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Inverse Problem for a Curved Quantum Guide

Laure Cardoulis*

Michel Cristofol†

Abstract

In this paper, we consider the Dirichlet Laplacian operator $-\Delta$ on a curved quantum guide in \mathbb{R}^n ($n = 2, 3$) with an asymptotically straight reference curve. We give uniqueness results for the inverse problem associated to the reconstruction of the curvature by using either observations of spectral data or a boot-strapping method.

keywords: Inverse Problem, Quantum Guide, Curvature

1 Introduction and main results in dimension $n = 2$

The spectral properties of curved quantum guides have been studied intensively for several years, because of their applications in quantum mechanics, electron motion. We can cite among several papers [6], [8], [9], [4], [5], [3] ...

However, inverse problems associated with curved quantum guides have not been studied to our knowledge, except in [2]. Our aim is to establish uniqueness results for the inverse problem of the reconstruction of the curvature of the quantum guide: the data of one eigenpair determines uniquely the curvature up to its sign and similar results are obtained by considering the knowledge of a solution of Poisson's equation in the guide.

We consider the Laplacian operator on a non trivially curved quantum guide $\Omega \subset \mathbb{R}^2$ which is not self-intersecting, with Dirichlet boundary conditions, denoted by $-\Delta_D^\Omega$. We proceed as in [6]. We denote by $\Gamma = (\Gamma_1, \Gamma_2)$ the function C^3 -smooth (see [3, Remark 5]) which characterizes the reference curve and by $N = (N_1, N_2)$ the outgoing normal to the boundary of Ω . We denote by d the fixed width of Ω and by $\Omega_0 := \mathbb{R} \times]-d/2, d/2[$. Each point (x, y) of Ω is described by the curvilinear coordinates (s, u) as follows:

$$\hat{f} : \Omega_0 \longrightarrow \Omega \quad \text{with} \quad (x, y) = \hat{f}(s, u) = \Gamma(s) + uN(s). \quad (1.1)$$

We assume $\Gamma_1'(s)^2 + \Gamma_2'(s)^2 = 1$ and we recall that the signed curvature γ of Γ is defined by:

$$\gamma(s) = -\Gamma_1''(s)\Gamma_2'(s) + \Gamma_2''(s)\Gamma_1'(s), \quad (1.2)$$

named so because $|\gamma(s)|$ represents the curvature of the reference curve at s . We recall that a guide is called simply-bent if γ does not change sign in \mathbb{R} . We assume throughout this article that:

Assumption 1.1. *i) \hat{f} is injective.*

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- ii) $\gamma \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\gamma \not\equiv 0$, (i.e. Ω is non-trivially curved).
- iii) $\frac{d}{2} < \frac{1}{\|\gamma\|_\infty}$, where $\|\gamma\|_\infty := \sup_{s \in \mathbb{R}} |\gamma(s)| = \|\gamma\|_{L^\infty(\mathbb{R})}$.
- iv) $\gamma(s) \rightarrow 0$ as $|s| \rightarrow +\infty$ (i.e. Ω is asymptotically straight).

Note that, by the inverse function theorem, the map \hat{f} (defined by (1.1)) is a local diffeomorphism provided $1 - u\gamma(s) \neq 0$, for all u, s , which is guaranteed by Assumption 1.1 and since \hat{f} is assumed to be injective, the map \hat{f} is a global diffeomorphism. Note also that $1 - u\gamma(s) > 0$ for all u and s . (More precisely, $0 < 1 - \frac{d}{2}\|\gamma\|_\infty \leq 1 - u\gamma(s) \leq 1 + \frac{d}{2}\|\gamma\|_\infty$ for all u, s .) The curvilinear coordinates (s, u) are locally orthogonal, so by virtue of the Frenet-Serret formulae, the metric in Ω is expressed with respect to them through a diagonal metric tensor (e.g. [9])

$$(g_{ij}) = \begin{pmatrix} (1 - u\gamma(s))^2 & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.3)$$

The transition to the curvilinear coordinates represents an isometric map of $L^2(\Omega)$ to $L^2(\Omega_0, g^{1/2} dsdu)$ where

$$(g(s, u))^{1/2} := 1 - u\gamma(s) \quad (1.4)$$

is the Jacobian $\frac{\partial(x, y)}{\partial(s, u)}$. So we can replace the Laplacian operator $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ by the Laplace-Beltrami operator H_g acting on $L^2(\Omega_0, g^{1/2} dsdu)$ relative to the given metric tensor (g_{ij}) (see (1.3) and (1.4)) where:

$$H_g := -g^{-1/2} \partial_s (g^{-1/2} \partial_s) - g^{-1/2} \partial_u (g^{1/2} \partial_u). \quad (1.5)$$

