



**HAL**  
open science

# Variance Asymptotics and Scaling Limits for Random Polytopes

Pierre Calka, J. E. Yukich

► **To cite this version:**

Pierre Calka, J. E. Yukich. Variance Asymptotics and Scaling Limits for Random Polytopes. *Advances in Mathematics*, 2017, 304, pp.1-55. hal-01263125

**HAL Id: hal-01263125**

**<https://hal.science/hal-01263125>**

Submitted on 27 Jan 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Variance Asymptotics and Scaling Limits for Random Polytopes

Pierre Calka\*, J. E. Yukich\*\*

January 27, 2016

## Abstract

Let  $K$  be a convex set in  $\mathbb{R}^d$  and let  $K_\lambda$  be the convex hull of a homogeneous Poisson point process  $\mathcal{P}_\lambda$  of intensity  $\lambda$  on  $K$ . When  $K$  is a simple polytope, we establish scaling limits as  $\lambda \rightarrow \infty$  for the boundary of  $K_\lambda$  in a vicinity of a vertex of  $K$  and we give variance asymptotics for the volume and  $k$ -face functional of  $K_\lambda$ ,  $k \in \{0, 1, \dots, d-1\}$ , resolving an open question posed in [18]. The scaling limit of the boundary of  $K_\lambda$  and the variance asymptotics are described in terms of a germ-grain model consisting of cone-like grains pinned to the extreme points of a Poisson point process on  $\mathbb{R}^{d-1} \times \mathbb{R}$  having intensity  $\sqrt{d}e^{dh}dh dv$ .

## 1 Main results

Let  $K$  be a convex subset of  $\mathbb{R}^d$  with non-empty interior. For all  $\lambda \in [1, \infty)$ , let  $\mathcal{P}_\lambda$  denote a homogeneous Poisson point process of intensity  $\lambda$  on  $K$ . Let  $K_\lambda$  be the polytope defined by the convex hull of  $\mathcal{P}_\lambda$ , with  $f_k(K_\lambda)$  denoting the number of  $k$ -faces of  $K_\lambda$ ,  $k \in \{0, \dots, d-1\}$ . The study of the random polytope  $K_\lambda$  has a long and rich history going back at least until 1864 and we refer to the surveys of Weil and Wieacker [18] and Reitzner [13] for details. Major papers of Rényi and Sulanke [14, 15] have played a seminal role in the subject. When  $K$  has a smooth boundary  $\partial K$ , it has been only recently shown by Reitzner [11] that  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  satisfy a central limit

---

*American Mathematical Society 2010 subject classifications.* Primary 60F05, 52A20; Secondary 60D05, 52A23

*Key words and phrases.* Random polytopes, germ-grain models, stabilization

\* Research partially supported by French ANR grant PRESAGE.

\*\* Research supported in part by NSF grant DMS-1406410.

theorem as  $\lambda \rightarrow \infty$ . More recently, for  $\partial K$  smooth, the second order properties and scaling limits of  $\partial K_\lambda$  have been established in [7, 9, 17].

When  $K$  is itself a convex polytope, the analysis of  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  appears more challenging. The lack of regularity in  $\partial K$  as well as the lack of rotational symmetry in  $K$  present additional technical obstacles. Still, the central limit theorem for  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  was shown in two remarkable papers of Bárány and Reitzner [4, 5], who also establish rates of normal convergence for these functionals. They do not consider scaling limits of  $\partial K_\lambda$  and though they obtain sharp lower bounds for  $\text{Var} f_k(K_\lambda)$  and  $\text{Var} \text{Vol}(K_\lambda)$ , their results stop short of determining precise variance asymptotics for  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  as  $\lambda \rightarrow \infty$ , an open problem going back to the 1993 survey of Weil and Wieacker (p. 1431 of [18]). When  $K$  is a simple polytope, we resolve this problem in Theorems 1.3 and 1.4, expressing variance asymptotics in terms of scaling limit functionals of the germ-grain model consisting of cone-like grains pinned to the ‘extreme’ points of the Poisson point process  $\mathcal{P}$  on  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity

$$d\mathcal{P}((v, h)) := \sqrt{d} e^{dh} dh dv, \quad (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}. \quad (1.1)$$

Along the way, we show that the scaling limit of  $\partial K_\lambda$  near any vertex of  $K$  coincides with the boundary of this germ-grain model.

Our results share some striking similarities with their asymptotic counterparts for convex hull functionals of i.i.d. uniform samples in the unit ball as well as for i.i.d. Gaussian samples in  $\mathbb{R}^d$ , as given in [9] and [8], respectively. The remarkable qualitative similarities, made precise in Remark (ii) below, help unify both the second order analysis of random polytopes as well as the scaling limit analysis of their boundaries.

Before stating our results we require some additional terminology. Henceforth we assume that  $K$  is a simple polytope, namely one whose vertices are adjacent to  $d$  facets (faces of dimension  $d - 1$ ). Let  $V := \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_{i=1}^d x_i = 0\}$  and for every  $v \in V$  and  $1 \leq i \leq d$ , we let  $l_i(v)$  be the  $i$ -th coordinate of  $v$  in the standard basis with respect to  $\mathbb{R}^d$  and  $l(v)$  the vector  $(l_1(v), \dots, l_d(v))$  in  $\mathbb{R}^d$ . Put

$$G(v) := \log\left(\frac{1}{d} \sum_{i=1}^d e^{l_i(v)}\right), \quad v \in V. \quad (1.2)$$

The graph of  $G$  has a cone-like structure, as shown in Lemma 4.5, and gives rise to the cone-like grain

$$\Pi^\downarrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -G(v)\} \quad (1.3)$$

opening in the down direction. For  $w := (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$  we put  $\Pi^\downarrow(w) := w \oplus \Pi^\downarrow$ , where  $\oplus$  denotes Minkowski addition. Given a locally finite set  $\mathcal{X}$  in  $\mathbb{R}^d$ , the maximal

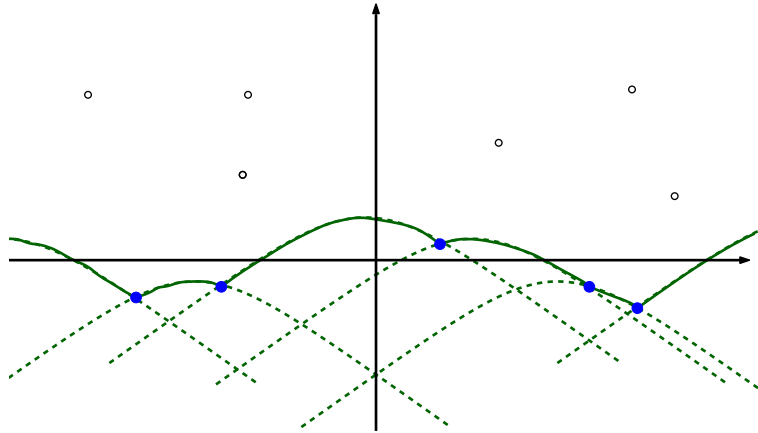


Figure 1: The point process  $\text{Ext}(\mathcal{X})$  (blue); the boundary  $\partial\Phi(\mathcal{X})$  of the down-grains containing  $\text{Ext}(\mathcal{X})$  (green).

union of grains  $\Pi^\downarrow(w), w \in \mathbb{R}^{d-1} \times \mathbb{R}$ , whose interior contains no point of  $\mathcal{X}$  is

$$\Phi(\mathcal{X}) := \bigcup_{\substack{w \in \mathbb{R}^{d-1} \times \mathbb{R} \\ \mathcal{X} \cap \text{int}(\Pi^\downarrow(w)) = \emptyset}} \Pi^\downarrow(w). \quad (1.4)$$

Remove points of  $\mathcal{X}$  not belonging to  $\partial(\Phi(\mathcal{X}))$  and call the resulting thinned point set  $\text{Ext}(\mathcal{X})$ . As shown in Figure 1,  $\partial(\Phi(\mathcal{X}))$  is a union of inverted cone-like surfaces ‘pinned’ to or ‘suspended’ from  $\text{Ext}(\mathcal{X})$ .

Extending the logarithmic function to  $(0, \infty)^d$  by the formula  $\log(z_1, \dots, z_d) = (\log z_1, \dots, \log z_d)$ , we consider for all  $\lambda \in [1, \infty)$

$$T^{(\lambda)} : \begin{cases} (0, \infty)^d & \longrightarrow V \times \mathbb{R} \\ (z_1, \dots, z_d) & \longmapsto \left( p_V(\log z), \frac{1}{d}(\log \lambda + \sum_{i=1}^d \log z_i) \right) \end{cases} \quad (1.5)$$

Here  $p_V : \mathbb{R}^d \rightarrow \mathbb{R}$  denotes the orthogonal projection onto  $V$ . Postponing the motivation behind  $T^{(\lambda)}$  until Section 4, we state our main results. Let  $K' := [0, \Delta_d]^d$  where  $\Delta_d \in [1, \infty)$  is a suitably large constant depending only on  $d$ , to be further specified in the sequel (cf. Lemma 7.1). Without loss of generality, re-scaling  $K$  if necessary, we make a volume preserving affine transformation such that the origin is a vertex of  $K$ ,  $K' \subset K$ , and  $K$  is contained in a multiple of  $K'$ . Put

$$\delta_0 := \delta_0(\lambda) := \exp(-(\log \lambda)^{1/d}) \quad (1.6)$$

and let  $Q_0 := [0, \delta_0]^d$ .

**Theorem 1.1** *Under the transformation  $T^{(\lambda)}$ , the extreme points of  $K_\lambda \cap Q_0$  converge in distribution to the thinned process  $\text{Ext}(\mathcal{P})$  as  $\lambda \rightarrow \infty$ .*

Let  $B_d(x, r)$  be the closed  $d$ -dimensional Euclidean ball centered at  $x \in \mathbb{R}^d$  and with radius  $r \in (0, \infty)$ .  $\mathcal{C}(B_d(x, r))$  is the space of continuous functions on  $B_d(x, r)$  equipped with the supremum norm. Let  $\mathbf{0}$  denote a point at the origin of  $\mathbb{R}^d$  or  $\mathbb{R}^{d-1}$ , according to context.

**Theorem 1.2** *Fix  $L \in (0, \infty)$ . As  $\lambda \rightarrow \infty$ , the re-scaled boundary  $T^{(\lambda)}((\partial K_\lambda) \cap Q_0)$  converges in probability to  $\partial(\Phi(\mathcal{P}))$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ .*

The transformation  $T^{(\lambda)}$  induces scaling limit  $k$ -face and volume functionals governing the large  $\lambda$  behavior of  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  as seen in the next results.

**Theorem 1.3** *For all  $k \in \{0, 1, \dots, d-1\}$ , there exists a constant  $F_{k,d} \in (0, \infty)$ , defined in terms of averages of covariances of a scaling limit  $k$ -face functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} \text{Var} f_k(K_\lambda) = F_{k,d} \cdot f_0(K). \quad (1.7)$$

**Theorem 1.4** *There exists a constant  $V_d \in (0, \infty)$ , defined in terms of averages of covariances of a scaling limit volume functional on  $\mathcal{P}$ , such that*

$$\lim_{\lambda \rightarrow \infty} \lambda^2 (\log \lambda)^{-(d-1)} \text{Var} \text{Vol}(K_\lambda) = V_d \cdot f_0(K). \quad (1.8)$$

*Remarks.* (i) *On the scaling transform.* Baryshnikov's far reaching work [6] uses the scaling transform  $T^{(\lambda)}$ , though in a different guise, to establish Gaussian fluctuations and variance asymptotics for the number of Pareto extreme points in a random sample. Baryshnikov (cf. Section 2.2.5 of [6]) also mentions that  $T^{(\lambda)}$  could be used to establish the asymptotic normality of  $f_0(K_\lambda)$ , but he does not provide the details. The present paper, in addition to establishing the scaling limit of  $\partial(K_\lambda)$  and the asymptotics of  $\text{Var} f_k(K_\lambda)$  and  $\text{Var} \text{Vol}(K_\lambda)$ , makes a three-fold contribution going beyond that in [6]. First, we establish a new, if not crucial, interpretation of the action of the scaling transform in terms of a dual process defined via cone-like grains called petals. In the setting of convex hulls of i.i.d. samples in the unit ball, the dual process has previously featured as a parabolic growth process [9, 17]. Second, we establish a qualitative link with scaling transforms used previously for different models of random polytopes; see remark (ii) below and Section 4.1. Lastly, the transform suitably re-scales the *floating bodies* for  $K$ , showing that their re-scaled images play a central role in the re-scaled convex hull geometry. Floating bodies usefully approximate random polytopes, as in

[4, 5], but here we show that their re-scaled images also play a key role in asymptotic analysis. The approach surrounding the transform  $T^{(\lambda)}$ , together with the counterpart transforms in [8, 9, 17], help unify the asymptotic analysis of random polytopes.

(ii) *Theorems 1.1 and 1.2 - related work.* The re-scaled point process  $T^{(\lambda)}(\mathcal{P}_\lambda)$  converges to the point process  $\mathcal{P}$  as seen in Lemma 4.2. The part of the re-scaled boundary of  $K_\lambda$  which is close to a vertex of  $K$  converges to a festoon of inverted cone-like hypersurfaces pinned to  $\text{Ext}(\mathcal{P})$ . The situation with  $K$  a unit ball involves quantitative differences and similarities. When  $K$  is the unit  $d$ -dimensional ball, in the large  $\lambda$  limit, the relevant scaling transform carries  $\mathcal{P}_\lambda$  into a homogeneous Poisson point process on the upper half-space  $\mathbb{R}^{d-1} \times \mathbb{R}^+$  and it carries the boundary of  $K_\lambda$  into a festoon of *parabolic* hypersurfaces [9, 17]. On the other hand, if the input is a Poisson point process  $\tilde{\mathcal{P}}_\lambda$  having Gaussian intensity  $\lambda\phi(x)dx$ , with  $\phi$  being the standard normal density on  $\mathbb{R}^d$ , then, as  $\lambda \rightarrow \infty$ , the relevant scaling transform carries  $\tilde{\mathcal{P}}_\lambda$  into a non-homogeneous Poisson point process  $\tilde{\mathcal{P}}$  on  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity density  $e^h dh dv$  and it carries the boundary of the convex hull of  $\tilde{\mathcal{P}}_\lambda$  into a festoon of parabolic hypersurfaces pinned against the extreme points of  $\tilde{\mathcal{P}}$  [8].

(iii) *Theorems 1.3 and 1.4.* Variance asymptotics (1.7) and (1.8) do not depend on the volume of  $K$ , but only on the number of its vertices. Breakthrough papers of Bárány and Reitzner [4, 5] establish precise growth rates for  $\text{Var}f_k(K_\lambda)$  and  $\text{VarVol}(K_\lambda)$ . While these works do not give a closed form expression for the asymptotic constants  $F_{k,d}$  and  $V_d$ , they do insure their strict positivity. We anticipate that methods given here yield expectation and variance asymptotics for non-homogenous Poisson point processes having intensity density  $\lambda\kappa$ , with  $\kappa : K \rightarrow \mathbb{R}^+$  bounded away from zero and infinity and continuous on  $\partial K$ .

(iv) *The locally defined transform  $T^{(\lambda)}$ .* The map  $T^{(\lambda)}$  is local in that it is defined with respect to  $\mathbf{0}$ , assumed to be a vertex of  $K$ . We are unable to find a suitable global transform for all of  $K$ . On the other hand, when  $K$  has rotational symmetry, e.g. when  $K$  is the  $d$ -dimensional ball, then we may globally map  $K$  into  $\mathbb{R}^{d-1} \times \mathbb{R}^+$  as in [9, 17]. The existence of a global scaling transform brings multiple benefits, leading to a more regularized re-scaled structure in  $\mathbb{R}^{d-1} \times \mathbb{R}$ , including stationarity as  $\lambda \rightarrow \infty$  and local independence (stabilization) with respect to spatial coordinates. When  $K$  lacks rotational symmetry, as is the case in this paper, then our methods do not yield any such global scaling transform. Roughly speaking, the obstruction to finding a global scaling transform goes as follows. The transform given here, like those in [8, 9], relies on the construction of a one-parameter family of  $(d-1)$ -dimensional surfaces interior to  $K$  (boundaries of associated floating bodies for  $K$ ), in which the height coordinate

is a function of the corresponding parameter and the space coordinate is given by a mapping from a subset of  $\mathbb{R}^{d-1}$  to the surface belonging to the one-parameter family. It is in general difficult to construct a global mapping from  $\mathbb{R}^{d-1}$  to a  $(d-1)$ -dimensional manifold, and thus difficult to find a global scaling transform.

(v) *Approximate additivity of the variance.* The lack of a global scaling transform necessitates showing that  $\text{Var}f_k(K_\lambda)$ ,  $k \in \{0, 1, \dots, d-1\}$ , and  $\text{VarVol}(K_\lambda)$  are well approximated by the sum of variances of contributions arising from small neighborhoods around each vertex of  $K$ . We show this decoupling of the variance over the vertex set of  $K$  by refining the dependency graph arguments in [4] and applying these arguments to a dyadic collection of Macbeath regions. These non-trivial technical obstacles are not present when  $K$  is the unit ball [9].

(vi) *Extension to general polytopes.* The fundamental work of Bárány and Buchta [2] shows that the extreme points of a general polytope  $K$  concentrate in regions defined by each flag of  $K$ . These ‘flag regions’, themselves simple polytopes, could individually be treated by the methods of this paper. If one could show (a) negligibility of covariances of contributions to  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  arising from input on distinct flag regions as well as (b) negligibility of contributions to  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  arising from input on the complements of flag regions, then variance would be additive with respect to flags. This would extend our results to general polytopes and it would align our second order results with the first order results of [2], which shows that expectation asymptotics are additive with respect to flags. However showing additivity of variance with respect to flags seems to be a separate project which would either require a scaling transform more general than  $T^{(\lambda)}$  or a non-trivial extension of the methods of Section 3.

This paper is organized as follows. Section 2 introduces scaling limit functionals of germ-grain models having cone-like grains. These scaling limit functionals feature in Theorem 2.1, which establishes expectation and variance asymptotics for the empirical measures for the volume and  $k$ -face functionals, thus extending Theorems 1.3 and 1.4. In Section 3 we establish two propositions which prepare for an effective use of the crucial transformation  $T^{(\lambda)}$ . We show that the extreme points near a vertex of  $K$  have a preferred normal direction and that the variance of the  $k$ -face and volume functional decouples over the vertices of  $K$ . Section 4 studies  $K_\lambda$  near a fixed vertex of  $K$  and establishes that the image of  $\mathcal{P}_\lambda$  under  $T^{(\lambda)}$  converges in distribution to  $\mathcal{P}$  and that  $T^{(\lambda)}$  defines re-scaled  $k$ -face and volume functionals. We show that the scaling transform  $T^{(\lambda)}$  maps the Euclidean convex hull geometry into ‘cone-like’ convex geometry in  $\mathbb{R}^{d-1} \times \mathbb{R}$  and that the extreme points near a vertex of  $K$  are with high probability characterized in terms of the geometry of *so-called* petals. Section 5 establishes that the re-scaled  $k$ -face and volume functionals localize in space, which is crucial to showing

convergence of their means and covariances to the respective means and covariances of their scaling limits. Section 6 provides proofs of the main results whereas Section 7, an Appendix, contains proofs of several lemmas.

## 2 General results

Here we consider functionals of the germ-grain model  $\Phi(\mathcal{P})$  which are central to the description of the scaling limits of the  $k$ -face and volume functionals  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$ . We use their second order correlations to establish variance asymptotics for the empirical measures induced by the  $k$ -face and volume functionals. This extends Theorems 1.3 and 1.4 to the setting of measures and yields formulas for the constants  $F_{k,d}$  and  $V_d$  in (1.7) and (1.8), respectively. Denote points in  $\mathbb{R}^{d-1} \times \mathbb{R}$  by  $w := (v, h)$ .

**2.1. Empirical  $k$ -face and volume measures.** Given a finite point set  $\mathcal{X} \subset \mathbb{R}^d$ , let  $\text{co}(\mathcal{X})$  be its convex hull.

**Definition 2.1** *Given  $k \in \{0, 1, \dots, d-1\}$  and  $x$  a vertex of  $\text{co}(\mathcal{X})$ , define the  $k$ -face functional  $\xi_k(x, \mathcal{X})$  to be the product of  $(k+1)^{-1}$  and the number of  $k$ -faces of  $\text{co}(\mathcal{X})$  which contain  $x$ . Otherwise we put  $\xi_k(x, \mathcal{X}) = 0$ . Thus the total number of  $k$ -faces in  $\text{co}(\mathcal{X})$  is  $\sum_{x \in \mathcal{X}} \xi_k(x, \mathcal{X})$ . Letting  $\delta_x$  be the unit point mass at  $x$ , the empirical  $k$ -face measure for  $\mathcal{P}_\lambda$  is*

$$\mu_\lambda^{\xi_k} := \sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda) \delta_x. \quad (2.1)$$

We now consider the defect volume of  $K_\lambda$  with respect to  $K$  in a cubical neighborhood of  $\mathbf{0}$ . Recall from (1.6) that  $\delta_0 := \delta_0(\lambda) := \exp(-(\log \lambda)^{1/d})$  and  $Q_0 := [0, \delta_0]^d$ .

Let  $\mathcal{X} \subset [0, \infty)^d$  be finite. Given  $x$  a vertex of  $\text{co}(\mathcal{X})$ , let  $\mathcal{F}^+(x, \mathcal{X})$  be the (possibly empty) collection of facets in  $\text{co}(\mathcal{X})$ , included in  $Q_0$ , containing  $x$  and having outer normals in  $(-\infty, 0]^d$ . Let  $\text{cone}(x, \mathcal{X}) := \{ry : r > 0, y \in \mathcal{F}^+(x, \mathcal{X})\}$  be the (possibly empty) cone generated by  $\mathcal{F}^+(x, \mathcal{X})$ .

We first define the volume score for points in  $\mathcal{P}_\lambda \cap Q_0$ . If  $x$  is vertex of  $K_\lambda \cap Q_0$  with  $\text{cone}(x, \mathcal{P}_\lambda) \neq \emptyset$ , then define the defect volume functional

$$\xi_V(x, \mathcal{P}_\lambda) := d^{-1} \lambda \text{Vol}(\text{cone}(x, \mathcal{P}_\lambda) \cap (K \setminus K_\lambda)). \quad (2.2)$$

Otherwise put  $\xi_V(x, \mathcal{P}_\lambda) = 0$ . In general, the union  $\bigcup_x (\text{cone}(x, \mathcal{P}_\lambda) \cap (K \setminus K_\lambda))$ , with  $x$  ranging over the vertices of  $K_\lambda \cap Q_0$ , does not equal  $(K \setminus K_\lambda) \cap Q_0$ . See Figure 2.



