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January 2016

*Dedicated to Pavel Exner on the occasion of his 70th birthday*

**Abstract**

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of $\mathbb{R}^3$ which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

**classification** Primary 81Q10; Secondary 47F05.

**keywords** Waveguide, mixed boundary conditions, twisting.

## 1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_0 := \mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial\Omega_0$.
The cross section $\omega$ is supposed to be an open bounded connected subset of $\mathbb{R}^2$ of diameter $d > 0$ which is not rotationally invariant. Moreover $\omega$ is supposed to have smooth boundary $\partial\omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by $\mathcal{N}$ the Neumann window. It is an open bounded subset of the boundary $\partial\Omega_0$. Let $\mathcal{D}$ be its complement set in $\partial\Omega_0$. When $\mathcal{N}$ is an annulus of size $l > 0$ we will denote it by,

$$\mathcal{A}_a(l) := I_a(l) \times \partial\omega, I_a(l) := (a, l + a), a \in \mathbb{R}.$$

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Consider first the self-adjoint operator $H_0^{\mathcal{N}}$ associated to the following quadratic form. Let $D(Q^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_0) \ \mid \psi_{\lceil \mathcal{D}} = 0\}$ and for $\psi \in D(Q^{\mathcal{N}})$,

$$Q^{\mathcal{N}}(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

i.e. the Laplace operator defined on $\Omega_0$ with Neumann boundary conditions (NBC) on $\mathcal{N}$ and Dirichlet boundary conditions (DBC) on $\mathcal{D}$ [5, 11].

It is actually shown in the Section 2 of this paper that if $\mathcal{N}$ contains an annulus of size $l$ large enough then $H_0^{\mathcal{N}}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if $\mathcal{N}$ contains an annulus of any size $l > 0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_0^{\mathcal{N}}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function $\theta$ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose $\theta \in C_c^1(\mathbb{R})$ and introduce the diffeomorphism

$$\begin{aligned} \mathcal{L} : \Omega_0 \quad &\longrightarrow \quad \mathbb{R}^3 \\ (s, t_2, t_3) \quad &\longmapsto \quad \Big(s, t_2 \cos\theta(s) - t_3 \sin\theta(s), t_2 \sin\theta(s) + t_3 \cos\theta(s)\Big). \end{aligned} \tag{1}$$

The twisted tube is given by $\Omega_\theta := \mathcal{L}(\Omega_0)$. Let $D(Q_\theta^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_\theta) \ \mid \psi_{\lceil \mathcal{L}(\mathcal{D})} = 0\}$ and consider the following quadratic form

$$Q_\theta^{\mathcal{N}}(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \ \psi \in D(Q_\theta^{\mathcal{N}}). \tag{2}$$

Through unitary equivalence, we then have to consider

$$q_\theta^{\mathcal{N}}(\psi) \quad := \quad Q_\theta^{\mathcal{N}}(\psi o \mathcal{L}^{-1}) = \| \nabla' \psi \|^2 + \| \partial_s \psi + \dot{\theta} \partial_\tau \psi \|^2, \tag{3}$$

$\psi \in D(q_\theta^{\mathcal{N}}) := \{\psi \in \mathcal{H}^1(\Omega_0) \ \mid \psi_{\lceil \mathcal{D}} = 0\}$ and where

$$\nabla' := {}^t\left(\partial_{t_2}, \partial_{t_3}\right), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}. \tag{4}$$

Denote by $H_\theta^{\mathcal{N}}$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H_\theta^{\mathcal{N}}) = \{\psi \in D(q_\theta^{\mathcal{N}}), \quad H_\theta^{\mathcal{N}} \psi \in L^2(\Omega_0) \quad \frac{\partial \psi}{\partial n} \lceil_{\mathcal{N}} = 0\}$ with

$$H_\theta^{\mathcal{N}} \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_s)^2) \psi, \tag{5}$$

where the transverse Laplacian $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = \mathcal{A}_a(l), l > 0$, we will denote these forms respectively as $Q_\theta^l, q_\theta^l$ and the corresponding operator as $H_\theta^l$ and if $\mathcal{N} = \emptyset$ we denote the associated operator by $H_\theta$.

