COMBINATORIAL METHODS FOR INTERVAL EXCHANGE TRANSFORMATIONS

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ABSTRACT. This is a survey on the big questions about interval exchanges (minimality, unique ergodicity, weak mixing, simplicity) with emphasis on how they can be tackled by mainly combinatorial methods.

Interval exchange transformations, defined in Definition 1 below, constitute a famous class of dynamical systems; they were introduced by V. Oseledec [25], and have been extensively studied by many famous authors; up to now, the main results in this swifly-evolving field can be found in the two excellent courses [33] and [34]. To study interval exchanges, three kind of methods can be used: by definition, these systems are *one-dimensional*, and the first results on them naturally used one-dimensional techniques; then the strongest results on interval exchanges have been obtained by lifting the transformation to *higher dimensions* and using deep geometric methods. However, many of these results have been reproved by using *zero-dimensional* methods; these use the codings of orbits to replace the original dynamical system by a symbolic dynamical system, as in Definition 4 below.

Now, most of the existing texts, including the two courses mentioned above, focus on the geometric methods; the present survey wants to emphasize what can be achieved by the two other kinds of methods, which have both a strong flavour of combinatorics. The one-dimensional methods yield the basic results, some of which the reader will find in Section 2 below, but also the famous Keane counter-examples described in Section 4, and a very nice new result of M. Boshernitzan which is the object of our Section 6; Sections 3 and 5 are devoted to the zero-dimensional methods; the necessary definitions of word combinatorics, symbolic and measurable dynamics are given in Section 1. All those sections are also retracing the colourful history of the theory of interval exchanges, made with big conjectures brilliantly solved after long waits; thus we finish the paper by explaining in Section 7 the last big open question in the domain.

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1. PRELIMINARIES

1.1. Interval exchange transformations.

Definition 1. *Let $r \geq 2$. Let Λ_r be the set of vectors $(\lambda_1, ...\lambda_r)$ in $\mathbb{R}^r$ such that $0 \leq \lambda_i \leq 1$ for all i and $\Sigma_{i=1}^r \lambda_i = 1$. An* r-interval exchange transformation*, or iet for short, is given by a vector $\lambda \in \Lambda_r$ and a permutation π of $\{1, 2, \dots, r\}$. The map $T_{\lambda,\pi}$ is the piecewise translation defined by partitioning the interval $X = [0, 1[$ into r sub-intervals of lengths $\lambda_1, \lambda_2, \dots, \lambda_r$ and rearranging*

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them according to the permutation π; or, formally,

$$T_{\lambda,\pi}x = x + \sum_{\pi^{-1}j<\pi^{-1}i} \lambda_j - \sum_{j<i} \lambda_j$$

when x is in the interval

$$X_i = \left[\sum_{j<i} \lambda_j, \sum_{j\leq i} \lambda_j\right[.$$

Throughout this paper, we denote $T_{\lambda,\pi}$ simply by T when there is no ambiguity; we call β_i, $1 \leq i \leq r-1$, the i-th discontinuity of T^{-1}, namely $\beta_i = \sum_{\pi^{-1}j\leq\pi^{-1}i} \lambda_j$, while γ_i is the i-th discontinuity of T, namely $\gamma_i = \sum_{j\leq i} \lambda_j$. We shall use also $\gamma_0 = 0$, $\gamma_r = 1$. Then X_i is the interval $[\gamma_{i-1}, \gamma_i[$.

Warning: roughly half the texts on interval exchanges re-order the subintervals by π^{-1}; as it is not always clear to which half a given text belongs, we insist that the present definition corresponds to the following ordering of TX_i: from left to right, $TX_{\pi(1)}, ...TX_{\pi(r)}$. It makes sense to re-order also the X_i, thus defining T by two permutations π_0 and π_1 (though of course sometimes π_0^{-1} and π_1^{-1} are used...); this is done in most recent texts, such as [33] [34], but would not be really useful in the present combinatorial context.

Note that $T_{\lambda,\pi}$ is not continuous, and thus if we apply the definitions strictly $(X, T_{\lambda,\pi})$ is not a topological dynamical system; there are ways to get rid of this problem, for example by the natural coding defined in Definition 5 below; but in the present context, it is enough for us that notions like minimailty and unique ergodicity (see Definition 4 below) can be defined for interval exchanges.

1.2. **Word combinatorics.**

Definition 2. *We look at finite* words *on a finite alphabet $\mathcal{A}$. A word $w_1...w_k$, is of* length *k and we write $|w| = k$. The* concatenation *of two words w and w' is denoted by ww'. The* empty word *is the unique word of length zero.*
A word $w = w_1...w_k$ occurs at place *i in a word $v = v_1...v_s$ or an infinite sequence $v = v_1v_2...$ if $w_1 = v_i$, ...$w_k = v_{i+k-1}$. We say that w is a* factor *of v. When it is finite, we denote by $N(w, v)$ the number of occurrences of w in v.*
The empty word *is a factor of any v.* Prefixes *and* suffixes *are defined in the usual way.*
A language *L is a set of words such that if w is in L, all its factors are in L, and wb is in L for at least one letter b of $\mathcal{A}$.*
A language L is uniformly recurrent *if for each w in L there exists n such that w occurs in each word of length n of L.*
The language $L(u)$ of an infinite sequence is the set of all its finite factors.

Definition 3. *Let L be a fixed language. A word w is* right special, *resp.* left special *if there exist at least two different letters x such that xw, resp. wx, is in L.*
The complexity *of L is the function p_L which to each positive integer n associates the number of different words of length n in L.*
The Rauzy graph *of length n of L is the graph whose vertices are the words of length n in L, with an edge $w \to w'$ if there exists a word v of length $n-1$ such that $w = av$, $w' = vb$, and $avb \in L$.*

Note that the Rauzy graphs should not be confused with the *Rauzy diagrams* used in [33][34] to describe the induction on interval exchanges.

1.3. **Dynamical systems.**

Definition 4. *The* symbolic dynamical system *associated to a language L is the one-sided shift $S(x_0x_1x_2...) = x_1x_2...$ on the subset X_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $s < t$, $x_s...x_t$ is in L.*
For a word $w = w_1...w_k$ in L, the cylinder $[w]$ *is the set $\{x \in X_L; x_0 = w_1, ...x_{k-1} = w_k\}$.*
(X_L, S) *is* minimal *if L is uniformly recurrent.*
(X_L, S) *is* uniquely ergodic *if there is one S-invariant probability measure μ; then the* frequency *of the word w is the measure $\mu[w]$.*

Starting from any dynamical system (X, T) (in most cases, geometric in origin), we can get a symbolic dynamical system:

Definition 5. *For a transformation T defined on a set X, partitioned into X_1, ... X_r, and a point x in X, its* trajectory *is the infinite sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = i$ if $T^n x$ falls into X_i, $1 \le i \le r$.*
The language $L(T)$ is the set of all finite factors of its trajectories.
The coding *of (X, T) by the partition $\{X_1, ...X_r\}$ is the symbolic dynamical system $(X_{L(T)}, S)$.*
The natural coding *of an r-interval exchange is its coding by the partition into the intervals X_i, $1 \le i \le r$ defined above.*

If the transformation T is minimal (i.e. every orbit is dense), all its trajectories have the same finite factors, and the language $L(T)$ is uniformly recurrent; the special words depend on the language and not on the individual trajectories; thus they are defined by any trajectory of T. If there is no periodic orbit, every word w is a factor of a bispecial word; hence the bispecial words determine the finite factors of the trajectories, and thus the symbolic dynamical system $(X_{L(T)}, S)$.