We rewrite H_g (defined by (1.5)) into a Schrödinger-type operator acting on $L^2(\Omega_0, dsdu)$. Indeed, using the unitary transformation

$$U_g : L^2(\Omega_0, g^{1/2} dsdu) \xrightarrow{\psi} L^2(\Omega_0, dsdu) \\ \psi \mapsto g^{1/4} \psi \quad (1.6)$$

setting

$$H_\gamma := U_g H_g U_g^{-1},$$

we get

$$H_\gamma = -\partial_s (c_\gamma(s, u) \partial_s) - \partial_u^2 + V_\gamma(s, u) \quad (1.7)$$

with

$$c_\gamma(s, u) = \frac{1}{(1 - u\gamma(s))^2} \quad (1.8)$$

and

$$V_\gamma(s, u) = -\frac{\gamma^2(s)}{4(1 - u\gamma(s))^2} - \frac{u\gamma''(s)}{2(1 - u\gamma(s))^3} - \frac{5u^2\gamma'^2(s)}{4(1 - u\gamma(s))^4}. \quad (1.9)$$

We will assume throughout all this paper that the following assumption is satisfied:

Assumption 1.2. $\gamma \in C^2(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for each $k = 0, 1, 2$ where $\gamma^{(k)}$ denotes the k^{th} derivative of γ .

Remarks: Since Ω is non trivially-curved and asymptotically straight, the operator $-\Delta_D^\Omega$ has at least one eigenvalue of finite multiplicity below its essential spectrum (see [3], [9] ; see also [6] under the additional assumptions that the width d is sufficiently small and the curvature γ is rapidly decaying at infinity ; see [8] under the assumption that the curvature γ has a compact

support).

Furthermore, note that such operator H_γ admits bound states and that the minimum eigenvalue λ_1 is simple and associated with a positive eigenfunction ϕ_1 (see [7, Sec.8.17]). Then, note that by [11, Theorem 7.1] any eigenfunction of H_γ is continuous and by [1, Remark 25 p.182] any eigenfunction of H_γ belongs to $H^2(\Omega_0)$.

Finally, note also that (λ, ϕ) is an eigenpair (i.e. an eigenfunction associated with its eigenvalue) of the operator H_γ acting on $L^2(\Omega_0, dsdu)$ means that $(\lambda, U_g^{-1}\phi)$ is an eigenpair of $-\Delta_D^\Omega$ acting on $L^2(\Omega)$. So the data of one eigenfunction of the operator H_γ is equivalent to the data of one eigenfunction of $-\Delta_D^\Omega$.

We first prove that the data of one eigenpair determines uniquely the curvature.

Theorem 1.1. *Let Ω be the curved guide in \mathbb{R}^2 defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1, 1.2. Let H_γ be the operator defined by (1.7) and (λ, ϕ) be an eigenpair of H_γ .*

Then

$$\gamma^2(s) = -4 \frac{\Delta\phi(s, 0)}{\phi(s, 0)} - 4\lambda$$

for all s when $\phi(s, 0) \neq 0$.

Note that the condition $\phi(s, 0) \neq 0$ in Theorem 1.1 is satisfied for the positive eigenfunction ϕ_1 and for all $s \in \mathbb{R}$. Then, we prove under

Assumption 1.3. $\gamma \in C^5(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for each $k = 0, \dots, 5$,

that one weak solution ϕ of the problem

$$\begin{cases} H_\gamma\phi = f & \text{in } \Omega_0 \\ \phi = 0 & \text{on } \partial\Omega_0 \end{cases} \quad (1.10)$$

(where f is a known given function) is in fact a classical solution and the data of ϕ determines uniquely the curvature γ .

Theorem 1.2. *Let Ω be the curved guide in \mathbb{R}^2 defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1 and 1.3. Let H_γ be the operator defined by (1.7). Let $f \in H^3(\Omega_0) \cap C(\Omega_0)$ and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of (1.10).*

Then we have $\gamma^2(s) = -4 \frac{\Delta\phi(s, 0)}{\phi(s, 0)} - 4 \frac{f(s, 0)}{\phi(s, 0)}$ for all s when $\phi(s, 0) \neq 0$

In the case of a simply-bent guide (i.e. when γ does not change sign in \mathbb{R}), we can restrain the hypotheses upon the regularity of γ . We obtain the following result:

Theorem 1.3. *Let Ω be the curved guide in \mathbb{R}^2 defined as above. Let γ be the signed curvature defined by (1.2) and satisfying Assumptions 1.1 and 1.2. We assume also that γ is a nonnegative function. Let H_γ be the operator defined by (1.7). Let $f \in L^2(\Omega_0)$ be a non null function and let ϕ be a weak solution in $H_0^1(\Omega_0)$ of (1.10). Assume that there exists a positive constant M such that $|f(s, u)| \leq M|\phi(s, u)|$ almost everywhere in Ω_0 . Then (f, ϕ) determines uniquely the curvature γ .*

Note that the above result is still valid for a nonpositive function γ .

