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► **To cite this version:**

| Philippe Briet, Hiba Hammedi. Twisted waveguide with a Neumann window. 2016. hal-01264262v1

**HAL Id: hal-01264262**

**<https://hal.science/hal-01264262v1>**

Preprint submitted on 29 Jan 2016 (v1), last revised 24 Apr 2017 (v3)

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# Twisted waveguide with a Neumann window

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January 2016

## Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of  $\mathbb{R}^3$  which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

Primary 81Q10; Secondary 47F05.

Keywords: Waveguide, mixed boundary conditions, twisting.

## 1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain  $\Omega_0 := \mathbb{R} \times \omega$  having a Neumann window on its boundary  $\partial\Omega_0$ .

The cross section  $\omega$  is supposed to be an open bounded connected subset of  $\mathbb{R}^2$  of diameter  $d > 0$  which is not rotationally invariant. Moreover  $\omega$  is supposed to have smooth boundary  $\partial\omega$ .

We would expect that the Laplace operator associated to such a straight tube to have bound states [8].

Let us introduce some notations. Denote by  $\mathcal{N}$  the Neumann window. It is an open bounded subset of the boundary  $\partial\Omega_0$ . Let  $\mathcal{D}$  be its complement set in  $\partial\Omega_0$ . When  $\mathcal{N}$  is an annulus of size  $l > 0$  we will denote it by,

$$\mathcal{A}_a(l) := I_a(l) \times \partial\omega, I_a(l) := (a, l + a), a \in \mathbb{R}.$$

Consider first the self-adjoint operator  $H_0^{\mathcal{N}}$  associated to the following quadratic form. Let  $D(Q^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_0) \mid \psi|_{\mathcal{D}} = 0\}$  and for  $\psi \in D(Q^{\mathcal{N}})$ ,

$$Q^{\mathcal{N}}(\psi) = \int_{\Omega_0} |\nabla\psi|^2 dx$$

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i.e. the Laplace operator defined on  $\Omega_0$  with Neumann boundary conditions (NBC) on  $\mathcal{N}$  and Dirichlet boundary conditions (DBC) on  $\mathcal{D}$  [5, 11].

It is actually shown in the Section 2 of this paper that if  $\mathcal{N}$  contains an annulus of size  $l$  large enough then  $H_0^{\mathcal{N}}$  has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if  $\mathcal{N}$  contains an annulus of any size  $l > 0$ .

The question we are interested in is the following: is it possible that the discrete spectrum of  $H_0^{\mathcal{N}}$  disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function  $\theta$  having a derivative  $\dot{\theta}$  with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].

Let us now define the twisting [4, 7]. Choose  $\theta \in C_c^1(\mathbb{R})$  and introduce the diffeomorphism

$$\begin{aligned} \mathcal{L} : \Omega_0 &\longrightarrow \mathbb{R}^3 \\ (s, t_2, t_3) &\longmapsto \left( s, t_2 \cos \theta(s) - t_3 \sin \theta(s), t_2 \sin \theta(s) + t_3 \cos \theta(s) \right). \end{aligned} \quad (1)$$

The twisted tube is given by  $\Omega_\theta := \mathcal{L}(\Omega_0)$ . Let  $D(Q_\theta^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_\theta) \mid \psi|_{\mathcal{L}(\mathcal{D})} = 0\}$  and consider the following quadratic form

$$Q_\theta^{\mathcal{N}}(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \quad \psi \in D(Q_\theta^{\mathcal{N}}). \quad (2)$$

Through unitary equivalence, we then have to consider

$$q_\theta^{\mathcal{N}}(\psi) := Q_\theta^{\mathcal{N}}(\psi \circ \mathcal{L}^{-1}) = \|\nabla' \psi\|^2 + \|\partial_s \psi + \dot{\theta} \partial_\tau \psi\|^2, \quad (3)$$

$\psi \in D(q_\theta^{\mathcal{N}}) := \{\psi \in \mathcal{H}^1(\Omega_0) \mid \psi|_{\mathcal{D}} = 0\}$  and where

$$\nabla' := {}^t(\partial_{t_2}, \partial_{t_3}), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}. \quad (4)$$

Denote by  $H_\theta^{\mathcal{N}}$  the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let  $D(H_\theta^{\mathcal{N}}) = \{\psi \in D(q_\theta^{\mathcal{N}}), \quad H_\theta^{\mathcal{N}} \psi \in L^2(\Omega_0) \mid \frac{\partial \psi}{\partial n}|_{\mathcal{N}} = 0\}$  with

$$H_\theta^{\mathcal{N}} \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_s)^2) \psi, \quad (5)$$

where the transverse Laplacian  $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$ . If  $\mathcal{N} = \mathcal{A}_a(l), l > 0$ , we will denote these forms respectively as  $Q_\theta^l, q_\theta^l$  and the corresponding operator as  $H_\theta^l$  and if  $\mathcal{N} = \emptyset$  we denote the associated operator by  $H_\theta$ .