If $(X_{L(T)}, S)$ is the natural coding of an interval exchange $T_{\lambda,\pi}$, it is not topologically conjugate to $([0, 1[, T_{\lambda,\pi})$, but it shares all its properties of minimality and unique ergodicity, and any invariant measure for one of these systems can be carried to the other one.

Definition 6. *The* induced map *of any transformation T on a set Y is the map $y \to T^{r(y)}y$ where, for $y \in Y$, $r(y)$ is the smallest $r \ge 1$ such that $T^r y$ is in Y (in all cases considered in this paper, $r(y)$ is finite).*

Definition 7. *Let (X, T, μ) be a finite measure-preserving dynamical system. It is* ergodic *if every invariant subset A (i.e. such that $\mu(T^{-1}A\Delta A) = 0$) is trivial, i.e. has zero or full measure.*

A complex number θ is an eigenvalue *of T (denoted multiplicatively) if there exists a non-constant f in $\mathcal{L}^2(X,\mathbb{C})$ such that $f \circ T = \theta f$ in $\mathcal{L}^2(X,\mathbb{C})$; f is then an* eigenfunction *for the eigenvalue θ. $\theta = 1$ is not an eigenvalue iff T is ergodic. T is* weakly mixing *if it has no eigenvalue.*

We shall use the famous *ergodic theorem*, attributed to Birkhoff (1931) though the Russian school prefers to call it the Khinchin theorem:

Proposition 1. *If (X, T, μ) is ergodic, g a function in $\mathcal{L}_1(X)$, then when $N \to +\infty$*

$$\frac{1}{N}\sum_{n=0}^{N-1} g \circ T^n \to \mu(g) \quad \text{in } \mathcal{L}_1 \text{ and almost everywhere.}$$

Definition 8. *In (X, T, μ), a (Rokhlin)* tower *is a collection of disjoint measurable sets called* levels *F, TF, ..., $T^{h-1}F$.*

2. MINIMALITY

The question of *minimality* of interval exchanges is an old one, for which we find useful to give a quick reminder. The following fundamental lemma is proved in [21] and [19]; note that it was first stated as an independent lemma in [18], but with the improved, though unfortunately false, bound $r+1$.

Lemma 2. *The induced map U_J of an r-interval exchange T on an interval J is an s-interval exchange for some $s \leq r+2$.*

Proof We look at the (at most) $r+1$ points made by the γ_i and the two endpoints of J; if x is any of these points, let $s(x)$ be the largest negative integer s such that $T^s x$ is in the interior of J; we partition J by the (at most) $r+1$ points $T^{s(x)}x$. Then on each of these (at most) $r+2$ intervals U_J is continuous and is of the form T^j for a constant j. □

Definition 9. *$T_{\lambda,\pi}$ satisfies the* i.d.o.c. property [21] *if the negative orbits of the discontinuity points γ_i, $1 \leq i \leq r-1$, are infinite and disjoint.*

Proposition 3. [21] *The i.d.o.c. condition implies minimality.*

Proof
We show first that there is no periodic point: if $T^m x = x$, let b be the $T^n\gamma_j$ nearest to x on the left, for $0 \leq j \leq r-1$, $0 \leq n \leq m-1$. Then $T^m b = b$ as each T^i, $1 \leq i \leq m-1$, is an isometry on $[b, x]$; this contradicts the i.d.o.c. unless $b = T^a 0$, and 0 is itself an image of a discontinuity if $\pi 1 \neq 1$, while $\pi 1 = 1$ contradicts also the i.d.o.c. (in different ways, depending whether $\pi 2 \neq 2$, or $\pi 2 = 2$ and $\pi 3 \neq 3$, etc...).

Given an interval J, we make the *induction castle* of J: J is partitioned into s subintervals J_t, $1 \leq t \leq s$, the $T^j J_t$, $1 \leq t \leq s$, $0 \leq j \leq h_t - 1$, are disjoint intervals, $T^{h_t} J_t = U_J J_t \subset J$. Let Y be $\cup_{1 \leq t \leq s} \cup_{0 \leq j \leq h_t - 1} T^j J_t$; Y is a union of intervals, let G be the union of their left ends. Then $TY = Y$, and for $x \in G$, either $Tx \in G$ or $x = \gamma_j$ for some $0 \leq j \leq r-1$. Because G is finite and T aperiodic, for all $x \in G$ there exists n such that $T^n x = \gamma_j$ for some $0 \leq j \leq r-1$. Similarly for $x \in G$, either $T^{-1}x \in G$ or $T^{-1}x = \beta_j$ for some $1 \leq j \leq r-1$, and there exists m such that $T^{-m}x = \beta_j$ for some $1 \leq j \leq r-1$. The only possibility for x which does not contradict the i.d.o.c. is $x = \gamma_0$, thus $Y = X$.

Now if the orbit of x is not dense, its closure does not intersect an interval J, which contradicts the fact that the induction castle of J fills X. □

It is well known and proved in [21] that, if the permutation π is *irreducible* ($\pi\{0, ...l\} \neq \{0, ...l\}$ if $l \neq r$), then *total irrationality* (the λ_i satisfy no rational relation except $\Sigma\lambda_i = 1$) implies the i.d.o.c. condition. But the i.d.o.c. condition is strictly weaker than total irrationality: for three intervals it means that λ_1 and λ_2 do not satisfy any rational relation of the forms $p\lambda_1 + q\lambda_2 = p - q$, $p\lambda_1 + q\lambda_2 = p - q + 1$, or $p\lambda_1 + q\lambda_2 = p - q - 1$, for p and q integers. Also, the i.d.o.c. condition is not equivalent to minimality, here is a counter-example from [33]: $k = 4$, $\pi = (4321)$ (i.e. $\pi 1 = 4$, $\pi 2 = 3$, etc...), $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$, $\frac{\lambda_1}{\lambda_2} = \frac{\lambda_3}{\lambda_4}$ is irrational.

The i.d.o.c. condition ensures that each trajectory u is uniformly recurrent, (X_u, S) is minimal, and the language $L(u)$ is the same for all the trajectories.

