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PFAFFIAN REPRESENTATIONS OF CUBIC SURFACES

FABIO TANTURRI

ABSTRACT. Let \mathbb{K} be a field of characteristic zero. We describe an algorithm which requires a homogeneous polynomial F of degree three in $\mathbb{K}[x_0, x_1, x_2, x_3]$ and a zero \mathbf{a} of F in $\mathbb{P}_{\mathbb{K}}^3$ and ensures a linear Pfaffian representation of $V(F)$ with entries in $\mathbb{K}[x_0, x_1, x_2, x_3]$, under mild assumptions on F and \mathbf{a} . We use this result to give an explicit construction of (and to prove the existence of) a linear Pfaffian representation of $V(F)$, with entries in $\mathbb{K}'[x_0, x_1, x_2, x_3]$, being \mathbb{K}' an algebraic extension of \mathbb{K} of degree at most six. An explicit example of such a construction is given.

1. INTRODUCTION

Let \mathbb{K} be a field of characteristic zero and let X be the hypersurface in $\mathbb{P}_{\mathbb{K}}^n$ defined by a polynomial $F \in \mathbb{K}[x_0, x_1, \dots, x_n]$. One may ask whether the polynomial F^k is the determinant of a matrix M with entries in $\mathbb{K}[x_0, x_1, \dots, x_n]$, for some integer k .

For $k = 1$, such a matrix M is said to be a *determinantal representation* of X . If the entries are linear forms, then the determinantal representation is said to be *linear*. Linear determinantal representations of curves and surfaces of small degree are a classical subject and date back to the middle of nineteenth century; see for example [4], [14] for a historical account.

A relevant class of matrices with determinant F^2 are Pfaffian representations, that is, skew-symmetric matrices whose Pfaffian is F , up to constants. Let us recall the following definition.

Definition 1.0.1 (Pfaffian). Let $T = (t_{ij})$ be a skew-symmetric matrix of even size $2n$ with entries in a ring R . Then its determinant is the square of an element in R , called the *Pfaffian* of T .

If we denote by T_{ij} the square matrix of order $(2n - 2)$ obtained by deleting from T the i -th and j -th rows and columns, the Pfaffian is defined recursively as

$$(1) \quad \text{Pf}(T) = \begin{cases} \sum_{j < 2n} (-1)^j t_{2n,j} \text{Pf}(T_{2n,j}) & \text{if } n \geq 2 \\ t_{12} & \text{if } n = 1. \end{cases}$$

Pfaffian representations are a generalization of determinantal representations, since from a determinantal representation M we get a Pfaffian representation

$$\left(\begin{array}{c|c} 0 & M \\ \hline -M & 0 \end{array} \right).$$

The references about Pfaffian representations are very recent, even though some general results were probably well-known to the experts before. In [4], Beauville collects many results about determinantal and Pfaffian representations, giving criteria for the existence of linear Pfaffian representations of plane curves, surfaces,

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threefolds and fourfolds. The fact that a generic cubic threefold can be written as a linear Pfaffian had been proved by Adler [2, Apx.V], with $\mathbb{K} = \bar{\mathbb{K}}$. With the same method used by Adler, in [19] it is proved that a general quartic threefold admits a linear Pfaffian representation. A non-computer-assisted proof of this fact can be found in [5].

Again in the case $\mathbb{K} = \bar{\mathbb{K}}$, linear Pfaffian representations of plane curves and their elementary transformations are the subject of [9] and [8]; in [16] and [10], respectively almost quadratic and almost linear Pfaffian representations of surfaces are considered. In [12] it is proved that every smooth quartic surface admits a linear Pfaffian representation, a result which strengthens the Beauville-Schreyer's one in [4].

In this paper we will use the following two definitions.

Definition 1.0.2. Let $F \in \mathbb{K}[x_0, x_1, \dots, x_n]$ define the hypersurface X and let \mathbb{K}' be a field containing \mathbb{K} . A linear Pfaffian \mathbb{K}' -representation of X is a skew-symmetric matrix whose Pfaffian is F , up to constants, and whose entries are linear forms in $\mathbb{K}'[x_0, x_1, \dots, x_n]$.

Definition 1.0.3 (\mathbb{K} -point). If a point $\mathbf{a} \in \mathbb{P}_{\mathbb{K}}^n$ admits a representative $\underline{a} \in \mathbb{A}_{\mathbb{K}}^{n+1}$, then it will be called a \mathbb{K} -point.

By convention, hypersurfaces will be considered in $\mathbb{P}_{\bar{\mathbb{K}}}^n$, being $\bar{\mathbb{K}}$ the algebraic closure of \mathbb{K} . In this way, X is non-empty even if its defining polynomial $F \in \mathbb{K}[x_0, x_1, \dots, x_n]$ has no zero in $\mathbb{A}_{\bar{\mathbb{K}}}^{n+1}$, that is, if X has no \mathbb{K} -points.

According to these notations, in [4] Beauville provided a proof of the following theorem:

Theorem 1.0.4. Let \mathbb{S} be a surface of degree d in $\mathbb{P}_{\bar{\mathbb{K}}}^3$, without singular \mathbb{K} -points. The following conditions are equivalent:

- (1) \mathbb{S} admits a linear Pfaffian \mathbb{K} -representation;
- (2) $\mathbb{S} \cap \mathbb{P}_{\bar{\mathbb{K}}}^3$ contains a finite, reduced, arithmetically Gorenstein subscheme Z of index $2d - 5$, not contained in any surface of degree $d - 2$.

Moreover, the degree of Z is $\frac{1}{6}d(d - 1)(2d - 1)$.

Here a finite, reduced subscheme Z of degree c in $\mathbb{P}_{\bar{\mathbb{K}}}^n$, with ideal $I_Z \subset \mathbb{K}[x_0, x_1, \dots, x_n]$, is said to be arithmetically Gorenstein (AG for short) if $\mathbb{K}[x_0, x_1, \dots, x_n]/I$ is a Gorenstein ring. For such a scheme, its *index* is the (unique) integer N such that

$$(2) \quad \dim (R/I_Z)_p + \dim (R/I_Z)_{N-p} = c \quad \text{for all } p \in Z.$$

The proof of Theorem 1.0.4 is based on considering the rank-two vector bundle $\text{coker}(M)$ and its scheme Z associated via the Hartshorne-Serre correspondence. As remarked by Beauville, another way to prove the existence of a Pfaffian representation is via the Buchsbaum-Eisenbud Structure Theorem in [7], which we state after the following definition.

Definition 1.0.5 (depth, Gorenstein ideal). Let I be an ideal in the ring R . Let M be an R -module. Then $\text{depth}(I, M)$ is the length of a maximal regular M -sequence contained in I .

The ideal I is said to be Gorenstein if

$$(3) \quad \text{depth}(I, R) = \text{hd}(R/I) = k \quad \text{and} \quad \text{Ext}_R^k(R/I, R) \cong R/I$$

for some $k \in \mathbb{N}$, where hd denotes the homological dimension.

Theorem 1.0.6 (Buchsbaum-Eisenbud Structure Theorem).

- (1) Let $n \geq 3$ be an odd integer, and let \mathcal{M} be a free module of rank n over a Noetherian local ring R with maximal ideal J . Let $f : \mathcal{M} \rightarrow \mathcal{M}^*$ be an alternating map of rank $n - 1$ whose image is contained in $J \cdot \mathcal{M}^*$ and let $I = \text{Pf}_{n-1}(f)$ be the ideal generated by the $(n - 1) \times (n - 1)$ Pfaffians of the matrix representing f . If $\text{depth}(I, R) = 3$, then I is Gorenstein, and the minimal number of generators of I is n .
- (2) Every Gorenstein ideal I of R with $\text{depth}(I, R) = 3$ arises as in 1..

