HAL
open science

Twisted waveguide with a Neumann window

Philippe Briet, Hiba Hammedi

To cite this version:

Philippe Briet, Hiba Hammedi. Twisted waveguide with a Neumann window. Functional Analysis and Operator Theory for Quantum Physics : The Pavel Exner Anniversary Volume, European Mathematical Society, pp.161-175, 2017, EMS Series of Congress Reports, 978-3-03719-175-0. 10.4171/175-1/8 . hal-01264262v3

HAL Id: hal-01264262

https://hal.science/hal-01264262v3

Submitted on 24 Apr 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Twisted waveguide with a Neumann window

Philippe Briet, Hiba Hammedi *

Aix-Marseille Université, CNRS, CPT, UMR 7332, Case 907, 13288 Marseille and Université de Toulon, CNRS, CPT, UMR 7332 83957, La Garde, France;

January 2016

Dedicated to Pavel Exner on the occasion of his 70th birthday

Abstract

This paper is concerned with the study of the existence/non-existence of the discrete spectrum of the Laplace operator on a domain of $\mathbb{R}^3$ which consists in a twisted tube. This operator is defined by means of mixed boundary conditions. Here we impose Neumann Boundary conditions on a bounded open subset of the boundary of the domain (the Neumann window) and Dirichlet boundary conditions elsewhere.

classification Primary 81Q10; Secondary 47F05.

keywords Waveguide, mixed boundary conditions, twisting.

1 Introduction

In this work, we would like to study the influence of a geometric twisting on trapped modes which occur in certain waveguides. Here the waveguide consists in a straight tubular domain $\Omega_0 := \mathbb{R} \times \omega$ having a Neumann window on its boundary $\partial\Omega_0$.
The cross section ω is supposed to be an open bounded connected subset of $\mathbb{R}^2$ of diameter $d > 0$ which is not rotationally invariant. Moreover ω is supposed to have smooth boundary $\partial\omega$.

It can be shown that the Laplace operator associated to such a straight tube has bound states [8].

Let us introduce some notations. Denote by $\mathcal{N}$ the Neumann window. It is an open bounded subset of the boundary $\partial\Omega_0$. Let $\mathcal{D}$ be its complement set in $\partial\Omega_0$. When $\mathcal{N}$ is an annulus of size $l > 0$ we will denote it by,

$$\mathcal{A}_a(l) := I_a(l) \times \partial\omega, I_a(l) := (a, l + a), a \in \mathbb{R}.$$

*briet@univ-tln.fr, hammedi@univ-tln.fr.

Consider first the self-adjoint operator $H_0^{\mathcal{N}}$ associated to the following quadratic form. Let $D(Q^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_0) \ \mid \psi_{\lceil \mathcal{D}} = 0\}$ and for $\psi \in D(Q^{\mathcal{N}})$,

$$Q^{\mathcal{N}}(\psi) = \int_{\Omega_0} |\nabla \psi|^2 dx$$

i.e. the Laplace operator defined on Ω_0 with Neumann boundary conditions (NBC) on $\mathcal{N}$ and Dirichlet boundary conditions (DBC) on $\mathcal{D}$ [5, 11].

It is actually shown in the Section 2 of this paper that if $\mathcal{N}$ contains an annulus of size l large enough then $H_0^{\mathcal{N}}$ has at least one discrete eigenvalue. In fact it is proved in [8] that this holds true if $\mathcal{N}$ contains an annulus of any size $l > 0$.

The question we are interested in is the following: is it possible that the discrete spectrum of $H_0^{\mathcal{N}}$ disappears when we apply a geometric twisting on the guide? This question is motivated by the results of [6, 10] where it is shown that this phenomenon occurs in some bent tubes when they are subjected to a twisting defined from an angle function θ having a derivative $\dot{\theta}$ with a compact support. In this paper we consider the situation described above which is very different from the one of [6, 10].
Let us now define the twisting [4, 7]. Choose $\theta \in C_c^1(\mathbb{R})$ and introduce the diffeomorphism

$$\begin{aligned} \mathcal{L} : \Omega_0 &\longrightarrow \mathbb{R}^3 \\ (s, t_2, t_3) &\longmapsto \Big(s, t_2 \cos\theta(s) - t_3 \sin\theta(s), t_2 \sin\theta(s) + t_3 \cos\theta(s)\Big). \end{aligned} \tag{1}$$

The twisted tube is given by $\Omega_\theta := \mathcal{L}(\Omega_0)$. Let $D(Q_\theta^{\mathcal{N}}) = \{\psi \in \mathcal{H}^1(\Omega_\theta) \ \mid \psi_{\lceil \mathcal{L}(\mathcal{D})} = 0\}$ and consider the following quadratic form

$$Q_\theta^{\mathcal{N}}(\psi) := \int_{\Omega_\theta} |\nabla \psi|^2 dx, \ \psi \in D(Q_\theta^{\mathcal{N}}). \tag{2}$$

Through unitary equivalence, we then have to consider

$$q_\theta^{\mathcal{N}}(\psi) \ := \ Q_\theta^{\mathcal{N}}(\psi o \mathcal{L}^{-1}) = \| \nabla' \psi \|^2 + \| \partial_s \psi + \dot{\theta} \partial_\tau \psi \|^2, \tag{3}$$