Then the main result of this paper is

**Theorem 1.1.** *i) Under conditions stated above on  $\omega$  and  $\theta$ , there exists  $l_{min} := l_{min}(\omega, d) > 0$  such as if for some  $a \in \mathbb{R}$  and  $l > l_{min}$ ,  $\mathcal{N} \supset \mathcal{A}_a(l)$  then*

$$\sigma_d(H_\theta^{\mathcal{N}}) \neq \emptyset. \quad (6)$$

*ii) Suppose in addition that  $\theta$  has a bounded second derivative. Then there exists  $d_{max} := d_{max}(\theta, \omega) > 0$  such that for all  $0 < d \leq d_{max}$  there exists  $l_{max} := l_{max}(\omega, d, \theta)$  such as for all  $0 < l \leq l_{max}$ , if  $\mathcal{N} \subset \mathcal{A}_a(l)$  and  $\text{supp}(\dot{\theta}) \cap I_a(l) = \emptyset$  for some  $a \in \mathbb{R}$  then*

$$\sigma_d(H_\theta^{\mathcal{N}}) = \emptyset. \quad (7)$$

Roughly speaking this result implies that for  $d$  small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

## 2 Existence of bound states

First we prove the following. Denote by  $E_1, E_2, \dots$  the eigenvalues (transverse modes) of the Laplacian  $-\Delta_\omega$  defined on  $L^2(\omega)$  with DBC on  $\partial\omega$ . Let  $\chi_1, \chi_2, \dots$  be the associated eigenfunctions. Then we have

**Proposition 2.1.**  $\sigma_{ess}(H_\theta^N) = [E_1, \infty)$ .

*Proof.* We know that  $\sigma(H_\theta) = [E_1, \infty)$  see e.g. [2]. But by usual arguments [12],  $H_\theta^N \leq H_\theta$ , then

$$[E_1, \infty) \subset \sigma_{ess}(H_\theta^N). \quad (8)$$

Let  $a' \in \mathbb{R}$  and  $l' > 0$  large enough such that  $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial\omega$  and  $\text{supp}(\theta) \subset I_{a'}(l')$ . Let  $\tilde{H}_\theta^{l'}$  be the operator defined as in (5) but with additional Neumann boundary conditions on  $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$ . So  $H_\theta^N \geq \tilde{H}_\theta^{l'}$  and then  $\sigma_{ess}(H_\theta^N) \subset \sigma_{ess}(\tilde{H}_\theta^{l'})$  [12].

But  $\tilde{H}_\theta^{l'} = \tilde{H}_i \oplus \tilde{H}_e$ . The interior operator  $\tilde{H}_i$  is the corresponding operator defined on  $L^2(I_{a'}(l') \times \omega)$  with NBC on  $\{a'\} \times \omega \cup \{a' + l'\} \times \omega \cup (a, a + l) \times \partial\omega$  and DBC elsewhere on the boundary of  $\mathcal{A}_{a'}(l')$ . By general arguments of [12] it has only discrete spectrum consequently  $\sigma_{ess}(\tilde{H}_\theta^{l'}) = \sigma_{ess}(\tilde{H}_e)$ .

Now the exterior operator  $\tilde{H}_e$  is defined on  $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$  with DBC on  $(-\infty, a') \times \partial\omega \cup (a' + l', \infty) \times \partial\omega$  and NBC on  $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$ . Since  $\theta = 0$  for  $x < a'$  and  $x > a' + l'$ , it is easy to see that

$$\tilde{H}_e = \bigoplus_{n \geq 1} (-\partial^2 + E_n)(\chi_n, \cdot)\chi_n.$$

Hence  $\sigma(\tilde{H}_e) = \sigma_{ess}(\tilde{H}_e) = [E_1, +\infty)$ . □

The Theorem 1.1 i) follows from

**Proposition 2.2.** *Under conditions on  $\omega$  and  $\theta$  given above, there exists  $l_{min} := l_{min}(\omega, d) > 0$  such as for all  $l > l_{min}$  we have*