Indeed, identifying R with $\mathbb{K}[x_0, x_1, x_2, x_3]$, an AG subscheme Z as those arising in Theorem 1.0.4 satisfies the hypotheses of (2) in Theorem 1.0.6: Z has a Gorenstein homogenous ideal I_Z by definition and by a theorem of Serre [3]. The fact that $\text{depth}(I_Z, R) = 3$ follows from (3) and $\text{hd}(R/I_Z) = 3$, which is true since the homogeneous coordinate ring of a finite set of points is Cohen-Macaulay and from the Auslander-Buchsbaum formula [15, ex. 18.15, ex. 19.8].

Given Z as in Theorem 1.0.4, one can apply Theorem 1.0.6: I_Z is generated by the $(2d - 2) \times (2d - 2)$ principal Pfaffians extracted from a skew-symmetric $(2d - 1) \times (2d - 1)$ matrix T with linear forms as entries. Then the surface admits a Pfaffian \mathbb{K} -representation

$$(4) \quad \left(\begin{array}{c|c} T & -C^t \\ \hline C & 0 \end{array} \right),$$

where C is a suitable $1 \times (2d - 1)$ matrix with linear forms as entries, which can be found by formula (1) (see also subsection 2.2).

In this paper we focus on case $d = 3$. If $\mathbb{K} = \bar{\mathbb{K}}$, then by [13] a set of five points in $\mathbb{P}_{\mathbb{K}}^3$ is an AG scheme if and only if they are in general position, i.e. no four of them are on a plane. This fact, together with Theorem 1.0.6, implies

Corollary 1.0.7. If $\mathbb{K} = \bar{\mathbb{K}}$, every smooth cubic surface in $\mathbb{P}_{\mathbb{K}}^3$ admits a linear Pfaffian representation [4].

This result has been generalized in [17] as follows.

Proposition 1.0.8. If $\mathbb{K} = \bar{\mathbb{K}}$, every cubic surface in $\mathbb{P}_{\mathbb{K}}^3$ admits a linear Pfaffian representation.

We study how to construct *explicitly* a linear Pfaffian \mathbb{K} -representation, where \mathbb{K} is not necessarily algebraically closed, starting from the least amount of initial data possible. We will show that, in general, it is sufficient to know a \mathbb{K} -point on \mathbb{S} .

Our contribution is the following: we prove

Theorem 1.0.9. Let \mathbb{S} be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$. Given a \mathbb{K} -point \mathbf{a}^1 , which is not a T -point — in the sense of Definition 3.1.1 — it is possible to construct explicitly a linear Pfaffian \mathbb{K} -representation of \mathbb{S} .

The same method can be used to prove a weaker result, if \mathbf{a}^1 is not given:

Proposition 1.0.10. Let \mathbb{S} be a cubic surface, neither reducible nor a cone, whose equation is $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$. Then it is possible to construct explicitly a Pfaffian \mathbb{K}' -representation of \mathbb{S} , where \mathbb{K}' is an algebraic extension of degree $[\mathbb{K}' : \mathbb{K}] \leq 3$. Moreover, if $\mathbb{K} \subseteq \mathbb{R}$, then also \mathbb{K}' can be chosen so.

On one hand, these results strengthen one implication of Theorem 1.0.4 and give a bound for the degree of algebraic extension required to produce a linear Pfaffian representation. On the other hand, they are constructive: it is possible to

implement an algorithm which produces a linear Pfaffian representation, provided the requested inputs.

After discussing the cases of reducible surfaces and cones, we are able to prove Theorem 1.0.11, which strengthens Proposition 1.0.8.

Theorem 1.0.11. *Every cubic surface in $\mathbb{P}_{\mathbb{K}}^3$, with equation $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$, admits a Pfaffian \mathbb{K}' -representation, \mathbb{K}' being an algebraic extension of \mathbb{K} of degree $[\mathbb{K}' : \mathbb{K}] \leq 6$.*

Moreover, it is possible to explicitly realize such a representation.

This paper is structured as follows: in section 2, we retrace the proof of Theorem 1.0.6 and we use it to construct a skew-symmetric matrix \mathbb{T} as in (8), whose Pfaffians generate the ideal of the four fundamental points and the unit point in \mathbb{P}^3 . This enables us to produce Algorithm 2.2.1, whose inputs are five points in general position on a surface \mathbb{S} and whose output is a linear Pfaffian representation of \mathbb{S} .

In section 3, we make use of the tangent plane process, a classical argument (see, for example, [23]); starting from a \mathbb{K} -point \mathbf{a}^1 on an irreducible surface which is not a cone, we show that it is always possible to find four other points on the surface such that all the five points are in general position, provided that \mathbf{a}^1 satisfies a mild condition.

In section 4 we summarize the previous results in Theorem 1.0.9 and Proposition 1.0.10. Then we discuss the case of reducible surfaces and the case of cones, so to prove Theorem 1.0.11. An example of the construction of a Pfaffian representation is finally given.

2. FROM FIVE POINTS TO A PFAFFIAN REPRESENTATION

In this section, we make explicit the construction of the proof of Theorem 1.0.6, in the particular case of the ideal I of the four fundamental points and the unit point

$$(5) \quad [1 : 0 : 0 : 0], [0 : 1 : 0 : 0], [0 : 0 : 1 : 0], [0 : 0 : 0 : 1], [1 : 1 : 1 : 1]$$

in $\mathbb{P}_{\mathbb{Q}}^3$. This produces the skew-symmetric matrix \mathbb{T} in (8), whose Pfaffians generate I ; we will make use of \mathbb{T} to implement Algorithm 2.2.1, which produces a linear Pfaffian \mathbb{K} -representation of a cubic surface \mathbb{S} starting from five \mathbb{K} -points in general position on \mathbb{S} .

From now on, we will consider only *linear* Pfaffian representations.

2.1. An explicit construction. For the sake of completeness, we recall briefly the constructions made in [7] in the proof of Theorem 1.0.6.

Let R be the ring of polynomials $\mathbb{K}[x_0, x_1, x_2, x_3]$ and let I be a Gorenstein ideal with $\text{depth}(I, R) = 3$. From a minimal free resolution of I

$$(6) \quad \underline{F} : \quad 0 \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \longrightarrow R/I \longrightarrow 0,$$

where $F_0 \cong R \cong F_3$, it is possible to make a change of basis in F_1 such that the map $F_2 \longrightarrow F_1$ is alternating. This can be found by equipping this resolution with a graded commutative algebra, the symmetric square of \underline{F}

$$s_2(\underline{F}) = (\underline{F} \otimes \underline{F})/M,$$

where M is the graded submodule of $\underline{F} \otimes \underline{F}$ generated by the elements of the set

$$\left\{ f \otimes g - (-1)^{(\deg f)(\deg g)} g \otimes f \mid f, g \text{ homogeneous elements of } \underline{F} \right\}.$$

By convention, an element f has degree i if and only if it belongs to F_i ; the degree of $(f \otimes g)$ is simply $\deg(f) + \deg(g)$. The differential is inherited from \underline{F} as follows:

$$d(f \otimes g) = d f \otimes g + (-1)^{\deg f} f \otimes d g.$$

The symmetric square $s_2(\underline{F})$ is a complex of projective R -modules, canonically isomorphic to \underline{F} in degree 0 and 1. Therefore, there exists a map of complexes $\Phi : s_2(\underline{F}) \longrightarrow \underline{F}$ which lifts up these two isomorphisms and it can be chosen so that the restrictions of Φ to $F_0 \otimes F_k$ are the isomorphisms $F_0 \otimes F_k \cong \overline{F_k}$. The multiplication in $s_2(\underline{F})$ is given by $f \cdot g = \Phi(f \otimes g)$, where $f \otimes g$ is the class of $f \otimes g$ modulo M . Since $F_3 \cong R$, this multiplication induces a map $F_k \otimes F_{3-k} \longrightarrow R$, which turns to be a perfect pairing. This can be viewed as an isomorphism between F_1 and F_2^* , which makes the composition $F_2 \longrightarrow F_1 \longrightarrow F_2^*$ an alternating map.