This paper is organized as follows: In Section 2, we prove Theorems 1.1, 1.2 and 1.3. In Sections 3 and 4, we extend our results to the case of a curved quantum guide defined in \mathbb{R}^3 .

2 Proofs of Theorems 1.1, 1.2 and 1.3

2.1 Proof of Theorem 1.1

Recall that ϕ is an eigenfunction of H_γ , belonging to $H^2(\Omega_0)$. Since ϕ is continuous and $H_\gamma\phi = \lambda\phi$, then $H_\gamma\phi$ is continuous too. Thus, noticing that $c_\gamma(s, 0) = 1$, we deduce the continuity of the function $(s, 0) \mapsto \Delta\phi(s, 0)$ and from (1.7) to (1.9), we get:

$$-\Delta\phi(s, 0) - \frac{\gamma^2(s)}{4}\phi(s, 0) = \lambda\phi(s, 0)$$

and equivalently,

$$\gamma^2(s) = -4\frac{\Delta\phi(s, 0)}{\phi(s, 0)} - 4\lambda \text{ if } \phi(s, 0) \neq 0.$$

2.2 Proof of Theorem 1.2

First, we recall from [1, Remark 25 p.182] the following lemma.

Lemma 2.1. *For a second-order elliptic operator defined in a domain $\omega \subset \mathbb{R}^n$, if $\phi \in H_0^1(\omega)$ satisfies*

$$\int_\omega \sum_{i,j} a_{ij} \frac{\partial\phi}{\partial x_i} \frac{\partial\psi}{\partial x_j} = \int_\omega f\psi \text{ for all } \psi \in H_0^1(\omega)$$

then if ω is of class C^2

$$(f \in L^2(\omega), a_{ij} \in C^1(\overline{\omega}), D^\alpha a_{ij} \in L^\infty(\omega) \text{ for all } i, j \text{ and for all } \alpha, |\alpha| \leq 1)$$

$$\text{imply } (\phi \in H^2(\omega))$$

and for $m \geq 1$, if ω is of class C^{m+2}

$$(f \in H^m(\omega), a_{ij} \in C^{m+1}(\overline{\omega}), D^\alpha a_{ij} \in L^\infty(\omega) \text{ for all } i, j \text{ and for all } \alpha, |\alpha| \leq m+1)$$

$$\text{imply } (\phi \in H^{m+2}(\omega)).$$

Now we can prove the Theorem 1.2.

We have $H_\gamma\phi = f$, so

$$\int_{\Omega_0} [c_\gamma(\partial_s\phi)(\partial_s\psi) + (\partial_u\phi)(\partial_u\psi)] = \int_{\Omega_0} [f - V_\gamma\phi]\psi \text{ for all } \psi \in H_0^1(\Omega_0) \quad (2.1)$$

with c_γ defined by (1.8) and V_γ defined by (1.9).

Using Assumption 1.3, since $\gamma^{(k)} \in L^\infty(\Omega_0)$ for $k = 0, 1, 2$ then $V_\gamma \in L^\infty(\Omega_0)$ and $f - V_\gamma\phi \in L^2(\Omega_0)$. From the hypotheses $\gamma \in C^1(\mathbb{R})$ and $\gamma' \in L^\infty(\mathbb{R})$, we get that $c_\gamma \in C^1(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for any α , $|\alpha| \leq 1$, and so, using Lemma 2.1 for the equation (2.1), we obtain that $\phi \in H^2(\Omega_0)$.

By the same way, we get that $f - V_\gamma\phi \in H^1(\Omega_0)$, $c_\gamma \in C^2(\overline{\Omega_0})$ and $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for any α , $|\alpha| \leq 2$ (from $\gamma \in C^3(\mathbb{R})$, $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for any $k = 0, \dots, 3$). Using Lemma 2.1, we obtain that $\phi \in H^3(\Omega_0)$.

We apply again the Lemma 2.1 to get that $\phi \in H^4(\Omega_0)$ (since $f - V_\gamma\phi \in H^2(\Omega_0)$, $c_\gamma \in C^3(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for all α , $|\alpha| \leq 3$, from the hypotheses $\gamma \in C^4(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$

for $k = 0, \dots, 4$).

Finally, using Assumption 1.3 and Lemma 2.1, we obtain that $\phi \in H^5(\Omega_0)$.

Due to the regularity of Ω_0 , we have $\phi \in H^5(\mathbb{R}^2)$ and $\Delta\phi \in H^3(\mathbb{R}^2)$. Since $\nabla(\Delta\phi) \in (H^2(\mathbb{R}^2))^2$ and $H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$, we can deduce that $\Delta\phi$ is continuous (see [1, Remark 8 p.154]).