Proposition 4. *The language of the natural coding of an r-interval exchange satisfying the i.d.o.c. condition has complexity $(r-1)n+1$ for all $n \geq 0$.*

Proof
The cylinder $[w_1...w_n]$ is the set $\cap_{i=0}^{n-1} T^{-i} X_{w_{i+1}}$. By induction on n, these are either empty sets or intervals, and the partition of X into nonempty cylinders of length n is the partition of X by the points $T^{-i}\gamma_j$, $1 \leq j \leq r-1$, $0 \leq i \leq n-1$. The i.d.o.c. condition ensures that all these points are different. □

Note that the left special words of length n are the prefixes of length n of the trajectories of the β_i, $1 \leq i \leq r-1$. Thus when n is large enough, there are $r-1$ left special words of length n, with two extensions for each one - and the same for right special words by looking at T^{-1}.

3. Invariant measures

The question of *unique ergodicity* was the first big problem to be considered for interval exchanges; before tackling it, we look at what information can be obtained, by purely combinatorial considerations, on invariant measures for interval exchanges. A similar study on more general classes of systems can be found in [16].

We shall need first the fundamental

Lemma 5. $p_L(n+1) - p_L(n) = \Sigma_{w \in RS_n}(\#D(w) - 1)$ *where* RS_n *is the set of right special words of length* n *and* $D(w)$ *is the set of letters* a *such that* wa *is in* L.

Proof
$p_L(n+1) = \Sigma_{w \in L_n} \#D(w)$, thus $p_L(n+1) - p_L(n) = \Sigma_{w \in L_n}(\#D(w) - 1)$, and $\#D(w) - 1 = 0$ whenever w is not in RS_n. □

A consequence of Lemma 5 is a famous result of M. Morse and G. Hedlund [24]:

Proposition 6. *If* $p_L(n) \leq n$ *for at least one* n, *then* L *is the union of a finite number of* $L(u^j)$ *where each infinite sequence* u^j *is ultimately periodic, (namely, there exist positive integers* n_j *and* t_j *such that* $u^j_{n+t_j} = u^j_n$ *for all* $n > n_j$*), and* $p_L(n)$ *is bounded.*

Proof
Then either $p(1) = 1$ or there exists m with $p(m+1) = p(m)$. There is no right special word of length m, and a loop in each connected component of the Rauzy graph. □

The following proposition is the first of many contributions from M. Boshernitzan to this survey:

Proposition 7. [3] *A minimal symbolic system such that* $p(n+1) - p(n) = r$ *for all* n *has at most* $r-1$ *ergodic invariant measures.*

Proof
We begin by showing that there are at most r invariant measures. By Lemma 5, for each n there are at most r right special words of length n; we denote them by $d_{n,1}, ...d_{n,r}$, possibly with repetitions. For a word w, we define $\nu_{n,i}(w) = \lim_{t \to +\infty} \frac{N(w, d^t_{n,i})}{t|d_{n,i}|}$. By taking subsequences, we ensure the $\nu_{n,i}$ converge to a probability μ_i on X_L, $1 \leq i \leq r$.

We remark that if n is large enough *every word in* L *of length at least* $(r+2)n$ *contains one of the* $d_{n,i}$, because, in such a word, words of length n can occur at $(r+1)n$ places at least, hence the words at two of these occurences must be equal, and if there is no right special words among them this creates a loop in the Rauzy graph thus X_L contains ultimately periodic sequences, which is impossible for a minimal system of complexity $rn + s$.

Let μ be an ergodic invariant probability on X_L. By the ergodic theorem, we choose an $x \in X_L$ such that for every $w \in L$, $\mu([w]) = \lim_{t\to+\infty} \frac{N(w,x_0...x_{t-1})}{t}$. We fix n and cut x into disjoint words of length $(r+2)n$, each of which contains a $d_{n,i}$.Thus there exists $t(n)$ such that $d_{n,t(n)}$ occurs in the j-th word for a set of j of upper density at least $\frac{1}{r}$, and we choose t such that $t(n) = t$ for infinitely many n. For those n and m large, $N(w, x_0...x_{m(r+2)n-1} \geq \frac{m}{2r}N(w, d_{n,t})$. Dividing by the lengths and letting m then n tend to infinity we get $\mu([w]) \geq \frac{1}{2r(r+2)}\mu_t([w])$. Thus $\mu = c\mu_t + (1-c)\mu'$ for some positive measure μ' and, as ergodic measures are extremal, $\mu = \mu_t$.

To improve the bound, we notice that if for infinitely many n there are at most $r-1$ right special words of length n, the above reasoning implies that there are at most $r-1$ ergodic invariant measures. Thus we can assume that for each n large enough there are exactly r right special words, and thus by Lemma 5 each of them can be extended by two letters only. We shall show now that there exist K and $1 \leq j \leq r$ such that for infinitely many n *every word in L of length at least* Kn *contains one of the* $d_{n,i}, i \neq j$; then again the above reasoning implies our result.

Indeed, we suppose the above assertion is not satisfied; then for every j, and n large enough, there is a path in the Rauzy graph of length n, going from $d_{n,j}$ to $d_{n,j}$, not meeting any $d_{n,i}$ except at the two ends; by minimality and because there are only two letters extending $d_{n,j}$, this is the only path satisfying these conditions, and our hypothesis implies that it can be followed $q_{n,j}$ times consecutively, with $q_{n,j}$ tending to infinity with n. By following this path, we define a word $g_{n,j}$ such that $d_{n,j}g_{n,j}$ begins and ends with $d_{n,j}$, with no other occurrence of any $d_{n,i}$, and the word $g_{n,j}^{q_{n,j}}$ occurs in L; our hypothesis implies also that $\frac{q_{n,j}|g_{n,j}|}{n}$ tends to infinity with n. If we take $q_{n,j}$ maximal, then $g_{n,j}^{q_{n,j}-1}$ is right special, thus is identified with some $d_{n_1,i}$; by unicity of the path, $g_{n_1,i}$ is identified with $g_{n,j}$, thus $q_{n_1,i}$ with $q_{n,j}$; but then $\frac{q_{n_1,i}|g_{n_1,i}|}{n_1} \leq \frac{q_{n,j}}{q_{n,j}-1}$ is smaller than 2 for some arbitrarily large n_1, which contradicts our hypothesis. □

This result is not optimal: the hypothesis can be weakened [3], and the optimal bound for the number of invariant measures for an r- interval exchange is $[\frac{r}{2}]$ [17][29]. But it is enough for our purpose, as it implies that, under the i.d.o.c. condition, three-interval exchanges are uniquely ergodic, and four-interval exchanges have at most two invariant ergodic measures.