Let us consider the special case where I is the ideal of the points (5). We have the free resolution (6), with $F_1 \cong R^5 \cong F_2$. We have to develop $\Phi_3 : s_2(\underline{F})_3 \longrightarrow F_3$ in the diagram (7)

$$(7) \quad \begin{array}{ccccccccccc} \dots & \longrightarrow & s_2(\underline{F})_3 & \xrightarrow{d'_3} & s_2(\underline{F})_2 & \xrightarrow{d'_2} & s_2(\underline{F})_1 & \xrightarrow{d'_1} & s_2(\underline{F})_0 & \xrightarrow{\pi} & R/I & \longrightarrow & 0 \\ & & \downarrow \Phi_3 & & \downarrow \Phi_2 & & \downarrow \Phi_1 & & \downarrow \Phi_0 & & & & \\ 0 & \longrightarrow & F_3 & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{\pi} & R/I & \longrightarrow & 0 \end{array}$$

We choose the ordered basis of $s_2(\underline{F})_2 \cong (F_0 \otimes F_2) \oplus (\wedge^2 F_1)$ to be formed by the classes modulo M of $1 \otimes f_1^2, 1 \otimes f_2^2, \dots, 1 \otimes f_5^2, f_1^1 \otimes f_2^1, f_1^1 \otimes f_3^1, \dots, f_4^1 \otimes f_5^1$, where the f_i^1 's are a basis of F_1 and the f_j^2 's are a basis of F_2 . A similar convention is fixed for $s_2(\underline{F})_3 \cong (F_0 \otimes F_3) \oplus (F_1 \otimes F_2)$.

After a computation with [CoCoA], we consider the maps of diagram (7) to be

$$d_3 = \begin{pmatrix} x_0x_1 - x_1x_3 \\ x_1x_2 - x_2x_3 \\ -x_0x_2 + x_1x_2 \\ -x_1x_3 + x_2x_3 \\ x_0x_3 - x_1x_3 \end{pmatrix}, \quad d_1^t = d_1'^t = \begin{pmatrix} x_1x_3 - x_2x_3 \\ x_0x_3 - x_2x_3 \\ x_1x_2 - x_2x_3 \\ x_0x_2 - x_2x_3 \\ x_0x_1 - x_2x_3 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} -x_2 & x_0 & 0 & 0 & x_2 \\ x_2 & -x_1 & x_1 & 0 & 0 \\ x_3 & -x_3 & x_3 & x_0 - x_3 & 0 \\ -x_3 & x_3 & 0 & -x_1 + x_3 & x_1 \\ 0 & 0 & -x_3 & 0 & -x_2 \end{pmatrix}.$$

The isomorphisms Φ_0 and Φ_1 are represented by identity matrices. With straightforward computations we get the matrices d_2' and d_3' . By trials, we can lift up Φ_1 by finding matrices Φ_2 and Φ_3 such that the diagrams

$$\begin{array}{ccc} & s_2(\underline{F})_2 & \\ & \swarrow \Phi_2 & \downarrow \Phi_1 \circ d_2' \\ F_2 & \xrightarrow{d_2} & \text{Im}(d_2) \longrightarrow 0 \end{array} \quad \begin{array}{ccc} & s_2(\underline{F})_3 & \\ & \swarrow \Phi_3 & \downarrow \Phi_2 \circ d_3' \\ F_3 & \xrightarrow{d_3} & \text{Im}(d_3) \longrightarrow 0 \end{array}$$

commute. A possible choice for Φ_2 is

$$\left(I_5 \left| \begin{array}{ccccccccc} -x_3 & x_1 & 0 & 0 & x_3 & x_3 - x_0 & x_3 & 0 & -x_1 & 0 \\ -x_3 & 0 & -x_2 & -x_1 & 0 & -x_2 & 0 & 0 & 0 & x_2 \\ 0 & -x_2 & -x_2 & -x_1 & -x_2 & -x_2 & -x_0 & 0 & x_2 & x_2 \\ 0 & 0 & 0 & 0 & x_3 & x_3 & x_3 & -x_2 & -x_1 & 0 \\ 0 & x_3 & x_3 & x_3 & x_3 & x_3 & x_3 & 0 & -x_1 & -x_0 \end{array} \right. \right).$$

This choice is indeed the unique with linear forms as entries in the right block, since the syzygies are of degree two. The map Φ_3 turns to be

$$(I_1 \mid 0 \ 0 \ 0 \ -1 \ 0 \ 0 \ 0 \ 0 \ -1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 1 \ 0 \ 0 \ -1 \ 0).$$

The isomorphism resulting from Φ_3 is

$$F_1 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}} F_2^*$$

and, with respect to this change of basis, the map d_2 turns to be alternating, represented by the skew-symmetric matrix

$$(8) \quad \mathbb{T} = \begin{pmatrix} 0 & 0 & -x_3 & 0 & -x_2 \\ 0 & 0 & x_3 & x_0 - x_1 & x_1 \\ x_3 & -x_3 & 0 & x_1 - x_3 & -x_1 \\ 0 & -x_0 + x_1 & -x_1 + x_3 & 0 & 0 \\ x_2 & -x_1 & x_1 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that the 4×4 principal Pfaffians of \mathbb{T} — listed in (9) — are exactly the five generators of I , that is, the entries of d_1 .

2.2. From five points to a Pfaffian representation: an algorithm. The procedure just shown can be applied as long as we have the ideal of a set X of five points in general position on a cubic surface \mathbb{S} . Due to the classical fact that two sets of five points in general position in \mathbb{P}^3 are projectively equivalent, instead of repeating the previous construction it is also possible to realize a Pfaffian representation in the following way.

By solving a linear system, we can find the matrix A of the projectivity which maps X to the five points (5). Replacing x_0, x_1, x_2, x_3 in (8) with the columns of the matrix $(x_0 \ x_1 \ x_2 \ x_3) \cdot A^t$, we get a matrix T whose Pfaffians P_i generate the ideal of X .

Finding a Pfaffian representation is then straightforward: if $\mathbb{S} = V(F)$, then F belongs to the ideal of X . Therefore, one can find five linear forms L_i such that $F = \sum_{i=1}^5 (-1)^{i+1} L_i P_i$. Setting $C = (L_1 \ L_2 \ L_3 \ L_4 \ L_5)$ and by (1), we get a Pfaffian representation of the form (4).

We summarize the whole procedure in Algorithm 2.2.1, presented in pseudocode, where $\mathbb{T} = \mathbb{T}(x_0, x_1, x_2, x_3)$ in (8) is seen as a matrix depending on four variables, the Pfaffians of which are

$$(9) \quad \begin{aligned} \text{Pf}_1(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_1(x_0 - x_3) \\ \text{Pf}_2(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_2(x_3 - x_1) \\ \text{Pf}_3(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_2(x_1 - x_0) \\ \text{Pf}_4(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_3(x_1 - x_2) \\ \text{Pf}_5(\mathbb{T})(x_0, x_1, x_2, x_3) &= x_3(x_0 - x_1). \end{aligned}$$

Remark 2.2.2. Algorithm 2.2.1 involves only linear equations. If the five given points are \mathbb{K} -points, as well as $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$, then the output Pfaffian representation of $\mathbb{S} = V(F)$ is a \mathbb{K} -representation too, for a suitable choice of the representatives of the points.