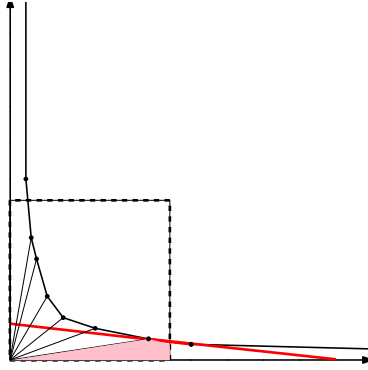


Figure 2: The part of the defect volume not counted in the sum of scores (pink); the boundary of the cap containing that part (red).

We now extend the definition of the volume score so that is defined for all points in  $\mathcal{P}_\lambda$ . This goes as follows. Let  $\mathcal{V}_K := \{\mathcal{V}_i\}$  denote the vertices of  $K$  and recall that we assume  $\mathbf{0} \in \mathcal{V}_K$ . Re-scaling  $K$  if necessary, for each vertex  $\mathcal{V}_i \in \mathcal{V}_K \setminus \{\mathbf{0}\}$ , we define an associated volume preserving affine transformation  $a_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , with  $a_i(\mathcal{V}_i) = \mathbf{0}$ , and such that the facets of  $a_i(K)$  containing  $\mathbf{0}$  also contain the facets of  $K' := [0, \Delta_d]^d$  belonging to the coordinate hyperplanes. For any  $\delta \in (0, \Delta_d)$ , define the parallelepiped  $p_d(\mathcal{V}_i, \delta) := a_i^{-1}([0, \delta]^d)$ . Put  $\mathcal{P}_\lambda(\delta) := \mathcal{P}_\lambda \cap \bigcup_{1 \leq i \leq f_0(K)} p_d(\mathcal{V}_i, \delta)$ . For any convex polytope  $K$  and any face  $F$  of  $K$ , we denote by  $C_F(K)$  the cone of outer normal vectors to  $F$ . For any  $x \in p_d(\mathcal{V}_i, \delta_0)$  define  $\mathcal{F}^+(x, \mathcal{P}_\lambda)$  to be the intersection of  $p_d(\mathcal{V}_i, \delta_0)$  and the (possibly empty) collection of facets in  $\text{co}(\mathcal{P}_\lambda)$ , included in  $p_d(\mathcal{V}_i, \delta_0)$ , containing  $x$  and having outer normals in  $C_{\mathcal{V}_i}(K)$ . We then put  $\text{cone}(x, \mathcal{P}_\lambda) = \{\mathcal{V}_i \oplus r(y - \mathcal{V}_i) : r > 0, y \in \mathcal{F}^+(x, \mathcal{P}_\lambda)\}$ . If  $x$  is vertex of  $K_\lambda \cap p_d(\mathcal{V}_i, \delta_0)$  with  $\text{cone}(x, \mathcal{P}_\lambda) \neq \emptyset$ , define the defect volume functional via the formula (2.2) and put  $\xi_V(x, \mathcal{P}_\lambda) = 0$  otherwise.

Finally, we define the volume score for points in  $\mathcal{P}_\lambda \setminus \mathcal{P}_\lambda(\delta_0)$ . Let  $\mathcal{F}_k(x) := \mathcal{F}_k(x, K_\lambda)$  be the (possibly empty) collection of  $k$ -dimensional faces in  $K_\lambda$  which contain  $x$ . For  $x$  a vertex of  $K_\lambda$  and  $x \notin \mathcal{P}_\lambda(\delta_0)$ , define the defect volume score

$$\xi_V(x, \mathcal{P}_\lambda) := \sum_{k=0}^{d-1} \sum_{F \in \mathcal{F}_k(x)} \frac{\lambda}{\text{card}\{F \cap (\mathcal{P}_\lambda \setminus \mathcal{P}_\lambda(\delta_0))\}} \text{Vol}((F \oplus C_F(K_\lambda)) \cap K \cap (D(\mathcal{P}_\lambda))^c), \quad (2.3)$$

where  $D(\mathcal{P}_\lambda) := \bigcup_{x \in \mathcal{P}_\lambda(\delta_0)} \text{cone}(x, \mathcal{P}_\lambda)$ . Otherwise, put  $\xi_V(x, \mathcal{P}_\lambda) = 0$ .

**Definition 2.2** Define the empirical defect volume measure of  $\mathcal{P}_\lambda$  by

$$\mu_\lambda^{\xi_V} := \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda) \delta_x, \quad (2.4)$$

where  $\xi_V(x, \mathcal{P}_\lambda)$  is given by (2.2) or (2.3) depending on whether  $x \in \mathcal{P}_\lambda(\delta_0)$  or not.

Notice that in general  $\sum_{x \in \mathcal{P}_\lambda \cap Q_0} \xi_V(x, \mathcal{P}_\lambda) \leq \lambda \text{Vol}(Q_0 \cap (K \setminus K_\lambda))$  and likewise we have  $\langle 1, \mu_\lambda^{\xi_V} \rangle \leq \lambda \text{Vol}(K \setminus K_\lambda)$ .

**2.2. Scaling limit  $k$ -face and volume functionals.** A set of  $(k + 1)$  extreme points  $\{w_1, \dots, w_{k+1}\} \subset \text{Ext}(\mathcal{P})$ , generates a  $k$ -dimensional *face* of the festoon  $\partial(\Phi(\mathcal{P}))$  if there exists a translate  $\tilde{\Pi}^\downarrow$  of  $\Pi^\downarrow$  such that  $\{w_1, \dots, w_{k+1}\} = \tilde{\Pi}^\downarrow \cap \text{Ext}(\mathcal{P})$ . When  $k = d - 1$  the face is a *hyperface*.

**Definition 2.3** Let  $w \in \text{Ext}(\mathcal{P})$ . Define the scaling limit  $k$ -face functional  $\xi_k^{(\infty)}(w, \mathcal{P})$ ,  $k \in \{0, 1, \dots, d-1\}$ , to be the product of  $(k+1)^{-1}$  and the number of  $k$ -dimensional faces of the festoon  $\partial(\Phi(\mathcal{P}))$  which contain  $w$ . The scaling limit defect volume functional is

$$\xi_V^{(\infty)}(w, \mathcal{P}) := \frac{1}{d\sqrt{d}} \int_{v \in \text{Cyl}(w)} \exp\{d \cdot \partial(\Phi(\mathcal{P}))(v)\} dv,$$

where  $\text{Cyl}(w)$  denotes the projection onto  $\mathbb{R}^{d-1}$  of the hyperfaces of  $\partial(\Phi(\mathcal{P}))$  containing  $w$ . When  $w \notin \text{Ext}(\mathcal{P})$  we put  $\xi_k^{(\infty)}(w, \mathcal{P}) = 0$  and  $\xi_V^{(\infty)}(w, \mathcal{P}) = 0$ .

Let  $\Xi$  denote the collection of functionals  $\xi_k$ ,  $k \in \{0, 1, \dots, d-1\}$ , together with  $\xi_V$ . Let  $\Xi^{(\infty)}$  denote the collection of scaling limits  $\xi_k^{(\infty)}$ ,  $k \in \{0, 1, \dots, d-1\}$ , together with  $\xi_V^{(\infty)}$ . A main feature of our approach (cf. Lemma 5.4) is that on a high probability set, the elements of  $\Xi^{(\infty)}$  are scaling limits of re-scaled elements of  $\Xi$ .

**2.3. Limit theory for empirical  $k$ -face and volume measures.** Define the following second order correlation functions for  $\xi^{(\infty)} \in \Xi^{(\infty)}$ .

**Definition 2.4** For all  $w_1, w_2 \in \mathbb{R}^d$  and  $\xi^{(\infty)} \in \Xi^{(\infty)}$  put

$$c^{\xi^{(\infty)}}(w_1, w_2) := c^{\xi^{(\infty)}}(w_1, w_2, \mathcal{P}) := \tag{2.5}$$

$$\mathbb{E} \xi^{(\infty)}(w_1, \mathcal{P} \cup \{w_2\}) \xi^{(\infty)}(w_2, \mathcal{P} \cup \{w_1\}) - \mathbb{E} \xi^{(\infty)}(w_1, \mathcal{P}) \mathbb{E} \xi^{(\infty)}(w_2, \mathcal{P})$$

and

$$\begin{aligned} \sigma^2(\xi^{(\infty)}) &:= \sqrt{d} \int_{-\infty}^{\infty} \mathbb{E} \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P})^2 e^{dh_0} dh_0 \\ &+ d \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{\xi^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)) e^{d(h_0+h_1)} dh_1 dv_1 dh_0. \end{aligned} \tag{2.6}$$

Let  $\mathcal{C}(K)$  be the class of bounded functions on  $K$  which are continuous on  $\mathcal{V}_K$ . Given  $g \in \mathcal{C}(K)$  let  $\langle g, \mu_\lambda^\xi \rangle$  denote its integral with respect to  $\mu_\lambda^\xi$ . Consider the regular  $d$ -dimensional simplex of edge length  $\sqrt{2}d$  given by

$$S(d) := \{(x_1, \dots, x_d) \in (-\infty, 1]^d : \sum_{i=1}^d x_i = 0\}. \tag{2.7}$$

The following general result is proved in Section 6.

**Theorem 2.1** *For all  $\xi \in \Xi$  and  $g \in \mathcal{C}(K)$  we have*

$$\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} \mathbb{E}[\langle g, \mu_\lambda^\xi \rangle] = d^{-d+(3/2)} \text{Vol}_d(S(d)) \int_{-\infty}^{\infty} \mathbb{E} \xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P}) e^{dh_0} dh_0 \sum_{\mathcal{V}_i \in \mathcal{V}_K} g(\mathcal{V}_i) \quad (2.8)$$

and

$$\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} \text{Var}[\langle g, \mu_\lambda^\xi \rangle] = d^{-d+1} \text{Vol}_d(S(d)) \sigma^2(\xi^{(\infty)}) \sum_{\mathcal{V}_i \in \mathcal{V}_K} g^2(\mathcal{V}_i). \quad (2.9)$$

**2.4. Deducing Theorems 1.3 and 1.4.** We may deduce the limits (1.7) and (1.8) from Theorem 2.1 as follows. Set  $\xi$  to be  $\xi_k$  and put  $g \equiv 1$ . Then  $\langle 1, \mu_\lambda^{\xi_k} \rangle = f_k(K_\lambda)$  and so (1.7) follows from (2.9), setting  $F_{k,d}$  to be  $d^{-d+1} \text{Vol}_d(S(d)) \sigma^2(\xi_k^{(\infty)})$ . Next, set  $\xi$  to be  $\xi_V$ . By Lemma 3.7(b) we have

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}\langle 1, \mu_\lambda^{\xi_V} \rangle}{(\log \lambda)^{d-1}} = \lim_{\lambda \rightarrow \infty} \frac{\lambda^2 \text{Var}(\text{Vol} K_\lambda)}{(\log \lambda)^{d-1}}.$$

Thus (1.8) follows from (2.9) where we set  $V_d$  to be  $d^{-d+1} \text{Vol}_d(S(d)) \sigma^2(\xi_V^{(\infty)})$ .

### 3 Decomposition of the variances

Before discussing the scaling transform  $T^{(\lambda)}$  we need some key simplifications. We show here that variance asymptotics for  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  are determined by the behavior of these functionals on points near any fixed vertex of  $K$ , assumed without loss of generality to be  $\mathbf{0}$ ; *that is to say the point set  $\mathcal{P}_\lambda \cap Q_0$  determines variance asymptotics.* It is far from clear that this should be the case, as covariances of scores on subsets of  $\mathcal{P}_\lambda$  near adjacent vertices of  $K$  might be non-negligible. Secondly, variances of scores on subsets of  $\mathcal{P}_\lambda$  ‘between’ adjacent vertices of  $K$  might also be non-negligible. The purpose of this section is to address these two issues via Proposition 3.2, showing the negligibility of the afore-mentioned quantities. This paves the way for an effective use of  $T^{(\lambda)}$ , which is well defined on  $\mathcal{P}_\lambda \cap Q_0$ , and which we use in Section 4 to re-scale the scores  $\xi \in \Xi$ , the input  $\mathcal{P}_\lambda$ , as well as  $Q_0 \cap \partial K_\lambda$ .

**Definition 3.1** *If the collection  $C_F(K_\lambda \cap Q_0)$  of outward normals to a face  $F$  in  $K_\lambda \cap Q_0$  all belong to the normal cone  $C_{\mathbf{0}}(K) := \{u = (u_1, \dots, u_d) \in (-\infty, 0]^d\}$ , then  $F$  is called a ‘cone-extreme’ face.*

Before stating Proposition 3.2 we require an auxiliary result, whose proof is deferred to the Appendix. We first assert that there is a high probability event  $A_\lambda$ , to be defined in Section 3.1, such that on  $A_\lambda$  all faces of  $K_\lambda \cap Q_0$  are cone-extreme. It is precisely the cone-extreme faces which are amenable to analysis under the transformation  $T^{(\lambda)}$ .

**Proposition 3.1** *There is an event  $A_\lambda$  with  $P[A_\lambda^c] \leq C(\log \lambda)^{-4d^2}$ , such that on  $A_\lambda$  we have  $C_F(K_\lambda \cap Q_0) \subset C_0(K)$  for any face  $F$  of  $K_\lambda \cap Q_0$ .*

Sections 4 and 5 develop the geometry and scaling properties of cone-extreme faces. In particular Lemma 4.3 identifies their collective image under  $T^{(\lambda)}$  with a festoon of inverted cone-like surfaces.

Next, for each  $\xi \in \Xi$ , put

$$Z := Z_\lambda := \sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda). \quad (3.1)$$

The contribution to the total score from points in  $\mathcal{P}_\lambda \cap p_d(\mathcal{V}_i, \delta)$  is

$$Z_i := Z_i(\delta) := \sum_{x \in \mathcal{P}_\lambda \cap p_d(\mathcal{V}_i, \delta)} \xi(x, \mathcal{P}_\lambda), \quad 1 \leq i \leq f_0(K). \quad (3.2)$$

We now choose  $\delta := \delta(\lambda)$  large enough so that  $\sum_{i=1}^{f_0(K)} Z_i(\delta)$  captures the bulk of the total score  $Z$ , but small enough so that  $Z_i(\delta)$  are independent random variables, or at least conditionally so, given the event  $A_\lambda$  of Proposition 3.1. The next proposition tells us that it suffices to set  $\delta$  to be  $\delta_0$  and it shows that  $\text{Var} Z$  is essentially a sum of variances of scores induced by points in  $\mathcal{P}_\lambda$  near each vertex of  $K$ . Given two sequences of scalars  $\alpha_\lambda$  and  $\beta_\lambda$ ,  $\lambda > 0$ , we write  $\alpha_\lambda = o(\beta_\lambda)$  if  $\alpha_\lambda/\beta_\lambda \rightarrow 0$  as  $\lambda \rightarrow \infty$ .

**Proposition 3.2** *For all  $\xi \in \Xi$  we have*

$$\mathbb{E}[Z\mathbf{1}(A_\lambda)] = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \mathbb{E}[Z_i(\delta_0)\mathbf{1}(A_\lambda)] + o(\mathbb{E}[Z]) \quad (3.3)$$

and

$$\text{Var}[Z\mathbf{1}(A_\lambda)] = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \text{Var}[Z_i(\delta_0)\mathbf{1}(A_\lambda)] + o(\text{Var}[Z]) = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \text{Var}[Z_i(\delta_0)] + o(\text{Var}[Z]). \quad (3.4)$$

Proposition 3.2 shows that to prove Theorems 1.3 and 1.4, it is enough to establish the variance of the  $k$ -face and volume functional for that part of  $K_\lambda$  included in  $Q_0$ . The identity (3.3) is essentially a re-phrasing of Theorems 3 and 4 in [2], which show

that  $\mathbb{E}[Z]$  is a sum of expectations of scores induced by points in  $\mathcal{P}_\lambda$  near each vertex of  $K$  (and more generally, near each flag of  $K$  when  $K$  is an arbitrary convex polytope). The methods of [2] do not appear to extend to variances.

To prove these two propositions, we shall rely heavily on a construction of dyadic Macbeath regions. The rest of this section is devoted to the proof of Proposition 3.2. The set-up of the next three subsections closely parallels that of the breakthrough paper [4].

**3.1. A critical annulus and a high probability set.** As in [4], define  $v : K \mapsto \mathbb{R}$  by

$$v(z) := \min\{V(K \cap H) : H \text{ is a half-space and } z \text{ in } H\}.$$

There should be no confusion between the function  $v$ , used in this section and in the Appendix, and the point  $v$ , denoting a generic point in  $\mathbb{R}^{d-1}$ , used in subsequent sections. For  $t \in [0, \infty)$ , let  $K(v = t)$  be the boundary of the floating body  $\{z \in K : v(z) \geq t\}$ , which we abbreviate as  $K(v \geq t)$ . Recall  $K' := [0, \Delta_d]^d$ , with  $\Delta_d \in [1, \infty)$  to be specified. Lemma 7.1 in the Appendix shows that for  $\Delta_d$  large, the floating bodies for  $K$  and  $K'$  coincide in  $[0, 1/2]^d$ . Following [4], put

$$s := s_\lambda := \frac{1}{\lambda(\log \lambda)^\beta}, \quad T := \frac{\alpha \log \log \lambda}{\lambda}, \quad T^* := d6^d T \quad (3.5)$$

with  $\beta := 4d^2 + d - 1$ ,  $\alpha := (6d)^d \beta$  (in this section,  $T$  denotes the scalar at (3.5) and there should be no confusion with  $T^{(\lambda)}$ ). Consider the annulus-like set

$$\mathcal{A}(s, T^*, K) := K(v \geq s) \setminus K(v \geq T^*).$$

By Lemma 5.2 of [4] there is an event  $A_\lambda := A_\lambda(K)$  such that on  $A_\lambda$  we have

$$\partial K_\lambda \subset \mathcal{A}(s, T^*, K), \text{ where } (\log \lambda)^{-(3d)^{d+2}} \leq P[A_\lambda^c] \leq C(\log \lambda)^{-4d^2}. \quad (3.6)$$

**3.2. Macbeath regions.** In this subsection we construct Macbeath regions near the origin. As we shall see, the construction serves as a prototype for constructing Macbeath regions near vertices  $\mathcal{V}_i \in \mathcal{V}_K \setminus \{\mathbf{0}\}$ .

For all  $z \in K$ , let  $M_K(z) := M_K(z, 1/2)$  be the Macbeath region (M-region for short) with center  $z$  and scale factor  $1/2$ , i.e.,

$$M_K(z) := M_K(z, 1/2) := z + \frac{1}{2}[(K - z) \cap (z - K)].$$

For  $z := (z_1, \dots, z_d) \in [0, 1/2]^d$  we have

$$M_{K'}(z) = \prod_{i=1}^d \left[ \frac{z_i}{2}, \frac{3z_i}{2} \right]. \quad (3.7)$$

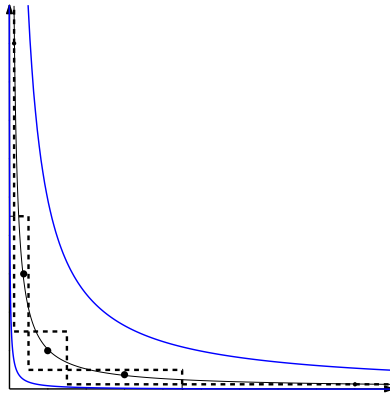


Figure 3: A saturated collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  of Macbeath regions

The inclusion  $K' \subseteq K$  gives for all  $z := (z_1, \dots, z_d) \in [0, 1/2]^d$

$$M_K(z) = \prod_{i=1}^d \left[ \frac{z_i}{2}, \frac{3z_i}{2} \right].$$

More generally, given  $\delta \in (0, 1/2)$  and integers  $k_i \in \mathbb{Z}$  with  $3^{k_i} \in (0, 1/(3\delta)]$ ,  $1 \leq i \leq d$ , the dyadic rectangular solids

$$\prod_{i=1}^d \left[ \frac{3^{k_i} \delta}{2}, \frac{3^{k_i+1} \delta}{2} \right] \quad (3.8)$$

coincide with the  $M$ -regions

$$M_{K'}((3^{k_1} \delta, \dots, 3^{k_d} \delta)) = M_K((3^{k_1} \delta, \dots, 3^{k_d} \delta)), \quad 3^{k_i} \in (0, 1/(3\delta)]. \quad (3.9)$$

Points  $z := (3^{k_1} \delta, \dots, 3^{k_d} \delta)$  are centers of dyadic solids. When  $\log_3 T/\delta^d \in \mathbb{Z}$ , then  $M_K(z)$  has center  $z$  belonging to  $K(v = T)$  as soon as  $\sum_{i=1}^d k_i = \log_3 T/\delta^d$ ; we shall use such  $M$ -regions to define a saturated system as in [4].

Henceforth, let  $\delta \in (0, 1/2)$  and with  $\log_3 T/\delta^d \in \mathbb{Z}$ . Consider the collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  of dyadic rectangular solids of the type (3.9) having centers on  $K(v = T) \cap [0, 1/2]^d$  (see Figure 3). The solids in  $\mathcal{M}_K(\mathbf{0}, \delta)$  do not cover  $K(v = T) \cap [0, 1/2]^d$  but they leave some parts uncovered. The uncovered part is too small to accommodate another  $M$ -region with center on  $K(v = T) \cap [0, 1/2]^d$ . In other words, the collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  of dyadic  $M$ -regions defined at (3.9) is maximal in that it can not be enlarged to include another  $M$ -region with center on  $K(v = T) \cap [0, 1/2]^d$ . The following is proved in the Appendix.

**Lemma 3.1** *The collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  of  $M$ -regions is maximal.*

We will use the collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  to control the spatial dependence of scores  $\xi \in \Xi$ . This is done via supersets of  $M$ -regions, described below.

**3.3. Supersets of  $M$ -regions.** The collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  generates a ‘dyadic staircase’, where the step width increases in a geometric progression according to its distance from a coordinate hyperplane  $H_l$ ,  $1 \leq l \leq d$ . Elements of  $\mathcal{M}_K(\mathbf{0}, \delta)$  are only pairwise interior-disjoint and not pairwise disjoint. Still, we claim that this collection allows us to reproduce the construction from the economic cap covering theorem (Theorem 2.5 of [4]) with the same outcome and to construct a partition of  $K(v \leq T^*)$  into supersets  $S'_j$  which are also pairwise interior-disjoint (see Figure 4). This goes as follows.

Each  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$  produces a superset, called an  $S$ -region in [4], in the following canonical way. For  $M$ -regions  $M_j$  meeting  $[0, (T^*)^{1/d}]^d$  we define the associated region  $S'_j := S'_j(M_j)$  to be the intersection of  $K(v \leq T^*)$  and the smallest cone with apex at  $((T^*)^{1/d}, \dots, (T^*)^{1/d})$  which contains  $M_j$ . We call these the ‘cone sets  $S'_j$ ’. The volume of every  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$  is  $\prod_{i=1}^d 3^{k_i} \delta = T$ , and thus the number of  $M$ -regions meeting  $[0, (T^*)^{1/d}]^d$  is bounded by a constant depending only on  $d$ . The ‘cone sets  $S'_j$ ’ are not contained in  $[0, (T^*)^{1/d}]^d$  when  $M_j$  itself is not contained in  $[0, (T^*)^{1/d}]^d$ . In this case, we replace the cone set  $S'_j$  with a so-called ‘cone-cylinder set’, defined to be the union of  $S'_j \cap [0, (T^*)^{1/d}]^d$  and the so-called ‘cylinder set’ generated by  $M_j \cap ([0, (T^*)^{1/d}]^d)^c$ , defined as follows.