Therefore we can conclude by using the continuity of the function

$$(s, 0) \mapsto -\partial_s(c_\gamma(s, 0)\partial_s\phi(s, 0)) - \partial_u^2\phi(s, 0) = f(s, 0) - V_\gamma(s, 0)\phi(s, 0).$$

Therefore, we get: $-\Delta\phi(s, 0) - \frac{\gamma^2(s)}{4}\phi(s, 0) = f(s, 0)$ and equivalently,

$$\gamma^2(s) = -4\frac{\Delta\phi(s, 0)}{\phi(s, 0)} - 4\frac{f(s, 0)}{\phi(s, 0)} \text{ if } \phi(s, 0) \neq 0.$$

2.3 Proof of Theorem 1.3

We prove here that (f, ϕ) determines uniquely γ when γ is a nonnegative function.

For that, assume that Ω_1 and Ω_2 are two quantum guides in \mathbb{R}^2 with same width d . We denote by γ_1 and γ_2 the curvatures respectively associated with Ω_1 and Ω_2 and we suppose that each γ_i satisfies Assumption 1.2 and is a nonnegative function. Assume that $H_{\gamma_1}\phi = f = H_{\gamma_2}\phi$. Then ϕ satisfies

$$-\partial_s((c_{\gamma_1}(s, u) - c_{\gamma_2}(s, u))\partial_s\phi(s, u)) + (V_{\gamma_1}(s, u) - V_{\gamma_2}(s, u))\phi(s, u) = 0. \quad (2.2)$$

Assume that $\gamma_1 \neq \gamma_2$.

Step 1. First, we consider the case where (for example) $\gamma_1(s) < \gamma_2(s)$ for all $s \in \mathbb{R}$.

Let $\epsilon > 0$, $\omega_\epsilon := \mathbb{R} \times I_\epsilon$ with $I_\epsilon =]-\epsilon, 0[$. Multiplying (2.2) by ϕ and integrating over ω_ϵ , we get:

$$\int_{\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)^2 - \int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)\phi\nu_s + \int_{\omega_\epsilon} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0. \quad (2.3)$$

Since $\epsilon \ll 1$, $V_{\gamma_i}(s, u) \simeq -\frac{\gamma_i^2(s)}{4}$ for $i = 1, 2$, and so $V_{\gamma_1}(s, u) - V_{\gamma_2}(s, u) > 0$ in ω_ϵ . Moreover, since

$$c_{\gamma_1}(s, u) - c_{\gamma_2}(s, u) = \frac{u(\gamma_1(s) - \gamma_2(s))(2 - u(\gamma_1(s) + \gamma_2(s)))}{(1 - u\gamma_1(s))^2(1 - u\gamma_2(s))^2}, \quad (2.4)$$

we have $c_{\gamma_1}(s, u) > c_{\gamma_2}(s, u)$ in ω_ϵ .

Since

$$\int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)\phi\nu_s = 0, \quad (2.5)$$

Thus from (2.3)-(2.5), we get

$$\int_{\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)^2 + \int_{\omega_\epsilon} (V_{\gamma_1} - V_{\gamma_2})\phi^2 = 0 \quad (2.6)$$

with $c_{\gamma_1} - c_{\gamma_2} > 0$ in ω_ϵ and $V_{\gamma_1} - V_{\gamma_2} > 0$ in ω_ϵ . We can deduce that $\phi = 0$ in ω_ϵ .

Using a unique continuation theorem (see [10, Theorem XIII.63 p.240]), from $H_\gamma\phi = f$, noting that $-\Delta(U_g^{-1}\phi) = U_g^{-1}f = g^{-1/4}f$, (recall that U_g is defined by (1.6)) and so by $|f| \leq M|\phi|$ we have $|\Delta(U_g^{-1}\phi)| \leq M|g^{-1/4}\phi|$ with $g > 0$ a.e., and we can deduce that $\phi = 0$ in Ω_0 . So we get a contradiction (since $H_\gamma\phi = f$ and f is assumed to be a non null function).

Step 2. From Step 1, we obtain that there exists at least one point $s_0 \in \mathbb{R}$ such that $\gamma_1(s_0) = \gamma_2(s_0)$. Since $\gamma_1 \neq \gamma_2$, we can choose $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that (for example) $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(s) < \gamma_2(s)$ for all $s \in]a, b[$ and $\gamma_1(b) = \gamma_2(b)$ if $b \in \mathbb{R}$. We proceed as in Step 1, considering, in this case, $\omega_\epsilon :=]a, b[\times I_\epsilon$. We study again the equation (2.3) and as in Step 1, we have

$$\int_{\partial\omega_\epsilon} (c_{\gamma_1} - c_{\gamma_2})(\partial_s\phi)\phi\nu_s = 0.$$