Remark 2.2.3. The matrix associated to the (non-homogeneous) linear system in line 12 of the algorithm is 20×20 ; it depends only on the projectivity applied in

Algorithm 2.2.1 from five points in general position to a Pfaffian representation

Require: $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$ and $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4, \mathbf{a}^5$ \mathbb{K} -points in general position on $\mathbb{S} = V(F)$

Ensure: M , a Pfaffian \mathbb{K} -representation of \mathbb{S} depending on some arbitrary parameters $\alpha_{i,j}$

- 1: **choose** a representative $\underline{a}^i = (a_0^i, a_1^i, a_2^i, a_3^i) \in \mathbb{A}_{\mathbb{K}}^4$ of \mathbf{a}^i for every $1 \leq i \leq 5$
- 2: **compute** the solution $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the linear system

$$\begin{pmatrix} a_0^1 & a_0^2 & a_0^3 & a_0^4 \\ a_1^1 & a_1^2 & a_1^3 & a_1^4 \\ a_2^1 & a_2^2 & a_2^3 & a_2^4 \\ a_3^1 & a_3^2 & a_3^3 & a_3^4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} a_0^5 \\ a_1^5 \\ a_2^5 \\ a_3^5 \end{pmatrix}$$

- 3: **compute** the change of basis matrix A from $(\lambda_i a^i)_{1 \leq i \leq 4}$ to the standard basis of $\mathbb{A}_{\mathbb{K}}^4$, so that

$$\lambda_i A \begin{pmatrix} a_0^i \\ a_1^i \\ a_2^i \\ a_3^i \end{pmatrix} = \begin{pmatrix} \delta_i^1 \\ \delta_i^2 \\ \delta_i^3 \\ \delta_i^4 \end{pmatrix} \quad \text{for every } 1 \leq i \leq 4$$

- 4: **for** $i = 1$ **to** 4 **do**

- 5: **set** z_{i-1} as the i -th row of the column vector $A \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

- 6: **set** $\mathbb{T}(x_0, x_1, x_2, x_3)$ as in (8)

- 7: **set** $T = \mathbb{T}(z_0, z_1, z_2, z_3)$

- 8: **for** $i = 1$ **to** 5 **do**

- 9: **set** $P_i = \text{Pf}_i(\mathbb{T})(z_0, z_1, z_2, z_3)$ as in (9)

- 10: **set** $L_i = \sum_{j=0}^3 \alpha_{i,j} x_j$

- 11: **set** $G = F - \sum_{i=1}^5 (-1)^{i+1} L_i P_i$

- 12: **compute** solutions of the linear system given by equaling the coefficients of G to zero, $\alpha_{i,j}$ as unknowns

- 13: **substitute** the solutions in L_i

- 14: **set** M as the matrix

$$\left(\begin{array}{ccccc|c} & & & & & L_1 \\ & & & & & L_2 \\ & & & & & L_3 \\ & & & & & L_4 \\ & & & & & L_5 \\ \hline -L_1 & -L_2 & -L_3 & -L_4 & -L_5 & 0 \end{array} \right)$$

line 9, not on the choice of F . Regardless to this projectivity, its rank is 15.

Since for any choice of $F \supset \{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4, \mathbf{a}^5\}$ a solution of this linear system does exist, the ‘‘Pfaffian representation depending on some parameters’’ ensured by Algorithm 2.2.1 turns to be a five-dimensional linear space of Pfaffian representations.

2.2.1. Classes of equivalent representations. We recall that two Pfaffian \mathbb{K} -representations M and M' are said to be equivalent if and only if there exists $X \in \text{GL}_{\mathbb{K}}(6)$ such that $M' = XMX^t$. Let $\text{coker}(M)$, $\text{coker}(M')$ be the cokernel sheaves of M, M' , seen as maps $\mathcal{O}_{\mathbb{P}^3}^6(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}^6$. Then from [4, (2.3)] it follows that M, M' are equivalent if and only if $\text{coker}(M) \cong \text{coker}(M')$.

In this way the study of equivalence classes of Pfaffian representations of a cubic

surface \mathbb{S} is strongly linked to the study of certain sheaves on \mathbb{S} .

Remark 2.2.4. Let Z be a fixed set of five points in general position on a surface \mathbb{S} without singular \mathbb{K} -points and consider the Pfaffian representations given by Algorithm 2.2.1, which are a five-dimensional linear space by Remark 2.2.3. It turns out that all these representations are equivalent. Indeed, by [4, (7.1)], up to automorphism there exists only one pair (E, s) , with E rank-two vector bundle on \mathbb{S} and $s \in H^0(\mathbb{S}, E)$, such that Z is the zero locus of s . In addition, these classes of pairs $[(E, s)]$ are in bijection with the equivalence classes of the pairs $[(M, \bar{s})]$, where $E = \text{coker}(M)$ and $\bar{s} \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^6)$ corresponds to s via the isomorphism $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}^6) \cong H^0(\mathbb{S}, E)$. It follows that all the representations produced in the algorithm belong to a unique equivalence class.

It is worth noting that, as Z varies among the possible sets of five \mathbb{K} -points in general position on a surface \mathbb{S} without singular \mathbb{K} -points, Algorithm 2.2.1 is surjective onto the possible Pfaffian \mathbb{K} -representations of \mathbb{S} , and therefore onto their equivalence classes. Indeed, as shown in [4, (7.2)], a general global section of $E = \text{coker}(M)$ has five points in general position as its zero locus Z and therefore M can be produced via the algorithm with input Z .

In [8], elementary transformations were used to construct non-equivalent Pfaffian representations of curves starting from a given one. This technique can be used in the case of surfaces as well.

Remark 2.2.5. The bijection between Pfaffian representations M and sheaves $E = \text{coker}(M)$ tells us more, when dealing with the algebraic closure $\bar{\mathbb{K}}$. Let $\mathcal{S}^3 / \text{GL}(6)$ be the set of equivalence classes of the 6×6 skew-symmetric matrices of linear forms in $\mathbb{P}_{\bar{\mathbb{K}}}^3$; let $\text{pf} : \mathcal{S}^3 / \text{GL}(6) \longrightarrow |\mathcal{O}_{\mathbb{P}_{\bar{\mathbb{K}}}^3}(3)|$ be the map which associates to a class $[M]$ the cubic surface in $\mathbb{P}_{\bar{\mathbb{K}}}^3$ with equation $\text{Pf}(M)$. As noticed in [17], for the general \mathbb{S} the fiber $\text{pf}^{-1}(\mathbb{S})$ can be identified with an open subset of the moduli space of simple rank-two vector bundles E on \mathbb{S} with $c_1(E) = \mathcal{O}_{\mathbb{S}}(2)$, $c_2(E) = 5$. Since the (projective) dimension of $\mathcal{S}^3 / \text{GL}(6)$ is $59 - 35 = 24$ and the dimension of $|\mathcal{O}_{\mathbb{P}_{\bar{\mathbb{K}}}^3}(3)|$ is 19, then for the general \mathbb{S} we have a five-dimensional space of essentially different Pfaffian $\bar{\mathbb{K}}$ -representations of \mathbb{S} .

The space $\mathcal{S}^3 / \text{GL}(6)$ has been recently considered in [18], in relation to the space of pairs (\mathbb{S}, Π) , being Π a complete pentahedron inscribed in \mathbb{S} .

3. CONSTRUCTING FIVE POINTS ON A SURFACE

Given an equation $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$, in general it is not easy to find a zero of F in $\mathbb{A}_{\bar{\mathbb{K}}}^4$. For example, if $\mathbb{K} = \mathbb{Q}$, the problem of the existence of rational points on cubic surfaces, reliable to diophantine equations, has been strongly faced in the last century (see, for example, [22], [23] and the more recent [20]). Our next aim is to weaken the required inputs of Algorithm 2.2.1.

3.1. From one point to five points. It is well known that from a *general* choice of a \mathbb{K} -point on a *general* cubic surface with equation in $\mathbb{K}[x_0, x_1, x_2, x_3]_3$ it is possible to find infinitely many others \mathbb{K} -points on the surface; this can be performed by using the tangent plane process, a classical argument (for example, see [23]). It starts by taking the tangent plane to the cubic surface \mathbb{S} at a smooth point P . $T_P \mathbb{S}$ cuts \mathbb{S} in a curve of degree three, for which P is a singular point. A line through P , lying on the tangent plane, intersects \mathbb{S} twice in P , while the third intersection is *generically* different and gives us another \mathbb{K} -point on \mathbb{S} .