For  $M$ -regions  $M_j$  with centers  $(3^{k_1} \delta, \dots, 3^{k_d} \delta)$  and such that  $M_j$  meets  $([0, (T^*)^{1/d}]^d)^c$ , define for  $1 \leq l \leq d$ , the cylinder

$$C_l(k_1, \dots, k_d) := \prod_{i=1}^{l-1} \left[ \frac{3^{k_i} \delta}{2}, \frac{3^{k_{i+1}} \delta}{2} \right] \times \mathbb{R} \times \prod_{i=l+1}^d \left[ \frac{3^{k_i} \delta}{2}, \frac{3^{k_{i+1}} \delta}{2} \right] \cap ([0, (T^*)^{1/d}]^d)^c.$$

Note that  $C_l(k_1, \dots, k_d)$  is the smallest cylinder containing  $M_j$  and oriented in the direction  $n_{H_l}$ , where  $n_{H_l}$  is a unit normal vector for the hyperplane  $H_l$ . The  $\tilde{S}_j$  region associated with  $M_j \cap ([0, (T^*)^{1/d}]^d)^c$  is

$$\tilde{S}_j := \tilde{S}_j((3^{k_1} \delta, \dots, 3^{k_d} \delta)) := \bigcup_{l: k_l = \min(k_1, \dots, k_d)} C_l(k_1, \dots, k_d) \cap K.$$

When  $k_l$  is the unique minimum,  $\tilde{S}_j$  consists solely of a single cylinder  $C_l$  and it simply extends  $M_j \cap ([0, (T^*)^{1/d}]^d)^c$  in the direction  $n_{H_l}$ . Note that  $n_{H_l}$  points in the direction of the facet of  $K'$  closest to  $M_j$ . The union of such  $\tilde{S}_j$  does not cover all of  $K(v \leq T^*) \cap ([0, 1/2]^d \setminus [0, (T^*)^{1/d}]^d)$ . The uncovered parts are rectangular regions produced by precisely one  $M$ -region having a cubical face. Consequently, we define  $S_j$

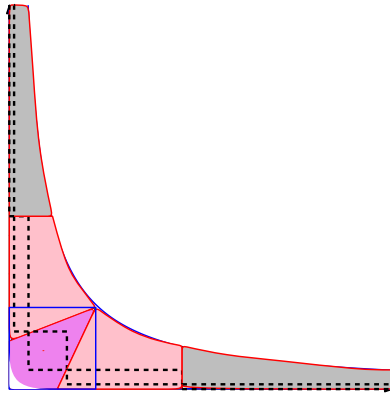


Figure 4: The supersets  $S'_j$  (purple, pink and grey) associated with the saturated system of Macbeath regions.

as the union of  $\tilde{S}_j$  and all rectangular regions produced by the ties in the minimum of  $k_1, \dots, k_d$ .

As on page 1518 of [4], we define the superset

$$S'_j := S_j \cap K(v \leq T^*). \quad (3.10)$$

We call these the ‘cylinder sets’. If  $S'_j$  has a facet  $F$  which meets  $\partial([0, (T^*)^{1/d}]^d)$  then we adjoin the cylinder set  $S'_j$  to the unique cone set containing  $F \cap [0, (T^*)^{1/d}]^d$  and we call the resulting set a cone-cylinder set. By construction, the sets  $S'_j$  are disjoint. Given this way of generating  $S'_j, 1 \leq j \leq \text{card}(\mathcal{M}_K(\mathbf{0}, \delta))$ , we may control its diameter in all directions  $n_{H_i}, 1 \leq i \leq d$ . The diameter of  $M_K((3^{k_1}\delta, \dots, 3^{k_d}\delta))$  in the direction  $n_{H_i}$  is

$$\frac{3^{k_i+1}\delta}{2} - \frac{3^{k_i}\delta}{2} = 3^{k_i}\delta.$$

The diameter of the corresponding  $S_j := S_j((3^{k_1}\delta, \dots, 3^{k_d}\delta))$  in the direction  $n_{H_i}, 1 \leq i \leq d$ , is thus at most  $c3^{k_i}\delta + 3^{k_i}\delta$ , with the first term accounting for the possible uncovered adjoined regions included in  $S_j$ , or, if the direction  $n_{H_i}$  were the direction  $i$  corresponding to the smallest  $k_i$  from the M-region, then the diameter would be the distance from the coordinate hyperplane to the pseudo-hyperboloid  $K(v = T^*)$  inside the cylinder  $C_i$  and thus would be bounded by  $c3^{k_i}\delta$ .

Let  $S'_1, \dots, S'_J$  be the supersets generated by  $M$ -regions meeting  $[0, 1/2]^d \setminus [0, (T^*)^{1/d}]^d$ . For  $0 < a < b < \infty$  and  $1 \leq i \leq d$ , we denote by  $H_i[a, b]$  the ‘parallel slab’ between hyperplanes  $H_i \oplus an_{H_i}$  and  $H_i \oplus bn_{H_i}$ . We also define for any bounded subset  $A$  of  $\mathbb{R}^d$ , the diameter  $\text{diam}_i(A)$  of  $A$  in the direction  $n_{H_i}$  as the width of the maximal parallel slab containing  $A$ . If  $\bigcup_{j=1}^J S'_j$  is connected and if it meets  $H_i[0, \delta]$ , then by the



diameter bound  $\text{diam}_i(S'_j((3^{k_1}\delta, \dots, 3^{k_d}\delta))) \leq c3^{k_i}\delta$ , valid for cone-sets, cone-cylinder sets and cylinder sets, we have

$$\text{diam}_i\left(\bigcup_{j=1}^{J'} S'_j\right) \leq c\delta(1 + 3 + \dots + 3^{J'-1}) = c'\delta 3^{J'}. \quad (3.11)$$

**3.4. Dependency graphs.** The above subsection describes a collection of supersets  $S'_j$  generated by the constituent  $M$ -regions in  $\mathcal{M}_K(\mathbf{0}, \delta)$ . These sets are either cone sets, cone-cylinder sets, or cylinder sets, depending on whether the  $M$ -region lies entirely in  $[0, (T^*)^{1/d}]^d$ , meets the boundary of  $[0, (T^*)^{1/d}]^d$ , or lies outside  $[0, (T^*)^{1/d}]^d$ . Given any vertex  $\mathcal{V}_i \in \mathcal{V}_K \setminus \{\mathbf{0}\}$  we may likewise construct a collection  $\mathcal{M}_K(\mathcal{V}_i, \delta)$  of dyadic  $M$ -regions in  $p_d(\mathcal{V}_i, 1/2)$  and use them to generate a corresponding collection of  $S'_j$  regions. Here  $p_d(\mathcal{V}_i, \delta)$  is the parallelepiped defined before (3.2) and without loss of generality, we may assume that the parallelepipeds  $p_d(\mathcal{V}_i, 1/2)$  are disjoint. We embed the union of  $\mathcal{M}_K(\mathcal{V}_i, \delta)$ ,  $i \leq f_0(K)$ , into a (possibly not unique) larger collection  $\mathcal{M}_K(m(T, \delta))$  of  $M$ -regions having cardinality  $m(T, \delta)$  and which is maximal for the entire surface  $K(v = T)$ . This is possible since among all possible collections containing the union of  $\mathcal{M}_K(\mathcal{V}_i, \delta)$ ,  $1 \leq i \leq f_0(K)$ , there is at least one which is maximal.

The integer  $m(T, \delta)$  may not be unique, but in any case it is bounded above and below by integers depending only on  $T$ , as shown in [4]. Next, let  $\mathcal{S}'(\delta) := \{S'_j\}_{j=1}^{m(T, \delta)}$  be the  $S'_j$  regions generated by the  $M$ -regions in  $\mathcal{M}_K(m(T, \delta))$ . The additional  $S'_j$  regions which are not associated with a dyadic  $M$ -region are defined exactly as in [4]. The collection  $\mathcal{S}'(\delta)$  partitions the annulus  $\mathcal{A}(s, T^*, K)$ . Notice that  $m(T, \delta)$  plays the role of  $m_\eta := m(T_\eta)$  in [4]. The choice of a saturated system on  $K(v = s)$  is not relevant for our discussion and may be chosen as in [4].

Now we are ready to consider a dependency graph  $\mathcal{G} := (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ , where  $\mathcal{V}_\mathcal{G} := \mathcal{S}'(\delta)$ . Following section 6 of [4], define  $L_j$ ,  $1 \leq j \leq m(T, \delta)$ , to be the union of all  $S'_k \in \mathcal{S}'(\delta)$  such that there are points

$$a \in S'_j \cap K(v \geq s), \quad b \in S'_k \cap K(v \geq s)$$

with the segment  $[a, b]$  disjoint from  $K(v \geq T^*)$ .  $L_j$  is not empty since it contains  $S'_j$ . Also,  $S'_k \subset L_j$  if and only if  $S'_j \subset L_k$ . Join vertices  $i, j \in \mathcal{V}_\mathcal{G}$  with an edge iff  $L_i$  and  $L_j$  contain at least one  $S'_k$  in common. Let  $\mathcal{E}_\mathcal{G}$  be the edges thus defined.

The first assertion of the next result is proved in the Appendix. The second assertion is Lemma 6.1 of [5]. Let  $L(\lambda) := T(K)^3(\log \log \lambda)^{3(d-1)}$ , where  $T(K)$  is the number of flags of  $K$ . Recall that we implicitly assume  $\delta \in (0, 1/2)$  satisfies  $\log_3 T/\delta^d \in \mathbb{Z}$ .

**Lemma 3.2** For any fixed  $\delta \in (0, 1/2)$  we have

a. The geometric properties of sets in  $\mathcal{S}'(\delta)$  fulfill the requirements of [[4], p. 1518, 5 lines before (5.4)], and

b. There is a constant  $c^* \in (0, \infty)$  such that for all  $1 \leq j \leq m(T, \delta)$  and all  $\lambda \in [1, \infty)$ , we have  $\text{card}\{k : S'_k \subset L_j\} \leq c^* L(\lambda)$ .

In other words  $L_j, 1 \leq j \leq m(T, \delta)$ , contains at most  $c^* L(\lambda)$  sets in  $\mathcal{S}'(\delta)$ . As shown in [4], neither  $L(\lambda)$  nor the maximal degree of  $\mathcal{G}$  is a function of  $\delta$ . By (3.11), if  $L_j$  has non-empty intersection with  $H_l[0, \delta]$ , then there exists a constant  $c_{\text{diam}} \in (0, \infty)$  such that its diameter in the direction  $n_{H_l}$  satisfies

$$\text{diam}_i(L_j) \leq c_{\text{diam}} \delta 3^{c^* L(\lambda)}. \quad (3.12)$$

**Lemma 3.3** If  $S'_j \subset [0, \delta]^d$  and if  $S'_i \cap H_l[c_{\text{diam}} \delta 3^{c^* L(\lambda)+1}, \text{diam}(K)] \neq \emptyset$  for all  $1 \leq l \leq d$ , then there is no edge in  $\mathcal{E}_{\mathcal{G}}$  between  $j$  and  $i$ .

*Proof.* Case (i). If there were such an  $S'_i$ , and if such an  $S'_i$  were generated by an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$ , then by the diameter bound (3.12), it would follow that  $L_i \subset H_l[2c_{\text{diam}} \delta 3^{c^* L(\lambda)}, \text{diam}(K)]$  whereas  $L_j \subset H_l[0, c_{\text{diam}} \delta 3^{c^* L(\lambda)}]$ . Thus  $L_i \cap L_j = \emptyset$ , showing that there is no edge between  $j$  and  $i$  in the case when  $S'_i$  is produced by an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$ . Case (ii). When  $S'_i$  is generated by an  $M$ -region in  $\mathcal{M}_K(m(T, \delta)) \setminus \mathcal{M}_K(\mathbf{0}, \delta)$  then we proceed by contradiction. If there were an edge between  $j$  and  $i$ , then there would exist an  $S'_l$  (on the path between  $S'_i$  and  $S'_j$ ) such that  $S'_l \cap H_l[2c_{\text{diam}} \delta 3^{c^* L(\lambda)}, \text{diam}(K)] \neq \emptyset$ ,  $S'_l$  is generated by an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$ , and there is an edge between  $l$  and  $i$ . This contradicts the first case of this proof.  $\square$

Next we recall Lemma 7.1 of [4] and the discussion on pages 1522-23 of [5]. Though this lemma is proved in [4] for the volume score  $\xi_V$ , its proof is general and applies to the scores  $\xi_k$  as well. This result provides conditions for independence of scores on disjoint sets with respect to the graph distance between the sets.

**Lemma 3.4** Let  $\xi \in \Xi$  and let  $\mathcal{W}_1$  and  $\mathcal{W}_2$  be disjoint subsets of  $\mathcal{V}_{\mathcal{G}}$  having no edge between them. Conditional on  $A_\lambda$ , the random variables

$$\sum_{x \in \mathcal{P}_\lambda \cap (\cup_{i \in \mathcal{W}_1} S'_i)} \xi(x, \mathcal{P}_\lambda) \quad \text{and} \quad \sum_{x \in \mathcal{P}_\lambda \cap (\cup_{i \in \mathcal{W}_2} S'_i)} \xi(x, \mathcal{P}_\lambda)$$

are independent.

The next result provides conditions for independence of scores on disjoint sets with respect to the Euclidean distance between the sets.

**Lemma 3.5** *There exists a constant  $c' \in (0, \infty)$  such that if  $S'_0 \in \mathcal{S}'(\delta)$  is a subset of  $[0, \delta]^d$  and if  $S' \in \mathcal{S}'(\delta)$  is at Euclidean distance at least  $c'\delta 3^{c^*L(\lambda)}$  from  $[0, \delta]^d$ , then conditional on  $A_\lambda$  the sum of the scores on  $S'_0$  and  $S'$  are independent.*

*Proof.* This is a consequence of Lemma 3.3. We can choose  $c' > 0$  such that if the Euclidean distance between  $S'_0$  and  $S'$  exceeds  $c'\delta 3^{c^*L(\lambda)}$  then the distance in any direction  $n_{H_i}$  is greater than  $c_{\text{diam}}\delta 3^{c^*L(\lambda)+1}$ . Consequently, by (3.12), the graph distance exceeds 2, because any edge from the dependency graph would intersect more than  $c^*L(\lambda)$  cylinder sets  $S'_j$ .  $\square$

Recall the definition of  $Z_i(\delta)$ ,  $1 \leq i \leq f_0(K)$ , at (3.2).

**Lemma 3.6** *Conditional on  $A_\lambda$ , the random variables  $Z_i(\delta)$ ,  $1 \leq i \leq f_0(K)$ , are independent for  $\lambda$  large enough whenever  $\delta := \delta(\lambda)$  satisfies  $\delta 3^{c^*L(\lambda)} = o(1)$ .*

*Proof.* Let  $S(\mathcal{V}_i) \in \mathcal{S}'(\delta)$  be a subset of  $p_d(\mathcal{V}_i, \delta) \cap K$ ,  $1 \leq i \leq f_0(K)$ . The Euclidean distance between  $S(\mathcal{V}_i)$  and  $S(\mathcal{V}_j)$ ,  $i \neq j$ , is bounded below by  $\|\mathcal{V}_i - \mathcal{V}_j\| - 2\delta$  which exceeds  $c'\delta 3^{c^*L(\lambda)}$ . Now apply Lemma 3.5.  $\square$

**3.5. Variance is additive over vertices of  $K$ .** Put  $\mathcal{A}(s, T^*, K, \delta) := \mathcal{A}(s, T^*, K) \setminus \bigcup_{i=1}^{f_0(K)} p_d(\mathcal{V}_i, \delta)$  and set

$$\mathcal{P}_\lambda(s, T^*, K, \delta) := \mathcal{P}_\lambda \cap \mathcal{A}(s, T^*, K, \delta).$$

Recall the definition of  $Z$  and  $Z_i$  at (3.1) and (3.2). Conditional on  $A_\lambda$ , we have for all  $\xi \in \Xi$

$$Z = Z_0 + \sum_{i=1}^{f_0(K)} Z_i, \tag{3.13}$$

where

$$Z_0 := Z_0(\delta) := \sum_{x \in \mathcal{P}_\lambda(s, T^*, K, \delta)} \xi(x, \mathcal{P}_\lambda) \tag{3.14}$$

is the contribution to  $Z$  from points in  $\mathcal{P}_\lambda$  which are far from  $\mathcal{V}_K$ .

Recall from (1.6) that  $\delta_0 := \exp(-(\log \lambda)^{1/d})$ . We now put  $\delta$  to be  $\delta_1 := r(\lambda, d)\delta_0$ , where  $r(\lambda, d) \in [1, 3^{1/d}]$  is chosen so that  $\log_3(T/\delta_1^d) \in \mathbb{Z}$ .

Roughly speaking, conditional on  $A_\lambda$ , we may bound the number of points in  $\mathcal{P}_\lambda(s, T^*, K, \delta_1)$  as well as magnitudes of scores arising from such points. In this way, the next lemma shows that the contribution to the total score arising from  $Z_0(\delta_1)$  is negligible. It also shows that the difference between the sum of the volume scores and the defect volume of  $K_\lambda$  is negligible. The proof is in the Appendix.

**Lemma 3.7** For  $\xi \in \Xi$  we have

- a.  $\text{Var}[Z_0(\delta_1)|A_\lambda] = o(\text{Var}[Z])$ .
- b.  $\text{Var}\left[\frac{1}{\lambda} \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda)\right] = \text{Var}[\text{Vol}(K_\lambda)] + o(\text{Var}[\text{Vol}(K_\lambda)])$ .

The next lemma, also proved in the Appendix, shows that the event  $A_\lambda^c$  contributes a negligible amount to the first and second order statistics of  $Z$  and  $Z_i, 1 \leq i \leq f_0(K)$ .

**Lemma 3.8** Let  $Z_i := Z_i(\delta_0)$  be as at (3.2). We have uniformly for  $1 \leq i \leq f_0(K)$ :

$$\max\{|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]|, |\mathbb{E}[Z_i] - \mathbb{E}[Z_i|A_\lambda]|, |\mathbb{E}[Z_i] - \mathbb{E}[Z_i \mathbf{1}(A_\lambda)]|\} = o(\mathbb{E}[Z]),$$

and

$$\begin{aligned} \max\{|\text{Var}[Z] - \text{Var}[Z \mathbf{1}(A_\lambda)]|, |\text{Var}[Z] - \text{Var}[Z|A_\lambda]|, \\ |\text{Var}[Z_i] - \text{Var}[Z_i|A_\lambda]|, |\text{Var}[Z_i] - \text{Var}[Z_i \mathbf{1}(A_\lambda)]|\} = o(\text{Var}[Z]). \end{aligned}$$

Finally we may prove the second main result of this section.

*Proof of Proposition 3.2.* By Lemma 3.8 it suffices to show

$$\text{Var}[Z|A_\lambda] = \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_0)|A_\lambda] + o(\text{Var}[Z]). \quad (3.15)$$

To do so, we proceed in two steps: (i) we first show

$$\text{Var}[Z|A_\lambda] = \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_1)|A_\lambda] + o(\text{Var}[Z]), \quad (3.16)$$

and (ii) then for every  $1 \leq i \leq f_0(K)$  we show

$$\text{Var}[Z_i(\delta_1)|A_\lambda] = \text{Var}[Z_i(\delta_0)|A_\lambda] + o(\text{Var}[Z]). \quad (3.17)$$

Let us show (3.16). Let  $\text{Cov}((X, Y)|A_\lambda)$  be short for  $\mathbb{E}[(X - \mathbb{E}[X|A_\lambda])(Y - \mathbb{E}[Y|A_\lambda])|A_\lambda]$ . Recalling (3.13), we have

$$\begin{aligned} \text{Var}[Z|A_\lambda] &= \text{Var}[Z_0(\delta_1) + \sum_i Z_i(\delta_1)|A_\lambda] \\ &= \text{Var}[Z_0(\delta_1)|A_\lambda] + \text{Var}\left[\sum_i Z_i(\delta_1)|A_\lambda\right] + 2\text{Cov}\left(\left(\sum_i Z_i(\delta_1), Z_0(\delta_1)\right)|A_\lambda\right) \\ &= \sum_i \text{Var}[Z_i(\delta_1)|A_\lambda] + 2\text{Cov}\left(\left(\sum_i Z_i(\delta_1), Z_0(\delta_1)\right)|A_\lambda\right) + o(\text{Var}[Z]), \end{aligned}$$

where the last equality follows from  $\delta_1 3^{e^{*L(\lambda)}} = o(1)$ , the conditional independence of  $Z_i(\delta_1), 1 \leq i \leq f_0(K)$ , as given by Lemma 3.6, as well as Lemma 3.7.

If random variables  $X$  and  $Y$  satisfy  $\max\{\text{Var}[X+Y|E], \text{Var}[Y|E]\} = O(\text{Var}[X+Y])$ , then writing  $X = (X+Y) - Y$ , it follows that  $\text{Var}[X|E] = O(\text{Var}[X+Y])$ . We have  $\max\{\text{Var}[\sum Z_i(\delta_1) + Z_0(\delta_1)|A_\lambda], \text{Var}[Z_0(\delta_1)|A_\lambda]\} = O(\text{Var}[Z])$  by [4] and by Lemma 3.7. It follows that  $\text{Var}[\sum Z_i(\delta_1)|A_\lambda] = O(\text{Var}[Z])$ . This estimate, Lemma 3.7 again, and the Cauchy-Schwarz inequality give

$$\begin{aligned} \text{Cov}\left(\left(\sum_i Z_i(\delta_1), Z_0(\delta_1)\right)|A_\lambda\right) &\leq \sqrt{\text{Var}\left[\sum_i Z_i(\delta_1)|A_\lambda\right]} \cdot \sqrt{\text{Var}[Z_0(\delta_1)|A_\lambda]} \\ &= O(\sqrt{\text{Var}[Z]})o(\sqrt{\text{Var}[Z]}) \\ &= o(\text{Var}[Z]). \end{aligned}$$

This yields the decomposition (3.16).