Then the main result of this paper is

**Theorem 1.1.** *i) Under conditions stated above on $\omega$ and $\theta$, there exists $l_{min} := l_{min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{min}$, $\mathcal{N} \supset \mathcal{A}_a(l)$ then*

$$\sigma_d(H_\theta^{\mathcal{N}}) \neq \emptyset. \tag{6}$$

*ii) Suppose $\theta$ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{max} := d_{max}(\theta, \omega) > 0$ such that for all $0 < d \leq d_{max}$ there exists $l_{max} := l_{max}(\omega, d, \theta)$ such as for all $0 < l \leq l_{max}$, if $\mathcal{N} \subset \mathcal{A}_a(l)$ and $supp(\dot{\theta}) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then*

$$\sigma_d(H_\theta^{\mathcal{N}}) = \emptyset. \tag{7}$$

Roughly speaking this result implies that for $d$ small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

# 2 Existence of bound states

First we prove the following. Denote by $E_1, E_2, ....$ the eigenvalues (transverse modes) of the Laplacian $-\Delta_\omega$ defined on $\mathrm{L}^2(\omega)$ with DBC on $\partial\omega$. Let $\chi_1, \chi_2, ...$ be the associated eigenfunctions. Then we have

**Proposition 2.1.** $\sigma_{ess}(H_\theta^{\mathcal{N}}) = [E_1, \infty)$.

*Proof.* We know that $\sigma(H_\theta) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $H_\theta^{\mathcal{N}} \leq H_\theta$, then

$$[E_1, \infty) \subset \sigma_{ess}(H_\theta^{\mathcal{N}}). \tag{8}$$

Let $a' \in \mathbb{R}$ and $l' > 0$ large enough such that $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial\omega$ and $supp(\dot{\theta}) \subset I_{a'}(l')$. Let $\tilde{H}_\theta^{l'}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. So $H_\theta^{\mathcal{N}} \geq \tilde{H}_\theta^{l'}$ and then $\sigma_{ess}(H_\theta^{\mathcal{N}}) \subset \sigma_{ess}(\tilde{H}_\theta^{l'})$ [12].

But $\tilde{H}_\theta^{l'} = \tilde{H}_i \oplus \tilde{H}_e$. The interior operator $\tilde{H}_i$ is the corresponding operator defined on $L^2(I_{a'}(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega \cup \mathcal{N}$ and DBC elsewhere on $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{ess}(\tilde{H}_\theta^{l'}) = \sigma_{ess}(\tilde{H}_e)$.
Now the exterior operator $\tilde{H}_e$ is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial\omega \cup (a' + l', \infty) \times \partial\omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for $x < a'$ and $x > a' + l'$, it is easy to see that

$$\tilde{H}_e = \underset{n \geq 1}{\oplus} (-\partial^2 + E_n)(\chi_n, .)\chi_n.$$

Hence $\sigma(\tilde{H}_e) = \sigma_{ess}(\tilde{H}_e) = [E_1, +\infty)$. □

The Theorem 1.1 *i)* follows from

**Proposition 2.2.** *Under conditions of the Theorem 1.1 i), there exists $l_{min} := l_{min}(\omega, d) > 0$ such as for all $l > l_{min}$ we have*

$$\sigma_d(H^l_\theta) \neq \emptyset. \tag{9}$$

*Proof.* Let $\varphi_{l,a}$ be the following function

$$\varphi_{l,a}(s) := \begin{cases} \frac{10}{l}(s-a), & \text{on } [a, a+\frac{l}{10});\\ 1, & \text{on } [a+\frac{l}{10}, a+\frac{9l}{10});\\ -\frac{10}{l}(s-l-a), & \text{on } [a+\frac{9l}{10}, a+l);\\ 0, & \text{elsewhere.}\end{cases}$$

It is easy to see that $\varphi_{l,a} \in D(q^l_\theta)$ and $\| \varphi_{l,a} \|^2 = \frac{13l}{15} \mid \omega \mid$. Let us calculate

$$q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 = \| \nabla' \varphi_{l,a} \|^2 + \| \dot{\theta}\partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a} \|^2 - E_1 \| \varphi_{l,a} \|^2 . \tag{10}$$

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$\| \dot{\theta}\partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a} \|^2 = \| \partial_s \varphi_{l,a} \|^2 = \frac{20}{l} \mid \omega \mid .$$