Note that unique ergodicity is a strong notion, because of the following classic result, which is stronger than the ergodic theorem and has a simple proof:

Proposition 8. *If (X, T) is uniquely ergodic, μ its invariant probability measure, g a continuous function on X, then when N tends to $+\infty$*

$$\frac{1}{N}\sum_{n=0}^{N-1} g \circ T^n \to \mu(g) \quad \textit{uniformly.}$$

Proof

Otherwise, there exist $\delta > 0$, f continuous, a sequence $n_k \to +\infty$ and a sequence of points x_{n_k} such that

$$\left|\frac{1}{n_k}\sum_{n=0}^{n_k-1} g(T^n x) - \mu(g)\right| > \delta.$$

By compacity, there exists a subsequence m_k of n_k such that for every continuous g,

$$\lim_{k\to+\infty}\frac{1}{m_k}\sum_{n=0}^{m_k-1} g\circ T^n$$

exists and defines a measure ν. ν is then a T-invariant probability, hence $\nu=\mu$, which contradicts the assumption. □

4. KEANE'S COUNTER-EXAMPLES

It was conjectured by M. Keane [21] that the i.d.o.c. condition implies unique ergodicity for every r; when it was made this conjecture had already been disproved by W. Veech [28]. Veech's counter-example was a five-interval exchange. Then Keane [22] lowered the number of intervals required for a counter-example to four, which is optimal in view of Proposition 7. But his paper uses very different techniques, and there appear for the first time two ideas which were to be named and systematically studied later: one is the *induction*, a different form of which will give the *Rauzy induction* and is the starting point of the geometric methods; the other one is the use of *matrices for adic systems*.

The remainder of this section is an adapted version of [22], where some proofs have been expanded and some terminology updated.

Lemma 9. *Let T be the 4-interval exchange with vector $(\lambda_1,...\lambda_4)$ and permutation π sending $1\to 4, 2\to 2, 3\to 1, 4\to 3$ (denoted by (4213)). Suppose*

- $\lambda_1<\lambda_4<\lambda_3$, $\lambda_4<\lambda_1+\lambda_2$,
- *for* $1\le k<m$ $T^{k-1}[\gamma_1,\beta_1[\subset X_2$, *then* $T^{m-1}[\gamma_1,\beta_1[\not\subset X_2$ *and* $T^{m-1}[\gamma_1,\beta_1[\subset X_2\cup X_3$,
- *for* $1\le k<p$ $T^k[\gamma_2,\gamma_2+\lambda_4[\subset X_3$, *then* $T^p[\gamma_2,\gamma_2+\lambda_4[\not\subset X_3$ *and* $T^p[\gamma_2,\gamma_2+\lambda_4[\subset X_3\cup X_4$.

Then the induced map U of T on $[\gamma_3,1[$ is a 4-interval exchange on an interval of length λ_4, with permutation (2431) and vector $(\lambda'_4,\lambda'_3,\lambda'_2,\lambda'_1)$ such that, if $\lambda'=(\lambda'_1,\lambda'_2,\lambda'_3,\lambda'_4)$, then

$$\lambda=A_{m,p}\lambda',$$

where m and p are nonnegative integers and $A_{m,p}$ is the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1\\ m-1 & m & 0 & 0\\ p & p & p-1 & p\\ 1 & 1 & 1 & 1\end{pmatrix}.$$

Proof

We make the induction castle of $X_4=[P_0,P_4[$.

Let $P_2=T^{-1}\gamma_1$; by T, $[P_0,P_2[$ goes to $[0,\gamma_1[=X_1$, which goes by T to $[\beta_2,\beta_3[\subset X_3$.

On the right of the picture, $T[P_2,P_4[=[\gamma_1,\beta_1[\subset X_2$. Let $P_3=T^{-m}\gamma_2$, thus m is the smallest k such that $T^{-k}\gamma_2$ is in $]P_2,P_4[$. For $1\le k<m$ $T^k[P_2,P_4[\subset X_2$, then $T^m[P_2,P_3[=[T^mP_2,\gamma_2[\subset X_2$, which is further sent by T to $[T^{m+1}P_2,\beta_2[\subset X_3$, while $T^m[P_3,P_4[=[\gamma_2,T^mP_4[\subset X_3$. Note that $T^{m+1}P_2=T^mP_4$ as the image of γ_1 to the right is β_1.

Thus $J=[\gamma_2,\gamma_2+\lambda_4[$ is the union of the successive intervals $T^m[P_3,P_4[$, $T^{m+1}[P_2,P_3[$, $T^2[P_0,P_2[$. And $P_1=T^{-p-2}\gamma_3$ is thus in $]P_0,P_2[$. Then p is the smallest k such that $T^{-k}\gamma_3$ is in $]\gamma_2,\gamma_2+\lambda_4[$, J, ...$T^{p-1}J$ are in X_3 while $T^p[\gamma_2,T^2P_1[\subset X_3$, $T^p[T^2P_1,\gamma_2+\lambda_4[\subset X_4$, and finally $T^{p+1}[\gamma_2,T^2P_1[\subset X_4$.

Thus we know the induced map U on X_4: on $[P_0, P_1[$ $U = T^{p+3}$, on $[P_1, P_2[$ $U = T^{p+2}$, on $[P_2, P_3[$ $U = T^{m+p+1}$, on $[P_3, P_4[$ $U = T^{m+p}$. The order of the image intervals is, from left to right, $U[P_1, P_2[$, $U[P_3, P_4[$, $U[P_2, P_3[$, $U[P_0, P_1[$.

Let λ'_{4-i} be the length of the interval $[P_i, P_{i+1}[$; then we get the required matrix equality, because, for example, the images $T^i[P_3, P_4[$, of length λ'_1, are in X_2 $m-1$ times then in X_3 p times before returning to X_4, etc... □

Lemma 10. *Let Ω be the open positive cone in $\mathbb{R}^4$. Then for any pair of positive integers (m, p) and any vector $\lambda \in A_{m,n}\Omega$, there exists a unique λ' such that $\lambda = A_{m,p}\lambda'$, and $T_{\lambda,\pi}$ satisfies all the requirements of Lemma 9 with these values of m and p.*

Proof

We check that $A_{m,p}$ is of determinant one and maps Ω into Ω, and the matrix equality implies the conditions on the lengths. □

Lemma 11. *For every infinite sequence of matrices A_{m_k,p_k}, the set $\cap_{k\in\mathbb{N}} A_{m_1,p_1}...A_{m_k,p_k}\Omega$ is nonempty.*

Proof

We check that any product of two successive A has all its entries strictly positive. Let $\overline{\Omega}$ be the closed positive cone in $\mathbb{R}^4$, $K_n = A_{m_1,p_1}...A_{m_n,p_n}\Omega$, $\overline{K_n} = A_{m_1,p_1}...A_{m_n,p_n}\overline{\Omega}$, $K'_n = \overline{K_n} \setminus \{0\}$; we have $K_n \subset K'_n \subset \overline{K_n}$. But if v is in $\overline{\Omega}$ with at least one strictly positive coordinate, then $A_{m_{n-1},p_{n-1}}.A_{m_n,p_n}v$ is in Ω, thus

$$\cap_{n\geq 1} K_n = \cap_{n\geq 1}\overline{K_n} \setminus \{0\} = \cap_{n\geq 1} K'_n.$$

Also, each K'_n is invariant by $v \to \lambda v$ for any scalar λ, thus the K'_n are decreasing compact sets in a projective space, thus their infinite intersection is non-empty; thus $\cap_{n\geq 1} K_n$ is non-empty. □

Proposition 12. *Let E be the set $\cap_{k\in\mathbb{N}} A_{m_1,p_1}...A_{m_k,p_k}\Omega$ normalized by $\lambda_1 + ...\lambda_4 = 1$. For every $\lambda \in E$, there exists a decreasing sequence of intervals J_k such that the induced map U_k of the four-interval exchange $T_{\lambda,\pi}$ on J_k is the four-interval exchange $T_{\lambda'_{(k)},\pi}$ (after renormalization, and with reversed order if k is odd), with $\lambda = A_{m_1,p_1}...A_{m_k,p_k}\lambda'_{(k)}$.*

$T_{\lambda,\pi}$ is minimal if the first coordinate of $\lambda'_{(2k+1)}$ or the last coordinate of $\lambda'_{(2k)}$ tend to 0 when $k \to +\infty$.