$$\sigma_d(H_\theta^l) \neq \emptyset. \quad (9)$$

*Proof.* Let  $\varphi_{l,a}$  be the following function

$$\varphi_{l,a}(s) := \begin{cases} \frac{10}{l}(s - a), & \text{on } [a, a + \frac{l}{10}); \\ 1, & \text{on } [a + \frac{l}{10}, a + \frac{9l}{10}); \\ -\frac{10}{l}(s - l - a), & \text{on } [a + \frac{9l}{10}, a + l); \\ 0, & \text{elsewhere.} \end{cases}$$

It is easy to see that  $\varphi_{l,a} \in D(q_\theta^l)$  and  $\|\varphi_{l,a}\|^2 = \frac{13l}{15} |\omega|$ . Let us calculate

$$q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 = \|\nabla' \varphi_{l,a}\|^2 + \|\dot{\theta} \partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a}\|^2 - E_1 \|\varphi_{l,a}\|^2. \quad (10)$$

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$\|\dot{\theta} \partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a}\|^2 = \|\partial_s \varphi_{l,a}\|^2 = \frac{20}{l} |\omega|.$$

Then

$$q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 = |\omega| \left( \frac{20}{l} - \frac{13l}{15} E_1 \right) \quad (11)$$

and thus if  $l \geq l_{min} := \sqrt{\frac{300}{13E_1}}$  we have  $q_\theta^l(\varphi_{l,a}) - E_1 \|\varphi_{l,a}\|^2 \leq 0$   $\square$

## 2.1 Proof of the Theorem 1.1 i)

Using the same notation as in the Theorem 1.1 i), then  $H_\theta^{\mathcal{N}} \leq H_\theta^l$ . Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

## 3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by  $\theta_m = \inf(\text{supp}(\dot{\theta}))$ ,  $\theta_M = \sup(\text{supp}(\dot{\theta}))$  and  $L = \theta_M - \theta_m$ . We first consider the case where the Neumann window is an annulus,  $\mathcal{A}_a(l) = I_a(l) \times \omega$ .

**Proposition 3.1.** *Suppose  $\mathcal{A}_a(l)$  is such that  $a \geq \theta_M$ . Assume also that  $\theta$  has a bounded second derivative. Then there exists  $d_{max} := d_{max}(\omega, \theta) > 0$ , such that for all  $0 < d \leq d_{max}$  there exists  $l_{max}(d, \theta, \omega) > 0$  such as for all  $0 < l \leq l_{max}$  we have*

$$\sigma_d(H_\theta^l) = \emptyset. \quad (12)$$

**Remark 3.2.** the case where  $l+a \leq \theta_m$  follows from same arguments developed below for the proof of this proposition.

This proof is based on the fact that under conditions of the Proposition 3.1, for every  $\psi \in D(q_\theta^l)$  it holds,

$$Q(\psi) := q_\theta^l(\psi) - E_1 \|\psi\|^2 \geq 0. \quad (13)$$

The proof of (13) involves several steps.

### 3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let  $g$  be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_a(l); \\ E_1, & \text{elsewhere.} \end{cases} \quad (14)$$

Choose  $p \in (\theta_m, \theta_M)$  s.t.  $\dot{\theta}(p) \neq 0$  and let

$$\rho(s) := \begin{cases} \frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases} \quad (15)$$

**Proposition 3.3.** *Under same conditions of the Proposition 3.1, then there exists a constant  $C > 0$  depending on  $p$  and  $\omega$  and  $\dot{\theta}$  such that for any  $\psi \in D(q_\theta^l)$ ,*

$$\|\nabla' \psi\|^2 + \|\dot{\theta} \partial_\tau \psi + \partial_s \psi\|^2 - \int_{\Omega_0} g(s) |\psi|^2 ds dt \geq C \int_{\Omega_0} \rho(s) |\psi|^2 ds dt. \quad (16)$$

We first show the following lemma. Denote by  $\Omega_p := (-\infty, p) \times \omega$ .

