

An explicit optimal input design for first order systems identification

Pascal DUFOUR^{1,3}, Madiha NADRI¹ and Jun QIAN^{1,2,3}

¹Université de Lyon, Lyon F-69003, Université Lyon 1, CNRS UMR 5007, Laboratory of Process Control and Chemical Engineering (LAGEP), Villeurbanne 69100, France

²Acsystème company (IT and Control engineering), Rennes, France

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 - **Difficulty**: how to take into account online the operational **constraints**.
- ⇒ **Our approach: online joined constrained OED and parameter estimation = model predictive control (MPC) + observer.**

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- Combines **observer design theory** and an on-line (**MPC**).
- Allows to specify **input and output constraints** to keep the process in a desired operating zone.

Some recalls about Qian et al. work (2/2)

- MPC Cost function based on the predicted Fisher Information Matrix M

$$\left\{ \begin{array}{l} F(\bar{y}_{\theta|k}, u_{|k}, y_p(k), \hat{x}(k), \hat{\theta}(k)) = \frac{1}{N_p} \sum_{l=k+1}^{k+N_p} M_{l|k} \\ u_{|k}^* = \arg \max_{u_{|k}} \left(J(u_{|k}) = \frac{\lambda_{\min}(F)}{\lambda_{\max}(F)} \right) \\ u_{|k} = \{u(k) \dots u(l) \dots u(k + N_p)\}, l \in [k, k + N_p]. \\ k(l) = \text{current(future) time index}, N_p = \text{prediction horizon} \end{array} \right. \quad (1)$$

- Handling constraints:

- Physical limitations of the inputs: $u_{\min} \leq u(k) \leq u_{\max}, \forall k$
- On the estimated states and/or the measured outputs y_p (dealing with safety, operating zone, production, ...):

$$g_{\min} \leq g(\hat{x}(k), y_p(k), u(k)) \leq g_{\max}, \forall k \quad (2)$$

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- Online input design = nonlinear constrained optimization problem = might be too time consuming in certain cases.

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Models and explicit step response

- First order model:

$$\begin{cases} \dot{x}(t) = -\frac{1}{\tau}(x(t) - Gu(t)), & t > t_k \\ x(t_k) = y_p(t_k), \end{cases} \quad (3)$$

x = measured state, u = measured input, θ (=the time constant $\tau > 0$) = unknown parameter, $G \in \mathbb{R}^*$ = known static gain, $y_p(t_k)$ = measured process output.

- Sensitivity model: with $x_\theta = \frac{\partial x}{\partial \tau}$

$$\begin{cases} \dot{x}_\theta(t) = \frac{1}{\tau^2}x(t) - \frac{1}{\tau}x_\theta(t) - \frac{G}{\tau^2}u(t), & t > t_k \\ x_\theta(t=0) = 0, \end{cases} \quad (4)$$

- Step response of sensitivity model: at current time t_k , over the prediction horizon N_p (control horizon=1),

$$x_\theta(t) = \left(x_\theta(t_k) + \frac{(y_p(t_k) - Gu(t_k))(t - t_k)}{\hat{\tau}^2(t_k)} \right) \exp\left(\frac{-(t - t_k)}{\hat{\tau}(t_k)}\right), \quad t > t_k. \quad (5)$$

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Observer design

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Augmented system

$$\begin{cases} \dot{x}_a(t) = A_a(y_p(t), u(t))x_a(t), & t > t_k \\ y_a(t) = C_a x_a(t), & t > t_k, \end{cases} \quad (6)$$

where :

$$A_a(y_p(t), u(t)) = \begin{pmatrix} 0 & -y_p(t) + Gu(t) \\ 0 & 0 \end{pmatrix}, \quad C_a = [1 \ 0]$$

$$x_a(t) = [x(t) \ \frac{1}{\tau}]^T.$$

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Extended Kalman filter (Besançon [2007])

$$\begin{cases} \dot{\hat{x}}_a(t) = A_a(y_p(t), u(t))\hat{x}_a(t) - \rho_o S_o^{-1}(t) C_a^T (C_a \hat{x}_a(t) - y_p(t)), & t > t_k \\ \dot{S}_o(t) = -\theta_o S_o(t) - A_a^T(y_p(t), u(t)) S_o(t) - S_o(t) A_a(y_p(t), u(t)) + \rho_o C_a^T C_a, & t > t_k \\ \hat{x}_a(t_k) = \begin{bmatrix} \hat{x}(t_k) \\ \hat{\tau}(t_k) \end{bmatrix} \end{cases} \quad (7)$$

where $\theta_o > 1$, $\rho_o > 0$ and $S_o(t=0)$ are the observer tuning parameters.