To prove (3.17) we introduce  $\delta'_1 := r'(\lambda, d)\delta_0$  where  $r'(\lambda, d) \in (3^{-1/d}, 1]$  is chosen so that  $\log_3(T/\delta_1'^d) \in \mathbb{Z}$ . Methods similar to the proof of Lemma 3.7 show that  $\text{Var}[Z_0(\delta'_1)|A_\lambda] = o(\text{Var}[Z])$  and  $\text{Var}[\sum_{x \in \mathcal{B}} \xi(x, \mathcal{P}_\lambda)|A_\lambda] = o(\text{Var}[Z])$ , with  $\mathcal{B}$  a subset of  $\mathcal{P}_\lambda(s, T^*, K, \delta'_1)$ . Note that  $\mathcal{B}_i := (p_d(\mathcal{V}_i, \delta_1) \setminus p_d(\mathcal{V}_i, \delta_0)) \cap \mathcal{P}_\lambda, 1 \leq i \leq f_0(K)$ , are subsets of  $\mathcal{P}_\lambda(s, T^*, K, \delta'_1)$ . Consequently,

$$\text{Var}[Z_i(\delta_1) - Z_i(\delta_0)|A_\lambda] = o(\text{Var}[Z]).$$

Moreover,

$$\text{Cov}((Z_i(\delta_1) - Z_i(\delta_0), Z_i(\delta_0))|A_\lambda) \leq \sqrt{\text{Var}[Z_i(\delta_1) - Z_i(\delta_0)|A_\lambda]} \sqrt{\text{Var}[Z_i(\delta_0)|A_\lambda]} = o(\text{Var}[Z]).$$

We deduce (3.17) from the two previous equalities. This completes the proof of Proposition 3.2.  $\square$

## 4 Re-scaled convex hull boundaries, $k$ -face, and volume functionals

Section 3 showed that variance asymptotics for  $f_k(K_\lambda)$  and  $\text{Vol}(K_\lambda)$  are determined by the respective behavior of  $\xi_k$  and  $\xi_V$  on  $\mathcal{P}_\lambda \cap Q_0$ . We discuss scaling transforms of  $\mathcal{P}_\lambda \cap Q_0, \partial K_\lambda \cap Q_0$ , as well as transforms for  $\xi_k$  and  $\xi_V$  restricted to input  $\mathcal{P}_\lambda \cap Q_0$ .

**4.1. Parallel between the scaling transform  $T^{(\lambda)}$  and those in previous work.** Scaling transforms lie at the heart of our asymptotic analysis. Before discussing the

technical details, we explain their relevant geometric aspects, comparing  $T^{(\lambda)}$  with counterparts in previous works on Gaussian polytopes [8], as well as random polytopes in the unit ball [9, 17] and in smooth convex bodies [7].

*Floating bodies and associated coordinates.* Seminal works of Bárány and Larman [3] and Bárány [1] show the importance of the deterministic approximation of the random polytope inside the mother body  $K$  by a floating body  $K(v \geq 1/\lambda)$ . Consequently, it makes sense to use the parametric surfaces  $\partial K(v \geq t/\lambda)$ ,  $t > 0$ , to associate to any point  $z \in K$  a *depth coordinate* which is the specific  $t$  such that  $z \in \partial K(v \geq t/\lambda)$  and a *spatial coordinate* indicating the position of  $z$  on the surface  $\partial K(v \geq t/\lambda)$ . When  $K$  is the unit ball, the floating bodies are balls  $B(\mathbf{0}, r)$ ,  $0 < r < 1$ , and coordinates coincide with the usual spherical coordinates. When  $K := (0, \infty)^d$ , the floating bodies are pseudo-hyperboloids, as seen in the next subsection. We could call the associated coordinates *cubical coordinates*. In the case of a general convex mother body  $K$ , there is not necessarily a natural way of globally defining a spatial coordinate, which explains *a posteriori* why we dealt with local spherical coordinates in [7].

*Extreme points and duality.* This paper, as well as [7, 8, 9], rely on a dual characterization of extreme points. Arguably, it is most natural to define an extreme point as *a point from the input on the boundary of the convex hull*. By duality, we may also assert that a point  $z_0$  from the input is extreme if it is included in a support hyperplane of the convex hull. In most cases, any hyperplane containing a fixed point  $z$  from the input is tangent to exactly one surface  $\partial K(v \geq t/\lambda)$  at one point of tangency. This suggests the idea of considering the *petal* of  $z$ , i.e. the subset  $S(z)$  of  $K$  whose boundary  $\partial S(z)$  consists of all points of tangency of hyperplanes containing  $z$ . In the case of the unit ball, and when the origin is inside the convex hull, the petal of  $z$  is the ball with diameter  $[0, z]$ . The collection of such balls associated to the points of the input constitutes the so-called *Voronoi flower* of the input with respect to the origin, which explains *a posteriori* the appellation *petal*. In the case of the orthant  $(0, \infty)^d$ , when the point  $z$  is cone-extreme (recall Definition 3.1), its petal is defined in (4.3) below. This provides the second definition of an extreme point (*cone-extreme* in the case of the orthant): *A point from the input is extreme iff its petal is not covered by the petals from the other points from the input.*

*Scaling transformations.* As in [8, 9], this paper uses the set of coordinates induced by the floating bodies to build the scaling transformation. With the proper re-scaling of both the spatial and depth coordinates, we get a new picture in a product space  $\mathbb{R}^{d-1} \times \mathbb{R}$ , the height being the re-scaled depth coordinate. The duality of the two

definitions of the extreme points (or cone-extreme points in the case of the orthant) is even more apparent in the re-scaled picture. Indeed, the re-scaled random polytope may be described either via the re-scaled boundary of the convex hull given below by  $\partial\Phi(\mathcal{P}^{(\lambda)})$  at (4.10) or via the re-scaled boundary of the union of petals given by  $\partial\Psi(\mathcal{P}^{(\lambda)})$  at (4.11).

**4.2. A new characterization of cone-extreme points.** We consider surfaces

$$\mathcal{H}_t := \{(z_1, \dots, z_d) \in (0, \infty)^d : \prod_{i=1}^d z_i = t\}, \quad t > 0. \quad (4.1)$$

When  $d = 2$  each  $\mathcal{H}_t$  is a branch of a hyperbola and for this reason we will sometimes refer to  $\mathcal{H}_t$  as a pseudo-hyperboloid. The surfaces  $\mathcal{H}_t, t > 0$ , coincide with boundaries of floating bodies of the orthant  $[0, \infty)^d$ , as shown in Lemma 7.1, and play a key role in the description of cone-extreme points of input inside  $(0, \infty)^d$ .

For every  $z^{(0)} \in (0, \infty)^d$ , we denote by  $H(z^{(0)})$  the hyperplane tangent to the unique surface  $\mathcal{H}_t$  containing  $z^{(0)}$ . The gradient of the function  $f(z) = \prod_{i=1}^d z_i$  at  $z^{(0)}$  is  $t \frac{1}{z^{(0)}}$ , where  $\frac{1}{z^{(0)}} := (\frac{1}{z_1^{(0)}}, \dots, \frac{1}{z_d^{(0)}})$ . It follows that  $H(z^{(0)})$  is described by

$$H(z^{(0)}) := \{(z_1, \dots, z_d) \in \mathbb{R}^d : \sum_{i=1}^d \frac{z_i}{z_i^{(0)}} = d\}. \quad (4.2)$$

To every point  $z^{(0)} \in (0, \infty)^d$ , we associate the surface

$$\mathcal{S}(z^{(0)}) := \{z \in (0, \infty)^d : z^{(0)} \in H(z)\}.$$

The *petal* of  $z^{(0)}$  is the closed set  $\mathcal{S}^-(z^{(0)})$  of points above  $\mathcal{S}(z^{(0)})$ . Notice that  $\mathcal{S}^-(z^{(0)})$  is also the set of points  $z$  such that  $z^{(0)}$  lies ‘below’  $H(z)$ . Using (4.2), we have

$$\mathcal{S}^-(z^{(0)}) := \{(z_1, \dots, z_d) \in (0, \infty)^d : \sum_{i=1}^d \frac{z_i^{(0)}}{z_i} \leq d\}. \quad (4.3)$$

The next lemma characterizes cone-extreme points in terms of the geometry of petals (see Figure 5). We are not aware of an analogous characterization of extreme points which are not cone-extreme.

**Lemma 4.1** *Let  $\mathcal{X}$  be any point set in  $(0, \infty)^d$ . Then  $z^{(0)} \in \mathcal{X}$  is cone-extreme with respect to  $\mathcal{X}$  if and only if  $\mathcal{S}^-(z^{(0)})$  is not completely covered by  $\bigcup_{z \in \mathcal{X} \setminus \{z^{(0)}\}} \mathcal{S}^-(z)$ .*

*Proof.* Indeed,  $\mathcal{S}^-(z^{(0)})$  is not covered iff there exists  $z \in \mathcal{S}(z^{(0)})$  which is below each of the surfaces  $\mathcal{S}(z^{(0)})$ . This is equivalent to saying that the hyperplane  $H(z)$  is a support hyperplane of  $\text{co}(\mathcal{X})$  containing  $z^{(0)}$  and with outward normal in  $C_0(K)$ .  $\square$

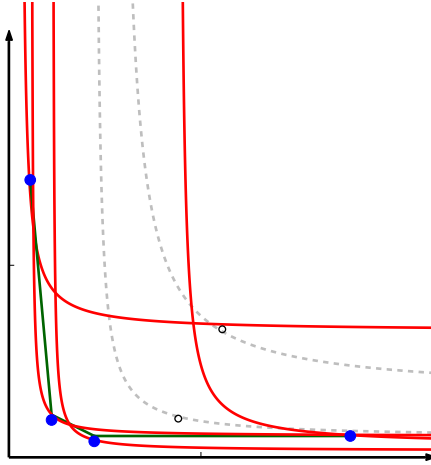


Figure 5: The point process  $\text{Ext}(\mathcal{P}_\lambda)$  (blue); the boundary of the associated petals containing the extreme points (red); the boundary of the convex hull (green). Points which are not extreme are apices of gray petals.

**4.3. The scaling transform  $T^{(\lambda)}$  at (1.5).** Here we describe the image under  $T^{(\lambda)}$  of  $\mathcal{P}_\lambda \cap Q_0$ , the image of half-spaces with bounding hyperplane  $H(z^{(0)})$ , as well as the image of petals. As a by-product, we find the image of a face of  $K_\lambda \cap Q_0$ .

Recall the definitions of  $V, l_i(v), l(v)$ , and  $p_V : \mathbb{R}^d \rightarrow V$  introduced before (1.2) and after (1.5), respectively. For any  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$  and any function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , we define the vector  $f(x) := (f(x_1), \dots, f(x_d))$ . Note that  $e^l : V \rightarrow \mathcal{H}_1$ .

Recall that  $w = (v, h)$  denotes a generic point in  $V \times \mathbb{R}$ . The inverse of  $T^{(\lambda)}$  is

$$[T^{(\lambda)}]^{-1} : \begin{cases} V \times \mathbb{R} & \longrightarrow (0, \infty)^d \\ (v, h) & \longmapsto \lambda^{-1/d} e^h e^{l(v)} \end{cases} \quad (4.4)$$

The expression for  $[T^{(\lambda)}]^{-1}$  is justified as follows. We have  $l(p_V(\log(z))) = \log z - \langle \log z, u^0 \rangle \frac{1}{d} u^0$  where  $u^0 := (1, \dots, 1)$  and  $\langle \cdot, \cdot \rangle$  is the usual scalar product on  $\mathbb{R}^d$ . Indeed,  $\langle \log z, u^0 \rangle \frac{1}{d} u^0$  is the projection of  $\log z$  onto the line directed by  $u^0$ . So  $\exp(l(p_V(\log z))) = \exp(-\frac{1}{d} \sum_{i=1}^d \log z_i) \cdot z = (z_1 \dots z_d)^{-1/d} z$  and so  $[T^{(\lambda)}]^{-1}(T^{(\lambda)})$  is the identity, as desired.

**Definition 4.1** For all  $\lambda \in [1, \infty)$  we put

$$W_\lambda := T^{(\lambda)}(Q_0) = \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \leq -l_i(v) + \log \lambda^{1/d} \delta_0, 1 \leq i \leq d\} \quad (4.5)$$

and

$$\mathcal{P}^{(\lambda)} := T^{(\lambda)}(\mathcal{P}_\lambda \cap Q_0). \quad (4.6)$$

When  $\lambda = \infty$  we identify  $W_\lambda$  with  $V \times \mathbb{R}$  and  $\mathcal{P}^{(\lambda)}$  with  $\mathcal{P}$  at (1.1).



**Lemma 4.2** *Let  $\mathcal{P}$  be the Poisson point process at (1.1). Then  $\mathcal{P}^{(\lambda)}$  is equal in distribution to  $\mathcal{P} \cap W_\lambda$  and  $\mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{D}} \mathcal{P}$  as  $\lambda \rightarrow \infty$ .*

*Remark.* Since  $d\mathcal{P}^{(\lambda)}$  is the image under  $T^{(\lambda)}$  of  $\lambda \text{Vol}_d$ , with  $\text{Vol}_d$  the  $d$ -dimensional volume measure on  $\mathbb{R}^d$ , this lemma says that  $T^{(\lambda)}(\lambda \text{Vol}_d) \xrightarrow{\mathcal{D}} d\mathcal{P}$ .

*Proof.*  $\mathcal{P}^{(\lambda)}$  is a Poisson point process with intensity measure  $T^{(\lambda)}(\lambda dz)$ . Endow  $V$  with a direct orthonormal basis  $\mathcal{B}$ . Using (4.4), the Jacobian of  $[T^{(\lambda)}]^{-1}$  with respect to the direct orthonormal basis of  $\mathbb{R}^d$  given by  $(\mathcal{B}, \frac{1}{\sqrt{d}}u^{(0)})$  equals

$$\sqrt{d}\lambda^{-1} \exp(dh) \exp\left(\sum_{i=1}^d l_i(v)\right) D,$$

where  $D$  is the determinant of the matrix for the change of basis from  $(\mathcal{B}, \frac{1}{\sqrt{d}}u^{(0)})$  to the standard basis of  $\mathbb{R}^d$ . Since both bases are direct and orthonormal, we have  $D = 1$ . Moreover, we notice that  $\sum_{i=1}^d l_i(v) = 0$  because  $v \in V$ .

Consequently,  $\mathcal{P}^{(\lambda)}$  has intensity measure with no  $\lambda$  dependency save that it is carried by the ‘pyramid-like’ set  $W_\lambda := T^{(\lambda)}(Q_0)$ . In other words,

$$d\mathcal{P}^{(\lambda)} = \sqrt{d} \exp(dh) \mathbf{1}((v, h) \in W_\lambda) dv dh. \quad (4.7)$$

Lemma 4.2 follows from (4.7) and the convergence  $W_\lambda \uparrow W = V \times \mathbb{R}$ .  $\square$

Having considered the behavior of the scaling transform  $T^{(\lambda)}$  on  $\mathcal{P}_\lambda$ , we now consider the image under  $T^{(\lambda)}$  of surfaces  $\mathcal{H}_{c/\lambda}$  with  $c > 0$ ,  $C_0$ -half-spaces, and petals. A  $C_0$ -half-space is one having an outward normal in  $C_0(K)$ , where  $C_0(K)$  is as in Definition 3.1. Note that a  $C_0$ -half-space is bounded by a hyperplane  $H(z^{(0)})$  for some  $z^{(0)} \in (0, \infty)^d$ . Denote by  $H^+(z^{(0)})$  the half-space bounded by  $H(z^{(0)})$  and containing  $\mathbf{0}$ .

Recall the definition of the down cone-like grain  $\Pi^\downarrow$  given at (1.3) and its translate  $\Pi^\downarrow(w) := w \oplus \Pi^\downarrow$ ,  $w = (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}$ . Define similarly the up cone-like grain

$$\Pi^\uparrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h \geq G(-v)\}, \quad (4.8)$$

and the translate  $\Pi^\uparrow(w) = w \oplus \Pi^\uparrow$ ,  $w \in \mathbb{R}^{d-1} \times \mathbb{R}$ .

The duality between up and down cone-like grains is expressed through the following equivalence: for all  $w, w' \in \mathbb{R}^{d-1} \times \mathbb{R}$ ,

$$w \in \Pi^\uparrow(w') \iff w' \in \Pi^\downarrow(w).$$

The next lemma shows that  $T^{(\lambda)}$  sends pseudo-hyperboloids to hyperplanes parallel to  $V$ ,  $C_0$ -half-spaces to down cone-like grains, and petals to up cone-like grains.

**Lemma 4.3** (i) For every  $c \in (0, \infty)$ , we have

$$T^{(\lambda)}(\mathcal{H}_{c/\lambda}) = V \times \left\{ \frac{1}{d} \log(c) \right\}.$$

(ii) For every  $C_0$ -half-space  $H^+(z^{(0)})$ ,  $z^{(0)} \in (0, \infty)^d$ , we have

$$T^{(\lambda)}(H^+(z^{(0)})) = \Pi^\downarrow(T^{(\lambda)}(z^{(0)})).$$

(iii) For every petal  $S^-(z^{(0)})$ ,  $z^{(0)} \in (0, \infty)^d$ , we have

$$T^{(\lambda)}(S^-(z^{(0)})) = \Pi^\uparrow(T^{(\lambda)}(z^{(0)})).$$

*Proof.* For every  $(v, h) \in V \times \mathbb{R}$ , we have by (4.4)

$$[T^{(\lambda)}]^{-1}(v, h) \in \mathcal{H}_{c/\lambda} \iff \prod_{i=1}^d (\lambda^{-1/d} e^h e^{l_i(v)}) = c/\lambda \iff e^{dh} = c$$

which shows (i).

Fix  $z^{(0)} \in (0, \infty)^d$  and put  $T^{(\lambda)}(z^{(0)}) := (v^{(0)}, h^{(0)})$ . We notice that

$$T^{(\lambda)}\left(\frac{1}{z^{(0)}}\right) = (-v^{(0)}, -h^{(0)} + 2 \log(\lambda^{1/d})). \quad (4.9)$$

Using the equation of  $H^+(z^{(0)})$  implied by (4.2), the formula for  $[T^{(\lambda)}]^{-1}$  at (4.4) and (4.9), we have for any  $(v, h) \in V \times \mathbb{R}$

$$\begin{aligned} [T^{(\lambda)}]^{-1}((v, h)) \in H^+(z^{(0)}) &\iff \langle [T^{(\lambda)}]^{-1}((v, h), \frac{1}{z^{(0)}}) \rangle \leq d \\ &\iff \lambda^{-1/d} e^h \langle \exp(l(v)), \frac{1}{z^{(0)}} \rangle \leq d \\ &\iff e^{h-h^{(0)}} \langle \exp(l(v)), \exp(l(-v^{(0)})) \rangle \leq d \\ &\iff h \leq h^{(0)} - \log\left(\frac{1}{d} \langle \exp(l(v - v^{(0)})), u^{(0)} \rangle\right). \end{aligned}$$

This last equivalence, coupled with the definition of  $G$  at (1.2), gives (ii). Similarly, (iii) is a consequence of the equation of the petal  $\mathcal{S}^-(z^{(0)})$  at (4.3), (4.4) and (4.9). Indeed, we have for every  $(v, h) \in V \times \mathbb{R}$

$$\begin{aligned} [T^{(\lambda)}]^{-1}((v, h)) \in \mathcal{S}^-(z^{(0)}) &\iff \langle \lambda^{1/d} e^{-h} e^{-l(v)}, \lambda^{-1/d} e^{h^{(0)}} e^{l(v^{(0)})} \rangle \leq d \\ &\iff e^{h^{(0)}-h} \langle e^{l(v^{(0)}-v)}, u^0 \rangle \leq d \\ &\iff h \geq h^{(0)} + G(v^{(0)} - v). \end{aligned}$$

This completes the proof of Lemma 4.3.  $\square$

**4.4. Re-scaled extreme points and scores.** It is time to define re-scaled scores  $\xi^{(\lambda)}$  on  $\mathcal{P}^{(\lambda)}$ . We use Proposition 3.1 to show that on the event  $A_\lambda$  given at (3.6), the re-scaled scores  $\xi^{(\lambda)}$  coincide with functionals  $\hat{\xi}^{(\lambda)}$  defined in terms of the geometry of the re-scaled convex hull boundary. This is facilitated with the following definitions.

**Definition 4.2** Write  $[\Pi^\uparrow(w)]^{(\lambda)}$  for  $\Pi^\uparrow(w) \cap W_\lambda$  and similarly for  $[\Pi^\downarrow(w)]^{(\lambda)}$ . Given  $\mathcal{P}^{(\lambda)}$ ,  $1 \leq \lambda \leq \infty$ , we define the  $\mathcal{P}^{(\lambda)}$ -hull as at (1.4) with  $\mathcal{X}$  set to  $\mathcal{P}^{(\lambda)}$ , namely

$$\Phi(\mathcal{P}^{(\lambda)}) := \bigcup_{\substack{w \in \mathbb{R}^{d-1} \times \mathbb{R} \\ \mathcal{P}^{(\lambda)} \cap \text{int}(\Pi^\downarrow(w)) = \emptyset}} [\Pi^\downarrow(w)]^{(\lambda)}. \quad (4.10)$$

We also put

$$\Psi(\mathcal{P}^{(\lambda)}) := \bigcup_{w \in \mathcal{P}^{(\lambda)}} [\Pi^\uparrow(w)]^{(\lambda)}. \quad (4.11)$$

Abusing notation, we let  $\text{Ext}(\mathcal{P}^{(\lambda)})$  be those points in  $\mathcal{P}^{(\lambda)}$  which are on the boundary of some down cone-like grain  $\Pi^\downarrow(w)$ ,  $w \in W_\lambda$ , and  $\text{int}(\Pi^\downarrow(w_1)) \cap \mathcal{P}^{(\lambda)} = \emptyset$ .

Equivalently, a point  $w_0 \in \mathcal{P}^{(\lambda)}$  is *extreme* with respect to  $\Psi(\mathcal{P}^{(\lambda)})$  if the grain  $\Pi^\uparrow(w_0)$  is not a subset of the union of the grains  $\Pi^\uparrow(w)$ ,  $w \in \mathcal{P}^{(\lambda)} \setminus \{w_0\}$  (see Figure 6). By Lemma 4.1, Lemma 4.3(iii) and Proposition 3.1, on the event  $A_\lambda$ , the extreme points in  $\mathcal{P}_\lambda \cap Q_0$  are transformed to  $\text{Ext}(\mathcal{P}^{(\lambda)})$ . By Lemma 4.3 we also have on the event  $A_\lambda$  that  $T^{(\lambda)}(\partial K_\lambda \cap Q_0) = \partial(\Phi(\mathcal{P}^{(\lambda)}))$ .

**Definition 4.3** For  $\lambda \in [1, \infty)$ , put  $\xi_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi_k([T^{(\lambda)}]^{-1}(w), \mathcal{P}_\lambda)$ . For  $w \in \text{Ext}(\mathcal{P}^{(\lambda)})$  and  $\lambda \in [1, \infty]$  we put

$$\hat{\xi}_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := (k+1)^{-1}[\text{number of } k\text{-faces of } \Phi(\mathcal{P}^{(\lambda)}) \text{ containing } w]. \quad (4.12)$$

For  $w \notin \text{Ext}(\mathcal{P}^{(\lambda)})$  we put  $\hat{\xi}_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = 0$ . Similarly, for  $\lambda \in [1, \infty)$  and  $w \in \text{Ext}(\mathcal{P}^{(\lambda)})$  we define  $\xi_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi_V([T^{(\lambda)}]^{-1}(w), \mathcal{P}_\lambda)$  and

$$\hat{\xi}_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \frac{1}{d} \int_{v \in \text{Cyl}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})} \int_{-\infty}^{\partial(\Phi(\mathcal{P}^{(\lambda)})(v))} \sqrt{d} e^{dh} dv dh, \quad (4.13)$$

where  $\text{Cyl}^{(\lambda)}(w) := \text{Cyl}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  denotes the projection onto  $V$  of the hyperfaces of  $\Phi(\mathcal{P}^{(\lambda)})$  containing  $w$ . When  $w \notin \text{Ext}(\mathcal{P}^{(\lambda)})$  we define  $\hat{\xi}_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = 0$ .