$\psi \in D(q_\theta^{\mathcal{N}}) := \{\psi \in \mathcal{H}^1(\Omega_0) \ \mid \psi_{\lceil \mathcal{D}} = 0\}$ and where

$$\nabla' := {}^t\left(\partial_{t_2}, \partial_{t_3}\right), \quad \partial_\tau := t_2 \partial_{t_3} - t_3 \partial_{t_2}. \tag{4}$$

Denote by $H_\theta^{\mathcal{N}}$ the associated self-adjoint operator. It is defined as follows (see [5, 11]). Let $D(H_\theta^{\mathcal{N}}) = \{\psi \in D(q_\theta^{\mathcal{N}}), \quad H_\theta^{\mathcal{N}} \psi \in L^2(\Omega_0) \quad \frac{\partial \psi}{\partial n} \lceil_{\mathcal{N}} = 0\}$ with

$$H_\theta^{\mathcal{N}} \psi = (-\Delta_\omega - (\dot{\theta} \partial_\tau + \partial_s)^2) \psi, \tag{5}$$

where the transverse Laplacian $\Delta_\omega := \partial_{t_2}^2 + \partial_{t_3}^2$. If $\mathcal{N} = \mathcal{A}_a(l), l > 0$, we will denote these forms respectively as Q_θ^l, q_θ^l and the corresponding operator as H_θ^l and if $\mathcal{N} = \emptyset$ we denote the associated operator by H_θ.

Then the main result of this paper is

Theorem 1.1. *i) Under conditions stated above on ω and θ, there exists $l_{min} := l_{min}(\omega, d) > 0$ such as if for some $a \in \mathbb{R}$ and $l > l_{min}$, $\mathcal{N} \supset \mathcal{A}_a(l)$ then*

$$\sigma_d(H_\theta^{\mathcal{N}}) \neq \emptyset. \tag{6}$$

ii) Suppose θ is a non zero function satisfying the same conditions as in i) and has a bounded second derivative. Then there exists $d_{max} := d_{max}(\theta, \omega) > 0$ such that for all $0 < d \leq d_{max}$ there exists $l_{max} := l_{max}(\omega, d, \theta)$ such as for all $0 < l \leq l_{max}$, if $\mathcal{N} \subset \mathcal{A}_a(l)$ and $supp(\dot{\theta}) \cap I_a(l) = \emptyset$ for some $a \in \mathbb{R}$ then

$$\sigma_d(H_\theta^{\mathcal{N}}) = \emptyset. \tag{7}$$

Roughly speaking this result implies that for d small enough, the discrete spectrum disappears when the width of the Neumann window decreases.

Let us describe briefly the content of the paper. In the Section 2 we give the proof of the Theorem 1.1 i). The section 3 is devoted to the proof of the second part of the Theorem 1.1, this proof needs several steps. In particular we first establish a local Hardy inequality. This allows us to reduce the problem to the analysis of a one dimensional Schrödinger operator from which the Theorem 1.1 ii) follows. Finally in the Appendix of the paper we give partial results we use in previous sections.

2 Existence of bound states

First we prove the following. Denote by $E_1, E_2, ...$ the eigenvalues (transverse modes) of the Laplacian $-\Delta_\omega$ defined on $\mathrm{L}^2(\omega)$ with DBC on $\partial\omega$. Let $\chi_1, \chi_2, ...$ be the associated eigenfunctions. Then we have

Proposition 2.1. $\sigma_{ess}(H_\theta^{\mathcal{N}}) = [E_1, \infty)$.

Proof. We know that $\sigma(H_\theta) = [E_1, \infty)$ see e.g. [2]. But by usual arguments [12], $H_\theta^{\mathcal{N}} \leq H_\theta$, then

$$[E_1, \infty) \subset \sigma_{ess}(H_\theta^{\mathcal{N}}). \tag{8}$$

Let $a' \in \mathbb{R}$ and $l' > 0$ large enough such that $\mathcal{N} \subset \mathcal{A}_{a'}(l') = I_{a'}(l') \times \partial\omega$ and $supp(\dot{\theta}) \subset I_{a'}(l')$. Let $\tilde{H}_\theta^{l'}$ be the operator defined as in (5) but with additional Neumann boundary conditions on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. So $H_\theta^{\mathcal{N}} \geq \tilde{H}_\theta^{l'}$ and then $\sigma_{ess}(H_\theta^{\mathcal{N}}) \subset \sigma_{ess}(\tilde{H}_\theta^{l'})$ [12].

But $\tilde{H}_\theta^{l'} = \tilde{H}_i \oplus \tilde{H}_e$. The interior operator $\tilde{H}_i$ is the corresponding operator defined on $L^2(I_{a'}(l') \times \omega)$ with NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega \cup \mathcal{N}$ and DBC elsewhere on $\mathcal{A}_{a'}(l')$. By general arguments of [12] it has only discrete spectrum consequently $\sigma_{ess}(\tilde{H}_\theta^{l'}) = \sigma_{ess}(\tilde{H}_e)$.
Now the exterior operator $\tilde{H}_e$ is defined on $L^2((-\infty, a') \times \omega \cup (a' + l', \infty) \times \omega)$ with DBC on $(-\infty, a') \times \partial\omega \cup (a' + l', \infty) \times \partial\omega$ and NBC on $\{a'\} \times \omega \cup \{a' + l'\} \times \omega$. Since $\theta = 0$ for $x < a'$ and $x > a' + l'$, it is easy to see that