Constrained optimal input design

- General MPC formulation (1) is written here as:

$$\max_{u_{inf} \leq u(t_k) \leq u^{sup}} J(u(t_k)) = \int_{t_k}^{t_k + N_p} (x_\theta(t))^2 dt \quad (8)$$

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- Determination of u_{inf} and u^{sup} from:
 - constraints on u and y_p :

$$\begin{cases} u_{inf,u} \leq u \leq u^{sup,u}, \forall t > 0 \\ y_{inf} \leq y_p \leq y_p^{sup}, \forall t > 0 \end{cases} \quad (9)$$

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- known static gain G :

$$\begin{cases} u^{sup,y} = \max\left(\frac{y_{inf}}{G}, \frac{y^{sup}}{G}\right) \\ u^{sup} = \min(u^{sup,y}, u^{sup,u}) \\ u_{inf,y} = \min\left(\frac{y_{inf}}{G}, \frac{y^{sup}}{G}\right) \\ u_{inf} = \max(u_{inf,y}, u_{inf,u}). \end{cases} \quad (10)$$

Explicit control law

- Optimization problem (8), convex in $u(t_k)$, becomes:

$$\max_{u_{inf} \leq u(t_k) \leq u^{sup}} J(u(t_k)) = \int_{t_k}^{t_k+N_p} f_2(t)u(t_k)^2 + f_1(t)u(t_k) + f_0(t)dt \quad (11)$$

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and may be rewritten as:

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leading to the **explicit** solution (see functions in the paper):

$$\begin{cases} u_{min}(t_k) = u_{min}(y_p(t_k), \hat{\tau}(t_k)) = \frac{1}{G} (x_{\theta}(t_k)\hat{\tau}(t_k)^2\gamma(\cdot) + y_p(t_k)) \\ \gamma(\cdot) = \gamma(N_p, T_s, \hat{\tau}(t_k), \nu(N_p, T_s, \hat{\tau}(t_k))) \end{cases} \quad (14)$$

Explicit control law

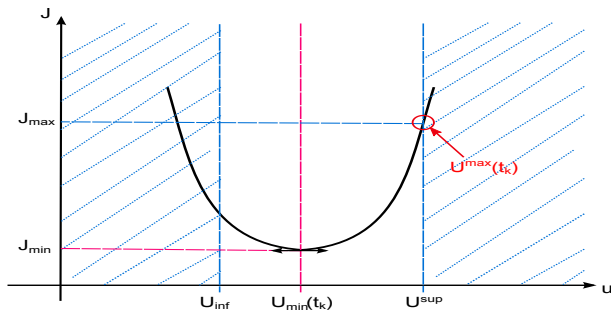


Figure: Quadratic criterion: geometric interpretation

- Explicitly defined optimal control law $u^*(t_k) = u^{max}(t_k)$:

$$u^{max}(t_k) = \begin{cases} u_{sup} & \text{if } u_{min}(t_k) < u_{inf} + \frac{u^{sup} - u_{inf}}{2} \\ u^{inf} & \text{else} \end{cases}$$

with u^{sup} and u_{inf} defined in (10), and $u_{min}(t_k)$ defined in (14).

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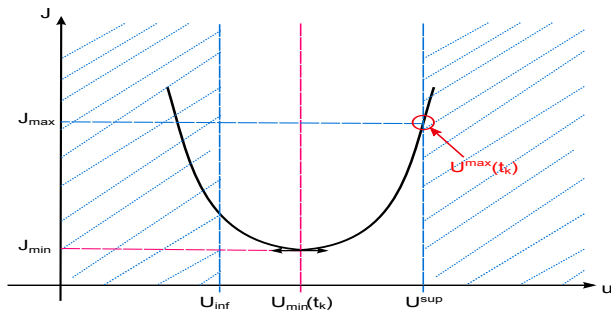


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with u^{\sup} and u_{\inf} defined in (10), and $u_{\min}(t_k)$ defined in (14).