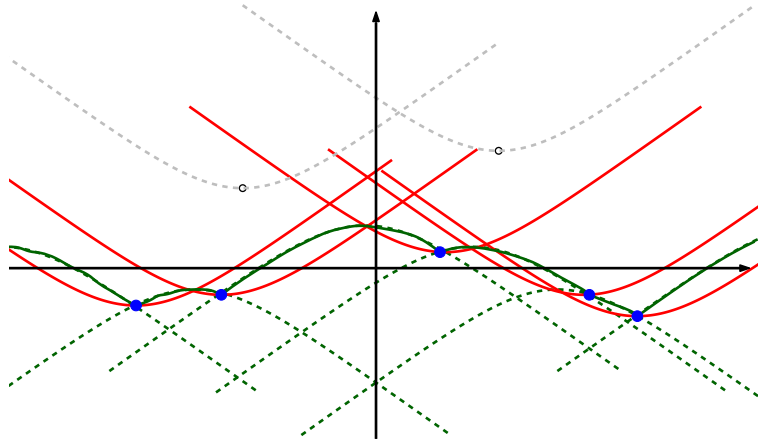


Figure 6: The point process  $\text{Ext}(\mathcal{P}^{(\lambda)})$  (blue); the boundary of the up-grains containing the extreme points (red); the boundary  $\partial(\Phi(\mathcal{P}^{(\lambda)}))$  of the down-grains containing  $\text{Ext}(\mathcal{P}^{(\lambda)})$  (green). Points which are not extreme are apices of gray up-grains.

By Proposition 3.1, on the event  $A_\lambda$ , the vertices of  $K_\lambda \cap Q_0$  coincide with cone-extreme points. Putting  $w = T^{(\lambda)}(x)$  gives

$$\xi_k(x, \mathcal{P}_\lambda) \mathbf{1}(A_\lambda) = \hat{\xi}_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda) = \xi_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda). \quad (4.14)$$

Lemma 3.8, together with (4.14), shows that the variance asymptotics for  $\sum_{x \in \mathcal{P}_\lambda} \xi_k(x, \mathcal{P}_\lambda)$  coincide with those for  $\sum_{x \in \mathcal{P}^{(\lambda)}} \hat{\xi}_k^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda)$ . We exploit this fundamental scaling identity in Section 6. Similarly, by Proposition 3.1, Lemma 4.1, and the remark after Lemma 4.2, we have the analog of (4.14), namely

$$\hat{\xi}_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda) = \xi_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda). \quad (4.15)$$

Lemma 3.8 shows that the variance asymptotics for  $\sum_{x \in \mathcal{P}^{(\lambda)}} \hat{\xi}_V^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda)$  coincide with those for  $\sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda)$ .

Given  $\lambda \in [1, \infty)$ , let  $\Xi^{(\lambda)}$  denote the collection of re-scaled functionals  $\xi_k^{(\lambda)}$ ,  $k \in \{0, 1, \dots, d-1\}$ , together with  $\xi_V^{(\lambda)}$ . Likewise, for  $\lambda \in [1, \infty]$ , we let  $\hat{\Xi}^{(\lambda)}$  denote the collection of functionals  $\hat{\xi}_k^{(\lambda)}$ ,  $k \in \{0, 1, \dots, d-1\}$ , together with  $\hat{\xi}_V^{(\lambda)}$ .

**4.5. Properties of the function  $G$  defined at (1.2).** We record two properties of  $G$  needed in the sequel. Notice that  $G$  is an even function only when  $d = 2$ .

**Lemma 4.4**  *$G$  is a positive convex function.*

*Proof.* By the convexity of the exponential function, for any  $v \in V$ , we have

$$G(v) \geq \log\left(\exp\left(\frac{1}{d} \sum_{i=1}^d l_i(v)\right)\right) = \log(1) = 0.$$

Let  $v, v' \in V$  and  $t \in [0, 1]$ . Hölder's inequality gives

$$\begin{aligned} G(tv + (1-t)v') &= \log\left(\sum_{i=1}^d \left[\frac{1}{d} \exp(l_i(v))\right]^t \left[\frac{1}{d} \exp(l_i(v'))\right]^{(1-t)}\right) \\ &\leq \log\left(\left[\sum_{i=1}^d \frac{1}{d} \exp(l_i(v))\right]^t \left[\sum_{i=1}^d \frac{1}{d} \exp(l_i(v'))\right]^{(1-t)}\right) \\ &= tG(v) + (1-t)G(v'). \end{aligned}$$

Thus  $G$  is convex, completing the proof of Lemma 4.4.  $\square$

The next lemma shows that the graph of  $G$  is sandwiched between circular cones.

**Lemma 4.5** *There exist  $c_1, c_2 \in (0, \infty)$  such that for every  $v \in V$ ,*

$$c_1\|v\| - \log d \leq G(v) \leq c_2\|v\|. \quad (4.16)$$

*Proof.* Since  $\max_{1 \leq i \leq d} |l_i(v)|$  is a norm on  $V$ , it is equivalent to the Euclidean norm  $\|\cdot\|$ . It follows that there are constants  $c_1, c_2 \in (0, \infty)$  such that  $c_1'\|v\| \leq \max_{1 \leq i \leq d} |l_i(v)| \leq c_2'\|v\|$  for all  $v \in V$ . We have for every  $v \in V$

$$G(v) = \log\left(\frac{1}{d} \sum_{i=1}^d \exp(l_i(v))\right) \leq \log\left(\exp\left(\max_{1 \leq i \leq d} |l_i(v)|\right)\right) \leq c_2'\|v\|.$$

Moreover, one of the  $l_i(v)$  is at least equal to  $\frac{1}{d-1} \max_{1 \leq i \leq d} |l_i(v)|$ . This implies that

$$G(v) \geq \log\left(\frac{1}{d} \exp\left(\frac{c_1'}{d-1}\|v\|\right)\right) \geq \frac{c_1'}{d-1}\|v\| - \log d,$$

which establishes (4.16).  $\square$

## 5 Properties of re-scaled $k$ -face and volume functionals

Section 4 introduced re-scaled functionals  $\hat{\xi}^{(\lambda)}$  of re-scaled input  $\mathcal{P}^{(\lambda)}$ . Here we establish localization properties of the functionals  $\hat{\xi}^{(\lambda)} \in \hat{\Xi}^{(\lambda)}$ , bounds on their moments, as well convergence of their one and two-point correlation functions.

**5.1. Stabilization.** We establish localization of the functionals  $\hat{\xi}^{(\lambda)} \in \hat{\Xi}^{(\lambda)}$  in both the space and time domains. Recalling that  $B_{d-1}(v, r)$  is the  $(d-1)$  dimensional ball centered at  $v \in \mathbb{R}^{d-1}$  with radius  $r$ , define the cylinder

$$C(v, r) := C_{d-1}(v, r) := B_{d-1}(v, r) \times \mathbb{R}. \quad (5.1)$$

We show that the boundaries of the germ-grain models  $\Psi(\mathcal{P}^{(\lambda)})$  and  $\Phi(\mathcal{P}^{(\lambda)})$ ,  $\lambda \in [1, \infty]$  defined at (4.11) and (4.10), respectively, are not far from  $V$ . Recall that  $\mathcal{P}^{(\lambda)}$ ,  $\lambda = \infty$ , is taken to be  $\mathcal{P}$ . If  $w \in \text{Ext}(\mathcal{P}^{(\lambda)})$  we put  $H(w) := H(w, \mathcal{P}^{(\lambda)})$  to be the maximal height coordinate (with respect to  $\mathbb{R}^{d-1}$ ) of an apex of a down cone-like grain belonging to  $\Phi(\mathcal{P}^{(\lambda)})$  and containing  $w$ . Otherwise, if  $w \notin \text{Ext}(\mathcal{P}^{(\lambda)})$  then we put  $H(w) = 0$ .

**Lemma 5.1** (a) *There is a constant  $c$  such that for all  $\lambda \in [1, \infty]$  and  $(v_0, h_0) \in W_\lambda$*

$$P[H((v_0, h_0), \mathcal{P}^{(\lambda)}) \geq t] \leq c \exp\left(-\frac{e^{t/c}}{c}\right), \quad t \geq h_0 \vee 0. \quad (5.2)$$

(b) *There is a constant  $c$  such that for all  $L \in (0, \infty)$  and  $\lambda \in [1, \infty]$*

$$P[\|\partial\Psi(\mathcal{P}^{(\lambda)}) \cap C(\mathbf{0}, L)\|_\infty > t] \leq cL^{2(d-1)} \exp\left(-\frac{t}{c}\right), \quad t \in (0, \infty). \quad (5.3)$$

*The bound (5.3) also holds for  $\partial(\Phi(\mathcal{P}^{(\lambda)}))$ .*

*Proof.* We first prove (5.2). Rewrite the event  $\{H((v_0, h_0), \mathcal{P}^{(\lambda)}) \geq t\}$  as

$$\begin{aligned} & \{H((v_0, h_0), \mathcal{P}^{(\lambda)}) \geq t\} \\ &= \{\exists w_1 := (v_1, h_1) \in \partial[\Pi^\uparrow((v_0, h_0))]^{(\lambda)} : h_1 \in [t, \infty), [\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset\}. \end{aligned}$$

First consider the case  $\lambda = \infty$ . Let  $w_1 := (v_1, h_1) \in \partial\Pi^\uparrow((v_0, h_0))$  with  $h_1 \in [t, \infty)$ . Recalling (4.16), the  $d\mathcal{P}$  measure of  $\Pi^\downarrow(w_1)$  is bounded below by the  $d\mathcal{P}$  measure of  $\{(v, h) : h \leq h_1 - c_2\|v - v_1\|\} \cap (\mathbb{R}^{d-1} \times [0, \infty))$ , which we generously bound below by

$$c \int_{\frac{h_1}{4}}^{\frac{h_1}{2}} e^{dh} (h_1 - h)^{d-1} dh \geq c' e^{h_1/c'}.$$

Here and elsewhere, unless noted otherwise,  $c$  and  $c'$  denote positive constants which are independent of other parameters except for dimension and whose value may change at each occurrence. Thus the probability that  $\Pi^\downarrow(w_1)$  does not contain points in  $\mathcal{P}$  is bounded above by  $c \exp(-e^{h_1/c})$ .

We now discretize  $\partial\Pi^\uparrow((v_0, h_0)) \cap (\mathbb{R}^{d-1} \times [t, \infty))$ . Notice that if  $w_1 := (v_1, h_1) \in \partial\Pi^\uparrow((v_0, h_0))$  then (4.16) gives  $h_1 = h_0 + G(v_0 - v_1) \geq h_0 + c_1\|v_1 - v_0\| - \log d$ , which yields  $\|v_0 - v_1\| \leq \frac{1}{c_1}[h_1 - h_0 + \log d]$ . This gives

$$P[H((v_0, h_0), \mathcal{P}) \geq t] \leq c \int_t^\infty (h_1 - h_0 + \log d)^{d-2} \exp(-ce^{h_1/c}) dh_1.$$

Thus (5.2) holds.

Next consider the case  $\lambda \in [1, \infty)$ . The above argument still holds as soon as we can show for any  $(v_1, h_1) \in W_\lambda$ , that the  $d\mathcal{P}$  measure of the intersection of  $W_\lambda$  with the down cone-like grain  $\{(v, h) : h \leq h_1 - c_2\|v - v_1\|\}$  is bounded below by  $c \exp(-e^{h_1/c}/c)$ . To do so, let  $C_{\min}$  be the largest circular cone included in the pyramid  $W_\lambda$  and with the same apex as  $W_\lambda$ . (Actually,  $C_{\min}$  does not depend on  $\lambda$  since for  $\lambda' > \lambda$ ,  $W_{\lambda'}$  is the image of  $W_\lambda$  by a translation.) Then any *vertical* cone with apex in  $W_\lambda$  is such that its intersection with  $W_\lambda$  is either the cone itself or contains a translate of  $C_{\min}$  with same apex. Consequently, the intersection  $\{(v, h) : h \leq h_1 - c_2\|v - v_1\|\} \cap W_\lambda$  contains another cone  $\{(v, h) : h \leq h_1 - c_3\|v - v_1\|\}$  with  $c_3$  depending only on  $d$ . Its  $d\mathcal{P}$  measure is then bounded below by  $c \exp(-e^{h_1/c})$ . The proof is concluded as in the case  $\lambda = \infty$ .

We now prove (5.3). We bound the probability of the events

$$E_3 := \{\partial\Psi(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : \|v\| \leq L, h > t\} \neq \emptyset\}$$

and

$$E_4 := \{\partial\Psi(\mathcal{P}^{(\lambda)}) \cap \{(v, h) : \|v\| \leq L, h < -t\} \neq \emptyset\}.$$

When in  $E_3$ , there is a point  $w_1 := (v_1, h_1)$  with  $h_1 \in [t, \infty)$ ,  $\|v_1\| \leq L$ , and such that  $[\Pi^\downarrow(w_1)]^{(\lambda)} \cap \mathcal{P}^{(\lambda)} = \emptyset$ . As in the proof of (5.2), there is a subset of  $C(\mathbf{0}, L)$  of volume one and on this subset the density of the  $d\mathcal{P}^{(\lambda)}$  measure exceeds  $c \exp(h_1/c)$ . Discretize  $\{(v, h) : \|v\| \leq L, h \in [t, \infty)\}$  into unit volume sub-cubes and bound cross-sectional areas by  $cL^{d-1}$  to obtain

$$P[E_3] \leq cL^{d-1} \int_{c_1 t/3}^\infty \exp(dh_1) \exp(-ce^{h_1/c}) dh_1 \leq cL^{d-1} \exp(-\frac{e^t}{c}).$$

On the event  $E_4$ , there exists a point  $(v_1, h_1)$  with  $\|v_1\| \leq L$  and  $h_1 \in (-\infty, -t]$  which is on the boundary of an up cone-like grain with apex in  $\mathcal{P}^{(\lambda)}$ . The apex of this up cone-like grain is contained in the union of all down cone-like grains with apex on  $B_{d-1}(\mathbf{0}, L) \times \{h_1\}$ . The  $d\mathcal{P}^{(\lambda)}$  measure of this union is bounded by  $cL^{d-1} \exp(h_1/c)$  (here we use that the union is a subset of the union of standard circular cones). Consequently, the probability that the union contains points from  $\mathcal{P}^{(\lambda)}$  is less than

$1 - \exp(-cL^{d-1}e^{h_1/c}) \leq cL^{d-1} \exp(h_1/c)$ . It remains to discretize and integrate over  $h_1 \in (-\infty, t)$ . This goes as follows.

Discretizing  $C(\mathbf{0}, L) \times (-\infty, -t]$  into unit volume subcubes and using the previous bound, we find that the probability there exists  $(v_1, h_1) \in \mathbb{R}^{d-1} \times (-\infty, -t]$  on the boundary of an up cone-like grain is bounded by

$$cL^{2(d-1)} \int_{-\infty}^{-t} e^{h_1/c} e^{dh_1} dh_1.$$

This establishes (5.3). Similar arguments apply to  $\partial(\Phi(\mathcal{P}^{(\lambda)}))$ .  $\square$

For  $(v_0, h_0) \in \text{Ext}(\mathcal{P}^{(\lambda)})$  and  $t \in \mathbb{R}$ , we define

$$\mathcal{U}^{(\lambda)}(v_0, h_0, t) := \bigcup_{w_1 \in [\Pi^\uparrow((v_0, h_0))]^{(\lambda)} \cap (\mathbb{R}^{d-1} \times (-\infty, t])} [\Pi^\downarrow(w_1)]^{(\lambda)}.$$

The score  $\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})$  depends only on the points of  $\mathcal{P}^{(\lambda)}$  inside  $\mathcal{U}^{(\lambda)}(v_0, h_0, H((v_0, h_0), \mathcal{P}^{(\lambda)}))$ , as this set contains all faces in  $\Phi(\mathcal{P}^{(\lambda)})$  which contain  $(v_0, h_0)$ . Put

$$R := R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)] := \inf\{r > 0 : \mathcal{P}^{(\lambda)} \cap \mathcal{U}^{(\lambda)}(v_0, h_0, H((v_0, h_0), \mathcal{P}^{(\lambda)})) \subset C(v_0, r)\}. \quad (5.4)$$

It follows from the definitions that

$$\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) = \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)), \quad r \in [R, \infty).$$

In other words, as in Section 6 of [9],  $R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)]$  is a radius of spatial stabilization for  $\hat{\xi}^{(\lambda)}$ . The next lemma shows that  $R$  is finite a.s. and in fact has exponentially decaying tails. Given  $c_1$  as in (4.16), we put for all  $h_0 \in \mathbb{R}$ ,

$$\tilde{h}_0 := \left(\frac{6}{c_1} \log d\right) \vee \left(-\frac{6}{c_1} h_0\right) \mathbf{1}(h_0 < 0). \quad (5.5)$$

**Lemma 5.2** *There is a constant  $c > 0$  such that for all  $\hat{\xi} \in \hat{\Xi}$ ,  $\lambda \in [1, \infty]$ ,  $(v_0, h_0) \in W_\lambda$ , and all  $t \in [\tilde{h}_0, \infty)$  we have*

$$P[R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)] > t] \leq c \exp\left(-\frac{t}{c}\right). \quad (5.6)$$

*Proof.* We show (5.6) for  $v_0 = \mathbf{0}$ , as the proof is analogous for arbitrary  $v_0$ . Put  $R := R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)]$  and write

$$P[R > t] \leq P[H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \geq \frac{c_1 t}{6}] + P[H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \in (-\infty, \frac{c_1 t}{6}], R \geq t].$$



Lemma 5.1(a) shows that the first term on the right-hand side is bounded by  $c \exp(-e^{t/c}/c)$ . Thus we only need to control the second term.

When  $H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \in (-\infty, c_1 t/6]$ , then  $\hat{\xi}^{(\lambda)}((\mathbf{0}, h_0))$  only depends on elements of  $\mathcal{P}^{(\lambda)}$  in

$$\mathcal{U} := \mathcal{U}^{(\lambda)}(\mathbf{0}, h_0, c_1 t/6).$$

Let  $w = (v, h) \in \mathcal{U}$  and  $w_1 = (v_1, h_1)$ ,  $h_1 \leq c_1 t/6$ , be such that  $\partial[\Pi^\downarrow(w_1)]^{(\lambda)}$  contains both  $(\mathbf{0}, h_0)$  and  $w$ . We assert that if  $h \in [-c_1 t/6, c_1 t/6]$ , then  $\|v\| \leq t$ . To see this we first note that  $\|v - v_1\| \leq t/2$ . This follows because  $h = h_1 - G(v - v_1)$ , which in view of (4.16) yields  $h \leq h_1 - c_1 \|v - v_1\| + \log d$ , that is to say

$$\|v - v_1\| \leq \frac{1}{c_1} [h_1 - h + \log d]. \quad (5.7)$$

Now  $\|v - v_1\| \leq t/2$  since all three quantities  $h_1$ ,  $-h$ , and  $\log d$  are bounded by  $c_1 t/6$ . Using  $h_1 = h_0 + G(-v_1) \geq h_0 + c_1 \|v_1\| - \log d$  we get  $\|v_1\| \leq t/2$  and thus  $\|v\| \leq t$  by the triangle inequality.

Consequently, if  $\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, -c_1 t/6]) = \emptyset$ , then only elements of  $\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-c_1 t/6, c_1 t/6])$  contribute to the score  $\hat{\xi}^{(\lambda)}((\mathbf{0}, h_0))$ , showing that in this case  $R^{\hat{\xi}^{(\lambda)}}[(\mathbf{0}, h_0)] \in (0, t]$ . Therefore

$$P[H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \in (-\infty, \frac{c_1 t}{6}], R \geq t] \leq P[\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, -c_1 t/6]) \neq \emptyset].$$

Notice that if  $v \in \mathcal{U}$  then  $\|v\| \leq \|v_1\| + \|v - v_1\| \leq \frac{t}{2} + \frac{t}{3} - \frac{h}{c_1}$ , where we use  $\|v_1\| \leq t/2$  and (5.7). Similar to the proof of Lemma 5.1, discretization methods yield

$$\begin{aligned} & d\mathcal{P}^{(\lambda)}(\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, \frac{-c_1 t}{6}])) \\ & \leq c \int_{-\infty}^{-c_1 t/6} e^{dh} \left(\frac{t}{2} + \frac{t}{3} - \frac{h}{c_1}\right)^{(d-1)} dh \leq c \exp\left(-\frac{t}{c}\right). \end{aligned}$$

It follows that

$$P[\mathcal{P}^{(\lambda)} \cap \mathcal{U} \cap (\mathbb{R}^{d-1} \times (-\infty, \frac{-c_1 t}{6}]) \neq \emptyset] \leq c \exp\left(-\frac{t}{c}\right),$$

as desired.  $\square$

**Lemma 5.3** *For all  $p \in [1, \infty)$  and  $\hat{\xi} \in \hat{\Xi}$ ,  $\hat{\xi}$  a  $k$ -face functional, there is a constant  $c > 0$  such that for all  $(v_0, h_0) \in W_\lambda$ ,  $\lambda \in [1, \infty]$ , we have*

$$\mathbb{E}[\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})^p] \leq c(|h_0|^c + 1) \exp\left(-\frac{e^{(h_0 \vee 0)/c}}{c}\right). \quad (5.8)$$

For all  $p \in [1, \infty)$  there is a constant  $c > 0$  such that for all  $(v_0, h_0) \in W_\lambda$ ,  $\lambda \in [1, \infty]$ , we have

$$\mathbb{E} [\hat{\xi}_V^{(\lambda)}((v_0, h_0)), \mathcal{P}^{(\lambda)}]^p \leq c(|h_0|^c + 1) \exp(c(h_0 \vee 0)) \exp\left(-\frac{e^{(h_0 \vee 0)/c}}{c}\right). \quad (5.9)$$

*Proof.* We first prove (5.8) for the  $k$ -face functional  $\hat{\xi}^{(\lambda)} := \hat{\xi}_k^{(\lambda)}$ ,  $k \in \{0, 1, \dots, d-1\}$ . We start by showing for all  $\lambda \in [1, \infty]$  and  $h_0 \in \mathbb{R}$

$$\sup_{v_0 \in \mathbb{R}^{d-1}} \mathbb{E} [\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})^p] \leq c(|h_0|^c + 1). \quad (5.10)$$

Let  $R := R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)]$  be as at (5.4) and  $N^{(\lambda)}((v_0, h_0))$  the cardinality of extreme points in  $C(v, R)$  which share a common facet with  $(v_0, h_0)$ . Clearly

$$\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \leq \frac{1}{k+1} \binom{N^{(\lambda)}((v_0, h_0))}{k}.$$