**Lemma 3.4.** *Under same conditions of the Proposition 3.3. Then for any  $\psi \in D(q_\theta^l)$  we have*

$$\int_{\Omega_p} |\nabla' \psi|^2 + |\dot{\theta} \partial_\tau \psi + \partial_s \psi|^2 - E_1 |\psi|^2 ds dt \geq C \int_{\Omega_p} \rho(s) |\psi|^2 ds dt. \quad (17)$$

In the following we will use notations suggested in [6]. For  $A \subset \mathbb{R}$  denote by  $\chi_A$  the characteristic function of  $A \times \omega$ . Let  $\psi \in D(q_\theta^l)$  and define,

$$\begin{aligned} q_1^A(\psi) &:= \|\chi_A \nabla' \psi\|^2 - E_1 \|\chi_A \psi\|^2, & q_2^A(\psi) &:= \|\chi_A \partial_s \psi\|^2, \\ q_3^A(\psi) &:= \|\chi_A \dot{\theta} \partial_\tau \psi\|^2, & q_{2,3}^A(\psi) &:= 2\Re(\partial_s \psi, \chi_A \dot{\theta} \partial_\tau \psi). \end{aligned} \quad (18)$$

Denote also by  $Q^A(\psi) = q_1^A(\psi) + q_2^A(\psi) + q_3^A(\psi) + q_{2,3}^A(\psi)$ . Here and hereafter we often use the fact that for any  $\psi \in D(q_\theta^l)$

$$q_1^A(\psi) \geq 0, \quad (19)$$

for every  $A \subset \mathbb{R}$  such that  $A \cap I_a(l) = \emptyset$ .

*Proof.* Choose  $r > 0$  such that  $\dot{\theta}(s) \neq 0$  for any  $s \in [p-r, p]$ . Let  $f$  be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty); \\ \frac{p-s}{r}, & \text{on } (p-r, p]; \\ 1, & \text{elsewhere.} \end{cases} \quad (20)$$

For any  $\psi \in D(q_\theta^l)$ , simple estimates lead to:

$$\begin{aligned} \int_{\Omega_p} \frac{|\psi(s, t)|^2}{1+(s-p)^2} ds dt &= \int_{\Omega_p} \frac{|\psi(s, t)f(s) + (1-f(s))\psi(s, t)|^2}{1+(s-p)^2} ds dt \\ &\leq 2 \left( \int_{\Omega_p} \frac{|f(s)\psi(s, t)|^2}{(s-p)^2} ds dt + \|\chi_{(p-r, p)} \psi\|^2 \right). \end{aligned} \quad (21)$$

Since  $f(p)\psi(p, \cdot) = 0$ , we can use the usual Hardy inequality (see e.g. [9]), then we get,

$$\int_{\Omega_p} \frac{|\psi(s, t)|^2}{1+(s-p)^2} ds dt \leq 8q_2^{(-\infty, p)}(f\psi) + 2\|\chi_{(p-r, p)} \psi\|^2. \quad (22)$$

Note that with our choice  $[p-r, p] \cap [a, a+l] = \emptyset$ . Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists  $\lambda_0 = \lambda_0(\theta, p, r) > 0$  s.t. for any  $\psi \in D(q_\theta^l)$  we have

$$\|\chi_{(p-r,p)}\psi\|^2 \leq \frac{1}{\lambda_0} Q^{(p-r,p)}(\psi) \leq \frac{1}{\lambda_0} Q^{(-\infty,p)}(\psi). \quad (23)$$

We now want to estimate the first term on the right hand side of (22). We have

$$q_2^{(-\infty,p)}(f\psi) = \int_{\Omega_p} |\partial_s(f\psi)|^2 ds dt = q_2^{(-\infty,\theta_m)}(f\psi) + q_2^{(\theta_m,p)}(f\psi). \quad (24)$$

Evidently since  $\dot{\theta} = 0$  and  $f = 1$  in  $(-\infty, \theta_m)$ , from (19), we have

$$q_2^{(-\infty,\theta_m)}(f\psi) \leq Q^{(-\infty,\theta_m)}(\psi). \quad (25)$$

In the other hand since  $f(p)\psi(p, \cdot) = 0$ , we can apply the Lemma 4.1 of the Appendix. So for any  $0 < \alpha < 1$  there exists  $\gamma_{\alpha,1} > 0$  such that

$$|q_{2,3}^{(\theta_m,p)}(f\psi)| \leq \gamma_{\alpha,1} q_1^{(\theta_m,p)}(f\psi) + \alpha q_2^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi). \quad (26)$$