- Control law: needs only online observer integration (=few computations).

Simulation results

- Prescribed constraints :

$$0 \leq u(k) \leq 1, \forall k; \quad (15)$$

$$0 \leq y(k) \leq 4, \forall k. \quad (16)$$

- Two different cases:

- open loop with PRBS input under input constraints (15);
- closed loop with explicit optimal input design under input constraints (15) and output constraints (16).

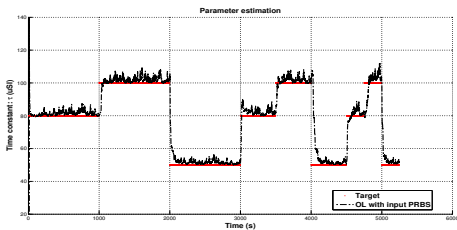
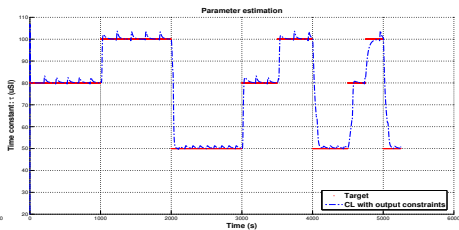
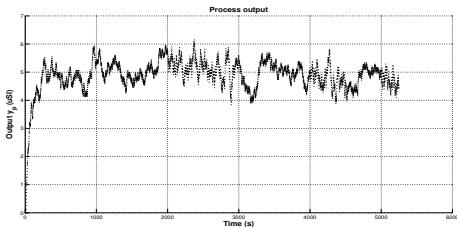
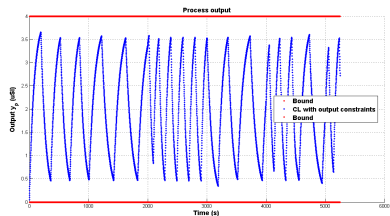
- Setting the adaptive horizon prediction:

$$N_p(t_k) T_s = \hat{\tau}(t_k).$$

Tableau: The time constant τ : time-variant target of the unknown parameter

Time (s)	0	1000	2000	3000	3500	4000	4500	4750	5000
θ (s)	80	100	50	80	100	50	80	100	50

Simulation results

(a) $\hat{\theta}$ in open loop(b) $\hat{\theta}$ in closed loop(a) y_p in open loop(b) y_p in closed loop

Simulation results

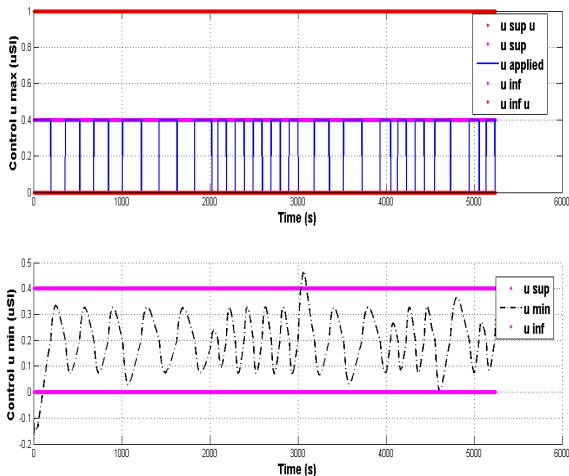


Figure: 1st order system: u^* (at the top) and u_{\min} (at the bottom) in close loop

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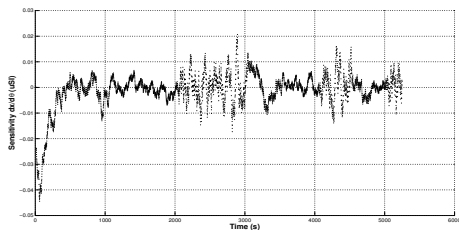
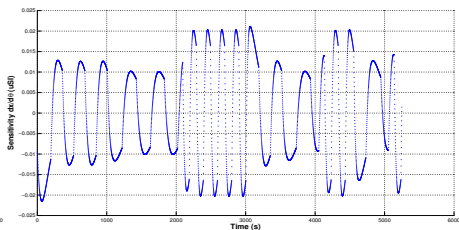
(a) x_θ in open loop(b) x_θ in closed loop