To show (5.10), given  $p \in [1, \infty)$ , it suffices to show there is a constant  $c := c(p, k, d)$  such that for  $\lambda \in [1, \infty]$

$$\mathbb{E} N^{(\lambda)}((v_0, h_0))^{pk} \leq c(|h_0|^c + 1). \quad (5.11)$$

By (1.1), for all  $r, \ell \in \mathbb{R}$  we have

$$d\mathcal{P}^{(\lambda)}(C(v_0, r) \cap (\mathbb{R}^{d-1} \times (-\infty, \ell))) \leq cr^{d-1}e^{d\ell}.$$

Consequently, with  $H := H((v_0, h_0), \mathcal{P}^{(\lambda)})$  as defined before Lemma 5.1 and with  $\text{Po}(\alpha)$  denoting a Poisson random variable with mean  $\alpha \in (0, \infty)$ , we have for  $\lambda \in [1, \infty]$

$$\begin{aligned} & \mathbb{E} N^{(\lambda)}((v_0, h_0))^{pk} \\ & \leq \mathbb{E} [\text{card}(\mathcal{P}^{(\lambda)} \cap [C(v, R) \cap (\mathbb{R}^{d-1} \times (-\infty, H))])^{pk}] \\ & = \sum_{i=0}^{\infty} \sum_{j=\lfloor h_0 \rfloor}^{\infty} \mathbb{E} [\text{Po}(d\mathcal{P}^{(\lambda)}(C(v, R) \cap (\mathbb{R}^{d-1} \times (-\infty, H))))^{pk} \mathbf{1}(i \leq R < i+1, j \leq H < j+1)] \\ & \leq \sum_{i=0}^{\infty} \sum_{j=\lfloor h_0 \rfloor}^{\infty} \mathbb{E} [\text{Po}(c(i+1)^{d-1}e^{(j+1)d})^{pk} \mathbf{1}(R \geq i, H \geq j)]. \end{aligned}$$

We shall repeatedly use the moment bound  $\mathbb{E} [\text{Po}(\alpha)^r] \leq c(r)\alpha^r$ ,  $r \in [1, \infty)$ . Using Hölder's inequality, we get

$$\mathbb{E} N^{(\lambda)}((v_0, h_0))^{pk} \leq c \sum_{i=0}^{\infty} \sum_{j=\lfloor h_0 \rfloor}^{\infty} (i+1)^{pk(d-1)/3} e^{(j+1)dpk/3} P[R \geq i]^{1/3} P[H \geq j]^{1/3}.$$

Splitting the sum on the  $i$  indices into  $i \in [0, \tilde{h}_0]$  and  $i \in [\tilde{h}_0, \infty]$ , with  $\tilde{h}_0$  defined at (5.5), and splitting the sum on the  $j$  indices into  $[h_0] \wedge 0, 0]$  and  $[0, \infty)$ , we get

$$EN^{(\lambda)}((v_0, h_0))^{pk} \leq S_1 + S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_1 &:= \sum_{i=0}^{\tilde{h}_0} \sum_{j=[h_0] \wedge 0}^0 (i+1)^{pk(d-1)/3} e^{(j+1)dpk/3} P[R \geq i]^{1/3} P[H \geq j]^{1/3} \\ S_2 &:= \sum_{i=\tilde{h}_0}^{\infty} \sum_{j=[h_0] \wedge 0}^0 (i+1)^{pk(d-1)/3} e^{(j+1)dpk/3} P[R \geq i]^{1/3} P[H \geq j]^{1/3} \\ S_3 &:= \sum_{i=0}^{\tilde{h}_0} \sum_{j=[h_0] \vee 0}^{\infty} (i+1)^{pk(d-1)/3} e^{(j+1)dpk/3} P[R \geq i]^{1/3} P[H \geq j]^{1/3} \\ S_4 &:= \sum_{i=\tilde{h}_0}^{\infty} \sum_{j=[h_0] \vee 0}^{\infty} (i+1)^{pk(d-1)/3} e^{(j+1)dpk/3} P[R \geq i]^{1/3} P[H \geq j]^{1/3}. \end{aligned}$$

Now we compute

$$S_1 \leq c \sum_{i=0}^{\tilde{h}_0} (i+1)^{pk(d-1)/3} \sum_{j=[h_0] \wedge 0}^0 \exp((j+1)dpk/3) \leq c(|h_0|^c + 1),$$

since the second sum is bounded by a constant and where  $c := c(p, k, d)$ . Next,

$$S_2 \leq c \sum_{i=\tilde{h}_0}^{\infty} (i+1)^{pk(d-1)/3} P[R \geq i]^{1/3} \sum_{j=[h_0] \wedge 0}^0 \exp((j+1)dpk/3) \leq c$$

where the first sum converges by the exponentially decaying tail bound for  $P[R \geq i]$ . Making use of the super exponentially decaying tail bound for  $P[H \geq j]$  we get

$$S_3 \leq c \sum_{i=0}^{\tilde{h}_0} (i+1)^{pk(d-1)/3} \sum_{j=[h_0] \vee 0}^{\infty} e^{(j+1)dpk/3} \exp(-e^{j/c}/3c) \leq c(|h_0|^c + 1).$$

Finally,

$$S_4 \leq c \sum_{i=\tilde{h}_0}^{\infty} (i+1)^{pk(d-1)/3} P[R \geq i]^{1/3} \sum_{j=[h_0] \vee 0}^{\infty} e^{(j+1)dpk/3} \exp(-e^{j/c}/3c) \leq c,$$

since both sums are bounded by a constant. Combining the bounds for  $S_1, S_2, S_3$  and  $S_4$  gives the required bound (5.11).

To deduce (5.8), we argue as follows. First consider the case  $h_0 \in [0, \infty)$ . By the Cauchy-Schwarz inequality and (5.10) we have

$$\begin{aligned} & \mathbb{E} [\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})^p] \\ & \leq (\mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})^{2p})^{1/2} P[\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) > 0]^{1/2} \\ & \leq c(|h_0| + 1)^c P[\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \neq 0]^{1/2}. \end{aligned}$$

The event  $\{\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \neq 0\}$  coincides with the event that  $(v_0, h_0)$  is extreme in  $\mathcal{P}^{(\lambda)}$  and we may now apply (5.2) for  $t = h_0$ , which is possible since we have assumed  $h_0$  is positive. This gives (5.8) for  $h_0 \in [0, \infty)$ . When  $h_0 \in (-\infty, 0)$  we bound  $P[\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) > 0]^{1/2}$  by  $c \exp(-e^0/c)$ ,  $c$  large, which shows (5.8) for  $h_0 \in (-\infty, 0)$ . This concludes the proof of (5.8) when  $\hat{\xi}$  is a  $k$ -face functional.

We now prove (5.8) for the volume functional  $\hat{\xi}_V$ . We start by proving the analog of (5.10). Without loss of generality we put  $(v_0, h_0) = (\mathbf{0}, h_0)$ . Recalling the definition of  $H := H((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})$  we have

$$\hat{\xi}_V^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)}) \leq \frac{1}{d} \int_{v \in \text{Cyl}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})} dv \int_{-\infty}^H e^{dh} dh.$$

Integrating, raising both sides to the  $p$ th power, taking expectations and applying the Cauchy-Schwarz inequality we get

$$\mathbb{E} \hat{\xi}_V^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})^p \leq C (\mathbb{E} (\text{VolCyl}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})^{2p})^{1/2} (\mathbb{E} e^{2pdH})^{1/2}.$$

Now

$$\mathbb{E} (\text{VolCyl}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})^{2p}) \leq \mathbb{E} (R^{\hat{\xi}_V^{(\lambda)}}[(\mathbf{0}, h)])^{2p(d-1)} \leq c(|h_0| + 1)^c$$

by Lemma 5.2. Lemma 5.1(a) and the formula  $\mathbb{E} X = \int_0^\infty P[X \geq t] dt$  imply that

$$\begin{aligned} \mathbb{E} e^{2pdH} &= \int_0^\infty P[e^{2pdH} > t] dt \\ &= \int_0^\infty P[2pdH > \log t] dt \\ &= \int_{t \leq \exp((h_0 \vee 0)2pd)} P[2pdH > \log t] dt + \int_{t \geq \exp((h_0 \vee 0)2pd)} P[2pdH > \log t] dt \\ &\leq \exp((h_0 \vee 0)2pd) + \int_{t \geq \exp((h_0 \vee 0)2pd)} P[H \geq \log t^{1/2pd}] dt \\ &\leq c \exp((h_0 \vee 0)2pd). \end{aligned}$$

Thus

$$\mathbb{E} \hat{\xi}_V^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P}^{(\lambda)})^p \leq c(|h_0| + 1)^c \exp(c(h_0 \vee 0)). \quad (5.12)$$

The bound (5.9) for  $\hat{\xi}_V^{(\lambda)}$  follows from (5.12) in the same way that (5.10) implies (5.8) for  $\hat{\xi}_k^{(\lambda)}$ . This completes the proof of Lemma 5.3.  $\square$

**5.2. Two point correlation function for  $\hat{\xi}^{(\lambda)}$ .** For all  $h \in \mathbb{R}$ ,  $(v_0, h_0), (v_1, h_1) \in W_\lambda$ , and  $\hat{\xi} \in \hat{\Xi}$  we extend the definition at (2.5) by putting for all  $\lambda \in [1, \infty]$

$$\begin{aligned} c^{(\lambda)}((v_0, h_0), (v_1, h_1)) &:= c^{\hat{\xi}^{(\lambda)}}((v_0, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)}) := & (5.13) \\ \mathbb{E} [\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cup (v_1, h_1)) \times \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup (v_0, h_0))] - \\ & \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \mathbb{E} \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}). \end{aligned}$$

The first part of the next lemma justifies the assertion that the functionals in  $\hat{\Xi}^{(\infty)}$  are scaling limits of their counterparts in  $\hat{\Xi}^{(\lambda)}$ .

**Lemma 5.4** (a) For all  $(v_0, h_0) \in \mathbb{R}^{d-1} \times \mathbb{R}$  and  $\hat{\xi} \in \hat{\Xi}$  we have

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) = \mathbb{E} \hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P}).$$

(b) For all  $h_0 \in \mathbb{R}$ ,  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$  and  $\hat{\xi} \in \hat{\Xi}$  we have

$$\lim_{\lambda \rightarrow \infty} c^{\hat{\xi}^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1)) = c^{\hat{\xi}^{(\infty)}}((\mathbf{0}, h_0), (v_1, h_1)).$$

*Proof.* We first prove (a). Suppose  $(v_0, h_0) \notin \text{Ext}(\mathcal{P})$ . Then  $(v_0, h_0) \notin \text{Ext}(\mathcal{P}^{(\lambda)})$ , showing that both sides vanish. Without loss of generality, let  $(v_0, h_0) \in \text{Ext}(\mathcal{P})$ . Put

$$B(v_0, h_0) := C(v_0, R^{\hat{\xi}^{(\infty)}}[(v_0, h_0), \mathcal{P}]) \cap (\mathbb{R}^{d-1} \times (-\infty, H((v_0, h_0), \mathcal{P}))).$$

We have

$$\hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P}) = \hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P} \cap B(v_0, h_0)).$$

For  $\lambda$  large we have  $B(v_0, h_0) \subset W_\lambda$ . For such  $\lambda$  it follows that

$$\hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P}) = \hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P} \cap W_\lambda) = \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P} \cap W_\lambda),$$

in other words

$$\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P} \cap W_\lambda) \rightarrow \hat{\xi}^{(\infty)}((v_0, h_0), \mathcal{P}) \quad a.s.$$

Convergence of expectations follows from the uniform integrability of  $\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P} \cap W_\lambda)$ , as shown in Lemma 5.3. This shows part (a). Part (b) follows from identical methods, since products of scores  $\hat{\xi}^{(\lambda)}((\mathbf{0}, h_0), \mathcal{P} \cap W_\lambda)$  and  $\hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P} \cap W_\lambda)$  a.s. converge to their  $\hat{\xi}^{(\infty)}$  counterparts and the products are uniformly integrable by Lemma 5.3.  $\square$

**Lemma 5.5** *Let  $c_1$  be as at (4.16). Let  $\hat{\xi} \in \hat{\Xi}$  be a  $k$ -face functional. There is a constant  $c_3 := c_3(\hat{\xi}, d) \in (0, \infty)$  such that for all  $\lambda \in [1, \infty]$  and  $(v_0, h_0), (v_1, h_1) \in W_\lambda$  satisfying*

$$\|v_1 - v_0\| \geq 2 \max \left( \frac{6}{c_1} \log d, -\frac{6}{c_1} h_0 \mathbf{1}(h_0 < 0), -\frac{6}{c_1} h_1 \mathbf{1}(h_1 < 0) \right) \quad (5.14)$$

we have

$$|c^{\hat{\xi}(\lambda)}((v_0, h_0), (v_1, h_1))| \leq c_3(|h_0|+1)^{c_3}(|h_1|+1)^{c_3} \exp \left( \frac{-1}{c_3} (\|v_1 - v_0\| + e^{h_0 \vee 0} + e^{h_1 \vee 0}) \right). \quad (5.15)$$

When  $\hat{\xi}$  is the volume functional  $\hat{\xi}_V$  we have

$$\begin{aligned} |c^{\hat{\xi}(\lambda)}((v_0, h_0), (v_1, h_1))| &\leq c_3(|h_0|+1)^{c_3}(|h_1|+1)^{c_3} \exp(c_4((h_0 \vee 0) + (h_1 \vee 0))) \\ &\quad \times \exp \left( \frac{-1}{c_3} (\|v_1 - v_0\| + e^{h_0 \vee 0} + e^{h_1 \vee 0}) \right). \end{aligned} \quad (5.16)$$

*Proof.* We prove this assuming that  $\hat{\xi}$  is the  $k$ -face functional, as the proof for the volume functional  $\hat{\xi}_V$  follows from identical methods. Put

$$\begin{aligned} X_\lambda &:= \hat{\xi}(\lambda)((v_0, h_0), \mathcal{P}^{(\lambda)} \cup (v_1, h_1)), \\ Y_\lambda &:= \hat{\xi}(\lambda)((v_1, h_1), \mathcal{P}^{(\lambda)} \cup (v_0, h_0)), \\ \tilde{X}_\lambda &:= \hat{\xi}(\lambda)((v_0, h_0), \mathcal{P}^{(\lambda)}) \quad \text{and} \quad \tilde{Y}_\lambda := \hat{\xi}(\lambda)((v_1, h_1), \mathcal{P}^{(\lambda)}). \end{aligned}$$

We have

$$c^{\hat{\xi}(\lambda)}((v_0, h_0), (v_1, h_1)) = \mathbb{E} X_\lambda Y_\lambda - \mathbb{E} \tilde{X}_\lambda \mathbb{E} \tilde{Y}_\lambda. \quad (5.17)$$

Put  $r := \|v_1 - v_0\|/2$  and let  $R^{\hat{\xi}(\lambda)}[(v_i, h_i)]$ ,  $i \in \{0, 1\}$ , be as at (5.4). Now

$$\begin{aligned} &|\mathbb{E} X_\lambda Y_\lambda - \mathbb{E} X_\lambda Y_\lambda \mathbf{1}(R^{\hat{\xi}(\lambda)}[(v_0, h_0)] \leq r, R^{\hat{\xi}(\lambda)}[(v_1, h_1)] \leq r)| \\ &\leq \mathbb{E} X_\lambda Y_\lambda [\mathbf{1}(R^{\hat{\xi}(\lambda)}[(v_0, h_0)] \geq r) + \mathbf{1}(R^{\hat{\xi}(\lambda)}[(v_1, h_1)] \geq r)]. \end{aligned}$$

Let  $v_1$  and  $v_0$  satisfy (5.14). Hölder's inequality and Lemma 5.2 imply that the right hand side of the above is bounded by

$$\begin{aligned} &\|X_\lambda\|_3 \|Y_\lambda\|_3 [P[R^{\hat{\xi}(\lambda)}[(v_0, h_0)] \geq r]^{1/3} + P[R^{\hat{\xi}(\lambda)}[(v_1, h_1)] \geq r]^{1/3}] \\ &\leq c(|h_0|+1)^c (|h_1|+1)^c \exp \left( \frac{-1}{c} (e^{h_0 \vee 0} + e^{h_1 \vee 0}) \right) \\ &\quad \times [P[R^{\hat{\xi}(\lambda)}[(v_0, h_0)] \geq r]^{1/3} + P[R^{\hat{\xi}(\lambda)}[(v_1, h_1)] \geq r]^{1/3}] \\ &\leq c(|h_0|+1)^c (|h_1|+1)^c \exp \left( \frac{-1}{c} (\|v_1 - v_0\| + e^{h_0 \vee 0} + e^{h_1 \vee 0}) \right). \end{aligned} \quad (5.18)$$

Now

$$\begin{aligned}
& \mathbb{E} X_\lambda Y_\lambda \mathbf{1}(R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)] \leq r, R^{\hat{\xi}^{(\lambda)}}[(v_1, h_1)] \leq r) \\
&= \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r)) \\
&\quad \times \mathbf{1}(R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)] \leq r, R^{\hat{\xi}^{(\lambda)}}[(v_1, h_1)] \leq r).
\end{aligned}$$

Following the above methods, the difference of

$$\mathbb{E} X_\lambda Y_\lambda \mathbf{1}(R^{\hat{\xi}^{(\lambda)}}[(v_0, h_0)] \leq r, R^{\hat{\xi}^{(\lambda)}}[(v_1, h_1)] \leq r)$$

and

$$\mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r))$$

is also bounded by (5.18). By independence we have

$$\begin{aligned}
& \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r)) \\
&= \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \mathbb{E} \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r)).
\end{aligned}$$

Thus we have shown

$$\begin{aligned}
& |\mathbb{E} X_\lambda Y_\lambda - \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \mathbb{E} \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r))| \\
&\leq c(|h_0| + 1)^c (|h_1| + 1)^c \exp\left(\frac{-1}{c}(\|v_1 - v_0\| + e^{h_0 v_0} + e^{h_1 v_0})\right).
\end{aligned}$$

Identical methods give

$$\begin{aligned}
& |\mathbb{E} \tilde{X}_\lambda \mathbb{E} \tilde{Y}_\lambda - \mathbb{E} \hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(v_0, r)) \mathbb{E} \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cap C(v_1, r))| \\
&\leq c(|h_0| + 1)^c (|h_1| + 1)^c \exp\left(\frac{-1}{c}(\|v_1 - v_0\| + e^{h_0 v_0} + e^{h_1 v_0})\right).
\end{aligned}$$

Combining the last two displays with (5.17), we get (5.15).  $\square$

Our last lemma shows that  $c^{\hat{\xi}^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))e^{dh_0}e^{dh_1}$  is bounded by an integrable function, a fact used in establishing variance asymptotics in the next section.

**Lemma 5.6** *For all  $\hat{\xi} \in \hat{\Xi}$  there is an integrable  $g : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^+$  such that for all  $\lambda \in [1, \infty]$  we have*

$$|c^{\hat{\xi}^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))|e^{dh_0}e^{dh_1} \leq g(h_0, v_1, h_1). \quad (5.19)$$

*Proof.* With  $c_1$  as at (4.16), define  $F : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}^+$  by

$$\begin{aligned} F(h_0, v_1, h_1) &:= c(|h_0| + 1)^c (|h_1| + 1)^c \exp(c_4((h_0 \vee 0) + (h_1 \vee 0))) \\ &\times \left( \exp\left(\frac{-1}{c}(\|v_1\| + e^{h_0 \vee 0} + e^{h_1 \vee 0})\right) \right. \\ &\quad \left. + \mathbf{1}(\|v_1\| \leq 2 \max\left(\frac{6}{c_1} \log d, -\frac{6}{c_1} h_0 \mathbf{1}(h_0 < 0), -\frac{6}{c_1} h_1 \mathbf{1}(h_1 < 0)\right)) \right), \end{aligned}$$

where  $c$  is a constant. If  $c$  is large enough, then Lemma 5.5 gives

$$|c^{\hat{\xi}^{(\lambda)}}((\mathbf{0}, h_0), (v_1, h_1))| e^{dh_0} e^{dh_1} \leq F(h_0, v_1, h_1) e^{dh_0} e^{dh_1}.$$

Put  $g(h_0, v_1, h_1) := F(h_0, v_1, h_1) e^{dh_0} e^{dh_1}$  and note that  $g$  is integrable as claimed.  $\square$

## 6 Proof of main results

**6.1. Proof of Theorems 1.1 and 1.2.** The next proposition immediately yields Theorem 1.2. It also yields Theorem 1.1, since it implies that the extreme points of  $K_\lambda \cap Q_0$  converge in law to  $\text{Ext}(\mathcal{P})$  as  $\lambda \rightarrow \infty$ . Recall that  $T^{(\lambda)}(\mathcal{P}_\lambda \cap Q_0) := \mathcal{P}^{(\lambda)}$  as at (4.6) and  $T^{(\lambda)}((\partial K_\lambda) \cap Q_0) := \partial(\Phi(\mathcal{P}^{(\lambda)}))$  on the event  $A_\lambda$ .