Let  $\gamma := \max(1, \gamma_{\alpha,1})$ . Then

$$\gamma^{-1} |q_{2,3}^{(\theta_m,p)}(f\psi)| \leq q_1^{(\theta_m,p)}(f\psi) + \alpha \gamma^{-1} q_2^{(\theta_m,p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m,p)}(f\psi). \quad (27)$$

Hence with the decomposition,  $q_{2,3}^{(\theta_m,p)} = \gamma^{-1} q_{2,3}^{(\theta_m,p)} + (1 - \gamma^{-1}) q_{2,3}^{(\theta_m,p)}$  and (27) we have,

$$Q^{(\theta_m,p)}(f\psi) \geq (1 - \gamma^{-1}) \left( q_2^{(\theta_m,p)}(f\psi) + q_{2,3}^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi) \right) + \gamma^{-1} (1 - \alpha) q_2^{(\theta_m,p)}(f\psi) \quad (28)$$

and since  $q_3^{(\theta_m,p)} + q_{2,3}^{(\theta_m,p)} + q_2^{(\theta_m,p)} \geq 0$ , we arrive at,

$$q_2^{(\theta_m,p)}(f\psi) \leq \frac{\gamma}{(1 - \alpha)} Q^{(\theta_m,p)}(f\psi). \quad (29)$$

Now by using that,  $q_1^{(\theta_m,p)}(f\psi) \leq q_1^{(\theta_m,p)}(\psi)$ ,

$$\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)(f\psi)\|^2 \leq 2(\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)\psi\|^2 + \frac{1}{r^2} \|\chi_{(p-r,p)}\psi\|^2)$$

and (23), we get,

$$q_2^{(\theta_m,p)}(f\psi) \leq \frac{2\gamma}{(1 - \alpha)} (Q^{(\theta_m,p)}(\psi) + \frac{1}{\lambda_0 r^2} Q^{(p-r,p)}(\psi)) \leq c' Q^{(\theta_m,p)}(\psi) \quad (30)$$

with  $c' = \frac{2\gamma}{(1 - \alpha)} (1 + \frac{1}{\lambda_0 r^2})$ . Then (25) and (30) imply

$$q_2^{(-\infty,p)}(f\psi) \leq (1 + c') Q^{(-\infty,p)}(\psi). \quad (31)$$

Hence (31) and (23) prove the lemma with

$$C^{-1} = 8(1 + c') + \frac{2}{\lambda_0}. \quad (32)$$

□

*Proof of the proposition 3.3.* To prove the proposition we note that for any  $\psi \in D(q_\theta^l)$  and for  $p' \in \mathbb{R}$  we have

$$\int_\omega \int_{p'}^\infty |\nabla' \psi|^2 + |\dot{\theta} \partial_\tau \psi + \partial_s \psi|^2 ds dt \geq \int_\omega \int_{p'}^\infty g(s) |\psi|^2 ds dt. \quad (33)$$

Then (33) with  $p' = p$  and Lemma 3.4 imply (16). □

### 3.2 Reduction to a one dimensional problem

We now want to prove the following result,

**Proposition 3.5.** *Under conditions of the Proposition 3.1, then a sufficient condition in order to get (13) is given by*

$$\int_{\mathbb{R}} |\psi'(s)|^2 + 2C\rho(s) |\psi(s)|^2 ds - 4E_1 \int_a^{a+l} |\psi(s)|^2 ds \geq 0, \quad (34)$$

for any  $\psi \in \mathcal{H}^1(\mathbb{R})$  where the constant  $C$  is defined in (32).

**Remark 3.6.** This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on  $L^2(\mathbb{R})$ ,

$$-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1 \mathbf{1}_{I_a(l)}. \quad (35)$$

where  $\mathbf{1}_{I_a(l)}$  is the characteristic function of  $I_a(l)$ .

*Proof.* Evidently we have

$$Q(\psi) = \frac{1}{2} \left( Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) |\psi|^2 ds dt + q_\theta^l(\psi) - \int_{\Omega_0} g(s) |\psi|^2 ds dt \right), \quad (36)$$

where  $g$  is defined in (14). By using (16), then

$$Q(\psi) \geq \frac{1}{2} \left( q_\theta^l(\psi) - E_1 \|\psi\|^2 + C \int_{\Omega_0} \rho(s) |\psi|^2 ds dt - E_1 \|\chi_{(a, a+l)} \psi\|^2 \right) \quad (37)$$