Indeed from (2.4) and $\gamma_1(a) = \gamma_2(a)$ we have $c_{\gamma_1}(a, u) = c_{\gamma_2}(a, u)$ and so

$$\int_{-\epsilon}^0 (c_{\gamma_1}(a, u) - c_{\gamma_2}(a, u))\partial_s\phi(a, u)\phi(a, u) du = 0.$$

By the same way if $b \in \mathbb{R}$, we also have $c_{\gamma_1}(b, u) = c_{\gamma_2}(b, u)$. Thus the equation (2.3) becomes (2.6) with $c_{\gamma_1} - c_{\gamma_2} > 0$ in ω_ϵ and $V_{\gamma_1} - V_{\gamma_2} > 0$ in ω_ϵ . So $\phi = 0$ in ω_ϵ and as in Step 1, by a unique continuation theorem, we obtain that $\phi = 0$ in Ω_0 . Therefore we get a contradiction.

Note that the previous theorem is true if we replace the hypothesis " γ is nonnegative " by the hypothesis " γ is nonpositive ". Indeed, in this last case, we just have to take $I_\epsilon =]0, \epsilon[$ and the proof rests valid.

3 Uniqueness result for a \mathbb{R}^3 -quantum guide

Now, we apply the same ideas for a tube Ω in \mathbb{R}^3 . We proceed here as in [3]. Let $s \mapsto \Gamma(s)$, $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$, be a curve in \mathbb{R}^3 . We assume that $\Gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ is a C^4 -smooth curve satisfying the following hypotheses

Assumption 3.1. Γ possesses a positively oriented Frenet frame $\{e_1, e_2, e_3\}$ with the properties that

- i) $e_1 = \Gamma'$,
- ii) $\forall i \in \{1, 2, 3\}$, $e_i \in C^1(\mathbb{R}, \mathbb{R}^3)$,
- iii) $\forall i \in \{1, 2\}$, $\forall s \in \mathbb{R}$, $e'_i(s)$ lies in the span of $e_1(s), \dots, e_{i+1}(s)$.

Recall that a sufficient condition to ensure the existence of the Frenet frame of Assumption 3.1 is to require that for all $s \in \mathbb{R}$ the vectors $\Gamma'(s), \Gamma''(s)$ are linearly independent. Then we define the moving frame $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ along Γ by following [3]. This moving frame better reflects the geometry of the curve and it is still called the Tang frame because it is a generalization of the Tang frame known from the theory of three-dimensional waveguides. Given a C^5 bounded open connected neighborhood ω of $(0, 0) \in \mathbb{R}^2$, let Ω_0 denote the straight tube $\mathbb{R} \times \omega$. We define the curved tube Ω of cross-section ω about Γ by

$$\Omega := \tilde{f}(\mathbb{R} \times \omega) = \tilde{f}(\Omega_0), \tilde{f}(s, u_2, u_3) := \Gamma(s) + \sum_{i=2}^3 u_i \sum_{j=2}^3 R_{ij}(s)e_j(s) = \Gamma(s) + \sum_{i=2}^3 u_i \tilde{e}_i(s) \quad (3.1)$$

with $u = (u_2, u_3) \in \omega$ and

$$R(s) := (R_{ij}(s))_{i,j \in \{2,3\}} = \begin{pmatrix} \cos(\theta(s)) & -\sin(\theta(s)) \\ \sin(\theta(s)) & \cos(\theta(s)) \end{pmatrix},$$

θ being a real-valued differentiable function such that $\theta'(s) = \tau(s)$ the torsion of Γ . This differential equation is a consequence of the definition of the moving Tang frame (see [3, Remark 3]).

Note that R is a rotation matrix in \mathbb{R}^2 chosen in such a way that (s, u_2, u_3) are orthogonal “coordinates” in Ω . Let k be the first curvature function of Ω . Recall that since $\Omega \subset \mathbb{R}^3$, k is a nonnegative function. We assume throughout all this section that the following hypothesis holds:

Assumption 3.2. *i) $k \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $a := \sup_{u \in \omega} \|u\|_{\mathbb{R}^2} < \frac{1}{\|k\|_\infty}$, $k(s) \rightarrow 0$ as $|s| \rightarrow +\infty$*
ii) Ω does not overlap.

The Assumption 3.2 assures that the map \tilde{f} (defined by (3.1)) is a diffeomorphism (see [3]) in order to identify Ω with the Riemannian manifold $(\Omega_0, (g_{ij}))$ where (g_{ij}) is the metric tensor induced by \tilde{f} , i.e. $(g_{ij}) := {}^t J(\tilde{f}) \cdot J(\tilde{f})$, ($J(\tilde{f})$ denoting the Jacobian matrix of \tilde{f}). Recall that $(g_{ij}) = \text{diag}(h^2, 1, 1)$ (see [3]) with

$$h(s, u_2, u_3) := 1 - k(s)(\cos(\theta(s))u_2 + \sin(\theta(s))u_3). \quad (3.2)$$