**Proposition 6.1** *Fix  $L \in (0, \infty)$ . We have that  $\partial\Psi(\mathcal{P}^{(\lambda)})$  converges in probability as  $\lambda \rightarrow \infty$  to  $\partial\Psi(\mathcal{P})$  in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ . Likewise,  $\partial\Phi(\mathcal{P}^{(\lambda)})$  converges in probability as  $\lambda \rightarrow \infty$  to  $\partial(\Phi(\mathcal{P}))$ .*

*Proof.* We prove the first convergence statement as follows. With  $L$  fixed, for all  $l \in [0, \infty)$  and  $\lambda \in [1, \infty)$ , let  $E(L, l, \lambda)$  be the event that the heights of  $\partial(\Psi(\mathcal{P}^{(\lambda)}))$  and  $\partial(\Psi(\mathcal{P}))$  belong to  $[-l, l]$  over the spatial region  $B_{d-1}(\mathbf{0}, L)$ . Lemma 5.1(b) shows that  $P[E(L, l, \lambda)^c]$  decays exponentially fast in  $l$ , uniformly in  $\lambda$ . It is enough to show, conditional on  $E(L, l, \lambda)$ , that  $\partial(\Psi(\mathcal{P}^{(\lambda)}))$  and  $\partial(\Psi(\mathcal{P}))$  coincide with high probability in the space  $\mathcal{C}(B_{d-1}(\mathbf{0}, L))$ ,  $\lambda$  large. Indeed, conditional on  $E(L, l, \lambda)$ ,  $\partial\Psi(\mathcal{P}) \cap (B_{d-1}(\mathbf{0}, L) \times [-l, l])$  depends only on points in

$$\mathcal{D} := \bigcup_{w \in B_{d-1}(\mathbf{0}, L) \times [-l, l]} \Pi^\downarrow(w).$$

Thus whenever we have equality of  $\mathcal{P} \cap \mathcal{D} \cap W_\lambda$  and  $\mathcal{P} \cap \mathcal{D}$ , it follows that  $\partial\Psi(\mathcal{P}^{(\lambda)})$  and  $\partial\Psi(\mathcal{P})$  coincide in  $\mathcal{C}_{d-1}[\mathbf{0}, L]$ . Since  $\mathcal{D} \setminus W_\lambda$  decreases to  $\emptyset$ , we have as  $\lambda$  goes to infinity

$$P[\mathcal{P} \cap \mathcal{D} \cap W_\lambda \neq \mathcal{P} \cap \mathcal{D}] = P[\mathcal{P} \cap (\mathcal{D} \setminus W_\lambda) \neq \emptyset] \leq dP[\mathcal{D} \setminus W_\lambda] \rightarrow 0.$$



This completes the proof of the first convergence statement. The proof of the second convergence statement is nearly identical and we leave the details to the reader.  $\square$

**6.2. Proof of Theorem 2.1.** When  $g \equiv 1$ , the decomposition (3.3) shows that it is enough to find expectation asymptotics for  $\mathbb{E}[\langle \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)]$  and multiply the result by  $f_0(K)$ . For arbitrary  $g \in \mathcal{C}(K)$ , an identical decomposition holds and so to show (2.8), it suffices to find  $\lim_{\lambda \rightarrow \infty} \mathbb{E}[\langle g \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)]$ . We have

$$\begin{aligned} & \mathbb{E}[\langle g \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)] \\ &= \int_{Q_0} g(x) \mathbb{E}[\xi(x, \mathcal{P}_\lambda) \mathbf{1}(A_\lambda)] \lambda dx \\ &= \sqrt{d} \int_{(v,h) \in W_\lambda} g([T^{(\lambda)}]^{-1}(v, h)) \mathbb{E}[\hat{\xi}^{(\lambda)}((v, h), \mathcal{P} \cap W_\lambda) \mathbf{1}(A_\lambda)] e^{dh} dh dv \\ &= \sqrt{d} \int_{(v,h) \in W_\lambda} g([T^{(\lambda)}]^{-1}(v, h)) \mathbb{E}[\hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_\lambda - v)) \mathbf{1}(A_\lambda)] e^{dh} dh dv \end{aligned}$$

where the second equality uses (4.14) and (4.15), whereas the last equality uses translation invariance of  $\hat{\xi}^{(\lambda)}$ . Scaling by  $(\log \lambda)^{d-1}$  and making the change of variable  $u = (\frac{1}{d} \log \lambda)^{-1} v$ ,  $dv = d^{-(d-1)} (\log \lambda)^{d-1} du$ , we obtain

$$\begin{aligned} & (\log \lambda)^{-(d-1)} \mathbb{E}[\langle g \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)] = \\ & d^{-d+3/2} \int_{(u,h) \in W_\lambda'} g([T^{(\lambda)}]^{-1}((\frac{1}{d} \log \lambda)u, h)) \mathbb{E}[\hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_\lambda' - u) \log \lambda^{1/d}) \mathbf{1}(A_\lambda)] e^{dh} dh du, \end{aligned} \tag{6.1}$$

where  $W_\lambda' := \{(\frac{1}{d} \log \lambda)^{-1} v, h\}; (v, h) \in W_\lambda\}$ . Here, for  $B \subset \mathbb{R}^{d-1} \times \mathbb{R}$  and  $s \in \mathbb{R}$ , we write  $sB := \{(sv, h) : (v, h) \in B\}$ . We now prove (2.8) via the following three steps.

(i) We first show the almost everywhere convergence

$$\lim_{\lambda \rightarrow \infty} \mathbf{1}((u, h) \in W_\lambda') g \left( [T^{(\lambda)}]^{-1} \left( \frac{1}{d} \log \lambda u, h \right) \right) = \mathbf{1}(u \in S(d)) g(\mathbf{0}), \tag{6.2}$$

where  $S(d)$  is defined at (2.7). Indeed, because of (4.5), the equation of  $W_\lambda'$  is  $\ell_i(u) \leq (1 + \frac{\log \delta_0 - h}{\log \lambda^{1/d}})$ ,  $1 \leq i \leq d$ . Consequently, in the limit as  $\lambda \rightarrow \infty$ , we have  $\ell_i(v) \leq 1$  for  $1 \leq i \leq d$ . In other words, the limit of  $W_\lambda'$  is a cylinder whose base is the intersection of  $V$  and the pyramid  $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \leq 1, 1 \leq i \leq d\}$ . This base is precisely  $S(d)$ .

Moreover in view of (4.4), we have

$$[T^{(\lambda)}]^{-1} \left( \frac{1}{d} \log \lambda u, h \right) = \lambda^{\frac{1}{d}(\ell(u)-1)} e^h.$$

If  $u \notin S(d)$ , then for  $\lambda$  large enough, the indicator function is equal to 0. If  $u \in \text{int}S(d)$  (where  $\text{int}$  denotes the interior), then  $l_i(u) < 1$  for  $1 \leq i \leq d$  and thus  $\lim_{\lambda \rightarrow \infty} [T^{(\lambda)}]^{-1}((\frac{1}{d} \log \lambda)u, h) = \mathbf{0}$ . By continuity of  $g$  we have

$$\lim_{\lambda \rightarrow \infty} g([T^{(\lambda)}]^{-1}((\frac{1}{d} \log \lambda)u, h)) = g(\mathbf{0}).$$

This shows (6.2).

(ii) We remove the indicator on the right hand side of (6.1) with small error:

$$\begin{aligned} (\log \lambda)^{-(d-1)} \mathbb{E} [\langle g \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)] &= d^{-d+3/2} \int g([T^{(\lambda)}]^{-1}((\frac{1}{d} \log \lambda)u, h)) \\ &\mathbb{E} [\hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_{\lambda'} - u) \log \lambda^{1/d})] \mathbf{1}((u, h) \in W_{\lambda'}) e^{dh} dh du + o(1). \end{aligned} \quad (6.3)$$

Indeed, by the Cauchy-Schwarz inequality and by moment bounds similar to those from Lemma 5.3, we have uniformly in  $u$  that

$$\begin{aligned} &\int \mathbb{E} [\hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_{\lambda'} - u) \log \lambda^{1/d}) \mathbf{1}(A_\lambda^c)] e^{dh} dh \\ &\leq \int (\mathbb{E} [\hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_{\lambda'} - u) \log \lambda^{1/d})]^2)^{1/2} P[A_\lambda^c]^{1/2} e^{dh} dh \\ &\leq c(\log \lambda)^{-2d^2}, \end{aligned}$$

where  $c$  is a constant not depending on  $u$ . Equality (6.3) follows from the estimate above, (6.2), and the dominated convergence theorem.

(iii) Since  $(W_{\lambda'} - u) \log \lambda^{1/d} \uparrow \mathbb{R}^d$  as  $\lambda \rightarrow \infty$ , an easy modification of the proof of Lemma 5.4 gives

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_{\lambda'} - u) \log \lambda^{1/d}) = \mathbb{E} \hat{\xi}^{(\infty)}((\mathbf{0}, h), \mathcal{P}). \quad (6.4)$$

Lemma 5.3 shows that  $\mathbb{E} \hat{\xi}^{(\lambda)}((\mathbf{0}, h), \mathcal{P} \cap (W_1 - \frac{1}{d}u) \log \lambda) e^{dh}$  is dominated by an integrable function on  $\mathbb{R}^{d-1} \times \mathbb{R}$ . Combining (6.2)- (6.4) yields (2.8) as desired.

Next we show variance asymptotics (2.9). By an easy extension of the decomposition (3.4) and Lemma 3.8, it suffices to find

$$\lim_{\lambda \rightarrow \infty} \text{Var} \sum_{x \in \mathcal{P}_\lambda \cap Q_0} \xi(x, \mathcal{P}_\lambda) g(x) \mathbf{1}(A_\lambda).$$

For  $g \in \mathcal{C}(K)$ , the Mecke-Slivnyak formula (Corollary 3.2.3 in [16]) gives

$$\text{Var}[\langle g \mathbf{1}(Q_0), \mu_\lambda^\xi \rangle \mathbf{1}(A_\lambda)] := I_1(\lambda) + I_2(\lambda), \quad (6.5)$$

where

$$I_1(\lambda) := \int_{Q_0} g(x)^2 \mathbb{E} [\xi(x, \mathcal{P}_\lambda)^2 \mathbf{1}(A_\lambda)] \lambda dx$$

and

$$I_2(\lambda) := \int_{Q_0} \int_{Q_0} g(x)g(y) [\mathbb{E} \xi(x, \mathcal{P}_\lambda \cup y) \xi(y, \mathcal{P}_\lambda \cup x) \mathbf{1}(A_\lambda) - \mathbb{E} \xi(x, \mathcal{P}_\lambda) \mathbf{1}(A_\lambda) \mathbb{E} \xi(y, \mathcal{P}_\lambda) \mathbf{1}(A_\lambda)] \lambda^2 dy dx.$$

Replacing  $g$  by  $g^2$  in the proof of expectation asymptotics, we obtain

$$\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} I_1(\lambda) = d^{-d+3/2} \text{Vol}_d(S(d)) \int_{-\infty}^{\infty} \mathbb{E} [\xi^{(\infty)}((\mathbf{0}, h_0), \mathcal{P})^2] e^{dh_0} dh_0 g^2(\mathbf{0}). \quad (6.6)$$

We next consider  $\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} I_2(\lambda)$ . Recalling (5.13) we have

$$\begin{aligned} & c^{\hat{\xi}^{(\lambda)} \mathbf{1}(A_\lambda)}((v_0, h_0), (v_1, h_1)) \\ &= \mathbb{E} [\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cup (v_1, h_1)) \times \hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup (v_0, h_0)) \mathbf{1}(A_\lambda)] \\ & \quad - \mathbb{E} [\hat{\xi}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda)] \mathbb{E} [\hat{\xi}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}) \mathbf{1}(A_\lambda)]. \end{aligned}$$

By (4.14) and (4.15), we may rewrite  $I_2(\lambda)$  in terms of  $\hat{\xi}^{(\lambda)}$ :

$$\begin{aligned} I_2(\lambda) &= d \int_{(v_0, h_0) \in W_\lambda} \int_{(v_1, h_1) \in W_\lambda} g([T^{(\lambda)}]^{-1}(v_0, h_0)) g([T^{(\lambda)}]^{-1}(v_1, h_1)) \\ & \quad \cdot c^{\hat{\xi}^{(\lambda)} \mathbf{1}(A_\lambda)}((v_0, h_0), (v_1, h_1)) e^{dh_0} e^{dh_1} dh_0 dh_1 dv_0 dv_1. \end{aligned}$$

Translation invariance of  $\hat{\xi}^{(\lambda)}$  yields

$$\begin{aligned} I_2(\lambda) &= d \int_{(v_0, h_0) \in W_\lambda} \int_{(v_1, h_1) \in W_\lambda} g([T^{(\lambda)}]^{-1}(v_0, h_0)) g([T^{(\lambda)}]^{-1}(v_1, h_1)) \\ & \quad \cdot c^{\hat{\xi}^{(\lambda)} \mathbf{1}(A_\lambda)}((\mathbf{0}, h_0), (v_1 - v_0, h_1), \mathcal{P} \cap (W_\lambda - v_0)) e^{dh_0} e^{dh_1} dh_0 dh_1 dv_1 dv_0. \end{aligned}$$

Again, we make the change of variable  $u = (\frac{1}{d} \log \lambda)^{-1} v_0$ . This gives

$$\begin{aligned} (\log \lambda)^{-(d-1)} I_2(\lambda) &= d^{-d+2} \int_{(u, h_0) \in W_\lambda'} \int_{(v_1, h_1) \in W_\lambda} g([T^{(\lambda)}]^{-1}(\frac{1}{d} \log \lambda \cdot u, h_0)) g([T^{(\lambda)}]^{-1}(v_1, h_1)) \\ & \quad \cdot c^{\hat{\xi}^{(\lambda)} \mathbf{1}(A_\lambda)} \left( (\mathbf{0}, h_0), (v_1 - \frac{1}{d} \log \lambda \cdot u, h_1), \mathcal{P} \cap (W_\lambda - \frac{1}{d} \log \lambda \cdot u) \right) e^{dh_0} e^{dh_1} dh_0 dh_1 dv_1 du. \end{aligned}$$

To conclude the proof of (2.9), it remains only to compute  $\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} I_2(\lambda)$ .

We proceed in four steps.

(i) Similarly to (6.2), we have

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \mathbf{1}((u, h_0) \in W_{\lambda'}) \mathbf{1}((v_1, h_1) \in W_{\lambda}) g([T^{(\lambda)}]^{-1}(\frac{1}{d} \log \lambda \cdot u, h_0)) g([T^{(\lambda)}]^{-1}(v_1, h_1)) \\ &= \mathbf{1}(u \in S(d)) g^2(\mathbf{0}). \end{aligned} \quad (6.7)$$

(ii) We remove the indicator from  $c^{\hat{\xi}^{(\lambda)}} \mathbf{1}(A_{\lambda})$  at the cost of a small additive error. Recall the definition of  $c^{\hat{\xi}^{(\lambda)}}((v_0, h_0), (v_1, h_1), \mathcal{P}^{(\lambda)})$  at (5.13). As in the proof of Lemmas 5.5 and 5.6 we show that the difference

$$\begin{aligned} & |c^{\hat{\xi}^{(\lambda)}} \mathbf{1}(A_{\lambda}) \left( (\mathbf{0}, h_0), (v_1 - \frac{1}{d} \log \lambda \cdot u, h_1), \mathcal{P} \cap (W_{\lambda} - \frac{1}{d} \log \lambda \cdot u) \right) \\ & - c^{\hat{\xi}^{(\lambda)}} \left( (\mathbf{0}, h_0), (v_1 - \frac{1}{d} \log \lambda \cdot u, h_1), \mathcal{P} \cap (W_{\lambda} - \frac{1}{d} \log \lambda \cdot u) \right) | e^{dh_0} e^{dh_1} \end{aligned}$$

is bounded above by  $P[A_{\lambda}^c]^{1/4} G(h_0, v_1, h_1)$  where  $G$  is a function which is integrable with respect to  $(h_0, v_1, h_1) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$  and which does not depend on  $u$ . Consequently, the integrated error is bounded, uniformly in  $u$ :

$$\begin{aligned} & \int_{h_0 \in \mathbb{R}} \int_{(v_1, h_1) \in W_{\lambda}} P[A_{\lambda}^c] G(h_0, v_1, h_1) dh_0 dh_1 dv_1 \leq C(\log \lambda)^{-d^2} \\ & = o((\log \lambda)^{(d-1)}). \end{aligned} \quad (6.8)$$

Combining (6.7) and (6.8) with the dominated convergence theorem, we find that the removal of the indicator does not modify the asymptotics of the variance.

(iii) Given a fixed  $u$ , for all  $(v_1, h_1) \in W_{\lambda}$ , we make the change of variable  $v' = v_1 - \frac{1}{d} \log \lambda \cdot u$ ,  $dv' = dv_1$ . This transforms

$$c^{\hat{\xi}^{(\lambda)}} \left( (\mathbf{0}, h_0), (v_1 - \frac{1}{d} \log \lambda \cdot u, h_1), \mathcal{P} \cap (W_{\lambda} - \frac{1}{d} \log \lambda \cdot u) \right) e^{dh_0} e^{dh_1}$$

into

$$c^{\hat{\xi}^{(\lambda)}} \left( (\mathbf{0}, h_0), (v', h_1), \mathcal{P} \cap (W_{\lambda'} - u) \log \lambda^{1/d} \right) e^{dh_0} e^{dh_1}.$$

By Lemma 5.6 the last expression is bounded by an integrable function of  $h_0, v'$  and  $h_1$ , uniformly in  $\lambda$ , and by Lemma 5.4 it converges to  $c^{\hat{\xi}^{(\infty)}}((\mathbf{0}, h_0), (v', h_1), \mathcal{P})$  as  $\lambda \rightarrow \infty$ .

(iv) We make the change of variable  $v' = v_1 - \frac{1}{d} \log \lambda \cdot u$ . The integration domain  $W_{\lambda}$  transforms to  $\{(v', h_0) \in (W_{\lambda'} - u) \log \lambda^{1/d}\}$ , which increases up to  $\mathbb{R}^{d-1} \times \mathbb{R}$ .

Combining observations (i)-(iv) with the dominated convergence theorem yields

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} I_2(\lambda) = \\ & d^{-d+2} \text{Vol}_d(S(d)) g^2(\mathbf{0}) \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{\hat{\xi}^{(\infty)}}((\mathbf{0}, h_0), (v', h_1)) e^{d(h_0+h_1)} dh_0 dv' dh_1. \end{aligned} \quad (6.9)$$

Combining (6.6) and (6.9) and recalling the definition of  $\sigma^2(\xi^{(\infty)})$  at (2.6) gives

$$\lim_{\lambda \rightarrow \infty} (\log \lambda)^{-(d-1)} \text{Var}[\langle g, \mu_\lambda^\xi \mathbf{1}(Q_0) \rangle] = d^{-d+1} \text{Vol}_d(S(d)) g^2(\mathbf{0}) \sigma^2(\xi^{(\infty)}).$$

We repeat this computation for each vertex of  $K$ . Proposition 3.2 yields (2.9), as desired.  $\square$

## 7 Appendix

We establish the unproved assertions of Section 3. Our first lemma shows that, near the origin, the boundary of the floating body for  $K$  is a pseudo-hyperboloid.

**Lemma 7.1** *There exists  $\Delta_d \in [1, \infty)$  depending only on  $d$  such that when  $K$  contains  $[\mathbf{0}, \Delta_d]^d$  and is contained in some multiple of that cube, then*

$$K(v = t) \cap \left[0, \frac{1}{2}\right]^d = \left\{ (z_1, \dots, z_d) \in \left[0, \frac{1}{2}\right]^d : \prod_{i=1}^d z_i = \frac{d!}{d^d} t \right\}, \quad t \in (0, \infty). \quad (7.1)$$

*Proof.* Put  $\tilde{t} := dt/d^d$ . Recall the definition of the surface  $\mathcal{H}_{\tilde{t}}$  at (4.1). We start by proving that for every  $z^{(0)} = (z_1^{(0)}, \dots, z_d^{(0)}) \in \mathcal{H}_{\tilde{t}}$ , we have

$$\text{Vol}([0, \infty)^d \cap H^+(z^{(0)})) = t \quad (7.2)$$

where we recall that  $H^+(z^{(0)})$  is the half-space containing the origin and bounded by the hyperplane tangent to  $\mathcal{H}_{\tilde{t}}$  at  $z^{(0)}$ . By (4.2), we have

$$H^+(z^{(0)}) := \left\{ (z_1, \dots, z_d) : \sum_{i=1}^d \frac{z_i}{z_i^{(0)}} \leq d \right\}. \quad (7.3)$$

Then

$$\begin{aligned} \text{Vol}([0, \infty)^d \cap H^+(z^{(0)})) &= \int_{[0, \infty)^d} \mathbf{1}\left(\sum_{i=1}^d \frac{z_i}{z_i^{(0)}} \leq d\right) dz_1 \cdots dz_d \\ &= \tilde{t} \int_{[0, \infty)^d} \mathbf{1}\left(\sum_{i=1}^d y_i \leq d\right) dy_1 \cdots dy_d = t \end{aligned}$$

where we use the change of variable  $y_i = \frac{z_i}{z_i^{(0)}}$ ,  $1 \leq i \leq d$ , and the identity  $\prod_{i=1}^d z_i^{(0)} = \tilde{t}$ . The proof of (7.2) is complete.

We now prove that the boundary of the floating body for  $[0, \infty)^d$  at level  $t$  satisfies

$$[0, \infty)^d(v = t) = \mathcal{H}_{\tilde{t}}. \quad (7.4)$$

Indeed, let  $H^+$  be a half-space with a boundary denoted by  $H$  such that  $H$  contains  $z^{(0)}$ . We may assume that  $H$  has a normal vector with all strictly positive coordinates (otherwise, we would have  $\text{Vol}([0, \infty)^d \cap H^+) = \infty$ ). Then  $H$  is tangent to exactly one  $\mathcal{H}_{\bar{s}}$  where  $s = \text{Vol}([0, \infty)^d \cap H^+)$ . Since  $z^{(0)} \in \mathcal{H}_{\bar{t}} \cap H$ , then necessarily  $t \in (0, s)$ . Consequently, we obtain  $z^{(0)} \in [0, \infty)^d(v = t)$ . In other words,  $\mathcal{H}_{\bar{t}} \subset [0, \infty)^d(v = t)$ . That this holds for any  $t \in (0, s]$ , combined with the fact that all  $\mathcal{H}_{\bar{t}}$ ,  $t > 0$ , and all  $[0, \infty)^d(v = t)$ ,  $t > 0$ , form a partition  $(0, \infty)^d$ , is enough to yield (7.4).

It remains to show the validity of (7.4) when replacing the orthant  $[0, \infty)^d$  with  $K$  whenever  $K$  contains a large enough cube  $[0, \Delta_d]^d$ . We remark that as soon as  $\Delta_d \geq d/2$ , we have for every  $t > 0$  and every  $z^{(0)} \in [0, \frac{1}{2}]^d \cap \mathcal{H}_{\bar{t}}$ , the equality

$$\text{Vol}([0, \infty)^d \cap H^+(z^{(0)})) = \text{Vol}(K \cap H^+(z^{(0)})) = t.$$

Indeed, we observe that  $[0, \infty)^d \cap H^+(z^{(0)}) \subset [0, \frac{d}{2}]^d \subset K$  because the equation (7.3) defining  $H^+(z^{(0)})$  implies that every  $z \in H^+(z^{(0)})$  satisfies  $z_i \leq z_i^{(0)} d \leq d/2$  for  $1 \leq i \leq d$ .

We now fix again  $z^{(0)} \in [0, \frac{1}{2}]^d \cap \mathcal{H}_{\bar{t}}$  and consider a half-space  $H^+$  with boundary  $H$  such that  $H$  contains  $z^{(0)}$ . We have to show that

$$\text{Vol}(K \cap H^+) \geq t. \tag{7.5}$$

When  $H^+ \cap (0, \infty)^d$  is a subset of  $[0, \Delta_d]^d$  then  $K \cap H^+ \supset [0, \Delta_d]^d \cap H^+$  and (7.5) follows. If  $H^+ \cap (0, \infty)^d$  is not a subset of  $[0, \Delta_d]^d$  then there is at least one point  $z = (z_1, \dots, z_d)$  from  $H$  with a coordinate greater than  $\Delta_d$ , say  $z_1$ . In particular,  $H^+ \cap [0, \Delta_d]^d$  contains a simplex which is the convex hull of  $(\{z_1^{(0)}\} \times \prod_{i=2}^d [0, z_i^{(0)}]) \cup \{z\}$ . Consequently,  $\text{Vol}(K \cap H^+)$  is bounded from below by  $c\Delta_d \prod_{i=2}^d z_i^{(0)}$  where  $c$  is a multiplicative constant depending only on  $d$ . When  $\Delta_d \geq d^d/2cd!$ , we have

$$c\Delta_d \prod_{i=2}^d z_i^{(0)} \geq \frac{d^d}{cd!} z_1 \prod_{i=2}^d z_i^{(0)} = t,$$

which completes the proof of Lemma 7.1. Notice that these arguments show that we can take  $\Delta_2 = 1$ .  $\square$

*Proof of Proposition 3.1.* Recall from Section 3.5 that  $\delta_1 := r(\lambda, d)\delta_0$ , where  $r(\lambda, d) \in [1, 3^{1/d})$  is chosen so that  $\log_3(T/\delta_1^d) \in \mathbb{Z}$ . We show the slightly stronger result that  $C_F(K_\lambda \cap [0, \delta_1]^d) \subset C_0(K)$  holds on  $A_\lambda$ . Assume there is a normal  $u \in C_F(K_\lambda \cap [0, \delta_1]^d)$  with  $u \notin C_0(K)$ . Thus there is some  $j \in \{1, \dots, d-1\}$  such that the first  $j$  coordinates of  $u$  are positive and the last  $(d-j)$  coordinates are negative.