$$\tilde{H}_e = \bigoplus_{n \geq 1} (-\partial^2 + E_n)(\chi_n, .)\chi_n.$$

Hence $\sigma(\tilde{H}_e) = \sigma_{ess}(\tilde{H}_e) = [E_1, +\infty)$. □

The Theorem 1.1 *i)* follows from

Proposition 2.2. *Under conditions of the Theorem 1.1 i), there exists* $l_{min} := l_{min}(\omega, d) > 0$ *such as for all* $l > l_{min}$ *we have*

$$\sigma_d(H_\theta^l) \neq \emptyset. \tag{9}$$

Proof. Let $\varphi_{l,a}$ be the following function

$$\varphi_{l,a}(s) := \begin{cases} \frac{10}{l}(s-a), & \text{on } [a, a+\frac{l}{10}); \\ 1, & \text{on } [a+\frac{l}{10}, a+\frac{9l}{10}); \\ -\frac{10}{l}(s-l-a), & \text{on } [a+\frac{9l}{10}, a+l); \\ 0, & \text{elsewhere.} \end{cases}$$

It is easy to see that $\varphi_{l,a} \in D(q_\theta^l)$ and $\| \varphi_{l,a} \|^2 = \frac{13l}{15} \mid \omega \mid$. Let us calculate

$$q_\theta^l(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 = \| \nabla' \varphi_{l,a} \|^2 + \| \dot{\theta}\partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a} \|^2 - E_1 \| \varphi_{l,a} \|^2 . \tag{10}$$

Evidently the first term on the r.h.s of (10) is zero. For the second term on the r.h.s of (10) we get,

$$\| \dot{\theta}\partial_\tau \varphi_{l,a} + \partial_s \varphi_{l,a} \|^2 = \| \partial_s \varphi_{l,a} \|^2 = \frac{20}{l} \mid \omega \mid .$$

Then

$$q_\theta^l(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 = \mid \omega \mid (\frac{20}{l} - \frac{13l}{15} E_1) \tag{11}$$

and thus if $l \geq l_{min} := \sqrt{\frac{300}{13E_1}}$ we have $q_\theta^l(\varphi_{l,a}) - E_1 \| \varphi_{l,a} \|^2 \leq 0$ □

2.1 Proof of the Theorem 1.1 $i)$

Using the same notation as in the Theorem 1.1 $i)$, then $H_\theta^{\mathcal{N}} \leq H_\theta^l$. Moreover these operators have the same essential spectrum, then by the min-max principle the assertion follows.

3 Absence of bound state

In this section we want to prove the second part of the Theorem 1.1. Denote by $\theta_m = \inf(\text{supp}(\dot{\theta}))$, $\theta_M = \sup(\text{supp}(\dot{\theta}))$ and $L = \theta_M - \theta_m$. Here $L > 0$. We first consider the case where the Neumann window is an annulus, $\mathcal{A}_a(l) = I_a(l) \times \omega$.

Proposition 3.1. *Suppose* $\mathcal{A}_a(l)$ *is such that* $a \geq \theta_M$. *Assume that conditions of the Theorem 1.1 ii) hold. Then there exists* $d_{max} := d_{max}(\omega, \theta) > 0$, *such that for all* $0 < d \leq d_{max}$ *there exists* $l_{max}(d, \theta, \omega) > 0$ *such as for all* $0 < l \leq l_{max}$ *we have*

$$\sigma_d(H_\theta^l) = \emptyset. \tag{12}$$

Remark 3.2. the case where $l + a \leq \theta_m$ follows from same arguments developed below.

This proof is based on the fact that under conditions of the Proposition 3.1, for every $\psi \in D(q_\theta^l)$ it holds,

$$Q(\psi) := q_\theta^l(\psi) - E_1 \| \psi \|^2 \geq 0. \tag{13}$$

The proof of (13) involves several steps.

3.1 A local Hardy inequality

The aim of this paragraph is to show a Hardy type inequality needed for the proof of the Proposition 3.1. It is the first step of the proof of (13). Let g be the following function

$$g(s) := \begin{cases} 0, & \text{on } I_a(l); \\ E_1, & \text{elsewhere.} \end{cases} \tag{14}$$

Choose $p \in (\theta_m, \theta_M)$ s.t. $\dot{\theta}(p) \neq 0$ and let

$$\rho(s) := \begin{cases} \frac{1}{1+(s-p)^2}, & \text{on } (-\infty, p]; \\ 0, & \text{elsewhere.} \end{cases} \tag{15}$$

Proposition 3.3. *Under same conditions of the Proposition 3.1, then there exists a constant $C > 0$ depending on p and ω and $\dot{\theta}$ such that for any $\psi \in D(q^l_\theta)$,*

$$\| \nabla'\psi \|^2 + \| \dot{\theta}\partial_\tau\psi + \partial_s\psi \|^2 - \int_{\Omega_0} g(s) \mid \psi \mid^2 dsdt \geq C \int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt. \tag{16}$$

We first show the following lemma. Denote by $\Omega_p := (-\infty, p) \times \omega$.