Then

$$q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 = \mid \omega \mid (\frac{20}{l} - \frac{13l}{15} E_1) \tag{11}$$

and thus if $l \geq l_{min} := \sqrt{\frac{300}{13E_1}}$ we have $q^l_\theta(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 \leq 0$ $\square$

### 2.1 Proof of the Theorem 1.1 $i)$

Using the same notation as in the Theorem 1.1 $i)$, then $H^{\mathcal{N}}_\theta \leq H^l_\theta$. Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

## 3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by $\theta_m = \inf(\text{supp}(\dot{\theta}))$, $\theta_M = \sup(\text{supp}(\dot{\theta}))$ and $L = \theta_M - \theta_m$. Here $L > 0$. We first consider the case where the Neumann window is an annulus, $\mathcal{A}_a(l) = I_a(l) \times \omega$.

**Proposition 3.1.** *Suppose $\mathcal{A}_a(l)$ is such that $a \geq \theta_M$. Assume that conditions of the Theorem 1.1 ii) hold. Then there exists $d_{max} := d_{max}(\omega, \theta) > 0$, such that for all $0 < d \leq d_{max}$ there exists $l_{max}(d, \theta, \omega) > 0$ such as for all $0 < l \leq l_{max}$ we have*

$$\sigma_d(H^l_\theta) = \emptyset. \tag{12}$$

**Remark 3.2.** the case where $l + a \leq \theta_m$ follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every $\psi \in D(q^l_\theta)$ it holds,

$$Q(\psi) := q^l_\theta(\psi) - E_1 \| \psi \|^2 \geq 0. \tag{13}$$

The proof of (13) involves several steps.

## 3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let $g$ be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_a(l); \\ E_1, & \text{elsewhere.} \end{cases} \tag{14}$$

Choose $p \in (\theta_m, \theta_M)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$\rho(s) := \begin{cases} \frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases} \tag{15}$$

**Proposition 3.3.** *Under same conditions of the Proposition 3.1, then there exists a constant $C > 0$ depending on $p$ and $\omega$ and $\dot{\theta}$ such that for any $\psi \in D(q_\theta^l)$,*

$$\| \nabla' \psi \|^2 + \| \dot{\theta} \partial_\tau \psi + \partial_s \psi \|^2 - \int_{\Omega_0} g(s) \mid \psi \mid^2 dsdt \geq C \int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt. \tag{16}$$

We first show the following lemma. Denote by $\Omega_p := (-\infty, p) \times \omega$.

**Lemma 3.4.** *Under same conditions of the Proposition 3.3. Then for any $\psi \in D(q_\theta^l)$ we have*

$$\int_{\Omega_p} \mid \nabla' \psi \mid^2 + \mid \dot{\theta} \partial_\tau \psi + \partial_s \psi \mid^2 - E_1 \mid \psi \mid^2 dsdt \geq C \int_{\Omega_p} \rho(s) \mid \psi \mid^2 dsdt. \tag{17}$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by $\chi_A$ the characteristic function of $A \times \omega$. Let $\psi \in D(q_\theta^l)$ and define,

$$\begin{aligned} q_1^A(\psi) &:= \| \chi_A \nabla' \psi \|^2 - E_1 \| \chi_A \psi \|^2, & q_2^A(\psi) &:= \| \chi_A \partial_s \psi \|^2, \\ q_3^A(\psi) &:= \| \chi_A \dot{\theta} \partial_\tau \psi \|^2, & q_{2,3}^A(\psi) &:= 2\Re(\partial_s \psi, \chi_A \dot{\theta} \partial_\tau \psi). \end{aligned} \tag{18}$$

Denote also by $Q^A(\psi) = q_1^A(\psi) + q_2^A(\psi) + q_3^A(\psi) + q_{2,3}^A(\psi)$. Here and hereafter we often use the fact that for any $\psi \in D(q_\theta^l)$

$$q_1^A(\psi) \geq 0, \tag{19}$$

for every $A \subset \mathbb{R}$ such that $A \cap I_a(l) = \emptyset$.