Proof

When we renormalize and reverse the order of the intervals the induced map U of Lemma 9 is exactly $T_{\lambda',\pi}$, and we iterate the construction. At each stage, the induction castle (for T), of J_k fills all the space, thus if it is made of small intervals we can make the reasoning of Proposition 3 to prove minimality. □

Proposition 13. *For any λ in E, the trajectories of $T_{\lambda,\pi}$ are* adic *words. Namely there exist finite words $B_{n,1}, \dots, B_{n,4}$ such that, for all n, all the trajectories are infinite concatenations of $B_{n,i}$, with*

- $B_{0,i} = i$, $1 \leq i \leq 4$,

- *for each $1 \le i \le 4$, there exist an integer $t(n,i) > 0$, and $t(n,i)$ integers $1 \le k_s(n,i) \le 4$ such that*
$$B_{n,i} = \Pi_{s=1}^{t(n,i)} B_{n-1,k_s(n,i)}$$

The matrix A_{m_n,p_n} has on its i-th line, $1 \le i \le 4$, and j-th column, $1 \le j \le 4$, the number of $1 \le s \le t(n,j)$ such that $k_s(n,j) = i$.

Each point μ in the set $\cap_{k\in\mathbb{N}} A_{m_1,p_1} \ldots A_{m_k,p_k}\Omega$ normalized by $\mu_1 + \ldots \mu_4 = 1$, defines an invariant probability measure on $([0,1[, T_{\lambda,\pi})$ such that $\mu(X_i) = \mu_i$; every invariant probability measure on $([0,1[, T_{\lambda,\pi})$ is of that form.

Proof

For $1 \le i \le 4$ we define $F_{n,i}$ to be the subinterval of J_n labelled i in the definition of U_n. By construction, for each given n, the $T^j F_{n,i}$, $1 \le i \le 4$, $0 \le j \le |h_{n,i}| - 1$ form a partition of $[0,1[$ into k Rokhlin towers; these partitions are increasing (the atoms of the $n+1$-th partition are subsets of atoms of the n-th partition), and, except possibly for a countable number of points, two points belonging to the same atom of the n-th partition for every n are the same; we say that the system $([0,1[, T_{\lambda,\pi})$ is of *rank at most* 4. Moreover, if $x \in F_{n,4}$, resp $F_{n,3}$, resp. $F_{n,2}$, resp. $F_{n,1}$, the nonnegative trajectory of x under U_{n-1}, for the natural coding given by Proposition 12, begins with 413^{p_n+1}, resp. 413^{p_n}, resp. $42^{m_n}3^{p_n}$, resp. $42^{m_n}3^{p_n-1}$. By iterating this process, if $x \in F_{n,i}$, the nonnegative trajectory of x under T begins with a word denoted by $B_{n,i}$. It follows from the definitions that $F_{n-1,i}$ is a union of images by T of the $F_{n,j}$, $1 \le j \le k$, and thus the $B_{n,i}$ are made by the above concatenation rules, and we check the matrix.

This is enough to ensure that a measure on $[0,1[$ is determined by its values on the atoms of these partitions, thus, if it is T-invariant, by its values on the $F_{n,i}$. We check also that if $v_n = (\mu(F_{n,1}), \ldots \mu(F_{n,k}))$, we get $v_{n-1} = A_{m_n,p_n} v_n$. Thus the measure μ is completely determined by the vector v_0. □

Proposition 14. *If $p_1 \ge 9$ and $3(p_n+1) \le m_n \le \frac{1}{2}(p_{n+1}+1)$ for all n, T is minimal and not uniquely ergodic.*

Proof

The condition of minimality is satisfied as $p_n \to +\infty$ and every coordinate of λ'_n is smaller than $\frac{1}{p_n-1}$.

We define $M_n = A_{m_n,p_n}$, $\tilde{M_n}x = \frac{M_n x}{|M_n x|}$ where $|y| = \sum_{i=1}^4 y_i$. Let $\tilde{B_k}$ be the mapping $\tilde{M_1} \ldots \tilde{M_k}$.

We note that for $i = 1$ or $i = 4$, and any x, $(M_n x)_i \le 1$ while $|M_n(x)| \ge p_n + 1$ thus $(\tilde{M_n}x)_i \le \frac{1}{p_n+1}$.

Suppose $2m_n \le p_{n+1} + 1$ for all n and let $e_3 = (0,0,1,0)$; we prove that then for $n \ge 1$, $(\tilde{B_n}e_3)_3 \ge 1 - \frac{3}{p_1+1}$. Let $x^{k+1} = e_3$, $x^{j-1} = \tilde{M_{j-1}}x^j$, for $2 \le j \le k+1$; then, in view of the previous result, it is enough to prove that $x_2^j \le \frac{1}{p_1+1}$ for $j = 1$, and we shall prove it by induction on j ; this is true for $j = k+1$, and under the induction hypothesis
$$x_2^j = \frac{(m_j-1)x_1^{j+1} + m_j x_2^{j+1}}{p_j + 1 + (m_j-1)x_1^{j+1} + m_j x_2^{j+1} + x_4^{j+1}} \le \frac{2m_j}{(p_j+1)(p_{j+1}+1)} \le \frac{1}{p_j+1}.$$

Suppose $m_n \ge 3(p_n+1)$ and let $e_2 = (0,1,0,0)$; we prove now that for $n \ge 1$, $(\tilde{B_n}e_2)_2 \ge \frac{1}{3}$. This is done in the same way as in the previous paragraph, by defining a sequence x_i, with the induction hypothesis $x_2^{j+1} \ge \frac{1}{3}$, We define now μ by a vector which is a limit (on a subsequence) of the $A_{m_1,p_1}...A_{m_k,p_k}e_2$, and ν in the same way with $A_{m_1,p_1}...A_{m_k,p_k}e_3$. Then if $\mu=\nu$, it would give measure at least $1-\frac{3}{n_1+1}$ to the set X_3 and at least $\frac{1}{3}$ to the set X_2 which is a contradiction as soon as $n_1\geq 9$. □

These examples can satisfy the requirement of total irrationality, which was not satisfied by Veech's examples: it is proved in [22] that for any given hyperplane H we can find sequences (m_n,p_n) satisfying the conditions of Proposition 14 and such that $\cap_{k\in\mathbb{N}}A_{m_1,p_1}...A_{m_k,p_k}\Omega$ does not intersect H, and we can avoid a countable family of hyperplanes.