Rewrite the expression of  $q_\theta^l$  given by (3) as follows:

$$q_\theta^l(\psi) = \|\nabla' \psi\|^2 + \|\partial_s \psi\|^2 + \|\dot{\theta} \partial_\tau \psi\|^2 + 2\Re(\partial_s \psi, \dot{\theta} \partial_\tau \psi). \quad (38)$$

We estimate the last term of the r.h.s. of (38). By using of the formula (49) of the Appendix,

$$|\langle q_{2,3}(\psi) | \psi \rangle| = |q_{2,3}^{(\theta_m, \theta_M)}(\psi)| \leq \gamma_{\frac{1}{2}, \frac{1}{2}} q_1^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2} q_2^{(\theta_m, \theta_M)}(\psi) + \frac{1}{2} q_3^{(\theta_m, \theta_M)}(\psi) \quad (39)$$



where

$$\gamma_{\frac{1}{2}, \frac{1}{2}} := \tilde{\gamma}_{\frac{1}{2}, \frac{1}{2}} + 4d^2 \|\dot{\theta}\|_\infty^2 \quad (40)$$

with  $\tilde{\gamma}_{\frac{1}{2}, \frac{1}{2}} := \max\left\{\frac{d\|\dot{\theta}\|_\infty\|\ddot{\theta}\|_\infty\sqrt{f(L)}}{\theta_0\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|_\infty^2 f(L)}{\lambda\theta_0^2}, 2d^2\|\ddot{\theta}\|_\infty^2 f(L)\right\}$  for some constant  $\lambda > 0$  depending only on the section  $\omega$  and  $f(L) := \max\{2 + \frac{16L^2}{r^2}, 4L^2\}$ .

Hence (38) together with (39) give:

$$q_\theta^l(\psi) \geq \|\nabla'\psi\|^2 + \frac{1}{2}\|\partial_s\psi\|^2 + \frac{1}{2}\|\dot{\theta}\partial_\tau\psi\|^2 - \gamma_{\frac{1}{2}, \frac{1}{2}}q_1^{(\theta_m, \theta_M)}(\psi). \quad (41)$$

In view of (19) we have

$$\|\nabla'\psi\|^2 - E_1\|\psi\|^2 \geq q_1^{(\theta_m, \theta_M)}(\psi) + q_1^{I_a(l)}(\psi) \geq q_1^{(\theta_m, \theta_M)}(\psi) - E_1\|\chi_{(a, a+l)}\psi\|^2.$$

Thus this last inequality together with (41) in (37) give

$$\begin{aligned} Q(\psi) &\geq \frac{1}{2}\left(\frac{1}{2}\|\partial_s\psi\|^2 + \frac{1}{2}\|\dot{\theta}\partial_\tau\psi\|^2 + C\int_{\Omega_0}\rho(s)|\psi|^2 ds - 2E_1\|\chi_{(a, l+a)}\psi\|^2\right. \\ &\quad \left.+ (1 - \gamma_{\frac{1}{2}, \frac{1}{2}})q_1^{(\theta_m, \theta_M)}(\psi)\right). \end{aligned}$$

Now if  $0 < d \leq d_{max}$  then  $\gamma_{\frac{1}{2}, \frac{1}{2}} \leq 1$  so the Proposition 3.5 follows.

### 3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

**Proposition 3.7.** *for all  $\varphi \in \mathcal{H}^1(\mathbb{R})$ , then there exists  $l_{max} > 0$  such that for any  $0 < l \leq l_{max}$  we have*

$$\int_{\mathbb{R}}|\varphi'(s)|^2 + 2C\rho(s)|\varphi(s)|^2 ds \geq 4E_1\int_{I_a(l)}|\varphi(s)|^2 ds. \quad (42)$$

*Proof.* Introduce the following function:

$$\Phi(s) := \begin{cases} (\frac{\pi}{2} + \arctan(s-p)), & \text{if } s < p; \\ \frac{\pi}{2}, & \text{if } s \geq p. \end{cases} \quad (43)$$

where  $p$  is the same real number as in (15). So clearly  $\Phi' = \rho$ . For any  $t \in I_a(l)$  and  $\varphi \in \mathcal{H}^1(\mathbb{R})$ , we have:

$$\begin{aligned} \frac{\pi}{2}\varphi(t) = \Phi(t)\varphi(t) &= \int_{-\infty}^t(\Phi(s)\varphi(s))' ds \\ &= \int_{-\infty}^t\rho(s)\varphi(s) ds + \int_{-\infty}^t\Phi(s)\varphi'(s) ds \end{aligned} \quad (44)$$

and since  $\rho(s) = 0$  for any  $s \in (p, \infty)$ , we get,

$$\frac{\pi}{2}\varphi(t) = \int_{-\infty}^p \rho(s)\varphi(s)ds + \int_{-\infty}^t \Phi(s)\varphi'(s)ds. \quad (45)$$