We want to get rid of this “generality”. Theorem 3.1.3 will show how, under reasonable hypotheses, the tangent plane process applied to a starting \mathbb{K} -point can be repeated to produce four other \mathbb{K} -points on \mathbb{S} , such that the five points are all together in general position. This will prove, under these hypotheses, that we only need a \mathbb{K} -point on \mathbb{S} to construct an explicit Pfaffian \mathbb{K} -representation.

Definition 3.1.1. A point $P \in \mathbb{S}$ will be called a T-point for \mathbb{S} if P is smooth for \mathbb{S} and $T_P \mathbb{S} \cap \mathbb{S}$ is set-theoretically union of lines.

Let us observe that the so-called Eckardt points, i.e. smooth points P with $T_P \mathbb{S} \cap \mathbb{S}$ made up of three lines through P , are T-points. Moreover, a smooth points P is a T-point if and only if $T_P \mathbb{S}$ is a tritangent plane. In general, for a T-point P one expects $T_P \mathbb{S} \cap \mathbb{S}$ to be union of three distinct lines, but it is possible to have one line with multiplicity three or two lines, one of them with multiplicity two.

The role of T-points will be clear in a while. Let us remark that, for a smooth point P which is not a T-point, $T_P \mathbb{S} \cap \mathbb{S}$ is either an irreducible cubic curve with P as a singular point, or union of a line through P and a smooth conic passing through P .

Remark 3.1.2. Let P be a T-point for \mathbb{S} . If $T_P \mathbb{S} \cap \mathbb{S}$ is a line r with multiplicity three, or union of a line r with multiplicity two and another line, then r is union of singular points for \mathbb{S} and T-points for \mathbb{S} sharing the same tangent plane.

Theorem 3.1.3. Let \mathbb{S} be an irreducible cubic surface which is not a cone, whose equation is $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$. Given a \mathbb{K} -point \mathbf{a}^1 on \mathbb{S} which is not a T-point — in the sense of Definition 3.1.1 — it is possible to explicitly construct four other \mathbb{K} -points on \mathbb{S} such that the five points together are in general position.

The constructive proof, which requires some steps and preliminary lemmas, will be the subject of next subsection. In subsection 4.1 we will see how this construction can be adapted if some of the hypotheses are missing.

3.2. Let us consider $F = F(x_0, x_1, x_2, x_3) \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$. Then we set, for every $\underline{a} = (a_0, a_1, a_2, a_3) \in \mathbb{A}_{\mathbb{K}}^4$:

- $P_{1,\underline{a}}(x_0, x_1, x_2, x_3) = \sum_{i=0}^3 a_i \frac{\partial F}{\partial x_i}$;
- $P_{2,\underline{a}}(x_0, x_1, x_2, x_3) = \sum_{i=0}^3 x_i \frac{\partial F}{\partial x_i}(\underline{a})$.

They are the equations of the first polar and the second polar of $\mathbf{a} = [a_0 : a_1 : a_2 : a_3]$ with respect to the surface $\mathbb{S} = V(F)$. If \mathbf{a} is smooth, $P_{2,\underline{a}}$ defines $T_{\mathbf{a}} \mathbb{S}$.

If $\underline{x} = (x_0, x_1, x_2, x_3)$, for every $\underline{a} \in \mathbb{A}_{\mathbb{K}}^4$ we have:

$$(10) \quad F(\underline{a} + t\underline{x}) = F(\underline{a}) + tP_{2,\underline{a}}(\underline{x}) + t^2P_{1,\underline{a}}(\underline{x}) + t^3F(\underline{x}).$$

We will consider the first and the second polar $V(P_{1,\mathbf{a}})$ and $V(P_{2,\mathbf{a}})$, for $\mathbf{a} \in \mathbb{P}_{\mathbb{K}}^3$, as hypersurfaces in $\mathbb{P}_{\mathbb{K}}^3$.

Lemma 3.2.1. Let \mathbf{a} be a singular point on a cubic surface \mathbb{S} , whose equation is $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$. Let us assume that \mathbb{S} is neither reducible, nor a cone. Then there are at most six lines through \mathbf{a} lying on \mathbb{S} .

Proof. By (10), if a point $\mathbf{x} \in \mathbb{S} \cap V(P_{1,\mathbf{a}})$, also the whole line through \mathbf{a} and \mathbf{x} does. $P_{1,\mathbf{a}}$ is not the zero polynomial since \mathbb{S} is not a cone, moreover F is irreducible: this means that the intersection $\mathbb{S} \cap V(P_{1,\mathbf{a}})$ is transversal. It is therefore a curve of degree six, union of lines through \mathbf{a} . \square

Lemma 3.2.2. Let \mathbb{S} be an irreducible, cubic surface which is not a cone and let us assume $\mathbf{a} \in \mathbb{S}$ is not a T-point.

- (1) If \mathbf{a} is smooth, then on $T_{\mathbf{a}}\mathbb{S}$ there are only finitely many T-points for \mathbb{S} . Moreover $V(P_{1,\mathbf{a}}) \cap T_{\mathbf{a}}\mathbb{S}$ is union of at most two lines through \mathbf{a} and any line through \mathbf{a} lying on \mathbb{S} lies also on $V(P_{1,\mathbf{a}}) \cap T_{\mathbf{a}}\mathbb{S}$.
- (2) If \mathbf{a} is singular, then point 1. still holds if we replace $T_{\mathbf{a}}\mathbb{S}$ with a plane π through \mathbf{a} , for all but finitely many choices of π .

Proof. We distinguish two classes of T-points: let us call \mathcal{A} the set of T-points P for which $T_P\mathbb{S} \cap \mathbb{S}$ is union of three distinct lines, \mathcal{A}' the set of T-points not in \mathcal{A} . Either \mathbb{S} contains finitely many lines or infinitely many ones. In the first case, note that \mathcal{A} is a finite set, since mutual intersections of lines on \mathbb{S} are finite in number; \mathcal{A}' is contained in a union of lines on \mathbb{S} , by Remark 3.1.2.

If \mathbb{S} contains infinitely many lines, then it is well-known (for example, [11]) that \mathbb{S} is either reducible, an irreducible cone or a ruled cubic with a double line. By hypotheses the first two cases cannot occur.

Moreover, a cubic surface with a double line which is not a cone is projectively equivalent to either $V(x_0^2x_3 + x_0x_1x_2 + x_1^3)$ or $V(x_0^2x_2 + x_1^2x_3)$ (see, for example, [1]). The study of these two cases leads to Table 1 and Table 2.

If \mathbb{S} is projectively equivalent to $V(x_0^2x_3 + x_0x_1x_2 + x_1^3)$, then Table 1 shows that

coordinates of \mathbf{a}	$T_{\mathbf{a}}\mathbb{S} \cap \mathbb{S}$ (if smooth)
$[1 : s : t : -s^3 - st]$	$\begin{cases} x_0(-2s^3 - st) + x_1(3s^2 + t) + sx_2 + x_3 = 0 \\ (x_0s - x_1)(2x_0^2s^2 - x_0x_1s + tx_0^2 - x_0x_2 - x_1^2) = 0 \end{cases}$
$[0 : 0 : s : t]$	singular

TABLE 1. points on $\mathbb{S} = V(x_0^2x_3 + x_0x_1x_2 + x_1^3)$.

coordinates of \mathbf{a}	restrictions	$T_{\mathbf{a}}\mathbb{S} \cap \mathbb{S}$ (if smooth)
$[1 : t : -t^2s : s]$	$s \neq 0 \neq t$	line and irreducible conic
$[1 : t : 0 : 0]$	$t \neq 0$	$\begin{cases} x_2 + x_3t^2 = 0 \\ x_3(x_0t \pm x_1) = 0 \end{cases}$
$[1 : 0 : 0 : s]$		$\begin{cases} x_2 = 0 \\ x_1^2x_3 = 0 \end{cases}$
$[0 : 1 : t : 0]$		$\begin{cases} x_3 = 0 \\ x_0^2x_2 = 0 \end{cases}$
$[0 : 0 : s : t]$		singular

TABLE 2. points on $\mathbb{S} = V(x_0^2x_2 + x_1^2x_3)$.

there are no T-points at all. If \mathbb{S} is projectively equivalent to $V(x_0^2x_2 + x_1^2x_3)$, then \mathcal{A} is contained in the line $[s : t : 0 : 0]$ and \mathcal{A}' is contained in the union of the lines $[s : 0 : 0 : t]$ and $[0 : s : t : 0]$, as shown in Table 2.