Note that Assumption 3.2 implies that $0 < 1 - a\|k\|_\infty \leq 1 - h(s, u_2, u_3) \leq 1 + a\|k\|_\infty$ for all $s \in \mathbb{R}$ and $u = (u_2, u_3) \in \omega$. Moreover, setting

$$g := h^2 \quad (3.3)$$

we can replace the Dirichlet Laplacian operator $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ by the Laplace-Beltrami operator K_g acting on $L^2(\Omega_0, hdsdu)$ relative to the metric tensor (g_{ij}) . We can rewrite K_g into a Schrödinger-type operator acting on $L^2(\Omega_0, dsdu)$. Indeed, using the unitary transformation

$$\begin{aligned} W_g : L^2(\Omega_0, hdsdu) &\longrightarrow L^2(\Omega_0, dsdu) \\ \psi &\longmapsto g^{1/4}\psi \end{aligned} \quad (3.4)$$

setting

$$H_k := W_g K_g W_g^{-1}, \quad (3.5)$$

we get

$$H_k = -\partial_s(h^{-2}\partial_s) - \partial_{u_2}^2 - \partial_{u_3}^2 + V_k \quad (3.6)$$

where ∂_s denotes the derivative relative to s and ∂_{u_i} denotes the derivative relative to u_i and with

$$V_k := -\frac{k^2}{4h^2} + \frac{\partial_s^2 h}{2h^3} - \frac{5(\partial_s h)^2}{4h^4}. \quad (3.7)$$

We assume also throughout all this section that the following hypotheses hold:

Assumption 3.3. *i) $k' \in L^\infty(\mathbb{R})$, $k'' \in L^\infty(\mathbb{R})$*
ii) $\theta \in C^2(\mathbb{R})$, $\theta' = \tau \in L^\infty(\mathbb{R})$, $\theta'' \in L^\infty(\mathbb{R})$.

Remarks: Note that, as for the 2-dimensional case, such operator H_k (defined by (3.2)-(3.7)) admits bound states and that the minimum eigenvalue λ_1 is simple and associated with a positive eigenfunction ϕ_1 (see [3, 7]). Still note that (λ, ϕ) is an eigenpair of the operator H_k acting on $L^2(\Omega_0, dsdu)$ means that $(\lambda, W_g^{-1}\phi)$ is an eigenpair of $-\Delta_D^\Omega$ acting on $L^2(\Omega)$ (with W_g defined by (3.4)). Finally, note that by [11, Theorem 7.1] any eigenfunction of H_k is continuous and by [1, Remark 25 p.182] any eigenfunction of H_k belongs to $H^2(\Omega_0)$.

As for the 2-dimensional case, first we prove that the data of one eigenpair determines uniquely the curvature.

Theorem 3.1. *Let Ω be the curved guide in \mathbb{R}^3 defined as above. Let k be the first curvature function of Ω . Assume that Assumptions 3.1 to 3.3 are satisfied. Let H_k be the operator defined by (3.2)-(3.7) and (λ, ϕ) be an eigenpair of H_k .*

Then $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\lambda$ for all s when $\phi(s,0,0) \neq 0$.

Then, under

Assumption 3.4. *i) $k \in C^5(\mathbb{R}), k^{(i)} \in L^\infty(\mathbb{R})$ for all $i = 0, \dots, 5$*

ii) $\theta \in C^5(\mathbb{R}), \theta^{(i)} \in L^\infty(\mathbb{R})$ for all $i = 1, \dots, 5$

where $k^{(i)}$ (resp. $\theta^{(i)}$) denotes the i -th derivative of k (resp. of θ), we obtain the following result:

Theorem 3.2. *Let Ω be the curved guide in \mathbb{R}^3 defined as above. Let k be the first curvature function of Ω . Assume that Assumptions 3.1 to 3.4 are satisfied. Let H_k be the operator defined by (3.2)-(3.7). Let $f \in H^3(\Omega_0) \cap C(\Omega_0)$ and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of $H_k\phi = f$ in Ω_0 .*

Then ϕ is a classical solution and $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\frac{f(s,0,0)}{\phi(s,0,0)}$ for all s when $\phi(s,0,0) \neq 0$.

Remarks: Recall that in \mathbb{R}^3 , k is a nonnegative function and that the condition imposed on ϕ ($\phi(s,0,0) \neq 0$) in Theorems 3.1 and 3.2 is satisfied by the positive eigenfunction ϕ_1 .

As for the two-dimensional case, we can restrain the hypotheses upon the regularity of the functions k and θ .