Lemma 3.4. *Under same conditions of the Proposition 3.3. Then for any $\psi \in D(q^l_\theta)$ we have*

$$\int_{\Omega_p} \mid \nabla'\psi \mid^2 + \mid \dot{\theta}\partial_\tau\psi + \partial_s\psi \mid^2 - E_1 \mid \psi \mid^2 dsdt \geq C \int_{\Omega_p} \rho(s) \mid \psi \mid^2 dsdt. \tag{17}$$

In the following we will use notations suggested in [6]. For $A \subset \mathbb{R}$ denote by χ_A the characteristic function of $A \times \omega$. Let $\psi \in D(q^l_\theta)$ and define,

$$\begin{aligned} &q^A_1(\psi) := \| \chi_A \nabla'\psi \|^2 - E_1 \| \chi_A \psi \|^2, && q^A_2(\psi) := \| \chi_A \partial_s\psi \|^2, \\ &q^A_3(\psi) := \| \chi_A \dot{\theta}\partial_\tau\psi \|^2, && q^A_{2,3}(\psi) := 2\Re(\partial_s\psi, \chi_A\dot{\theta}\partial_\tau\psi). \end{aligned} \tag{18}$$

Denote also by $Q^A(\psi) = q^A_1(\psi) + q^A_2(\psi) + q^A_3(\psi) + q^A_{2,3}(\psi)$. Here and hereafter we often use the fact that for any $\psi \in D(q^l_\theta)$

$$q^A_1(\psi) \geq 0, \tag{19}$$

for every $A \subset \mathbb{R}$ such that $A \cap I_a(l) = \emptyset$.

Proof. Choose $r > 0$ such that $\dot{\theta}(s) \neq 0$ for any $s \in [p-r, p]$. Let f be the following function:

$$f(s) := \begin{cases} 0, & \text{on } (p, \infty); \\ \frac{p-s}{r}, & \text{on } (p-r, p]; \\ 1, & \text{elsewhere.} \end{cases} \tag{20}$$

For any $\psi \in D(q^l_\theta)$, simple estimates lead to:

$$\begin{aligned} \int_{\Omega_p} \frac{\mid \psi(s,t) \mid^2}{1+(s-p)^2} dsdt &= \int_{\Omega_p} \frac{\mid \psi(s,t)f(s) + (1-f(s))\psi(s,t) \mid^2}{1+(s-p)^2} dsdt \\ &\leq 2\Big(\int_{\Omega_p} \frac{\mid f(s)\psi(s,t) \mid^2}{(s-p)^2} dsdt + \|\chi_{(p-r,p)}\psi\|^2 \Big). \end{aligned} \tag{21}$$

Since $f(p)\psi(p,.) = 0$, we can use the usual Hardy inequality (see e.g. [9]), then we get,

$$\int_{\Omega_p} \frac{\mid \psi(s,t) \mid^2}{1+(s-p)^2} dsdt \leq 8q_2^{(-\infty,p)}(f\psi) + 2\|\chi_{(p-r,p)}\psi\|^2. \tag{22}$$

Note that with our choice $[p-r,p] \cap [a,a+l] = \emptyset$. Hence to estimate the second term on the r.h.s of (22) we use the Theorem 6.5 of [10], then there exists $\lambda_0 = \lambda_0(\dot{\theta},p,r) > 0$ s.t. for any $\psi \in D(q_\theta^l)$ we have

$$\|\chi_{(p-r,p)}\psi\|^2 \leq \frac{1}{\lambda_0} Q^{(p-r,p)}(\psi) \leq \frac{1}{\lambda_0} Q^{(-\infty,p)}(\psi). \tag{23}$$

We now want to estimate the first term on the right hand side of (22). We have

$$q_2^{(-\infty,p)}(f\psi) = \int_{\Omega_p} \mid \partial_s(f\psi) \mid^2 dsdt = q_2^{(-\infty,\theta_m)}(f\psi) + q_2^{(\theta_m,p)}(f\psi). \tag{24}$$

Evidently since $\dot{\theta} = 0$ and $f = 1$ in $(-\infty,\theta_m)$, from (19), we have

$$q_2^{(-\infty,\theta_m)}(f\psi) \leq Q^{(-\infty,\theta_m)}(\psi). \tag{25}$$

In the other hand since $f(p)\psi(p,.) = 0$, we can apply the Lemma 4.1 of the Appendix. So for any $0 < \alpha < 1$ there exists $\gamma_{\alpha,1} > 0$ such that

$$\mid q_{2,3}^{(\theta_m,p)}(f\psi) \mid \leq \gamma_{\alpha,1} q_1^{(\theta_m,p)}(f\psi) + \alpha q_2^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi). \tag{26}$$

Let $\gamma := \max(1,\gamma_{\alpha,1})$. Then

$$\gamma^{-1} \mid q_{2,3}^{(\theta_m,p)}(f\psi) \mid \leq q_1^{(\theta_m,p)}(f\psi) + \alpha\gamma^{-1} q_2^{(\theta_m,p)}(f\psi) + \gamma^{-1} q_3^{(\theta_m,p)}(f\psi). \tag{27}$$