*Proof.* Choose $r > 0$ such that $\dot{\theta}(s) \neq 0$ for any $s \in [p-r, p]$. Let $f$ be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty); \\ \frac{p-s}{r}, & \text{on } (p-r, p]; \\ 1, & \text{elsewhere.} \end{cases} \tag{20}$$

For any $\psi \in D(q_\theta^l)$, simple estimates lead to:

$$\begin{aligned} \int_{\Omega_p} \frac{\mid \psi(s,t) \mid^2}{1+(s-p)^2} dsdt &= \int_{\Omega_p} \frac{\mid \psi(s,t) f(s) + (1-f(s))\psi(s,t) \mid^2}{1+(s-p)^2} dsdt \\ &\leq 2\Big( \int_{\Omega_p} \frac{\mid f(s)\psi(s,t) \mid^2}{(s-p)^2} dsdt + \|\chi_{(p-r,p)}\psi\|^2 \Big). \end{aligned} \tag{21}$$

Since $f(p)\psi(p,.)=0$, we can use the usual Hardy inequality (see e.g. [9]), then we get,

$$\int_{\Omega_p}\frac{|\psi(s,t)|^2}{1+(s-p)^2}dsdt \leq 8q_2^{(-\infty,p)}(f\psi)+2\|\chi_{(p-r,p)}\psi\|^2. \tag{22}$$

Note that with our choice $[p-r,p]\cap[a,a+l]=\emptyset$. Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists $\lambda_0=\lambda_0(\dot{\theta},p,r)>0$ s.t. for any $\psi\in D(q_\theta^l)$ we have

$$\|\chi_{(p-r,p)}\psi\|^2 \leq \frac{1}{\lambda_0}Q^{(p-r,p)}(\psi)\leq \frac{1}{\lambda_0}Q^{(-\infty,p)}(\psi). \tag{23}$$

We now want to estimate the first term on the right hand side of (22). We have

$$q_2^{(-\infty,p)}(f\psi)=\int_{\Omega_p}|\partial_s(f\psi)|^2\,dsdt = q_2^{(-\infty,\theta_m)}(f\psi)+q_2^{(\theta_m,p)}(f\psi). \tag{24}$$

Evidently since $\dot{\theta}=0$ and $f=1$ in $(-\infty,\theta_m)$, from (19), we have

$$q_2^{(-\infty,\theta_m)}(f\psi)\leq Q^{(-\infty,\theta_m)}(\psi). \tag{25}$$

In the other hand since $f(p)\psi(p,.)=0$, we can apply the Lemma 4.1 of the Appendix. So for any $0<\alpha<1$ there exists $\gamma_{\alpha,1}>0$ such that

$$|\,q_{2,3}^{(\theta_m,p)}(f\psi)\,|\leq \gamma_{\alpha,1}q_1^{(\theta_m,p)}(f\psi)+\alpha q_2^{(\theta_m,p)}(f\psi)+q_3^{(\theta_m,p)}(f\psi). \tag{26}$$

Let $\gamma:=\max(1,\gamma_{\alpha,1})$. Then

$$\gamma^{-1}\,|\,q_{2,3}^{(\theta_m,p)}(f\psi)\,|\leq q_1^{(\theta_m,p)}(f\psi)+\alpha\gamma^{-1}q_2^{(\theta_m,p)}(f\psi)+\gamma^{-1}q_3^{(\theta_m,p)}(f\psi). \tag{27}$$

Hence with the decomposition, $q_{2,3}^{(\theta_m,p)}=\gamma^{-1}q_{2,3}^{(\theta_m,p)}+(1-\gamma^{-1})q_{2,3}^{(\theta_m,p)}$ and (27) we have,

$$\begin{aligned}Q^{(\theta_m,p)}(f\psi) \geq{}& (1-\gamma^{-1})\Big(q_2^{(\theta_m,p)}(f\psi)+q_{2,3}^{(\theta_m,p)}(f\psi)+q_3^{(\theta_m,p)}(f\psi)\Big) \\ &+\gamma^{-1}(1-\alpha)q_2^{(\theta_m,p)}(f\psi)\end{aligned} \tag{28}$$

and since $q_3^{(\theta_m,p)}+q_{2,3}^{(\theta_m,p)}+q_2^{(\theta_m,p)}\geq 0$, we arrive at,

$$q_2^{(\theta_m,p)}(f\psi)\leq \frac{\gamma}{(1-\alpha)}Q^{(\theta_m,p)}(f\psi). \tag{29}$$