For a guide with a known torsion, we obtain the following result:

Theorem 3.3. *Let Ω be the curved guide in \mathbb{R}^3 defined as above. Let k be the first curvature function of Ω and let τ be the second curvature function (i.e. the torsion) of Ω . Denote by θ a primitive of τ and suppose that $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$. Assume that Assumptions 3.1 to 3.3 are satisfied. Let H_k be the operator defined by (3.2)-(3.7). Let $f \in L^2(\Omega_0)$ be a non null function and let $\phi \in H_0^1(\Omega_0)$ be a weak solution of $H_k\phi = f$ in Ω_0 . Assume that there exists a positive constant M such that $|f(s, u)| \leq M|\phi(s, u)|$ almost everywhere in Ω_0 .*

Then the data (f, ϕ) determines uniquely the first curvature function k if the torsion τ is given.

4 Proofs of Theorem 3.1, 3.2 and 3.3

4.1 Proof of Theorem 3.1

Recall that ϕ is an eigenfunction of H_k . Since ϕ is continuous, $H_k\phi = \lambda\phi$ and $\phi \in H^2(\Omega_0)$ then $H_k\phi$ is continuous. Therefore, for $u = (u_2, u_3) = (0, 0)$, we get: $-\Delta\phi(s, 0, 0) - \frac{k^2(s)}{4}\phi(s, 0, 0) = \lambda\phi(s, 0, 0)$ and equivalently, $k^2(s) = -4\frac{\Delta\phi(s,0,0)}{\phi(s,0,0)} - 4\lambda$ if $\phi(s, 0, 0) \neq 0$.

4.2 Proof of Theorem 3.2

We follow the proof of Theorem 1.2. We have $H_k\phi = f$ with $\phi \in H_0^1(\Omega_0)$. So

$$\int_{\Omega_0} [h^{-2}(\partial_s\phi)(\partial_s\psi) + (\partial_{u_2}\phi)(\partial_{u_2}\psi) + (\partial_{u_3}\phi)(\partial_{u_3}\psi)] = \int_{\Omega_0} [f - V_k\phi]\psi \text{ for all } \psi \in H_0^1(\Omega_0) \quad (4.1)$$

with h defined by (3.2) and V_k defined by (3.7).

From Assumptions 3.2 and 3.3, since $k, k', k'', \theta', \theta''$ are bounded, we deduce that $V_k \in L^\infty(\Omega_0)$. Therefore $f - V_k \phi \in L^2(\Omega_0)$. Moreover we have also $h^{-2} \in C^1(\overline{\Omega_0})$ and $D^\alpha(h^{-2}) \in L^\infty(\Omega_0)$ for any α , $|\alpha| \leq 1$. Thus, using Lemma 2.1 for the equation (4.1), we obtain that $\phi \in H^2(\Omega_0)$.

By the same way, we get that $f - V_k \phi \in H^1(\Omega_0)$, $h^{-2} \in C^2(\overline{\Omega_0})$ and $D^\alpha(h^{-2}) \in L^\infty(\Omega_0)$ for any α , $|\alpha| \leq 2$ (since $k \in C^3(\mathbb{R})$, $\theta \in C^3(\mathbb{R})$ and all of their derivatives are bounded). Using Lemma 2.1, we obtain that $\phi \in H^3(\Omega_0)$.

We apply again the Lemma 2.1 to get that $\phi \in H^4(\Omega_0)$ (since $f - V_\gamma \phi \in H^2(\Omega_0)$, $c_\gamma \in C^3(\overline{\Omega_0})$, $D^\alpha c_\gamma \in L^\infty(\Omega_0)$ for all α , $|\alpha| \leq 3$, from the hypotheses $\gamma \in C^4(\mathbb{R})$ and $\gamma^{(k)} \in L^\infty(\mathbb{R})$ for $k = 0, \dots, 4$).

Finally, using Assumption 3.4 and Lemma 2.1, we obtain that $\phi \in H^5(\Omega_0)$. Due to the regularity of Ω_0 (see [1, Note p.169]), we have $\phi \in H^5(\mathbb{R}^3)$ and $\Delta \phi \in H^3(\mathbb{R}^3)$. Since $\nabla(\Delta \phi) \in (H^2(\mathbb{R}^3))^3$ and $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, we can deduce that $\Delta \phi$ is continuous (see [1, Remark 8 p.154]).

Thus we conclude as in Theorem 1.2 and for $u = (u_2, u_3) = (0, 0)$, we get: $-\Delta \phi(s, 0, 0) - \frac{k^2(s)}{4} \phi(s, 0, 0) = f(s, 0, 0)$ and equivalently, $k^2(s) = -4 \frac{\Delta \phi(s, 0, 0)}{\phi(s, 0, 0)} - 4 \frac{f(s, 0, 0)}{\phi(s, 0, 0)}$ if $\phi(s, 0, 0) \neq 0$.

4.3 Proof of Theorem 3.3

We prove here that (f, ϕ, θ) determines uniquely k .