Tableau: Results analysis

Criteria	OL	CL
Maximum value taken by y_p	6.14	3.655
Minimum value taken by y_p	0.0	0.050
Mean sensitivity over $\left[\frac{T_{final}}{100}, T_{final} \right] \left(\frac{1}{s} \right)$	0.004	0.011
Mean time constant estimation error over $\left[\frac{T_{final}}{100}, T_{final} \right] (\%)$	4.62	3.60

Conclusions and perspectives

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 - + online the optimal design of experiment;
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- 👉 The input and output **constraints** specify the physical limitations imposed by the system and ensure the efficiency of the OED.
- The influence of the **prediction horizon** (the main tuning parameter) as to be studied in more details.

Contacts and discussion

Software

- ODOE4OPE (Optimal Design Of Experiments for Online Parameter Estimation)
Email: odoe4ope@univ-lyon1.fr; Website: odoe4ope.univ-lyon1.fr

Business contacts

- Acssysteme: Expertise in automation, signal processing, optimization, software developing,
...
Website: www.acssysteme.com/en
- LAGEP: Laboratory of Process Control and Chemical Engineering
Website: www-lagep.univ-lyon1.fr

Authors

- Pascal DUFOUR: www.tinyurl.com/dufourpascal
- Madiha NADRI: nadri@lagep.univ-lyon1.fr
- Jun QIAN: www.junqian.sitew.fr

Main ideas

Offline synthesis:

- determine $x_\theta(t_k)$ and $\hat{\theta}(t_k)$;
- get the predicted step response of the sensitivity model into the prediction horizon N_p ;
- determine the worst control law $u_{min}(t_k)$ (that minimize the MPC cost function), in order to define the explicit optimal control law (that maximize the MPC cost function) at each time t_k .

Online computations:

- at each current time t_k , get the process measure $y_p(t_k)$;
- determine the current estimate of the time constant $\hat{\tau}(t_k)$ based on the process input applied $u(t_k)$ and process output $y_p(t_k)$;
- compute the optimal control to apply $u^*(t_k) = u^{max}(t_k)$ based on the measure $y_p(t_k)$, the estimation $\hat{\tau}(t_k)$, the explicit control law $u_{min}(t_k)$ (defined offline) and the input bounds;
- apply $u^{max}(t_k)$ from $t_k + \epsilon$ to t_{k+1} ;
- Iterative procedure at the next sampling time.

Sensitivity model step response

- Laplace transform:

$$\begin{aligned} L\left(\frac{df}{dt}(t)\right) &= sL(f(t)) - f(t_k), \quad t > t_k; \\ L\left(\frac{df}{dt}(t)\right) &= sF(s) - f(t_k), \quad t > t_k. \end{aligned} \quad (17)$$

- Laplace transform of sensitivity model:

$$X_\theta(s) = \frac{x_\theta(t_k)}{s + \frac{1}{\tau}} + \frac{y_p(t_k) - GsU(s)}{\tau^2(s + \frac{1}{\tau})^2}, \quad (18)$$

- The step response of sensitivity model (in the Laplace domain) is:

$$X_\theta(s) = \frac{x_\theta(t_k)}{s + \frac{1}{\tau}} + \frac{y_p(t_k) - Gu(t_k)}{\tau^2(s + \frac{1}{\tau})^2}. \quad (19)$$

