in order to better supervise, diagnose or control it.

It may belong to a class in the continuous time domain:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t), t) \\ y(t) &= g(x(t), t)\end{aligned}$$

Different modelling approaches

Output measurements knowledge

Input measurements knowledge

Physical model structure

Known parameters

→ white box model

Different modelling approaches

Output measurements knowledge

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Different modelling approaches

Output measurements knowledge
Input measurements knowledge → white box model
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Physical model structure
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Output measurements knowledge
Input measurements knowledge → black box model
Assumed model structure
Unknown parameters

LTI system

Assume that the process has a linear dynamic described by

$$(\Sigma) \begin{cases} \dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \\ y(t) = C(\theta)x(t), \end{cases}$$

with:

- $u \in \mathbb{R}$ in $\mathcal{L}_{loc}^{\infty}(\mathbb{R}_+)$.
- $y \in \mathbb{R}$.
- $x \in \mathbb{R}^n$.
- $\theta \in \Theta \subset \mathbb{R}^q$.
- $A(\cdot), B(\cdot)$ and $C(\cdot)$ are sufficiently smooth and **known**.

Estimation problem

Goal

- Estimate on-line the state and parameters $(x(t), \theta)$ by knowing the inputs and outputs.

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Method

- Construct an asymptotic observer for the augmented system.

$$\begin{cases} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) \\ \dot{\theta} &= 0 \quad \leftarrow \text{added states} \\ y(t) &= C(\theta)x(t). \end{cases}$$

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Synthesis steps

- Introduce an extended dynamic system controlled by the known process inputs and outputs.

$$\dot{z} = \Lambda z + Ly, \quad \dot{w} = g(w, u), \quad z \in \mathbb{R}^r, \quad w \in \mathbb{R}^r.$$

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- Find a mapping $(x, \theta, w) \mapsto T(x, \theta, w)$ in C^1 which satisfies the following ODE

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- Implies the following equation

$$\dot{e}(t) = \Lambda e(t),$$

where

$$e(t) = z(t) - T(x(t), \theta, w(t))$$

Synthesis steps

- If Λ is a **Hurwitz** matrix then z is an estimate of T

$$\lim_{t \rightarrow +\infty} |z(t) - T(x(t), \theta, w(t))| = 0 .$$

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The nonlinear Luenberger observer is given as

$$\begin{aligned}\dot{z} &= \Lambda z + Ly \\ \dot{w} &= g(w, u) \\ (\hat{x}, \hat{\theta}) &= T^*(z(t), w(t))\end{aligned}$$

T^* : is the left inverse of T .

Difficulties

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- 1- Is there an explicit expression for T ?
- 2- Is T injective and full rank ?
- 3- Is there an explicit expression for T^* ?

Question 1: Existence of an explicit expression for T

Theorem 1

For any r -uplet of real negative elements $(\lambda_1, \dots, \lambda_r)$ such that

$$\lambda_i \notin \left(\bigcup_{\theta \in \Theta} \sigma\{A(\theta)\} \right) \quad i = 1, \dots, r$$

The mapping $T(x, \theta, w) = [T_1(x, \theta, w_1) \dots T_r(x, \theta, w_r)]^T$ is a solution. with

$$\begin{aligned} T_i: \mathbb{R}^n \times \Theta \times \mathbb{R} &\rightarrow \mathbb{R} \\ (x, \theta, w_i) &\rightarrow T_i(x, \theta, w_i) = C(\theta)(A(\theta) - \lambda_i I_n)^{-1}[x - B(\theta)w_i], \end{aligned}$$

and

$$\begin{aligned} g: \mathbb{R}^r \times \mathbb{R} &\mapsto \mathbb{R}^r \\ (w, u) &\mapsto g(w, u) = \Lambda w + Lu \end{aligned}$$

$$\Lambda = \text{Diag}\{\lambda_1, \dots, \lambda_r\}; \quad L = \mathbf{1}_r.$$

Question 2: Is T injective and full rank ?

$$T(x, \theta, w) = [T_1(x, \theta, w_1) \dots T_r(x, \theta, w_r)]^T$$

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Is it an injective and full rank function?

Answer

If the input makes the extended system **observable**, then by choosing the eigenvalues λ_i **sufficiently large** the function T is **injective** and **full rank** after a transient.

Question 2: Is T injective and full rank ?