Assume that Ω_1 and Ω_2 are two guides in \mathbb{R}^3 . We denote by k_1 and k_2 the first curvatures functions associated with Ω_1 and Ω_2 and we denote by θ a primitive of τ the common torsion of Ω_1 and Ω_2 . We suppose that k_1, k_2 and θ satisfy the Assumptions 3.2 and 3.3 and that $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$. Assume that $H_{k_1} \phi = f = H_{k_2} \phi$.

Then ϕ satisfies

$$\begin{aligned} & -\partial_s((h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3))\partial_s \phi(s, u_2, u_3)) \\ & + (V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3))\phi(s, u_2, u_3) = 0 \end{aligned} \quad (4.2)$$

where h_1 (associated with k_1) is defined by (3.2), V_{k_1} is defined by (3.7), h_2 (associated with k_2) is defined by (3.2) and V_{k_2} is defined by (3.7).

Assume that $k_1 \not\equiv k_2$.

Step 1. First, we consider the case where (for example) $k_1(s) < k_2(s)$ for all $s \in \mathbb{R}$. Recall that each k_i is a nonnegative function.

Let $\epsilon > 0$ and denote by $J_\epsilon :=]-\epsilon, 0[\times]-\epsilon, 0[$, $O_\epsilon := \mathbb{R} \times J_\epsilon$ with ϵ small enough to have $J_\epsilon \subset \omega$ (recall that $\Omega_0 = \mathbb{R} \times \omega$).

Multiplying (4.2) by ϕ and integrating over O_ϵ , we get:

$$\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)\phi \nu_s + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 = 0. \quad (4.3)$$

Since $\epsilon \ll 1$, $V_{k_i} \simeq -\frac{k_i^2(s)}{4}$ for $i = 1, 2$, and so $V_{k_1}(s, u_2, u_3) - V_{k_2}(s, u_2, u_3) > 0$ in O_ϵ .

Moreover, note that:

$$h_1^{-2}(s, u_2, u_3) - h_2^{-2}(s, u_2, u_3) = \frac{\alpha(s, u_2, u_3)(k_1(s) - k_2(s))(h_1(s, u_2, u_3) + h_2(s, u_2, u_3))}{h_1^2(s, u_2, u_3)h_2^2(s, u_2, u_3)} \quad (4.4)$$

with $\alpha(s, u_2, u_3) := \cos(\theta(s))u_2 + \sin(\theta(s))u_3$.

Since $(u_2, u_3) \in J_\epsilon$ and $0 \leq \theta(s) \leq \frac{\pi}{2}$ for all $s \in \mathbb{R}$, we have $\alpha(s, u_2, u_3) < 0$. Therefore, by (4.4), we deduce that $h_1^{-2} - h_2^{-2} > 0$ in O_ϵ .

Thus $\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 \geq 0$.

Note also that:

$$\int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)\phi \nu_s = 0. \quad (4.5)$$

Therefore, from (4.3) and (4.5) we get:

$$\int_{O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)^2 + \int_{O_\epsilon} (V_{k_1} - V_{k_2})\phi^2 = 0 \quad (4.6)$$

with $h_1^{-2} - h_2^{-2} > 0$ in O_ϵ and $V_{k_1} - V_{k_2} > 0$ in O_ϵ .

From (4.6) we can deduce that $\phi = 0$ in O_ϵ . Using a unique continuation theorem (see [10, Theorem XIII.63 p.240]), from $H_{k_1} \phi = f$, noting that $-\Delta(W_g^{-1}\phi) = W_g^{-1}f = g^{-1/4}f$, by $|f| \leq M|\phi|$ a.e. in Ω_0 , we can deduce that $\phi = 0$ in Ω_0 . So we get a contradiction since f is assumed to be a non null function.

Step2. From Step 1, we obtain that there exists at least one point $s_0 \in \mathbb{R}$ such that $k_1(s_0) = k_2(s_0)$. Since $k_1 \not\equiv k_2$, we can choose $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$ such that (for example) $k_1(a) = k_2(a)$, $k_1(s) < k_2(s)$ for all $s \in]a, b[$ and $k_1(b) = k_2(b)$ if $b \in \mathbb{R}$. We proceed as in Step 1, considering in this case $O_\epsilon :=]a, b[\times J_\epsilon$. From $k_1(a) = k_2(a)$, we get that $h_1^{-2}(a, u_2, u_3) = h_2^{-2}(a, u_2, u_3)$. Therefore we obtain $\int_{\partial O_\epsilon} (h_1^{-2} - h_2^{-2})(\partial_s \phi)\phi \nu_s = 0$. So (4.3) becomes (4.6) with $h_1^{-2} - h_2^{-2} > 0$ in O_ϵ and $V_{k_1} - V_{k_2} > 0$ in O_ϵ . So $\phi = 0$ in O_ϵ and as in Step 1, by a unique continuation theorem, we obtain that $\phi = 0$ in Ω_0 . Therefore we get a contradiction.

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