Determination of $u_{min}(t_k)$

The cost function is quadratic in its optimization argument $u(t_k)$, which can be rewritten explicitly:

$$\left\{ \begin{array}{l} \max_{u_{inf} \leq u(t_k) \leq u_{sup}} J(u(t_k)) = \int_{t_k}^{t_k+N_p} f_2(t)u(t_k)^2 + f_1(t)u(t_k) + f_0(t)dt \\ \text{où :} \\ f_2(t) = \exp\left(\frac{-2(t-t_k)}{\hat{\tau}(t_k)}\right) \left(\frac{G^2}{\hat{\tau}(t_k)^4} (t-t_k)^2\right) \in \mathcal{R}^+ \\ f_1(t) = \exp\left(\frac{-(t-t_k)}{\hat{\tau}(t_k)}\right) \left(-2(x_\theta(t_k) + \frac{y_p(t_k)}{\hat{\tau}(t_k)^2} (t-t_k)) \frac{G}{\hat{\tau}(t_k)^2} (t-t_k) \exp\left(\frac{-(t-t_k)}{\hat{\tau}(t_k)}\right)\right) \\ f_0(t) = \exp\left(\frac{-(t-t_k)}{\hat{\tau}(t_k)}\right) \left(x_\theta(t_k) + \frac{y_p(t_k)}{\hat{\tau}(t_k)^2} (t-t_k) \exp\left(\frac{-(t-t_k)}{\hat{\tau}(t_k)}\right)\right) \end{array} \right. \quad (20)$$

where $f_2(t) > 0$, the problem (11) is convex and may be rewritten showing characteristics of this parabola:

$$\max_{u_{inf} \leq u(t_k) \leq u_{sup}} J(u(t_k)) = c_\infty(t_k)(u(t_k) - u_{min}(t_k))^2 + J_{min}(t_k), \quad (21)$$

where c_∞ is a positive real (obtained from the integration $f_2(t)$ in (11)), and where $u_{min}(t_k)$ is the worst control $u(t_k)$ to apply, since it minimize at t_k the criteria $J(\cdot)$ (where $J(\cdot) = J_{min}(t_k) \geq 0$, since J is the L^2 norm of x_θ).

Determination of $u_{min}(t_k)$

Hence, $u_{min}(t_k)$ has to be determined through the first order optimality condition:

$$\frac{\partial J}{\partial u(t_k)} = 0 \text{ at } u(t_k) = u_{min}(t_k) \quad (22)$$

which leads to:

$$u_{min}(t_k) = \frac{-\int_{t_k}^{t_k+N_p} f_1(t) dt}{2 \int_{t_k}^{t_k+N_p} f_2(t) dt}. \quad (23)$$

This leads to determine $u_{min}(t_k)$:

$$u_{min}(t_k) = \frac{1}{G} y_p(t_k) + \frac{\hat{\tau}(t_k)^2}{G} \frac{\int_{t_k}^{t_k+N_p} (t - t_k) \exp\left(\frac{-2(t-t_k)}{\hat{\tau}(t_k)}\right) dt}{\int_{t_k}^{t_k+N_p} (t - t_k)^2 \exp\left(\frac{-2(t-t_k)}{\hat{\tau}(t_k)}\right) dt} x_\theta(t_k). \quad (24)$$

$$\Rightarrow \begin{cases} u_{min}(t_k) = \frac{1}{G} (\hat{\tau}(t_k)^2 \gamma(\cdot) x_\theta(t_k) + y_p(t_k)) \\ \text{with} \\ \gamma(N_p, T_s, \hat{\tau}(t_k), \nu(N_p, T_s, \hat{\tau}(t_k))) = \frac{2N_p T_s \nu(\cdot) + \hat{\tau}(t_k) (\nu(\cdot) - 1)}{2(N_p T_s)^2 \nu(\cdot) + 2\hat{\tau}(t_k) N_p T_s \nu(\cdot) + \hat{\tau}(t_k)^2 (\nu(\cdot) - 1)} \\ \nu(N_p, T_s, \hat{\tau}(t_k)) = \exp\left(\frac{-2N_p T_s}{\hat{\tau}(t_k)}\right). \end{cases} \quad (25)$$

Initial conditions and parameters

Tableau: Simulation conditions

	Parameter	Value (unit)
Time	T_{final}	5250 (s)
	T_s	1 (s)
Model and Process	G	10 (-)
	$y_p(0)$	0 (-)
	$\tau(0)$	80 (s)
Observer	ρ_o	1.02 (-)
	θ_o	0.05 (-)
	S_o	$\begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$ (-)
	$\hat{\tau}(0)$	20 (s)
Controller	$[u_{inf,u}, u^{sup,u}]$	[0, 1]
	$[y_{inf}, y^{sup}]$	[0, 4]