Then some straightforward estimates lead to,

$$\begin{aligned} \frac{\pi^2}{4}\varphi^2(t) &\leq 2\left(\left(\int_{-\infty}^p \rho(s)\varphi(s)ds\right)^2 + \left(\int_{-\infty}^t \Phi(s)\varphi'(s)ds\right)^2\right) \\ &\leq 2\left(\int_{-\infty}^p \rho(s)ds \int_{-\infty}^p \rho(s)\varphi^2(s)ds + \int_{-\infty}^t \Phi^2(s)ds \int_{-\infty}^t \varphi'^2(s)ds\right). \end{aligned} \quad (46)$$

By direct calculation  $\int_{-\infty}^p \rho(s)ds = \frac{\pi}{2}$  and  $\int_{-\infty}^p \Phi^2(s)ds + \int_p^t \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t-p)$ . Hence we get,

$$|\varphi(t)|^2 \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left(\frac{8 \ln 2}{\pi} + 2(t-p)\right) \int_{\mathbb{R}} |\varphi'(s)|^2 ds \quad (47)$$

We integrate both sides of (47) over  $I_a(l)$ , then

$$\begin{aligned} \int_{I_a(l)} |\varphi(t)|^2 dt &\leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \left(\frac{8 \ln 2}{\pi} + 2(a-p)l + l^2\right) \int_{\mathbb{R}} |\varphi'(s)|^2 ds \\ &\leq c'' \int_{\mathbb{R}} 2C\rho(s)\varphi^2(s) + |\varphi'(s)|^2 ds \end{aligned}$$

where  $c'' = 2l\left(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a-p\right) + l^2$ . Finally we get,

$$4E_1 \int_a^{l+a} |\varphi(t)|^2 dt \leq 4E_1 c'' \int_{\mathbb{R}} 2C\rho(s) |\varphi(s)|^2 + |\varphi'(s)|^2 ds. \quad (48)$$

So choose  $0 < l \leq l_{max}$  with

$$l_{max} := -\left(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a-p\right) + \sqrt{\left(\frac{1}{\pi C} + \frac{4 \ln 2}{\pi} + a-p\right)^2 + (4E_1)^{-1}}$$

then  $4E_1 c'' \leq 1$  and the proposition 3.7 follows.  $\square$

### 3.4 proof of the Theorem 1.1 ii)

Under assumptions of the Theorem 1.1 ii)  $H_\theta^N \geq H_\theta^l$ . These two operators have the same essential spectrum so the Theorem 1.1 ii) is proved by applying the Proposition 3.1 and the min-max principle.

## 4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all  $\psi \in D(q_\theta^l)$  we have for any  $\alpha, \beta > 0$  there exists  $\gamma_{\alpha, \beta} > 0$  such that:

$$|q_{2,3}(\psi)| \leq \gamma_{\alpha, \beta} q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi). \quad (49)$$

Then we have

**Lemma 4.1.** *Let  $p \in (\theta_m, \theta_M)$ . For all  $\psi \in D(q_\theta^l)$  such that  $\psi(p, \cdot) = 0$ , then for any  $\alpha, \beta > 0$  there exists  $\gamma_{\alpha, \beta} > 0$  such that:*

$$|q_{2,3}^{(\theta_m, p)}(\psi)| \leq \gamma_{\alpha, \beta} q_1^{(\theta_m, p)}(\psi) + \alpha q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi). \quad (50)$$

*Proof.* Let  $\psi \in D(q_\theta^l)$  such that  $\psi(p, \cdot) = 0$ . Then  $\psi \in \mathcal{H}_0^1(\Omega_p)$ . We know that we may first consider vectors  $\psi(s, t) = \chi_1(t)\phi(s, t)$ , where  $\phi \in C_0^\infty(\Omega_p)$ . For such a vector  $\psi$  we have,

$$\begin{aligned} q_1^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \chi_1 \nabla' \phi\|^2, & q_2^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \chi_1 \partial_s \phi\|^2 \\ q_3^{(\theta_m, p)}(\psi) &= \|\chi_{(\theta_m, p)} \dot{\theta}(\chi_1 \partial_\tau \phi + \phi \partial_\tau \chi_1)\|^2 \end{aligned} \quad (51)$$

and

$$q_{2,3}^{(\theta_m, p)}(\psi) = 2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi) + 2(\dot{\theta} \chi_{(\theta_m, p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) \quad (52)$$