Let the mapping $H_r : \Theta \rightarrow \mathbb{R}^{r \times n}$ be defined as

$$\theta \mapsto H_r(\theta) = \begin{bmatrix} C(\theta) \\ C(\theta)A(\theta) \\ \vdots \\ C(\theta)A(\theta)^{r-1} \end{bmatrix},$$

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and $G_r : \Theta \rightarrow \mathbb{R}^{r \times r}$ be defined as

$$\theta \mapsto G_r(\theta) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ C(\theta)B(\theta) & 0 & \cdots & \cdots & \cdots & 0 \\ C(\theta)A(\theta)B(\theta) & C(\theta)B(\theta) & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & 0 \\ C(\theta)A(\theta)^{r-2}B(\theta) & C(\theta)A(\theta)^{r-3}B(\theta) & \cdots & \cdots & C(\theta)B(\theta) & 0 \end{bmatrix},$$

then

$$\begin{bmatrix} y(t) & \dot{y}(t) & \cdots & y^{(r-1)}(t) \end{bmatrix} = H_r(\theta)x + G_r(\theta)v$$

Question 2: Is T injective and full rank ?

Assumption (Uniform differential observability)

There exist two compact subsets $\mathcal{C}_\theta \in \Theta$, $\mathcal{C}_x \in \mathbb{R}^n$, an integer r and U_r a bounded subset of \mathbb{R}^{r-1} such that the mapping \mathfrak{H}_r is injective and full rank

$$\mathfrak{H}_r(x, \theta, v) = H_r(\theta)x + G_r(\theta)v ,$$

for all (x, θ) and (x^*, θ^*) both in $\text{Cl}(\mathcal{C}_\theta) \times \text{Cl}(\mathcal{C}_x)$ and all v in U_r .

Or, there exist $L > 0$ such that

$$|\mathfrak{H}_r(x^*, \theta^*, v) - \mathfrak{H}_r(x, \theta, v)| \geq L \left\| \begin{bmatrix} x - x^* \\ \theta - \theta^* \end{bmatrix} \right\| .$$

Question 2: Is T injective and full rank ?

Theorem 2

- Assume the assumption holds. Let $u(\cdot)$ be a bounded $C^{r-2}([0, +\infty])$ function with bounded $r - 2$ first derivatives $\bar{u}^{(r-2)}(\cdot)$.
- For all $(\tilde{\lambda}_1, \dots, \tilde{\lambda}_r)$, for all positive time τ and all $t_1 \geq \tau$, if $\bar{u}(t_1)$ is in U_r then for all (x, θ) and (x^*, θ^*) in $\mathcal{C}_x \times \mathcal{C}_\theta$, T is injective and full rank.

Or, there exist two positive real numbers k^* and L_T such that for all $k > k^*$

$$|T(x, \theta, w(t_1)) - T(x^*, \theta^*, w(t_1))| \geq \frac{L_T}{k^r} \left\| \begin{bmatrix} x - x^* \\ \theta - \theta^* \end{bmatrix} \right\|.$$

With the mapping T is defined by taking $\lambda_i = k\tilde{\lambda}_i$.

Question 3: Existence of an explicit expression for T^*

The general form of T^* is the optimization of the following criteria

$$(\hat{x}(t), \hat{\theta}(t)) = \arg \min_{x, \theta} \|T(x(t), \theta, w(t)) - z(t)\|_2^2$$

$T(x, \theta, w)$ is nonlinear \Rightarrow iterative optimization methods.

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$T(x, \theta, w)$ is nonlinear \Rightarrow iterative optimization methods.

We are looking for an explicit expression of T^* .

Question 3: Existence of an explicit expression for T^*

Let the canonical **observable** structure

$$\begin{aligned}
 A(\theta) = A(\theta_a) &= \begin{bmatrix} -\theta_{a1} & 1 & 0 & \cdots & 0 \\ -\theta_{a2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ -\theta_{an} & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \\
 C(\theta) = C &= [1 \ 0 \ \cdots \ 0] \in \mathbb{R}^{1 \times n}, \\
 B(\theta) = B(\theta_b) &= [\theta_{b1} \ \theta_{b2} \ \cdots \ \theta_{bn}]^T \in \mathbb{R}^{n \times 1},
 \end{aligned}$$

All strictly proper linear SISO systems can be written in this form.