Hence with the decomposition, $q_{2,3}^{(\theta_m,p)} = \gamma^{-1} q_{2,3}^{(\theta_m,p)} + (1-\gamma^{-1}) q_{2,3}^{(\theta_m,p)}$ and (27) we have,

$$\begin{aligned} Q^{(\theta_m,p)}(f\psi) \geq (1-\gamma^{-1})\Big(q_2^{(\theta_m,p)}(f\psi) + q_{2,3}^{(\theta_m,p)}(f\psi) + q_3^{(\theta_m,p)}(f\psi)\Big) \\ + \gamma^{-1}(1-\alpha) q_2^{(\theta_m,p)}(f\psi) \end{aligned} \tag{28}$$

and since $q_3^{(\theta_m,p)} + q_{2,3}^{(\theta_m,p)} + q_2^{(\theta_m,p)} \geq 0$, we arrive at,

$$q_2^{(\theta_m,p)}(f\psi) \leq \frac{\gamma}{(1-\alpha)} Q^{(\theta_m,p)}(f\psi). \tag{29}$$

Now by using that, $q_1^{(\theta_m,p)}(f\psi) \leq q_1^{(\theta_m,p)}(\psi)$,

$$\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)(f\psi)\|^2 \leq 2(\|\chi_{(\theta_m,p)}(\partial_s + \dot{\theta}\partial_\tau)\psi\|^2 + \frac{1}{r^2}\|\chi_{(p-r,p)}\psi\|^2)$$

and (23), we get,

$$q_2^{(\theta_m,p)}(f\psi) \leq \frac{2\gamma}{(1-\alpha)}(Q^{(\theta_m,p)}(\psi) + \frac{1}{\lambda_0 r^2} Q^{(p-r,p)}(\psi)) \leq c' Q^{(\theta_m,p)}(\psi) \tag{30}$$

with $c' = \frac{2\gamma}{(1-\alpha)}(1+\frac{1}{\lambda_0 r^2})$. Then (25) and (30) imply

$$q_2^{(-\infty,p)}(f\psi) \leq (1+c')Q^{(-\infty,p)}(\psi). \tag{31}$$

Hence (31) and (23) prove the lemma with

$$C^{-1} = 8\big(1+c'\big) + \frac{2}{\lambda_0}. \tag{32}$$

□

Proof of the proposition 3.3. To prove the proposition we note that for any $\psi \in D(q_\theta^l)$ and for $p' \in \mathbb{R}$ we have

$$\int_\omega \int_{p'}^\infty \mid \nabla'\psi \mid^2 + \mid \dot\theta\partial_\tau\psi + \partial_s\psi \mid^2 dsdt \geq \int_\omega \int_{p'}^\infty g(s) \mid \psi \mid^2 dsdt. \tag{33}$$

Then (33) with $p' = p$ and Lemma 3.4 imply (16). □

3.2 Reduction to a one dimensional problem

We now want to prove the following result,

Proposition 3.5. *Under conditions of the Proposition 3.1, then a sufficient condition in order to get* (13) *is given by*

$$\int_{\mathbb{R}} \mid \psi'(s) \mid^2 + 2C\rho(s) \mid \psi(s) \mid^2 ds - 4E_1 \int_a^{a+l} \mid \psi(s) \mid^2 ds \geq 0, \tag{34}$$

for any $\psi \in \mathcal{H}^1(\mathbb{R})$ where the constant C is defined in (32).

Remark 3.6. This proposition means that the positivity needed here is given by the positivity of the effective one dimensional Schrödinger operator on $\mathrm{L}^2(\mathbb{R})$,

$$-\frac{d^2}{ds^2} + 2C\rho(s) - 4E_1\mathbf{1}_{I_a(l)}. \tag{35}$$

where $\mathbf{1}_{I_a(l)}$ is the characteristic function of $I_a(l)$.

Proof. Evidently we have

$$Q(\psi) = \frac{1}{2}\big(Q(\psi) - \int_{\Omega_0} (E_1 - g(s)) \mid \psi \mid^2 dsdt + q_\theta^l(\psi) - \int_{\Omega_0} g(s) \mid \psi \mid^2 dsdt\big), \tag{36}$$

where g is defined in (14). By using (16), then

$$Q(\psi) \geq \frac{1}{2}\Big(q_\theta^l(\psi) - E_1 \parallel \psi \parallel^2 + C\int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt - E_1 \parallel \chi_{(a,a+l)}\psi \parallel^2\Big) \tag{37}$$

Rewrite the expression of q_θ^l given by (3) as follows:

$$q_\theta^l(\psi) = \parallel \nabla'\psi \parallel^2 + \parallel \partial_s\psi \parallel^2 + \parallel \dot\theta\partial_\tau\psi \parallel^2 + 2\Re(\partial_s\psi, \dot\theta\partial_\tau\psi). \tag{38}$$

We estimate the last term of the r.h.s. of (38). By using the formula (49) of the Appendix,

$$| q_{2,3}(\psi) |=| q_{2,3}^{(\theta_m,\theta_M)}(\psi) |\leq \gamma_{\frac{1}{2},\frac{1}{2}} q_1^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_2^{(\theta_m,\theta_M)}(\psi) + \frac{1}{2} q_3^{(\theta_m,\theta_M)}(\psi) \tag{39}$$

where

$$\gamma_{\frac{1}{2},\frac{1}{2}} := \widetilde{\gamma}_{\frac{1}{2},\frac{1}{2}} + 4d^2 \parallel \dot{\theta} \parallel_\infty^2 \tag{40}$$

with $\widetilde{\gamma}_{\frac{1}{2},\frac{1}{2}} := max\Big\{ \frac{d\|\dot{\theta}\|_\infty \|\ddot{\theta}\|_\infty \sqrt{f(L)}}{\dot{\theta_0}\sqrt{\lambda}}, \frac{d^2\|\ddot{\theta}\|_\infty^2 f(L)}{\lambda \dot{\theta_0}^2}, 2d^2 \parallel \ddot{\theta} \parallel_\infty^2 f(L) \Big\}$ for some constant $\lambda > 0$ depending only on the section ω and $f(L) := \max\{2+\frac{16L^2}{r^2}, 4L^2\}$.