Now, let us assume \mathbf{a} is smooth. Since it is not a T-point, $T_{\mathbf{a}}\mathbb{S}$ cannot contain lines made up of T-points, so every such a line intersects $T_{\mathbf{a}}\mathbb{S}$ in one and only one point. Since they are finite in number, the first statement of 1. is proved.

For the second statement, let $\mathbf{x} \neq \mathbf{a}$ be a point in $\mathbb{P}_{\mathbb{K}}^3$ and let $Y = V(P_{1,\mathbf{a}}) \cap T_{\mathbf{a}}\mathbb{S}$. By (10), the point $\mathbf{x} \in Y$ if and only if either $F(\mathbf{a} + t\mathbf{x})$ is the zero polynomial or the line through \mathbf{a} and \mathbf{x} intersects \mathbb{S} only in \mathbf{a} . This means that, if $\mathbf{x} \in Y$, also

the whole line through it and \mathbf{a} is contained in Y ; the conclusion then holds if we prove that Y is a curve, that is, $V(P_{1,\mathbf{a}}) \not\subseteq T_{\mathbf{a}}\mathbb{S}$.

In fact, \mathbf{a} is not a T-point and so there exists a point \mathbf{y} on $\mathbb{S} \cap T_{\mathbf{a}}\mathbb{S}$ such that the line r through \mathbf{y} and \mathbf{a} does not lie on \mathbb{S} . The line r intersects \mathbb{S} in \mathbf{a} with multiplicity two and in \mathbf{y} with multiplicity one: this implies $\mathbf{y} \notin V(P_{1,\mathbf{a}})$. Part 1. of the lemma is proved.

If \mathbf{a} is singular, then by Lemma 3.2.1 only finitely many planes through \mathbf{a} contain a line on \mathbb{S} through \mathbf{a} . For any other choice π , the same argument of the smooth case holds, if we replace $T_{\mathbf{a}}\mathbb{S}$ with π . This proves part 2. of the lemma. \square

Proof of Theorem 3.1.3. We divide the proof into four steps.

Step 1: looking for the second point.

Either \mathbf{a}^1 is smooth or it is singular.

If \mathbf{a}^1 is smooth, then by hypotheses $\mathbb{S} \cap T_{\mathbf{a}^1}\mathbb{S}$ is a cubic curve, neither set-theoretically union of lines (\mathbf{a}^1 is not a T-point), nor the whole tangent plane (\mathbb{S} is irreducible).

Every line ℓ on $T_{\mathbf{a}^1}\mathbb{S}$ through \mathbf{a}^1 , but those contained in $T_{\mathbf{a}^1}\mathbb{S} \cap V(P_{1,\mathbf{a}^1})$ as in Lemma 3.2.2, has one and only one intersection with \mathbb{S} different from \mathbf{a}^1 . Here we do not care about any line on $T_{\mathbf{a}^1}\mathbb{S} \cap \mathbb{S}$ through \mathbf{a}^1 , since by Lemma 3.2.2 it would be contained in $T_{\mathbf{a}^1}\mathbb{S} \cap V(P_{1,\mathbf{a}^1})$ as well.

Fix a line ℓ ; the so-obtained \mathbf{a}^2 is smooth. Otherwise, ℓ would have multiplicity of intersection at least four with \mathbb{S} , and therefore $\ell \subset \mathbb{S}$, which is not.

Moreover, by Lemma 3.2.2, \mathbf{a}^2 can be a T-point only for finitely many choices of ℓ , and so these choices can be avoided.

By (10), in coordinates we have, having chosen a representative \underline{a}^1 for \mathbf{a}^1 ,

$$\underline{a}^2 = F(\underline{y}) \cdot \underline{a}^1 - P_{1,\underline{a}^1}(\underline{y}) \cdot \underline{y},$$

for any choice of $\underline{y} = (y_0, y_1, y_2, y_3)$ representing the class $\mathbf{y} \in T_{\mathbf{a}^1}\mathbb{S}$. Let us observe that $P_{1,\underline{a}^1}(\underline{y}) \neq 0$ and that \underline{a}^2 has coordinates in \mathbb{K} .

If \mathbf{a}^1 is singular, the previous argument can be repeated by replacing the role of $T_{\mathbf{a}^1}\mathbb{S}$ above with a plane π satisfying Lemma 3.2.2.

In both cases, we have constructed a smooth point \mathbf{a}^2 on \mathbb{S} , which is not a T-point.

Step 2: looking for the third point.

The tangent plane process can be repeated as in step 1 — smooth case — starting from \mathbf{a}^2 to construct next point \mathbf{a}^3 . Summarizing, every line on $T_{\mathbf{a}^2}\mathbb{S}$ through \mathbf{a}^2 with the exception of

- finitely many (by Lemma 3.2.2) lines through T-points,
- at most two lines in $T_{\mathbf{a}^2}\mathbb{S} \cap V(P_{1,\mathbf{a}^2})$ as in Lemma 3.2.2

has exactly one intersection with \mathbb{S} different from \mathbf{a}^2 , say \mathbf{a}^3 . It is smooth and not a T-point.

To state that \mathbf{a}^3 is in general position with \mathbf{a}^1 and \mathbf{a}^2 , we only need to verify that it does not lie on the line ℓ' through them. This is for free, since \mathbf{a}^3 belongs to $T_{\mathbf{a}^2}\mathbb{S}$ but \mathbf{a}^1 does not, otherwise $\ell' \subseteq \mathbb{S}$, which is not by construction.

Step 3: looking for the fourth point.

The tangent plane process can be repeated as in step 1 — smooth case — starting from \mathbf{a}^3 to construct next point \mathbf{a}^4 . We need to choose it not on the plane π_{123} containing \mathbf{a}^1 , \mathbf{a}^2 and \mathbf{a}^3 .

The planes $T_{\mathbf{a}^3} \mathbb{S}$ and π_{123} are distinct — for example, the first one does not contain \mathbf{a}^2 by construction — so their intersection is a line through \mathbf{a}^3 , say ℓ'' .

Claim. The system

$$(11) \quad \begin{cases} \mathbf{y} \in \mathbb{S} \\ \mathbf{y} \in T_{\mathbf{a}^3} \mathbb{S} \\ T_{\mathbf{y}} \mathbb{S} \ni \mathbf{a}^2 \end{cases}$$

which can be translated in homogeneous equations of degree 3, 1, 2 respectively, has finitely many solutions $\mathbf{y} \in \mathbb{P}_{\mathbb{K}}^3$.

Indeed, the system represents the intersection on the plane $T_{\mathbf{a}^3} \mathbb{S}$ between the cubic curve $\mathcal{C} = \mathbb{S} \cap T_{\mathbf{a}^3} \mathbb{S}$ and the conic \mathcal{Q} defined on $T_{\mathbf{a}^3} \mathbb{S}$ by the condition $T_{\mathbf{y}} \mathbb{S} \ni \mathbf{a}^2$. By construction, \mathbf{a}^3 is not a T-point and therefore \mathcal{C} is either irreducible or union of a line and an irreducible conic containing \mathbf{a}^3 ; \mathcal{Q} does not pass through \mathbf{a}^3 and so it cannot be contained in \mathcal{C} . This proves the claim.

The finitely many solutions of system (11) correspond to finitely many lines on $T_{\mathbf{a}^3} \mathbb{S}$ through \mathbf{a}^3 . Since we want $\mathbf{a}^2 \notin T_{\mathbf{a}^4} \mathbb{S}$, we will avoid them.