Question 3: Existence of an explicit expression for T^*

For all $i = 1, \dots, r$, we have

$$z_i(t) = T_i(\hat{x}(t), \hat{\theta}, w_i(t)) = C(A(\hat{\theta}) - \lambda_i I_n)^{-1} [\hat{x}(t) - B(\hat{\theta})w_i(t)]$$

By using the **Kronecker algebra**, we can get the following expression

$$z_i = \underbrace{\begin{bmatrix} z_i V_i^T & V_i^T & -(w_i^T \otimes V_i^T) \end{bmatrix}}_{P_i(z_i, w_i)} \begin{bmatrix} \hat{\theta}_a \\ \hat{x} \\ \hat{\theta}_b \end{bmatrix}$$

The solution is given by

$$(\hat{\theta}_a(t), \hat{x}(t), \hat{\theta}_b(t))^T = (P(z(t), w(t))^T P(z(t), w(t)))^{-1} P(z(t), w(t))^T z(t)$$

where: $P(z, w) = [P_1(z_1, w_1)^T, \dots, P_r(z_r, w_r)^T]^T \in \mathbb{R}^{r \times (2n+1)}$ and

$$V_i = \begin{bmatrix} \frac{1}{\lambda_i} & \dots & \frac{1}{\lambda_i^n} \end{bmatrix}^T \in \mathbb{R}^n, \quad r \geq 4n - 1$$

Question 3: Existence of an explicit expression for T^*

Question ?

Under which conditions on $T(x, \theta, w)$ and λ_i is the matrix P full rank column?

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Proposition

If the matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$ have the observable form, if the λ_i 's are different from $A(\theta)$ eigenvalues and if the dimension of T $r \geq 4n - 1$, then for any (z, x, θ, w) such that $z = T(x, \theta, w)$ and

$$\text{rank} \left(\frac{\partial T}{\partial (x, \theta)}(x, \theta, w) \right) = r$$

the matrix P is a full rank column.

Question 3: Existence of an explicit expression for T^*

Result

- When $z(t)$ is in $Im(T)$, the Luenberger observer

$$\begin{cases} \dot{z}(t) = \Lambda z(t) + Ly(t) \\ \dot{w}(t) = \Lambda w(t) + Lu(t) \\ (\hat{x}(t), \hat{\theta}(t)) = (P^T(w(t), z(t))P(w(t), z(t)))^{-1} P^T z(t), \end{cases}$$

is well defined.

Question 3: Existence of an explicit expression for T^*

Result

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is well defined.

- In the transient phase we can not guarantee that P is of full rank column. If it is numerically not the case, we keep the old values of the unknown parameters.

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Black box model

$$\dot{x}(t) = \begin{bmatrix} -\theta_{a1} & 1 \\ -\theta_{a2} & 0 \end{bmatrix} x(t) + \begin{bmatrix} \theta_{b1} \\ \theta_{b2} \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0] x(t)$$

We want to estimate θ_{a1} , θ_{a2} , θ_{b1} , θ_{b2} and state x_2 from knowledge of signals $u(t)$ and $y(t)$.

$(z(0), w(0)) = (0, 0)$
$(x_1(0), x_2(0), \theta_{a1}(0), \theta_{a2}(0), \theta_{b1}(0), \theta_{b2}(0)) = (0, 0.5, 2, 3, 1, -1)$
$(\hat{x}_1(0), \hat{x}_2(0), \hat{\theta}_{a1}(0), \hat{\theta}_{a2}(0), \hat{\theta}_{b1}(0), \hat{\theta}_{b2}(0)) = (0, 0, 0, 0, 0, 0)$

Table: Initial configuration of the system and the observer states.

Simulation without noise effect

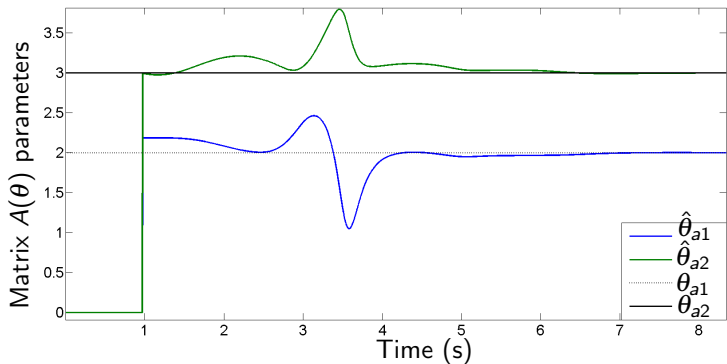


Figure: Estimation of parameters θ_{a1} and θ_{a2} .

Simulation without noise effect

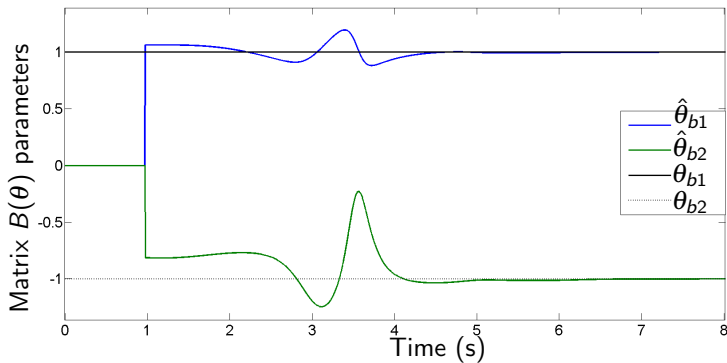


Figure: Estimation of parameters θ_{b1} and θ_{b2} .