Now by using that, $q_1^{(\theta_m,p)}(f\psi)\leq q_1^{(\theta_m,p)}(\psi)$,

$$\|\chi_{(\theta_m,p)}(\partial_s+\dot{\theta}\partial_\tau)(f\psi)\|^2\leq 2(\|\chi_{(\theta_m,p)}(\partial_s+\dot{\theta}\partial_\tau)\psi\|^2+\frac{1}{r^2}\|\chi_{(p-r,p)}\psi\|^2)$$

and (23), we get,

$$q_2^{(\theta_m,p)}(f\psi)\leq \frac{2\gamma}{(1-\alpha)}(Q^{(\theta_m,p)}(\psi)+\frac{1}{\lambda_0 r^2}Q^{(p-r,p)}(\psi))\leq c'Q^{(\theta_m,p)}(\psi) \tag{30}$$

with $c' = \frac{2\gamma}{(1-\alpha)}(1+\frac{1}{\lambda_0 r^2})$. Then (25) and (30) imply

$$q_2^{(-\infty,p)}(f\psi) \leq (1+c')Q^{(-\infty,p)}(\psi). \tag{31}$$

Hence (31) and (23) prove the lemma with

$$C^{-1} = 8\big(1+c'\big) + \frac{2}{\lambda_0}. \tag{32}$$

$\square$

*Proof of the proposition 3.3.* To prove the proposition we note that for any $\psi \in D(q_\theta^l)$ and for $p' \in \mathbb{R}$ we have

$$\int_\omega \int_{p'}^\infty \mid \nabla' \psi \mid^2 + \mid \dot{\theta}\partial_\tau \psi + \partial_s \psi \mid^2 dsdt \geq \int_\omega \int_{p'}^\infty g(s) \mid \psi \mid^2 dsdt. \tag{33}$$

Then (33) with $p' = p$ and Lemma 3.4 imply (16). $\square$

### 3.2 Reduction to a one dimensional problem

We now want to prove the following result,

**Proposition 3.5.** *Under conditions of the Proposition 3.1, then a sufficient condition in order to get* (13) *is given by*

$$\int_{\mathbb{R}} \mid \psi'(s) \mid^2 + 2C\rho(s) \mid \psi(s) \mid^2 ds - 4E_1 \int_a^{a+l} \mid \psi(s) \mid^2 ds \geq 0, \tag{34}$$

*for any $\psi \in \mathcal{H}^1(\mathbb{R})$ where the constant $C$ is defined in* (32).

**Remark 3.6.** This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on $\mathrm{L}^2(\mathbb{R})$,

$$-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 \mathbf{1}_{I_a(l)}. \tag{35}$$

where $\mathbf{1}_{I_a(l)}$ is the characteristic function of $I_a(l)$.

*Proof.* Evidently we have

$$Q(\psi) = \frac{1}{2}\big(Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) \mid \psi \mid^2 dsdt + q_\theta^l(\psi) - \int_{\Omega_0} g(s) \mid \psi \mid^2 dsdt\big), \tag{36}$$

where $g$ is defined in (14). By using (16), then

$$Q(\psi) \geq \frac{1}{2}\Big(q_\theta^l(\psi) - E_1 \parallel \psi \parallel^2 + C\int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt - E_1 \parallel \chi_{(a,a+l)}\psi \parallel^2 \Big) \tag{37}$$

Rewrite the expression of $q_\theta^l$ given by (3) as follows:

$$q_\theta^l(\psi) = \parallel \nabla'\psi \parallel^2 + \parallel \partial_s \psi \parallel^2 + \parallel \dot{\theta}\partial_\tau \psi \parallel^2 + 2\Re(\partial_s\psi, \dot{\theta}\partial_\tau\psi). \tag{38}$$