Hence (38) together with (39) give:

$$q_\theta^l(\psi) \geq \parallel \nabla'\psi \parallel^2 + \frac{1}{2} \parallel \partial_s \psi \parallel^2 + \frac{1}{2} \parallel \dot{\theta}\partial_\tau \psi \parallel^2 - \gamma_{\frac{1}{2},\frac{1}{2}} q_1^{(\theta_m,\theta_M)}(\psi). \tag{41}$$

In view of (19) we have

$$\parallel \nabla'\psi \parallel^2 - E_1 \parallel \psi\|^2 \geq q_1^{(\theta_m,\theta_M)}(\psi) + q_1^{I_a(l)}(\psi) \geq q_1^{(\theta_m,\theta_M)}(\psi) - E_1 \|\chi_{(a,a+l)}\psi\|^2.$$

Thus this last inequality together with (41) in (37) give

$$\begin{aligned} Q(\psi) \quad &\geq \quad \frac{1}{2}\Big(\frac{1}{2} \parallel \partial_s \psi \parallel^2 + \frac{1}{2} \parallel \dot{\theta}\partial_\tau \psi \parallel^2 + C \int_{\Omega_0} \rho(s) \mid \psi \mid^2 dsdt - 2E_1 \parallel \chi_{(a,l+a)}\psi \parallel^2 \\ &+ \quad (1-\gamma_{\frac{1}{2},\frac{1}{2}}) q_1^{(\theta_m \theta_M)}(\psi)\Big). \end{aligned}$$

Now if $0 < d \leq d_{max}$ then $\gamma_{\frac{1}{2},\frac{1}{2}} \leq 1$ so the Proposition 3.5 follows.

3.3 The one dimensional Schrödinger operator

In this part, under our conditions, we want to show that the one dimensional Schrödinger operator (35) is a positive operator. In view of the Proposition 3.5 this will imply the Proposition 3.1. Here we follow a similar strategy as in [1].

Proposition 3.7. *for all $\varphi \in \mathcal{H}^1(\mathbb{R})$, then there exists $l_{max} > 0$ such that for any $0 < l \leq l_{max}$ we have*

$$\int_{\mathbb{R}} \mid \varphi'(s) \mid^2 + 2C\rho(s) \mid \varphi(s) \mid^2 ds \geq 4E_1 \int_{I_a(l)} \mid \varphi(s) \mid^2 ds. \tag{42}$$

Proof. Introduce the following function:

$$\Phi(s) := \begin{cases} (\frac{\pi}{2} + \arctan(s-p)), & if\ s < p; \\ \frac{\pi}{2}, & if\ s \geq p. \end{cases} \tag{43}$$

where p is the same real number as in (15). So clearly $\Phi' = \rho$. For any $t \in I_a(l)$ and $\varphi \in \mathcal{H}^1(\mathbb{R})$, we have:

$$\begin{aligned} \frac{\pi}{2}\varphi(t) = \Phi(t)\varphi(t) \quad &= \quad \int_{-\infty}^t (\Phi(s)\varphi(s))' ds \\ &= \quad \int_{-\infty}^t \rho(s)\varphi(s) ds + \int_{-\infty}^t \Phi(s)\varphi'(s) ds \end{aligned} \tag{44}$$

and since $\rho(s) = 0$ for any $s \in (p, \infty)$, we get,

$$\frac{\pi}{2}\varphi(t) = \int_{-\infty}^{p} \rho(s)\varphi(s)ds + \int_{-\infty}^{t} \Phi(s)\varphi'(s)ds. \tag{45}$$

Then some straightforward estimates lead to,

$$\begin{aligned} \frac{\pi^2}{4}\varphi^2(t) &\leq 2\Big((\int_{-\infty}^{p} \rho(s)\varphi(s)ds)^2 + (\int_{-\infty}^{t} \Phi(s)\varphi'(s)ds)^2\Big) \qquad (46) \\ &\leq 2\Big(\int_{-\infty}^{p} \rho(s)ds \int_{-\infty}^{p} \rho(s)\varphi^2(s)ds + \int_{-\infty}^{t} \Phi^2(s)ds \int_{-\infty}^{t} \varphi'^2(s)ds\Big). \end{aligned}$$

By direct calculation $\int_{-\infty}^{p} \rho(s)ds = \frac{\pi}{2}$ and $\int_{-\infty}^{p} \Phi^2(s)ds + \int_{p}^{t} \Phi^2(s)ds = \pi \ln 2 + \frac{\pi^2}{4}(t-p)$. Hence we get,

$$\mid \varphi(t) \mid^2 \leq \frac{4}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \Big(\frac{8\ln 2}{\pi} + 2(t-p)\Big) \int_{\mathbb{R}} \mid \varphi'(s) \mid^2 ds \tag{47}$$