Summarizing, every line on $T_{\mathbf{a}^3} \mathbb{S}$ through \mathbf{a}^3 with the exception of

- finitely many lines through the solutions \mathbf{y} of system (11),
- ℓ'' ,
- finitely many (by Lemma 3.2.2) lines through T-points,
- at most two lines in $T_{\mathbf{a}^3} \mathbb{S} \cap V(P_{1,\mathbf{a}^3})$ as in Lemma 3.2.2

has exactly one intersection with \mathbb{S} different from \mathbf{a}^3 , say \mathbf{a}^4 . It is smooth and not a T-point, moreover $\mathbf{a}^2 \notin T_{\mathbf{a}^4} \mathbb{S}$.

Step 4: looking for the fifth point.

We can apply the usual tangent plane process to find \mathbf{a}^5 in general position with $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ and \mathbf{a}^4 . Let us call π_{ijk} the plane through different $\mathbf{a}^i, \mathbf{a}^j, \mathbf{a}^k$. The planes π_{134}, π_{234} and π_{124} intersect $T_{\mathbf{a}^4} \mathbb{S}$ into three lines through \mathbf{a}^4 : in fact they are four different planes, since $\mathbf{a}^2, \mathbf{a}^3 \notin T_{\mathbf{a}^4} \mathbb{S}$.

The line $\pi_{123} \cap T_{\mathbf{a}^4} \mathbb{S}$ cannot be contained in $T_{\mathbf{a}^4} \mathbb{S} \cap \mathbb{S}$, since $\mathbf{a}^4 \notin \pi_{123}$ and by construction \mathbf{a}^4 is not a T-point. This means that $\pi_{123} \cap T_{\mathbf{a}^4} \mathbb{S} \cap \mathbb{S}$ contains at most three points.

Summarizing, every line on $T_{\mathbf{a}^4} \mathbb{S}$ through \mathbf{a}^4 with the exception of

- three lines lying on the planes π_{134}, π_{234} and π_{124} ,
- at most three lines through the points in $\pi_{123} \cap T_{\mathbf{a}^4} \mathbb{S} \cap \mathbb{S}$,
- at most two lines in $T_{\mathbf{a}^4} \mathbb{S} \cap V(P_{1,\mathbf{a}^4})$ as in Lemma 3.2.2

has exactly one intersection with \mathbb{S} different from \mathbf{a}^4 , say \mathbf{a}^5 , in general position with $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4$.

□

Remark 3.2.3. Following the proof of Theorem 3.1.3, it is possible to implement an algorithm which requires a \mathbb{K} -point on \mathbb{S} , not a T-point, and ensures five \mathbb{K} -points in general position on \mathbb{S} . To test if a given point is a T-point or not, it is sufficient to check the reducibility of a polynomial of degree three in three variables, a task which can be easily performed with a software computation.

Remark 3.2.4. If \mathbb{S} is a smooth cubic surface, then any T-point P has $T_P \mathbb{S} \cap \mathbb{S}$ made up of three distinct lines. In such a situation, Theorem 3.1.3 can be proved with the weaker hypothesis: the starting point \mathbf{a}^1 is not an Eckardt point.

Remark 3.2.5. In the statement of Theorem 3.1.3 we require that \mathbf{a}^1 is not a T-point. Indeed, if \mathbf{a}^1 is Eckardt, then the tangent plane process fails at the very first step. If \mathbf{a}^1 is a non-Eckardt T-point, then the tangent plane process could give rise to either singular or other T-points, which can make one loose control in

subsequent steps.

In fact, this does happen in the following example: take $\mathbb{S} = V(x_0x_1x_3 + x_2^3 + x_2x_3^2)$ and $\mathbf{a}^1 = [0 : 0 : 0 : 1]$. The tangent plane process gives rise to points on the line $[s : t : 0 : 0]$, which are either singular or Eckardt points. The process then stops at the second step.

As pointed out by the referee, codimension three AG subschemes have been considered also in [21], where they are obtained as zero loci of sections of certain rank-three sheaves. In the case of five points in general position in $\mathbb{P}_{\mathbb{K}}^3$, it turns out that all such sets are the zero loci of appropriate sections of the bundle $\Omega_{\mathbb{P}^3}(3)$, which can be interpreted as four-tuple quadrics, that is, linear combinations (using linear forms as coefficients) of the syzygies of the map $(x_0 \ x_1 \ x_2 \ x_3)$. The membership of such a zero locus to a surface \mathbb{S} imposes conditions to the linear combination.

4. MAIN RESULTS AND FURTHER GENERALIZATIONS

In this last section, we firstly make use of Theorem 3.1.3 and Algorithm 2.2.1 to prove Theorem 1.0.9; if we drop the requirement of the starting point, then a weaker result holds (Proposition 1.0.10). After discussing the cases of reducible surfaces and cones, we state Theorem 1.0.11. A concrete example is finally given.

Proof of Theorem 1.0.9. Given \mathbf{a}^1 , one can apply Theorem 3.1.3 and construct four other \mathbb{K} -points $\mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4, \mathbf{a}^5$ on \mathbb{S} such that they are all together in general position. With these initial data, Algorithm 2.2.1 ensures a Pfaffian \mathbb{K} -representation of \mathbb{S} . \square

Remark 4.0.6. Let us work on $\bar{\mathbb{K}}$ and let \mathbb{S} be general. In Remark 2.2.4 we saw that the Pfaffian representations produced by Algorithm 2.2.1 are all equivalent, once fixed the inputs $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4, \mathbf{a}^5$. The constructive proof of Theorem 1.0.9 provides a new algorithm to construct many Pfaffian representations starting from just one point \mathbf{a}^1 : we claim that neither this algorithm is surjective onto the possible Pfaffian representations of \mathbb{S} , once fixed \mathbf{a}^1 .

Indeed, by Remark 2.2.5, the space of essentially different Pfaffian representations of \mathbb{S} is five-dimensional. Since we can suppose \mathbb{S} is smooth, \mathbf{a}^1 is not singular. The procedure described in the proof of Theorem 3.1.3 consists in taking a point on a plane cubic curve in each step, and so the space of sets of five points obtained starting from \mathbf{a}^1 is four-dimensional. The conclusion follows again by Remark 2.2.4.

Remark 4.0.7. The procedure lying beneath the proof of Theorem 1.0.9 involves only linear equations and can be implemented in a deterministic algorithm.

4.1. Weakening hypotheses.

4.1.1. No starting points. One of the hypotheses of Theorem 1.0.9 was a \mathbb{K} -point on \mathbb{S} . If this is not given, then one can manage to find a \mathbb{K}' -point \mathbf{a} , being \mathbb{K}' an algebraic extension of degree at most three, simply by solving a polynomial equation of degree three (given by intersecting \mathbb{S} with two arbitrary planes). For the general choice of these two planes, \mathbf{a} is not a T-point and so Theorem 1.0.9 applies. This proves Proposition 1.0.10.

4.1.2. Reducible surfaces. Let \mathbb{S} be a reducible cubic surface. Then \mathbb{S} is either union of three planes with equation π_1, π_2, π_3 or union of a plane π and a quadratic irreducible surface \mathcal{S} . In both cases, simple Pfaffian representations can be constructed, as we will show.

In the first case, a Pfaffian representation is given by $\left(\begin{array}{c|c} 0 & M \\ -M & 0 \end{array} \right)$, where

$$M = \begin{pmatrix} \pi_1 & 0 & 0 \\ 0 & \pi_2 & 0 \\ 0 & 0 & \pi_3 \end{pmatrix}.$$

In the second case, let us consider the matrix

$$\mathbb{T}' = \begin{pmatrix} 0 & -x_3 & -x_2 \\ x_3 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$

If $\mathcal{S} \ni [1 : 0 : 0 : 0]$, then we can find three linear forms L_1, L_2, L_3 such that an equation for \mathcal{S} is $\sum_{i=1}^3 (-1)^{i+1} L_i x_i$. A Pfaffian representation of \mathcal{S} is then given by

$$P = \left(\begin{array}{ccc|c} & & & L_1 \\ & \mathbb{T}' & & L_2 \\ & & & L_3 \\ \hline -L_1 & -L_2 & -L_3 & 0 \end{array} \right)$$

by formula (1).