Keane's examples have been generalized to any number $r\geq 4$ of intervals by J.-C. Yoccoz [34].

There is a duality between the lengths of the intervals and the values of the invariant measures on them: with the notations of Section 4, for any $\lambda\in E$, every invariant probability measure on $([0,1[,T_{\lambda,\pi})$ is defined from a vector $\mu\in E$ by giving measure μ_i to the i-th interval. Under the above conditions on the m_k, p_k, E is not reduced to a point but is a segment, whose two endpoints give the two invariant ergodic measures. If we choose λ to be in the interior of this segment, these two ergodic measures are absolutely continouous with respect to the Lebesgue measure but different from it; if we choose λ to be an endpoint, one ergodic measure is the Lebesgue measure and the other one is singular; a recent work of J. Chaika [6] has proved that this singular measure can have a support of arbitrarily small Hausdorff dimension.

5. Unique ergodicity after Boshernitzan

We call m the normalized Lebesgue measure on Λ_r, and ρ the Lebesgue measure on $[0,1[$.

Theorem 15. *For a given irreducible π, $T_{\lambda,\pi}$ is uniquely ergodic for m-almost every $\lambda\in\Lambda_r$.*

This result was the first big conjecture on interval exchanges, stated as a question by M. Keane in [22] and proved independently by W. Veech [31] and H. Masur [23]. The proofs of Veech and Masur use elaborate geometric methods; but a later proof of M. Boshernitzan uses mainly combinatorial methods; it is published in [2] but can be simplified (and made purely combinatorial) by using [4]. Thus we give here this simplified proof, with an updated vocabulary: in particular, the Rauzy graphs are used without being named im [2], and as far as we know this is the first published paper where they are mentioned.

Definition 10. *Let (X_L,S) be a minimal symbolic system. If μ is an S-invariant probability measure, for each natural integer n, we denote by $e_n(S,\mu)$ the smallest positive measure of the cylinders defined by words of length n of L.*

Proposition 16. [4] *If for some invariant probability measure μ, $ne_n(S,\mu)$ does not tend to 0 when n tends to $+\infty$, then the system (X_L,S) is uniquely ergodic.*

Proof

Then for infinitely many n we have $e_n\geq\frac{a}{n}$ and thus $p(n)\leq(C-1)n$, thus $\liminf_{n\to+\infty}p(n)-Cn=-\infty$. Thus on a subsequence $p(n+1)-C(n+1)\leq p(n)-Cn$ and $p(n+1)-C(n+1)\leq 0$. On this subsequence we have both $p(n)\leq p(n+1)\leq C(n+1)\leq C'n$ for n large enough and $p(n+1)-p(n)\leq C$ thus at most C right special words. By the reasoning of the first part of the proof of Proposition 7 we conclude that there is a finite number of ergodic invariant measures.

Thus if (X_L,S) is not uniquely ergodic there are two invariant ergodic measures μ_i and a finite word w such that $\mu_1[w]<\mu_2[w]$ and in the interval $]\mu_1[w],\mu_2[w][$ there is no $\nu[w]$ for ν any ergodic invariant measure. We fix $\mu_1[w]<t<s<\mu_2[w]$.

Let $E_n = \{x; \frac{N(w,x_0...x_{n-1})}{n} \in]t,s[\}$. As $\frac{N(w,x_0...x_{n-1})}{n} \to \nu[w]$ ν-almost everywhere, we get $\nu(E_n \quad \text{infinitely often}) = 0$, thus $\nu(E_n) \to 0$ when $n \to +\infty$, thus also $\mu(E_n) \to 0$ by convex combination.

By genericity, we choose x and y such that $\frac{N(w,x_0...x_{n-1})}{n} < t < s < \frac{N(w,y_0...y_{n-1})}{n}$ for n large enough. We choose in the Rauzy graph of length n a path from $x_0...x_{n-1} = v_1$ to $y_0...y_{n-1} = v_p$, through v_i, $2 \leq i \leq p-1$; it exists by minimality and can be chosen of minimal length thus without loops.

Then $|N(w,v_i) - N(w,v_{i+1})| \leq 1$. To go from below tn to above sn by moving by 1 at a time, we need to be at least $n(s-t)-2$ times between tn and sn. Thus $[v_i]$ is in E_n for at least $n(s-t)-2$ values of i, and the v_i are all distinct. We get $\mu(E_n) \geq (n(s-t)-2)e_n$ and thus $ne_n \to 0$, contradiction. □

For a given interval exchange $T_{\lambda,\pi}$, the Lebesgue measure ρ on the interval $[0,1[$ defines an invariant measure ρ' for its natural coding (X_u, S), and we denote by $e_n(T_{\lambda,\pi})$ the quantity $e_n(S,\rho')$ defined for this symbolic system.

Proposition 17. [2] *Let $U_{n,\epsilon}$ be the set of $\lambda \in \Lambda_r$ such that $e_n(T_{\lambda,\pi}) \leq \frac{\epsilon}{n}$. If ϵ is small enough, $m(U_{n,\epsilon}) \leq 3r^3\epsilon$.*

Proof

Let $G_n(\lambda)$ be the Rauzy graph of length n of the language of $T_{\lambda,\pi}$. As the complexity of any trajectory is $(r-1)n+1$, by Lemma 5 G_n has at most $3r-3$ branches, as a branch starts at a left or right special factor, and there are at most respectively $r-1$ and $2r-2$ in each case.

We recall that ψ is a *weight function* on a graph if it is positive on each vertex, the sum of its values on vertices in 1, and it can be extended to the edges such that for every vertex

$$\psi(w) = \sum_{\text{incoming edges}} \psi(e) = \sum_{\text{outcoming edges}} \psi(e).$$

We define a function ψ_λ on the vertices of $G_n(\lambda)$ by associating to the vertex $w_1...w_n$ the measure of the cylinder, $\rho[w_1...w_n]$; ψ_λ is a weight function on the graph $G_n(\lambda)$, and the weight of an edge $w_1...w_{n+1}$ is also $\rho[w_1...w_{n+1}]$.