Simulation without noise effect

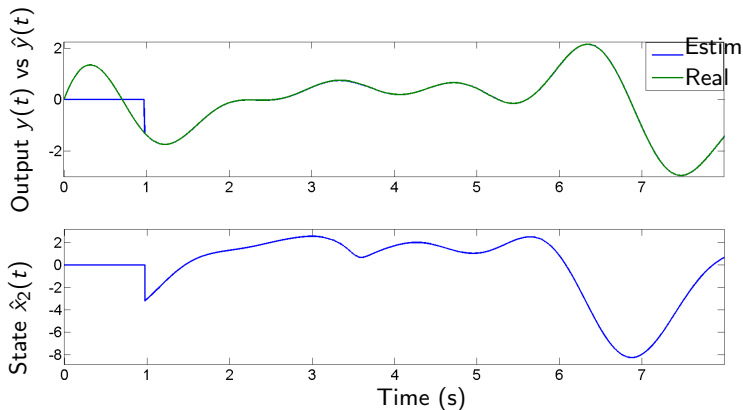


Figure: Comparison between estimated output $\hat{y}(t)$ and real output $y(t)$.

Simulation with output added noise of 5%

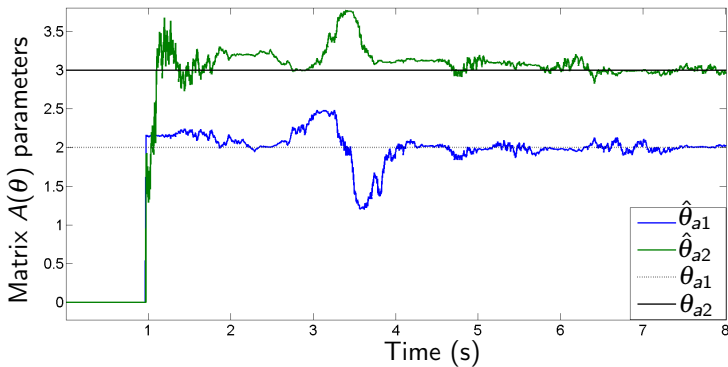


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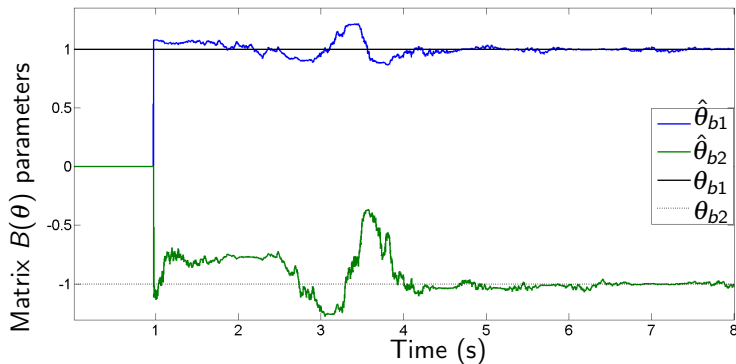


Figure: Estimation of parameters θ_{b1} and θ_{b2} .

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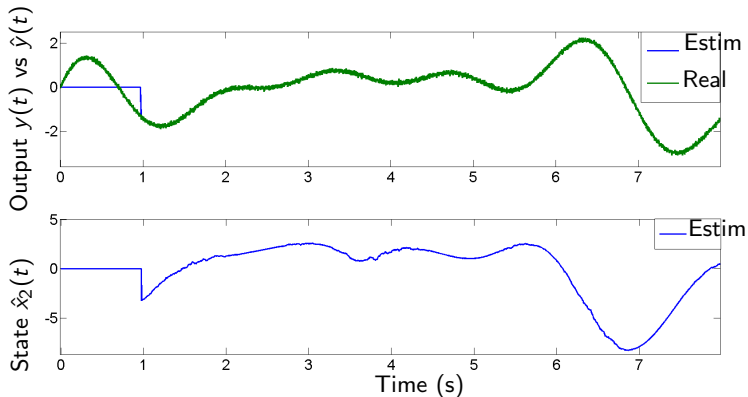


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Perspectives

- Numerical comparison with other approaches in literature.
- Study of the observer robustness with respect to noise.
- Study of the persistency excitation of input and observer order.
- Application of this approach on a real system.

THANK YOU