If $[1 : 0 : 0 : 0] \notin \mathcal{S}$, then it is sufficient to apply to x_1, x_2, x_3 in \mathbb{T}' the projectivity which maps a given point \mathbf{a} on \mathcal{S} to $[1 : 0 : 0 : 0]$, as described in subsection 2.2. Again by formula (1) one finds three linear forms and a Pfaffian representation P of \mathcal{S} as above.

A Pfaffian representation of \mathbb{S} is then given by

$$\left(\begin{array}{c|c|c} 0 & 0 & \pi \\ \hline 0 & P & 0 \\ \hline -\pi & 0 & 0 \end{array} \right).$$

Remark 4.1.1. Let $F \in \mathbb{K}[x_0, x_1, x_2, x_3]_3$ be an equation for the reducible surface \mathbb{S} . The Pfaffian representations just constructed are not \mathbb{K} -representations, in general. This is due to the fact that the splitting field of a polynomial of degree three is generally an algebraic extension of \mathbb{K} of degree six.

However, for such reducible surfaces we can state: it is possible to construct explicitly a Pfaffian \mathbb{K}' -representation, being \mathbb{K}' an algebraic extension of \mathbb{K} of degree at most six.

4.1.3. Cones. Let \mathbb{S} be an irreducible cone. If we suppose non-restrictively that its vertex is $[1 : 0 : 0 : 0]$, then \mathbb{S} is defined by an equation $F \in \mathbb{K}[x_1, x_2, x_3]$. Let us call \mathcal{C} the plane cubic curve defined by F in $\mathbb{P}_{\mathbb{K}}^3 \cap V(x_0)$.

As previously done, we can find a \mathbb{K}' -point \mathbf{a} on \mathcal{C} , being \mathbb{K}' an algebraic extension of \mathbb{K} , simply by solving a polynomial equation of degree three.

The construction of \mathbb{K}' -points on a plane cubic curve is a subject widely studied in literature (see for example [24]). Starting from a set X of \mathbb{K}' -points, it consists in considering tangent lines to the curve in each point of X , and secant lines through each pair of points of X ; the third intersection of such lines with \mathcal{C} is then set as a new element in X .

This process fails, for particular choices of $X = \{\mathbf{a}\}$: for example, if \mathbf{a} is an inflection point of the curve. For a general choice of \mathbf{a} , this process produces a lot of \mathbb{K}' -points on \mathcal{C} , and we can manage to find five points among them such that no three are collinear. Then the following proposition applies.

Proposition 4.1.2. Let \mathbb{S} be a cone over a plane cubic curve \mathcal{C} , with equation $F \in \mathbb{K}'[x_0, x_1, x_2, x_3]_3$. If there exist five \mathbb{K}' -points on \mathcal{C} such that no three of them are on a line, then there exist five \mathbb{K}' -points in general position on \mathbb{S} .

Proof. We can suppose the vertex is $[1 : 0 : 0 : 0]$, so that the equation of the plane curve (and the cone) is $C = C(x_1, x_2, x_3)$. Let $\underline{a}^i = (a_0^i, a_1^i, a_2^i, a_3^i)$ represent the five points. The vanishing of each of the 4×4 minors of the matrix

$$(12) \quad \begin{pmatrix} y_1 & a_1^1 & a_2^1 & a_3^1 \\ y_2 & a_1^2 & a_2^2 & a_3^2 \\ y_3 & a_1^3 & a_2^3 & a_3^3 \\ y_4 & a_1^4 & a_2^4 & a_3^4 \\ y_5 & a_1^5 & a_2^5 & a_3^5 \end{pmatrix}$$

imposes a non-trivial close condition to $\underline{y} \in \mathbb{A}_{\mathbb{K}'}^5$, since the 3×3 minors of the matrix obtained by deleting the first column in (12) are non-vanishing by hypotheses. So there exists \underline{y} satisfying none of these conditions and we get five points in general positions on \mathbb{S} . \square

Let us remark that also in the case of cones it is possible to implement an algorithm which requires an equation $F \in \mathbb{K}[x_0, x_1, x_2, x_3]$ for the surface \mathbb{S} and ensures a Pfaffian \mathbb{K}' -representation of \mathbb{S} , being $[\mathbb{K}' : \mathbb{K}] \leq 3$.

Summarizing, we can prove Theorem 1.0.11, which is a generalization of Proposition 1.0.8.

Proof of Theorem 1.0.11. It follows from Proposition 1.0.10, Remark 4.1.1 and from the discussion about cones made above. \square

4.2. An example. Let $F = x_0x_1^2 + x_1x_2^2 + x_2^3$ be the equation of \mathbb{S} , the unique cubic surface which does not admit a linear determinantal representation by [6], up to projectivity. Let us consider the point $\mathbf{a}^1 = [1 : 0 : 0 : 0]$, which is singular and therefore not a T-point. Then Theorem 1.0.9 applies, and we can construct explicitly a Pfaffian \mathbb{Q} -representation of \mathbb{S} .

According to the proof of Theorem 3.1.3, we choose the plane $x_3 = 0$, which does not cut \mathbb{S} in three lines. Considering the point $[1 : 1 : 0 : 0]$, the line through it and \mathbf{a}^1 intersects \mathbb{S} in $\mathbf{a}^2 = [0 : 1 : 0 : 0]$.

We have

$$T_{\mathbf{a}^2} \mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 = 0 \\ x_1x_2^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point on $x_0 = 0$, say $[0 : 0 : 1 : 1]$. The line through it and \mathbf{a}^2 intersects \mathbb{S} in $\mathbf{a}^3 = [0 : -1 : 1 : 1]$.

We have

$$T_{\mathbf{a}^3} \mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 + x_1 + 3x_2 - 2x_3 = 0 \\ -x_1^3 - 3x_1^2x_2 + 2x_1^2x_3 + x_1x_2^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point satisfying the first equation, say $[5 : 0 : -1 : 1]$. The line through it and \mathbf{a}^3 intersects \mathbb{S} in $\mathbf{a}^4 = [-10 : 1 : 1 : -3]$.

We have

$$T_{\mathbf{a}^4} \mathbb{S} \cap \mathbb{S} : \begin{cases} x_0 - 11x_1 + 3x_2 - 6x_3 = 0 \\ 11x_1^3 - 3x_1^2x_2 + 6x_1^2x_3 + x_1x_2^2 + x_2^3 = 0 \end{cases}$$

and so we choose a point satisfying the first equation, say $[40 : 2 : -2 : 2]$. The line through it and \mathbf{a}^4 intersects \mathbb{S} in $\mathbf{a}^5 = [95 : 1 : -6 : 11]$.

A Pfaffian \mathbb{Q} -representation can be obtained via Algorithm 2.2.1. For example,

simplifying denominators, we have $P = (p_{ij})$ with the following entries:

$$\begin{array}{ll}
 p_{12} = 0, & p_{13} = x_2 - x_3, \\
 p_{14} = 0, & p_{15} = 3x_2 + x_3, \\
 p_{16} = 1470x_1 + 686x_2 + 588x_3, & p_{23} = -x_2 + x_3, \\
 p_{24} = 34x_0 - 510x_1 - 170x_2 - 340x_3, & p_{25} = 2x_1 + x_2 + x_3, \\
 p_{26} = 1372x_1 + 588x_3, & p_{34} = 8670x_1 + 6120x_2 + 2550x_3, \\
 p_{35} = -34x_1 - 17x_2 - 17x_3, & p_{36} = -23324x_1 - 10829x_3, \\
 p_{45} = 0, & p_{46} = 774690x_1 - 624750x_2, \\
 p_{56} = -21658x_1 + 11662x_2 + 833x_3.
 \end{array}$$

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