We integrate both sides of (47) over $I_a(l)$, then

$$\begin{aligned} \int_{I_a(l)} \mid \varphi(t) \mid^2 dt &\leq \frac{4l}{\pi} \int_{\mathbb{R}} \rho(s)\varphi^2(s)ds + \Big((\frac{8\ln 2}{\pi} + 2(a-p))l + l^2\Big) \int_{\mathbb{R}} \mid \varphi'(s) \mid^2 ds \\ &\leq c'' \int_{\mathbb{R}} 2C\rho(s)\varphi^2(s) + \mid \varphi'(s) \mid^2 ds \end{aligned}$$

where $c'' = 2l(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p) + l^2$. Finally we get,

$$4E_1 \int_{a}^{l+a} \mid \varphi(t) \mid^2 dt \leq 4E_1 c'' \int_{\mathbb{R}} 2C\rho(s) \mid \varphi(s) \mid^2 + \mid \varphi'(s) \mid^2 ds. \tag{48}$$

So choose $0 < l \leq l_{max}$ with

$$l_{max} := -(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p) + \sqrt{(\frac{1}{\pi C} + \frac{4\ln 2}{\pi} + a - p)^2 + (4E_1)^{-1}}$$

then $4E_1 c'' \leq 1$ and the proposition 3.7 follows. □

3.4 proof of the Theorem 1.1 *ii*)

Under assumptions of the Theorem 1.1 *ii*) $H_\theta^{\mathcal{N}} \geq H_\theta^l$. These two operators have the same essential spectrum so the Theorem 1.1 *ii*) is proved by applying the Proposition 3.1 and the min-max principle.

4 Appendix

In this appendix we give a slight extension of the lemma 3 of [6] which states that under our conditions, for all $\psi \in D(q_\theta^l)$ we have for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:

$$\mid q_{2,3}(\psi) \mid \leq \gamma_{\alpha,\beta} q_1(\psi) + \alpha q_2(\psi) + \beta q_3(\psi). \tag{49}$$

Then we have

Lemma 4.1. *Let $p \in (\theta_m, \theta_M)$. For all $\psi \in D(q_\theta^l)$ such that $\psi(p,.) = 0$, then for any $\alpha, \beta > 0$ there exists $\gamma_{\alpha,\beta} > 0$ such that:*

$$| \, q_{2,3}^{(\theta_m,p)}(\psi) \, | \leq \gamma_{\alpha,\beta} q_1^{(\theta_m,p)}(\psi) + \alpha q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi). \tag{50}$$

Proof. Let $\psi \in D(q_\theta^l)$ such that $\psi(p,.) = 0$. Then $\psi \in \mathcal{H}_0^1(\Omega_p)$. We know that we may first consider vectors $\psi(s,t) = \chi_1(t)\phi(s,t)$, where $\phi \in C_0^\infty(\Omega_p)$. For such a vector ψ we have,

$$\begin{aligned} q_1^{(\theta_m,p)}(\psi) &= \| \, \chi_{(\theta_m,p)}\chi_1 \nabla' \phi \, \|^2, \quad q_2^{(\theta_m,p)}(\psi) = \| \, \chi_{(\theta_m,p)}\chi_1 \partial_s \phi \, \|^2 \\ q_3^{(\theta_m,p)}(\psi) &= \| \, \chi_{(\theta_m,p)} \dot\theta (\chi_1 \partial_\tau \phi + \phi \partial_\tau \chi_1) \, \|^2 \end{aligned} \tag{51}$$

and

$$q_{2,3}^{(\theta_m,p)}(\psi) = 2(\dot\theta \chi_{(\theta_m,p)}\chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi) + 2(\dot\theta \chi_{(\theta_m,p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) \tag{52}$$

By using simple estimates the first term on the r.h.s of (52) is estimated as :

$$| \, 2(\dot\theta \chi_{(\theta_m,p)}\chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi) \, | \leq 2 \, \| \, \dot\theta \, \|_\infty \| \, \chi_{(\theta_m,p)}\chi_1 \nabla' \phi \, \| \| \, \chi_{(\theta_m,p)}\chi_1 \partial_s \phi \, \|$$

then

$$| \, 2(\dot\theta \chi_{(\theta_m,p)}\chi_1 \partial_\tau \phi, \chi_1 \partial_s \phi) \, | \leq c_1 q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m,p)}(\psi), \tag{53}$$

where $c_1 := \frac{2}{\alpha} d^2 \, \| \, \dot\theta \, \|_\infty^2$ and $\alpha > 0$.
Integrating by parts twice and using the fact that $\dot\theta(\theta_m) = \phi(p,.) = 0$, the second term of the r.h.s of (52) is written as

$$2(\dot\theta \chi_{(\theta_m,p)} \phi \partial_\tau \chi_1, \chi_1 \partial_s \phi) = (\chi_{(\theta_m,p)} \ddot\theta \phi \chi_1, \chi_1 \partial_\tau \phi). \tag{54}$$