By using simple estimates the first term on the r.h.s of (52) is estimated as :

$$|2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi)| \leq 2 \|\dot{\theta}\|_\infty \|\chi_{(\theta_m, p)} \chi_1 \nabla' \phi\| \|\chi_{(\theta_m, p)} \chi_1 \partial_s \phi\|$$

then

$$|2(\dot{\theta} \chi_{(\theta_m, p)} \chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi)| \leq c_1 q_1^{(\theta_m, p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m, p)}(\psi), \quad (53)$$

where  $c_1 := \frac{2}{\alpha} d^2 \|\dot{\theta}\|_\infty^2$  and  $\alpha > 0$ .

Integrating by parts twice and using the fact that  $\dot{\theta}(\theta_m) = \phi(p, \cdot) = 0$ , the second term of the r.h.s of (52) is written as

$$2(\dot{\theta} \chi_{(\theta_m, p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) = (\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi). \quad (54)$$

Then the Cauchy Schwartz inequality implies,

$$|(\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi)|^2 \leq d^2 \|\ddot{\theta}\|_\infty^2 q_1^{(\theta_m, p)} \|\chi_{(\theta_m, p)} \chi_1 \phi\|^2. \quad (55)$$

Let  $p' \in \mathbb{R}$  and  $r' > 0$  such that  $(p' - r, p') \subset (\theta_m, p)$  and for  $s \in (p' - r, p')$ ,  $|\dot{\theta}(s)| \geq \dot{\theta}_0$  for some  $\dot{\theta}_0 > 0$ . As in the proof of the Lemma 3 of [6] we have,

$$\|\chi_{(\theta_m, p)} \chi_1 \phi\|^2 \leq c_2 \left( q_2^{(\theta_m, p)}(\psi) + \dot{\theta}_0^{-2} \|\chi_{(p' - r, p')} \dot{\theta} \chi_1 \phi\|^2 \right) \quad (56)$$

where  $c_2 := \max \left\{ 2 + 16 \frac{(p - \theta_m)^2}{r^2}, 4(p - \theta_m)^2 \right\}$ .

Moreover, for any  $s \in \mathbb{R}$ ,  $\dot{\theta}(s) \chi_1 \phi(s, \cdot) \in \mathcal{H}_0^1(\Omega_p)$ , then by using the Lemma 1 of [6] there exists  $\lambda > 0$  depending on  $\omega$  such that :

$$\|\chi_{(p' - r, p')} \dot{\theta} \chi_1 \phi\|^2 \leq \|\chi_{(\theta_m, p)} \dot{\theta} \chi_1 \phi\|^2 \leq \lambda^{-1} \left( q_3^{(\theta_m, p)}(\psi) + \|\dot{\theta}\|_\infty^2 q_1^{(\theta_m, p)}(\psi) \right). \quad (57)$$

Hence (56), (57) and (54) give

$$|(\chi_{(\theta_m, p)} \ddot{\theta} \phi \chi_1, \chi_1 \partial_\tau \phi)|^2 \leq \left( c_3 q_1^{(\theta_m, p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m, p)}(\psi) + \beta q_3^{(\theta_m, p)}(\psi) \right)^2 \quad (58)$$

where  $c_3 := \max \left\{ \frac{d \|\dot{\theta}\|_\infty \sqrt{c_2}}{\dot{\theta}_0 \sqrt{\lambda}}, \frac{d^2 \|\dot{\theta}\|_\infty^2 c_2}{\alpha}, \frac{d^2 \|\dot{\theta}\|_\infty^2 c_2}{2\beta \dot{\theta}_0^2 \lambda} \right\}$ . Then (53) and (58) imply (50) with  $\gamma_{\alpha, \beta} := c_1 + c_3$ .

Note that we can choose  $\chi_1 > 0$  on  $\omega$ . So that (50) holds for every  $\psi \in C_0^\infty(\Omega_p)$  and by a density argument this is even true for  $\psi \in \mathcal{H}_0^1(\Omega_p)$ .

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