We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix,

$$| q_{2,3}(\psi) |=| q_{2,3}^{(\theta_m,\theta_M)}(\psi) |\leq \gamma_{\frac{1}{2},\frac{1}{2}} q_1^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_2^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_3^{(\theta_m,\theta_M)}(\psi) \tag{39}$$

where

$$\gamma_{\frac{1}{2},\frac{1}{2}} := \widetilde{\gamma}_{\frac{1}{2},\frac{1}{2}} + 4d^2 \parallel \dot{\theta} \parallel_\infty^2 \tag{40}$$

with $\widetilde{\gamma}_{\frac{1}{2},\frac{1}{2}} := max\Big\{\frac{d\|\dot{\theta}\|_\infty \|\ddot{\theta}\|_\infty \sqrt{f(L)}}{\dot{\theta_0}\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|_\infty^2 f(L)}{\lambda \dot{\theta_0}^2}, 2d^2 \parallel \ddot{\theta} \parallel_\infty^2 f(L)\Big\}$ for some constant $\lambda > 0$ depending only on the section $\omega$ and $f(L) := \max\{2+\frac{16L^2}{r^2}, 4L^2\}$.

Hence (38) together with (39) give:

$$q_\theta^l(\psi) \geq \parallel \nabla'\psi \parallel^2 + \frac{1}{2} \parallel \partial_s\psi \parallel^2 + \frac{1}{2} \parallel \dot{\theta}\partial_\tau\psi \parallel^2 - \gamma_{\frac{1}{2},\frac{1}{2}} q_1^{(\theta_m,\theta_M)}(\psi). \tag{41}$$

In view of (19) we have

$$\parallel \nabla'\psi \parallel^2 - E_1 \parallel \psi\|^2 \geq q_1^{(\theta_m,\theta_M)}(\psi) + q_1^{I_a(l)}(\psi) \geq q_1^{(\theta_m,\theta_M)}(\psi) - E_1\|\chi_{(a,a+l)}\psi\|^2.$$

Thus this last inequality together with (41) in (37) give

$$\begin{aligned} Q(\psi) \quad &\geq \quad \frac{1}{2}\Big(\frac{1}{2} \parallel \partial_s\psi \parallel^2 + \frac{1}{2} \parallel \dot{\theta}\partial_\tau\psi \parallel^2 + C\int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt - 2E_1 \parallel \chi_{(a,l+a)}\psi \parallel^2 \\ &+ \quad (1-\gamma_{\frac{1}{2},\frac{1}{2}}) q_1^{(\theta_m\theta_M)}(\psi)\Big). \end{aligned}$$

Now if $0 < d \leq d_{max}$ then $\gamma_{\frac{1}{2},\frac{1}{2}} \leq 1$ so the Proposition 3.5 follows.

## 3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

**Proposition 3.7.** *for all $\varphi \in \mathcal{H}^1(\mathbb{R})$, then there exists $l_{max} > 0$ such that for any $0 < l \leq l_{max}$ we have*

$$\int_{\mathbb{R}} \mid \varphi'(s) \mid^2 + 2C\rho(s) \mid \varphi(s) \mid^2 ds \geq 4E_1 \int_{I_a(l)} \mid \varphi(s) \mid^2 ds. \tag{42}$$

*Proof.* Introduce the following function:

$$\Phi(s) := \left\{ \begin{array}{ll} (\frac{\pi}{2} + \arctan{(s-p)}), & if\ s < p; \\ \frac{\pi}{2}, & if\ s \geq p. \end{array} \right. \tag{43}$$

where $p$ is the same real number as in (15). So clearly $\Phi' = \rho$. For any $t \in I_a(l)$ and $\varphi \in \mathcal{H}^1(\mathbb{R})$, we have:

$$\begin{aligned} \frac{\pi}{2}\varphi(t) = \Phi(t)\varphi(t) \quad &= \quad \int_{-\infty}^t (\Phi(s)\varphi(s))' ds \\ &= \quad \int_{-\infty}^t \rho(s)\varphi(s)ds + \int_{-\infty}^t \Phi(s)\varphi'(s)ds \end{aligned} \tag{44}$$

and since $\rho(s) = 0$ for any $s \in (p, \infty)$, we get,

$$\frac{\pi}{2}\varphi(t) = \int_{-\infty}^{p} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds. \tag{45}$$