We fix now a Rauzy graph G of length n; let $\Lambda(G)$ be the set of $\lambda \in \Lambda_r$ such that $G_n(\lambda) = G$. For a given word $w = w_1...w_n$, $\psi_\lambda(w_1)$ is just λ_{w_1}; for all $\lambda \in \Lambda(G)$, all the Rauzy graphs $G_i(\lambda)$, $1 \leq i \leq n$, are fixed, and determine the way the measures of cylinders of length $i+1$ are computed from those of length i, through the defining equalities of the weight function ψ_λ on $G_i(\lambda)$; thus the numbers $\psi_\lambda(w_1...w_i)$, $1 < i \leq n$ can be computed inductively; they depend linearly on λ. Because $T_{\lambda,,\pi}$ preserves the measure ρ, $\psi_\lambda(w_1...w_n) = \psi_\lambda(w'_1...w'_n)$ if $w_1...w_n$ and $w'_1...w'_n$ are on the same branch of G; hence, for fixed λ, $\psi_\lambda(w_1...w_n)$ takes $1 \leq t \leq 3r-3$ values, which we denote by $\phi_1(\lambda)$, ..., $\phi_t(\lambda)$; the ϕ_j are linear functionals, $e_n(T_{\lambda,\pi})$ is just the smallest of the $\phi_j(\lambda)$, $1 \leq j \leq t$. Furthermore, again through the defining equalities of the successive weight functions on $G_i(\lambda)$, we can retrieve λ from the values $\psi_\lambda(w)$ on all the vertices of G; thus $\Lambda(G)$ is a convex set and every weight function ψ on G yields a $\lambda \in \Lambda(G)$ such that $\psi_\lambda = \psi$.

We want to estimate the measure $m(\{\lambda \in \Lambda(G); \phi_i(\lambda) \leq \frac{\epsilon}{n}\})$; for this, we use a general result for which we refer the reader to [3], Corollary 7.4: *If ϕ is the restriction of a linear functional to a* *convex set K of dimension d, taking values between 0 and A, then, if V denotes the volume,*

$$V(\pi^{-1}[0,B[) \leq \frac{dB}{A}V(K).$$

We apply it with $K = \Lambda(G)$, restricting ourselves to those with $m(\Lambda(G)) > 0$, $\phi = \phi_i$, $B = \frac{\epsilon}{n}$; the dimension is $r-1$, the volume is the Lebesgue measure; we need an estimate on A; for this, we claim that *for each vertex s of G, there exists a weight function such that $\psi(s) \geq \frac{1}{rn}$*. To do this, we choose a $\lambda \in \Lambda(G)$ such that $T_{\lambda,\pi}$ is minimal, which is possible as $m(\Lambda(G)) > 0$; this implies that G is strongly connected and thus we can find a loop $s_0 \to ..s_k \to s_0$ in G; by taking it of minimal length, we ensure it has no repetition. Then we define ψ' to be $\frac{1}{k+1}$ on the s_i and 0 on the other vertices; ψ' is not a weight function as it may be 0 on some vertices, but $\psi = (1-\delta)\psi + \delta\psi_\lambda$ is a weight function, and as $k \leq (r-1)n+1$ we can choose δ such that our claim is proved.

Thus we have $A \geq \frac{1}{rn}$, and thus, for all G with $m(\Lambda(G)) > 0$ and hence for all G,

$$m(\{\lambda \in \Lambda(G); \phi_i(\lambda) \leq \frac{\epsilon}{n}\}) \leq (r-1)r\epsilon m(\lambda(G)).$$

As $t \leq 3r-3$,

$$m(\{\lambda \in \Lambda(G); \min_{1\leq i\leq t} \phi_i(\lambda) \leq \frac{\epsilon}{n}\}) \leq 3(r-1)^2 r\epsilon m(\lambda(G)),$$

which implies the proposition. □

Proof of Theorem 15

For small $0 < \epsilon$ and $n \geq 1$, we put $V_{n,\epsilon} = \Lambda_r \setminus U_{n,\epsilon}$, and $V_\epsilon = \cap_{N\geq 1} \cup_{n>N} V_{n,\epsilon} \cap \{\lambda; T_{\lambda,\pi}$ i.d.o.c.$\}$.

If λ is in V_ϵ, there are infinitely many n such that $e_n(T_{\lambda,\pi}) \geq \frac{\epsilon}{n}$, hence $ne_n(T_{\lambda,\pi}) \not\to 0$ when $n \to +\infty$, and $T_{\lambda,\pi}$ is uniquely ergodic by Proposition 16. Thus $m(\{\lambda; T_{\lambda,\pi}$ is uniquely ergodic$\})$ is at least $m(V_\epsilon) \geq 1 - 3r^3\epsilon$, and thus is one as ϵ is arbitrary. □

The above proof does not use any of the geometric properties of interval exchanges; what it needs is only that it is a class of symbolic systems on an r-letter alphabet, of complexity at most sn for a fixed s, with a common invariant measure ρ such that the vector $(\rho[1], ...\rho[r])$ takes all possible values in Λ_r.

Note that explicit constructions of uniquely ergodic interval exchanges are not very difficult to make, for example by adapting the methods of [21], as exposed in Section 4 above, to get one invariant measure.

6. Weak mixing after Boshernitzan [5]

The second big conjecture on interval exchanges was about *weak mixing*: for any given primitive permutation π which is not a circular permutation (a circular permutation is such that $i \equiv \pi i \bmod r$, for all $1 \leq i \leq r$), $T_{\lambda,\pi}$ is weakly mixing for m-almost every $\lambda \in \Lambda_r$. This was stated as a question in [32] (W. Veech never makes conjectures) and resisted more than twenty years before being proved by A. Avila and G. Forni [1]. The recent result we give now is not known to be equivalent, but has at least a similar flavour and its proof is quite short; the proof we give is adapted, with modifications, from [5].

Definition 11. *For $0 < t < 1$, $\phi(t)$ is the largest $s > 0$ such that for infinitely many n there is a Rokhlin tower of total measure $4s$, made of $2n + 1$ intervals, with t in the middle of the middle level; $\phi(t)$ is defined to be 0 if no such s exists.*

Proposition 18. *If $T_{\lambda,\pi}$ is ergodic for the Lebesgue measure ρ, and $\phi(t) > 0$, the induced map of T on $[0, t[$ is weakly mixing for the Lebesgue measure on $[0, t[$.*

Proof

For infinitely many m, we build a tower of total measure $> a > 0$, made of $2m + 1$ intervals, with t in the middle of the middle level. The intersection of this tower with $[0, t[$ consists of about mt (by the ergodic theorem for T^{-1}) full levels below the one containing t, the left half of the level containing t, and about mt (by the ergodic theorem for T) full levels above the one containing t; we trim it to get exactly n levels below and above, called $Y_{n,k}$, for $-n \le k \le n$; this yields infinitely many values of n.

Suppose the induced map U om $[0, t[$ has an eigenfunction f for the eigenvalue θ. Because the $Y_{n,k}$ are small intervals, there exists a sequence of maps f_n such that $||f - f_n||_1 \to 0$ and f_n has a constant value $f_{n,k}$ on each $Y_{n,k}$. The $f_{n,k}$ can be taken of modulus 1.