Then the Cauchy Schwartz inequality implies,

$$| \, (\chi_{(\theta_m,p)} \ddot\theta \phi \chi_1, \chi_1 \partial_\tau \phi) \, |^2 \leq d^2 \, \| \, \ddot\theta \, \|_\infty^2 \, q_1^{(\theta_m,p)} \, \| \, \chi_{(\theta_m,p)}\chi_1 \phi \, \|^2 \, . \tag{55}$$

Let $p' \in \mathbb{R}$ and $r' > 0$ such that $(p' - r, p') \subset (\theta_m, p)$ and for $s \in (p' - r, p')$, $| \, \dot\theta(s) \, | \geq \dot\theta_0$ for some $\dot\theta_0 > 0$. As in the proof of the Lemma 3 of [6] we have,

$$\| \, \chi_{(\theta_m,p)}\chi_1 \phi \, \|^2 \leq c_2 \Big(q_2^{(\theta_m,p)}(\psi) + \dot\theta_0^{-2} \, \| \, \chi_{(p'-r,p')} \dot\theta \chi_1 \phi \, \|^2 \Big) \tag{56}$$

where $c_2 := \max \Big\{ 2 + 16 \frac{(p-\theta_m)^2}{r^2}, 4(p-\theta_m)^2 \Big\}$.
Moreover, for any $s \in \mathbb{R}$, $\dot\theta(s)\chi_1 \phi(s,.) \in \mathcal{H}_0^1(\Omega_p)$, then by using the Lemma 1 of [6] there exists $\lambda > 0$ depending on ω such that :

$$\| \, \chi_{(p'-r,p')} \dot\theta \chi_1 \phi \, \|^2 \leq \| \, \chi_{(\theta_m,p)} \dot\theta \chi_1 \phi \, \|^2 \leq \lambda^{-1} \Big(q_3^{(\theta_m,p)}(\psi) + \| \, \dot\theta \, \|_\infty^2 \, q_1^{(\theta_m,p)}(\psi) \Big). \tag{57}$$

Hence (56), (57) and (54) give

$$| \, (\chi_{(\theta_m,p)} \ddot\theta \phi \chi_1, \chi_1 \partial_\tau \phi) \, |^2 \leq \Big(c_3 q_1^{(\theta_m,p)}(\psi) + \frac{\alpha}{2} q_2^{(\theta_m,p)}(\psi) + \beta q_3^{(\theta_m,p)}(\psi) \Big)^2 \tag{58}$$

where $c_3 := \max \Big\{ \frac{d \|\ddot\theta\| \|\dot\theta\|_\infty \sqrt{c_2}}{\dot\theta_0 \sqrt{\lambda}}, \frac{d^2 \|\ddot\theta\|_\infty^2 c_2}{\alpha}, \frac{d^2 \|\ddot\theta\|_\infty^2 c_2}{2\beta \dot\theta_0^2 \lambda} \Big\}$. Then (53) and (58) imply (50) with $\gamma_{\alpha,\beta} := c_1 + c_3$.

Note that we can choose $\chi_1 > 0$ on ω. So that (50) holds for every $\psi \in C_0^\infty(\Omega_p)$ and by a density argument this is even true for $\psi \in \mathcal{H}_0^1(\Omega_p)$.

References

[1] D. Borisov, T. Ekholm, H. Kovařík, Spectrum of the Magnetic Schrödinger Operator in a Waveguide with Combined Boundary Conditions, *Ann. Henri Poincar*,**6** (2005), no. 2, 327–342.

[2] P. Briet, Spectral analysis for twisted waveguides, *Quatum probabbility and related topics* **27** (2011), 125–130.

[3] P. Briet, H. Hammedi, D. Krejčiřík, Hardy inequalities in globally twisted waveguides, *lett. Math. Phys.* **105** (2015), no. 7, 939–958.

[4] P. Briet, H. Kovařík, G. Raikov, E. Soccorsi, Eigenvalue asymptotics in a twisted waveguide, *Commun. in P.D.E.*, **34** (2009), 818–836.

[5] J. Dittrich, J. Kříž, Straight quantum waveguides with combined boundary conditions. Mathematical results in quantum mechanics (Taxco, 2001), 107–112, *Contemp. Math., 307, Amer. Math. Soc., Providence, RI, 2002.*

[6] T. Ekholm, H. Kovařík, D. Krejčiřík, A Hardy inequality in twisted waveguides, *Arch. Ration. Mech. Anal.*, **188** (2008), 245–264.

[7] P. Exner, H. Kovařík, Spectrum of the Schrödinger operator in a perturbed periodically twisted tube, *Lett. Math. Phys.* **73** (2005), 183–192.

[8] H.Hammedi, Analyse spectrale des guides d'ondes "twistés", Thèse de doctorat de Mathématique, Université de Toulon. (2016)

[9] G. H. Hardy, Note on a theorem of Hilbert, *Math, Zeit 6* (1920), 314–317.

[10] D. Krejčiřík, twisting versus bending in quantum waveguides, Analysis on Graphs and Applications (Cambridge 2007), Proc. Sympos. PureMath., *Amer. Math. Soc.*, Providence, RI, **77** (2008), 617–636.

[11] J. Kříž, Spectral properties of planar quantum waveguides with combined boundary conditions, P.H.D. Thesis, Charles University Prague, 2003.

[12] M. Reed, B. Simon, *Methods of Modern of Mathematical Physics, IV: Analysis of Operators*, Academic Press, New York-San Francisco-London, 1978.