Then some straightforward estimates lead to,

$$\begin{aligned} \frac{\pi^2}{4}\varphi^2(t) &\leq 2\Big( (\int_{-\infty}^{p} \rho(s)\varphi(s)ds)^2 + (\int_{-\infty}^{t} \Phi(s)\varphi'(s)ds)^2 \Big) \\ &\leq 2\Big( \int_{-\infty}^{p} \rho(s)ds \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \int_{-\infty}^{t} \Phi^2(s)ds \int_{-\infty}^{t} \varphi'^2(s)ds \Big). \end{aligned} \tag{46}$$

By direct calculation $\int_{-\infty}^{p} \rho(s)ds = \frac{\pi}{2}$ and $\int_{-\infty}^{p} \Phi^2(s)ds + \int_{p}^{t} \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t-p)$. Hence we get,

$$\mid \varphi(t) \mid^2 \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \Big( \frac{8 \ln 2}{\pi} + 2(t-p) \Big) \int_{\mathbb{R}} \mid \varphi'(s) \mid^2 ds \tag{47}$$

We integrate both sides of (47) over $I_a(l)$, then

$$\begin{aligned} \int_{I_a(l)} \mid \varphi(t) \mid^2 dt &\leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \Big( (\frac{8 \ln 2}{\pi} + 2(a-p))l + l^2 \Big) \int_{\mathbb{R}} \mid \varphi'(s) \mid^2 ds \\ &\leq c'' \int_{\mathbb{R}} 2C\rho(s)\varphi^2(s) + \mid \varphi'(s) \mid^2 ds \end{aligned}$$

where $c'' = 2l(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p) + l^2$. Finally we get,

$$4E_1 \int_{a}^{l+a} \mid \varphi(t) \mid^2 dt \leq 4E_1 c'' \int_{\mathbb{R}} 2C\rho(s) \mid \varphi(s) \mid^2 + \mid \varphi'(s) \mid^2 ds. \tag{48}$$

So choose $0 < l \leq l_{max}$ with

$$l_{max} := -(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p) + \sqrt{(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a - p)^2 + (4E_1)^{-1}}$$

then $4E_1 c'' \leq 1$ and the proposition 3.7 follows. $\square$

### 3.4 proof of the Theorem 1.1 *ii*)

Under assumptions of the Theorem 1.1 *ii*) $H_\theta^{\mathcal{N}} \geq H_\theta^l$. These two operators have the same essential spectrum so the Theorem 1.1 *ii*) is proved by applying the Proposition 3.1 and the min-max principle.

## 4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D(q_\theta^l)$ we have for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:

$$\mid q_{2,3}(\psi) \mid \leq \gamma_{\alpha,\beta} q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi). \tag{49}$$

Then we have

**Lemma 4.1.** *Let $p \in (\theta_m, \theta_M)$. For all $\psi \in D(q^l_\theta)$ such that $\psi(p,.) = 0$, then for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:*

$$| q^{(\theta_m,p)}_{2,3}(\psi) | \leq \gamma_{\alpha,\beta} q^{(\theta_m,p)}_1(\psi) + \alpha q^{(\theta_m,p)}_2(\psi) + \beta q^{(\theta_m,p)}_3(\psi). \tag{50}$$

*Proof.* Let $\psi \in D(q^l_\theta)$ such that $\psi(p,.) = 0$. Then $\psi \in \mathcal{H}^1_0(\Omega_p)$. We know that we may first consider vectors $\psi(s,t) = \chi_1(t)\phi(s,t)$, where $\phi \in C^\infty_0(\Omega_p)$. For such a vector $\psi$ we have,

$$\begin{aligned} q^{(\theta_m,p)}_1(\psi) &= \| \chi_{(\theta_m,p)}\chi_1\nabla'\phi \|^2, \quad q^{(\theta_m,p)}_2(\psi) = \| \chi_{(\theta_m,p)}\chi_1\partial_s\phi \|^2 \\ q^{(\theta_m,p)}_3(\psi) &= \| \chi_{(\theta_m,p)}\dot{\theta}(\chi_1\partial_\tau\phi + \phi\partial_\tau\chi_1) \|^2 \end{aligned} \tag{51}$$

and

$$q^{(\theta_m,p)}_{2,3}(\psi) = 2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi, \chi_1\partial_s\phi) + 2(\dot{\theta}\chi_{(\theta_m,p)}\phi\partial_\tau\chi_1, \chi_1\partial_s\phi) \tag{52}$$