Let Z be the tower made with the left halves of the $Y_{n,k}$, $-n \le k \le -1$; it has total measure at least $\frac{a}{5}$. If r_n is the translation taking the left half to the right half of each $Y_{n,k}$, and if x is in a level of Z, we have $f_n(U^{n+1}x) = f_n(U^n r_n x)$. Then, for any given ϵ and n large enough,

$$\rho(Z)|\theta - 1| = \sum_{k=-n}^{-1} |\theta^{n+1} f_{n,k} - \theta^n f_{n,k}|\rho(Y_{n,k}) = \int_Z |\theta^{n+1} f_n(x) - \theta^n n f_n(r_n x)|dx \le$$

$$2\epsilon + \int_Z |\theta^{n+1} f_(x) - \theta^n f(r_n x)|dx = 2\epsilon + \int_Z |f(U^{n+1}x) - f(U^n r_n x)|dx \le$$

$$4\epsilon + \int_Z |f_n(U^{n+1}x) - f_n(U^n r_n x)|dx = 4\epsilon.$$

Letting n tend to infinity, we get $\theta = 1$. But U is ergodic, as, if there is a non-trivial invariant set for U, we can use the induction castle to build a non-trivial invariant set for T; thus $\theta = 1$ is excluded. □

Proposition 19. *If $T_{\lambda,\pi}$ is minimal and ergodic for the Lebesgue measure, the set of t such that $\phi(t) > 0$ is residual and of full Lebesgue measure.*

Proof

By making the induction castle of a small subinterval, for any given N we can make towers of at least N intervals of total measure at least $\frac{1}{r+2}$. Let Y_n be a sequence of such towers, with h_n intervals, and Z_n be the middle ninth (=middle third in length and height) of Y_n. Let Z be $\{Z_n \quad \text{infinitely often}\}$. Z is residual and $\rho(Z)$ is at least $\frac{1}{10(r+2)}$, while if z is in Z_n there is a Rokhlin tower of total measure at least $\frac{1}{6(r+2)}$, made of $h_n \to +\infty$ intervals, with z in the middle of the middle level.

As the set of t such that $\phi(t) > 0$ is T-invariant, we conclude by ergodicity. □

Note that Proposition 19 does not give a way to find suitable values of t; explicit constructions (as opposed to existence theorems of sets of full measure) of weakly mixing r-interval exchanges are rather scarce: the only ones we have been able to find in the literature are for $r = 3$ [20][13], $r = 4$ [15][27], and $r = 6$ [27], while [11] gives a construction for every value of r and the permutation $\pi i = r + 1 - i$, and [12] gives a construction for every value of r and a quite different permutation.

7. Simplicity: the last frontier?

The third big conjecture on interval exchanges is still open, and we end this paper by stating it as Question 1 below, with the necessary historical background.

One recurrent preoccupation of ergodicians in the last twenty years has been with joinings: the notion of *self-joinings* of a system has been introduced by D. Rudolph in [26], to generalize some useful invariants of measure-theoretic isomorphism such as the factor algebra and the centralizer.

Definition 12. *A* self-joining *(of order two) of a system* (X, T, μ) *is any measure* ν *on* $X \times X$*, invariant under* $T \times T$*, for which both marginals are* μ*.*

Definition 13. *An ergodic system* (X, T, μ) *has* minimal self-joinings *(of order two) if any ergodic self-joining (of order two)* ν *is either the product measure* $\mu \times \mu$ *or a* diagonal *measure defined by* $\nu(A \times B) = \mu(A \cap T^i B)$ *for an integer* i*.*

A transformation which has minimal self-joinings has trivial centralizer and no nontrivial proper factor, and can be used to build a so-called *counter-example machine* [26] with surprising properties. The first example of a transformation with minimal self-joinings was given in [26], and a little later the famous *Chacon map* was shown in [8] also to have minimal self-joinings. However, both these examples may seem built on purpose, and they have no "natural", i.e. geometric, realization. Then geometric examples of transformations with minimal self-joinings were sought in the category of interval exchanges. And indeed A. del Junco [7] built a one-parameter family of three-interval exchanges, depending on an irrational γ, and proved that whenever this γ has bounded partial quotients in its continued fraction expansion the system has minimal self-joinings (the interested reader is warned that he will *not* find the terms "three-interval exchange" or "minimal self-joinings" in del Junco's paper; the systems which he describes as two-point extensions of rotations are indeed three-interval exchanges, and the notion of "simplicity" he proves is only slightly weaker than the original notion of minimal self-joinings, and has been standing as the current definition of "minimal self-joinings " since [9]).

But in the meantime, W. Veech [32] had shown that almost all interval exchanges (in the sense: for a fixed permutation, for Lebesgue-almost all values of the lengths of the intervals) are rigid:

Definition 14. *A system* (X, T, μ) *is* rigid *if there exists a sequence* $s_n \to \infty$ *such that for any measurable set* A

$$\mu(T^{s_n} A \Delta A) \to 0.$$

Simple systems have uncountable centralizers and cannot have minimal self-joinings; thus Veech devised in [30] a weakened notion of minimal self-joinings to allow for a nontrivial centralizer; the new notion, which Veech called "property S", is now known as *simplicity* (in the sense of Veech):

Definition 15. *An ergodic system* (X, T, μ) *is* simple of order two *if any ergodic self-joining of order two* ν *is either the product measure* $\mu \times \mu$ *or a measure defined by* $\nu(A \times B) = \mu(A \cap S^{-1} B)$ *for some measurable transformation* S *commuting with* T*.*

Simplicity is strong enough to keep many of the properties of systems with minimal self-joinings, though proving this required a lot of work [10] [30]. And Veech asked the following question (4.9 of [30])

Question 1. *Are almost all interval exchange transformations simple?*

The notion of simplicity having been devized just for that, the tacit conjecture is that indeed almost all interval exchanges are weakly mixing, simple and rigid.

But, while Veech's question stood unanswered, examples of simple transformations remained very scarce: there were of course the systems with minimal self-joinings, and some systems without minimal self-joinings but naturally related to these systems (such as the time-one map of a flow which, as a flow, has minimal self-joinings); at last in [9], a natural generalization of the Chacon map was (very cunningly!) shown to be simple and rigid. It remains to this day the main explicit example of a simple and rigid map; some more three-interval exchanges, beside del Junco's, were proved to have minimal self-joinings [14]; and at long last some three- [14] and four- [15] interval exchanges were proved to be simple and rigid, but they are proved to be so because they are measure-theoretically isomorphic to the del Junco-Rudolph map.

Thus an answer to Veech's question seems still to be a very difficult problem, which also fell a little out of fashion; the result of Avila-Forni was a necessary step towards a positive answer, but their proof of weak mixing does not seem to imply anything in the direction of simplicity. However, the result of Boshernitzan in Section 6 above may give a faint glimmer of new hope, as it proves weak mixing by the "Chacon trick" of building two towers of heights differing by one, and it is this trick which implies the weak mixing of the Chacon and del Junco-Rudolph maps, but also the minimal self-joinings of the Chacon map and (after long manipulations using the full knowledge of the system, and not only local towers), the simplicity of the del Junco-Rudolph map.

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