By using simple estimates the first term on the r.h.s of (52) is estimated as :

$$| 2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi, \chi_1\partial_s\phi) | \leq 2 \| \dot{\theta} \|_\infty \| \chi_{(\theta_m,p)}\chi_1\nabla'\phi \| \| \chi_{(\theta_m,p)}\chi_1\partial_s\phi \|$$

then

$$| 2(\dot{\theta}\chi_{(\theta_m,p)}\chi_1\partial_\tau\phi, \chi_1\partial_s\phi) | \leq c_1 q^{(\theta_m,p)}_1(\psi) + \frac{\alpha}{2} q^{(\theta_m,p)}_2(\psi), \tag{53}$$

where $c_1 := \frac{2}{\alpha} d^2 \| \dot{\theta} \|^2_\infty$ and $\alpha > 0$.
Integrating by parts twice and using the fact that $\dot{\theta}(\theta_m) = \phi(p,.) = 0$, the second term of the r.h.s of (52) is written as

$$2(\dot{\theta}\chi_{(\theta_m,p)}\phi\partial_\tau\chi_1, \chi_1\partial_s\phi) = (\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1, \chi_1\partial_\tau\phi). \tag{54}$$

Then the Cauchy Schwartz inequality implies,

$$| (\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1, \chi_1\partial_\tau\phi) |^2 \leq d^2 \| \ddot{\theta} \|^2_\infty \, q^{(\theta_m,p)}_1 \, \| \chi_{(\theta_m,p)}\chi_1\phi \|^2 . \tag{55}$$

Let $p' \in \mathbb{R}$ and $r' > 0$ such that $(p' - r, p') \subset (\theta_m, p)$ and for $s \in (p' - r, p')$, $| \dot{\theta}(s) | \geq \dot{\theta}_0$ for some $\dot{\theta}_0 > 0$. As in the proof of the Lemma 3 of [6] we have,

$$\| \chi_{(\theta_m,p)}\chi_1\phi \|^2 \leq c_2 \Big( q^{(\theta_m,p)}_2(\psi) + \dot{\theta}^{-2}_0 \| \chi_{(p'-r,p')}\dot{\theta}\chi_1\phi \|^2 \Big) \tag{56}$$

where $c_2 := \max\Big\{ 2 + 16\frac{(p-\theta_m)^2}{r^2}, 4(p-\theta_m)^2 \Big\}$.
Moreover, for any $s \in \mathbb{R}$, $\dot{\theta}(s)\chi_1\phi(s,.) \in \mathcal{H}^1_0(\Omega_p)$, then by using the Lemma 1 of [6] there exists $\lambda > 0$ depending on $\omega$ such that :

$$\| \chi_{(p'-r,p')}\dot{\theta}\chi_1\phi \|^2 \leq \| \chi_{(\theta_m,p)}\dot{\theta}\chi_1\phi \|^2 \leq \lambda^{-1} \Big( q^{(\theta_m,p)}_3(\psi) + \| \dot{\theta} \|^2_\infty \, q^{(\theta_m,p)}_1(\psi) \Big). \tag{57}$$

Hence (56), (57) and (54) give

$$| (\chi_{(\theta_m,p)}\ddot{\theta}\phi\chi_1, \chi_1\partial_\tau\phi) |^2 \leq \Big( c_3 q^{(\theta_m,p)}_1(\psi) + \frac{\alpha}{2} q^{(\theta_m,p)}_2(\psi) + \beta q^{(\theta_m,p)}_3(\psi) \Big)^2 \tag{58}$$

where $c_3 := \max\Big\{ \frac{d\|\ddot{\theta}\|\|\dot{\theta}\|_\infty\sqrt{c_2}}{\dot{\theta}_0\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|^2_\infty c_2}{\alpha}, \frac{d^2\|\ddot{\theta}\|^2_\infty c_2}{2\beta\dot{\theta}^2_0\lambda} \Big\}$. Then (53) and (58) imply (50) with $\gamma_{\alpha,\beta} := c_1 + c_3$.

Note that we can choose $\chi_1 > 0$ on $\omega$. So that (50) holds for every $\psi \in C^\infty_0(\Omega_p)$ and by a density argument this is even true for $\psi \in \mathcal{H}^1_0(\Omega_p)$.

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