Let  $H_u$  be the support hyperplane containing  $F$  and let  $z \in H_u \cap [0, \delta_1]^d \cap \mathcal{A}(s, T^*, K)$ . The existence of  $z$  is guaranteed on the event  $A_\lambda$ . The definition of

$u$  shows that  $\langle z' - z, u \rangle \leq 0$  must hold for all  $z' \in K_\lambda$ . However this is not case and we assert there is a  $z' := (z'_1, \dots, z'_d) \in \mathcal{P}_\lambda$  such that

$$z'_1 - z_1 > 0, \dots, z'_j - z_j > 0; \quad z'_{j+1} - z_{j+1} < 0, \dots, z'_d - z_d < 0. \quad (7.6)$$

We prove (7.6) as follows.

Since  $z \in [0, \delta_1]^d$ , the point  $z$  must belong to an  $M$ -region  $\Pi_{i=1}^d [3^{k_i} \delta_1/2, 3^{k_i+1} \delta_1/2]$ . We shall show there exist integers  $k'_1, \dots, k'_d$  with  $k'_i \leq \log_3(\delta_1^{-1})$ ,  $1 \leq i \leq d$ , such that

$$k'_i > k_i \text{ for } 1 \leq i \leq j \text{ whereas } k'_i < k_i \text{ for } j+1 \leq i \leq d, \quad (7.7)$$

and  $\sum_{i=1}^d k'_i = \log_3(T/\delta_1^d)$ . The last equality implies that the  $M$ -region

$$M := \prod_{i=1}^d \left[ \frac{3^{k'_i} \delta_1}{2}, \frac{3^{k'_i+1} \delta_1}{2} \right]$$

is an element of  $\mathcal{M}_K(\mathbf{0}, \delta_1)$ . On  $A_\lambda$  we know that  $M$  contains at least one element of  $\mathcal{P}_\lambda \cap \mathcal{A}(s, T^*, K) \cap [0, 1/2]^d$ , say  $z'$ , with coordinates  $z'_i$ ,  $1 \leq i \leq d$ , satisfying (7.6). This shows  $\langle z' - z, u \rangle \geq 0$  as desired. To show (7.7) we consider two cases. Note that  $\sum_{i=1}^d k_i \in [\log_3 s/\delta_1^d, \log_3 T^*/\delta_1^d]$ .

Case (i).  $\sum_{i=1}^d k_i \in [\log_3 s/\delta_1^d, \log_3 T/\delta_1^d]$ . We choose  $k'_i < k_i$  for  $j+1 \leq i \leq d$  such that

$$j \leq \sum_{i=j+1}^d (k_i - k'_i) \leq d.$$

Then for  $1 \leq i \leq j$ , we choose  $k'_i \in [k_i + 1, \log_3 \delta_1^{-1}]$  such that

$$\sum_{i=1}^j (k'_i - k_i) = \left( \log_3 T/\delta_1^d - \sum_{i=1}^d k_i \right) + \sum_{i=j+1}^d (k_i - k'_i).$$

Such  $k'_i$  exist since  $\log_3 T/\delta_1^d - \sum_{i=1}^d k_i$  is bounded by the maximum allowable value in the range of  $k'_i$ , that is to say it is bounded by  $\log_3 T/\delta_1^d - \sum_{i=1}^d k_i = o(\log_3 \delta_1^{-1})$ . Thus (7.7) holds in this situation.

Case (ii).  $\sum_{i=1}^d k_i \in [\log_3 T/\delta_1^d, \log_3 T^*/\delta_1^d]$ . We choose  $k'_i > k_i$  for  $1 \leq i \leq j$  such that

$$(d - j) \leq \sum_{i=1}^j (k'_i - k_i) \leq d.$$

Then for  $j+1 \leq i \leq d$ , we choose  $k'_i \leq k_i - 1$  such that

$$\sum_{i=j+1}^d (k_i - k'_i) = \left( \sum_{i=1}^d k_i - \log_3 T/\delta_1^d \right) + \sum_{i=1}^j (k'_i - k_i).$$

This shows (7.7), completing the proof of Proposition 3.1.  $\square$

*Proof of Lemma 3.1.* We start with a preliminary observation about sums of scalars. Given  $u_1, \dots, u_d \in \mathbb{R}$  and  $\sum_{i=1}^d u_i = k$ , with  $k$  an integer, we assert there exists  $v_1, \dots, v_d$  such that  $u_i \leq v_i < u_i + 1$  for every  $1 \leq i \leq d$  and  $\sum_{i=1}^d \lfloor v_i \rfloor = k$ . We prove this assertion for  $k = 0$  as the proof is similar for any other integer  $k$ . We can see that

$$\sum_{i=1}^d \lfloor u_i \rfloor \geq \sum_{i: u_i \text{ integer}} u_i + \sum_{i: u_i \text{ not integer}} (u_i - 1) = -\#\{i : u_i \text{ not integer}\}.$$

That number is at most equal to  $-d$ . Let us say that this number is equal to  $-k$ . Then it suffices to modify exactly  $k$  of the  $u_i$  which are not integers into  $v_i$  with  $u_i \leq v_i < u_i + 1$  so that the integer part will grow by 1 exactly.

Given this assertion, we now prove Lemma 3.1. Any point  $(z_1, \dots, z_d)$  of  $[0, 1]^d$  is coded by a  $d$ -tuple of integers  $(k_1, \dots, k_d)$  such that  $\frac{1}{2}\delta 3^{k_i} \leq z_i \leq \frac{1}{2}\delta 3^{k_i+1}$ , i.e.

$$k_i = \lfloor \log_3(\delta^{-1} 2z_i) \rfloor. \quad (7.8)$$

The point  $(z_1, \dots, z_d)$  belongs to an  $M$ -region in the collection  $\mathcal{M}_K(\mathbf{0}, \delta)$  iff  $k_1 + \dots + k_d = \log_3(T/\delta^d)$ . Indeed,  $(k_1 + \dots + k_d = \log_3(T/\delta^d))$  means that  $(z_1, \dots, z_d) \in M((3^{k_1}\delta, \dots, 3^{k_d}\delta))$ , which is an  $M$ -region centered at a point on  $K(v = T)$ , where we recall that  $\log_3(T/\delta^d) \in \mathbb{Z}$  by assumption.

Now, let  $(z_1^{(0)}, \dots, z_d^{(0)}) \in K(v = T)$ . Let us prove that  $M((z_1^{(0)}, \dots, z_d^{(0)}))$  intersects an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$ . To do this, we rewrite the equation of  $M((z_1^{(0)}, \dots, z_d^{(0)}))$  in terms of  $(k_1, \dots, k_d)$ -coordinates. We observe

$$\begin{aligned} (z_1, \dots, z_d) &\in M((z_1^{(0)}, \dots, z_d^{(0)})) \\ \iff \frac{z_i^{(0)}}{2} \leq z_i \leq \frac{3}{2}z_i^{(0)} \quad \forall 1 \leq i \leq d \\ \iff \log_3(\delta^{-1} z_i^{(0)}) \leq \log_3(\delta^{-1} 2z_i) < \log_3(\delta^{-1} z_i^{(0)}) + 1 \quad \forall 1 \leq i \leq d. \end{aligned} \quad (7.9)$$

Since  $(z_1^{(0)}, \dots, z_d^{(0)}) \in K(v = T)$ , this implies  $\prod_{i=1}^d z_i^{(0)} = T$ . We appeal to our assertion, setting  $u_i := \log_3(\delta^{-1} z_i^{(0)})$  and  $k := \log_3(T/\delta^d)$ . We choose  $v_1, \dots, v_d$  such that  $\sum_{i=1}^d \lfloor v_i \rfloor = k = \log_3(T/\delta^d)$  and  $u_i \leq v_i < u_i + 1, 1 \leq i \leq d$ . Then we take  $\tilde{z}_i$  such that  $v_i = \log_3(\delta^{-1} 2\tilde{z}_i)$  and we put  $\tilde{k}_i := \lfloor \log_3(\delta^{-1} 2\tilde{z}_i) \rfloor$ . Now  $(\tilde{z}_1, \dots, \tilde{z}_d) \in M((z_1^{(0)}, \dots, z_d^{(0)}))$  because  $(\tilde{z}_1, \dots, \tilde{z}_d)$  satisfies (7.9). On the other hand,  $(\tilde{z}_1, \dots, \tilde{z}_d)$  belongs to an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$  because  $\sum_{i=1}^d \tilde{k}_i = \sum_{i=1}^d \lfloor v_i \rfloor = \log_3(T/\delta^d)$ . Thus  $M((z_1^{(0)}, \dots, z_d^{(0)}))$  intersects an  $M$ -region in  $\mathcal{M}_K(\mathbf{0}, \delta)$ , showing the desired maximality of  $\mathcal{M}_K(\mathbf{0}, \delta)$ .  $\square$



*Proof of Lemma 3.2.* Similar to Section 2 of [4], we use the notation  $C(z^{(0)}) = K \cap H^+(z^{(0)})$  to signify a cap of  $K$  at  $z^{(0)}$ , where  $H^+(z^{(0)})$  is at (7.3). Also, if  $(3^{k_1}\delta, \dots, 3^{k_d}\delta)$  denotes the center of the  $M$ -region  $M_j$ , then for  $\gamma > 0$  we define  $K_j^\gamma := C^{6\gamma}((3^{k_1}\delta, \dots, 3^{k_d}\delta))$ , where  $C^{6\gamma}(z^{(0)})$  is the enlarged cap

$$C^{6\gamma}(z^{(0)}) := \{(z_1, \dots, z_d) : \sum_{i=1}^d \frac{z_i}{z_i^{(0)}} \leq 6d\gamma\}. \quad (7.10)$$

Looking closely at Section 5 of [4], it suffices to show the set inclusions

- (i)  $S'_j \subset K_j^\gamma$  where  $\gamma \in (\frac{1}{6d}(2^{d-1}d6^d + \frac{3}{2}(d-1)), \infty)$  and
- (ii)  $K'_j \subset S_j$ , where  $K'_j := M_K((3^{k_1}\delta, \dots, 3^{k_d}\delta)) \cap C((3^{k_1}\delta, \dots, 3^{k_d}\delta))$ .

When  $S'_j$  is either a cone set or cone-cylinder set, the cardinality of which is bounded independently of  $\lambda$ , these set inclusions are satisfied for large  $\gamma$ . The only challenge is to show these inclusions for the cylinder sets  $S'_j$ .

We start by showing the first inclusion. Let  $z^{(0)} \in K(v = T)$ . The aim is to show that the explicit regions  $S'_j$  defined at (3.10) satisfy the requirement from [4], i.e. that there exists an explicit  $\gamma$  depending only on dimension  $d$  such that  $S'_j \subset K_j^\gamma$  for every  $j$ . Actually, our explicit value of  $\gamma$  will be larger than the one used in [4] (see display before (5.4) therein) but we claim that this does not affect any of the results from [4] and in particular it does not modify the construction of the dependency graph.

We now describe the set  $S'_j$  constructed from the  $M$ -region containing  $z^{(0)}$ . Recalling (3.10), assume  $z_d^{(0)} = \min_{1 \leq i \leq d} z_i^{(0)}$  and that there is no tie for sake of simplicity (the case of a tie would be treated analogously). We have

$$S'_j = K(v \leq T^*) \cap [\frac{1}{2}z_1^{(0)}, \frac{3}{2}z_1^{(0)}] \times \dots \times [\frac{1}{2}z_{d-1}^{(0)}, \frac{3}{2}z_{d-1}^{(0)}] \times \mathbb{R}.$$

Putting  $\tilde{T}^* := d!/d^dT^*$ , the height of  $S'_j$  above the point  $(z_1, \dots, z_{d-1}, 0)$  is

$$z_d = \frac{\tilde{T}^*}{\prod_{i=1}^{d-1} z_i}. \quad (7.11)$$

In particular, (7.10) and (7.11) imply that  $S'_j \subset K_j^\gamma$  as soon as for every  $(z_1, \dots, z_{d-1}) \in [\frac{1}{2}z_1^{(0)}, \frac{3}{2}z_1^{(0)}] \times \dots \times [\frac{1}{2}z_{d-1}^{(0)}, \frac{3}{2}z_{d-1}^{(0)}]$ , we have

$$\frac{\tilde{T}^*}{\prod_{i=1}^{d-1} z_i} \leq z_d^{(0)}(6d\gamma - \sum_{i=1}^{d-1} \frac{z_i}{z_i^{(0)}}). \quad (7.12)$$

Noticing that on the one hand,

$$\frac{\tilde{T}^*}{\prod_{i=1}^{d-1} z_i} \leq \frac{2^{d-1}d6^d d! T}{d^d \prod_{i=1}^{d-1} z_i^{(0)}} = 2^{d-1}d6^d z_d^{(0)}$$

and that on the other hand,

$$z_d^{(0)}(6d\gamma - \sum_{i=1}^{d-1} \frac{z_i}{z_i^{(0)}}) \geq z_d^{(0)}(6d\gamma - \frac{3}{2}(d-1)),$$

we conclude that (7.12) is satisfied as soon as  $\gamma \in [\frac{1}{6d}(2^{d-1}d6^d + \frac{3}{2}(d-1)), \infty)$ .

We show now the second inclusion (ii). In particular, because of its definition,  $S_j$  contains  $M_{K'}((3^{k_1}\delta, \dots, 3^{k_d}\delta)) \cap C((3^{k_1}\delta, \dots, 3^{k_d}\delta)) = K'_j$ . This concludes the proof of Lemma 3.2.  $\square$

*Proof of Lemma 3.7.* We first assert that

$$\text{Vol}(\mathcal{A}(s, T^*, K, \delta_1)) = O(\log \log \lambda (\log \lambda)^{d-2+1/d} \lambda^{-1}). \quad (7.13)$$

Indeed, we notice that  $\mathcal{A}(s, T^*, K, \delta_1) \subset K(v \leq T^*)$ . Next we apply the bound in display (4.1) of [2] with the  $\varepsilon$  and  $\varphi$  of that bound set to  $T^*$  and a constant multiple of  $\delta_1$ , respectively. This is possible because without loss of generality the parallelepiped  $p_d(v_i, \delta_1) := a_i^{-1}([0, \delta_1]^d)$  contains the intersection of  $K$  with a slab of thickness proportional to  $\delta_1$ . Note that  $\varphi^d \geq \text{const} \cdot \varepsilon$ , which gives (7.13).

We prove part (a) of Lemma 3.7 for the  $k$ -face functional  $\xi_k$  and then treat the volume functional  $\xi_V$ . We first show  $\text{Var}[Z_0(\delta_1)\mathbf{1}(A_\lambda)] = o(\text{Var}Z)$ , which goes as follows. Put  $\tilde{\xi}(x, \mathcal{P}_\lambda) := \xi_k(x, \mathcal{P}_\lambda)\mathbf{1}(A_\lambda)$ . Letting  $\mathcal{P}_\lambda^y := \mathcal{P}_\lambda \cup \{y\}$  we have

$$\text{Var} \sum_{x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)} \tilde{\xi}(x, \mathcal{P}_\lambda) = V_1 + V_2, \quad (7.14)$$

where

$$V_1 := \mathbb{E} \sum_{x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)} \tilde{\xi}(x, \mathcal{P}_\lambda)^2$$

and

$$V_2 := \mathbb{E} \sum_{x, y \in \mathcal{P}_\lambda(s, T^*, K, \delta_1); x \neq y} [\tilde{\xi}(x, \mathcal{P}_\lambda^y)\tilde{\xi}(y, \mathcal{P}_\lambda^x) - \mathbb{E}\tilde{\xi}(x, \mathcal{P}_\lambda)\mathbb{E}\tilde{\xi}(y, \mathcal{P}_\lambda)].$$

We bound  $V_1$  as follows. Each  $x \in \mathcal{A}(s, T^*, K, \delta_1)$  belongs to some  $S'_i$  region in the collection  $\{S'_j\}_{j=1}^{m(T, \delta_1)}$ . Let  $\mathcal{S}_x$  denote the union of those  $S'_j$  such that there is an edge between  $i$  and  $j$ . By Theorem 6.2 of [4], we have  $\text{card}(\mathcal{S}_x) \leq D(\lambda)$ , where  $D(\lambda) = O(\log \log \lambda^{6(d-1)})$  is the maximal degree of the dependency graph  $(\mathcal{V}_G, \mathcal{E}_G)$ , where  $\mathcal{V}_G := \mathcal{S}'(\delta_1)$ . On  $A_\lambda$  we have  $\max_{j \leq m(T, \delta_1)} \text{card}(S_j \cap \mathcal{P}_\lambda) \leq c(d) \log \log \lambda$ , as explained two lines after display (5.4) in [4]. It follows that on  $A_\lambda$  at most  $O((\log \log \lambda)^{6(d-1)+1})$  points in  $\mathcal{P}_\lambda$  can potentially contribute to a  $k$ -face containing  $x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)$ . By

McMullen's bound [10], the number of  $k$ -faces on an  $n$  point set is bounded by  $Cn^{d/2}$ . The score at  $x$  thus satisfies

$$\sup_{x,y \in \mathcal{A}(s, T^*, K, \delta_1)} |\tilde{\xi}(x, \mathcal{P}_\lambda^y)| = O((\log \log \lambda)^{(6(d-1)+1)d/2}). \quad (7.15)$$

Combining (7.13), (7.15), and using the Slivnyak - Mecke formula, we find the desired bound for the first term in (7.14):

$$\begin{aligned} V_1 &= \lambda \int_{\mathcal{A}(s, T^*, K, \delta_1)} \mathbb{E} [\tilde{\xi}(x, \mathcal{P}_\lambda)^2] dx \\ &= O(\lambda \text{Vol}(\mathcal{A}(s, T^*, K, \delta_1)) (\log \log \lambda)^{(6(d-1)+1)d}) \\ &= o((\log \lambda)^{d-1}) \\ &= o(\text{Var}[Z]). \end{aligned}$$

Now we bound  $V_2$ . We treat separately the sum over  $x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)$  and  $y \notin S_x$  and the sum over  $x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)$  and  $y \in S_x$ . When  $x \in \mathcal{A}(s, T^*, K)$  and  $y \notin S_x$ , we have

$$\mathbb{E} [\xi_k(x, \mathcal{P}_\lambda^y) \xi_k(y, \mathcal{P}_\lambda^x) | A_\lambda] - \mathbb{E} [\xi_k(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E} [\xi_k(y, \mathcal{P}_\lambda) | A_\lambda] = 0.$$

Consequently,

$$\begin{aligned} &\mathbb{E} [\tilde{\xi}(x, \mathcal{P}_\lambda^y) \tilde{\xi}(y, \mathcal{P}_\lambda^x)] - \mathbb{E} [\tilde{\xi}(x, \mathcal{P}_\lambda)] \mathbb{E} [\tilde{\xi}(y, \mathcal{P}_\lambda)] \\ &= \mathbb{E} [\xi_k(x, \mathcal{P}_\lambda^y) \xi_k(y, \mathcal{P}_\lambda^x) | A_\lambda] P[A_\lambda] - \mathbb{E} [\xi_k(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E} [\xi_k(y, \mathcal{P}_\lambda) | A_\lambda] P[A_\lambda]^2 \\ &= \mathbb{E} [\xi_k(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E} [\xi_k(y, \mathcal{P}_\lambda) | A_\lambda] P[A_\lambda] P[A_\lambda^c]. \end{aligned} \quad (7.16)$$

Combining (3.6), (7.15) and (7.16) and applying the Slivnyak-Mecke formula, we get

$$\mathbb{E} \left[ \sum_{x,y \in \mathcal{P}_\lambda(s, T^*, K, \delta_1); y \notin S_x} [\xi_k(x, \mathcal{P}_\lambda^y) \xi_k(y, \mathcal{P}_\lambda^x) - \mathbb{E} \xi_k(x, \mathcal{P}_\lambda) \mathbb{E} \xi_k(y, \mathcal{P}_\lambda)] \right] = o(\text{Var}Z).$$

Now we prove that

$$\mathbb{E} \sum_{x,y \in \mathcal{P}_\lambda(s, T^*, K, \delta_1); y \in S_x} [\tilde{\xi}(x, \mathcal{P}_\lambda^y) \tilde{\xi}(y, \mathcal{P}_\lambda^x) - \mathbb{E} \tilde{\xi}(x, \mathcal{P}_\lambda) \mathbb{E} \tilde{\xi}(y, \mathcal{P}_\lambda)] = o(\text{Var}Z). \quad (7.17)$$

By (7.15) we also have

$$\sup_{x,y} |\tilde{\xi}(x, \mathcal{P}_\lambda^y) \tilde{\xi}(y, \mathcal{P}_\lambda^x) - \mathbb{E} \tilde{\xi}(x, \mathcal{P}_\lambda) \mathbb{E} \tilde{\xi}(y, \mathcal{P}_\lambda)| = O((\log \log \lambda)^{(6(d-1)+1)d}). \quad (7.18)$$

Moreover, we deduce from (5.4) in [4] that

$$\sup_{x \in \mathcal{A}(s, T^*, K)} \text{Vol}(\mathcal{S}_x) \leq \sup_{x \in \mathcal{A}(s, T^*, K)} \text{card}(\mathcal{S}_x) \cdot \sup_{S'_j \in S'(\delta_1)} \text{Vol}(S'_j) = O\left(\frac{\log \log \lambda^{6(d-1)+1}}{\lambda}\right). \quad (7.19)$$

Consequently, using the Slivnyak - Mecke formula, (7.13), (7.18) and (7.19), we get

$$\begin{aligned} & \mathbb{E} \sum_{x, y \in \mathcal{P}_\lambda(s, T^*, K, \delta_1); y \in \mathcal{S}_x} [\tilde{\xi}(x, \mathcal{P}_\lambda^y) \tilde{\xi}(y, \mathcal{P}_\lambda^x) - \mathbb{E} \tilde{\xi}(x, \mathcal{P}_\lambda) \mathbb{E} \tilde{\xi}(y, \mathcal{P}_\lambda)] \\ &= O(\lambda^2 \text{Vol}(\mathcal{A}(s, T^*, K, \delta_1)) \sup_{x \in \mathcal{A}(s, T^*, K)} \text{Vol}(\mathcal{S}_x) (\log \log \lambda)^{(6(d-1)+1)d}) \\ &= o(\text{Var} Z). \end{aligned}$$

Now we show  $\text{Var}[Z_0(\delta_1)|A_\lambda] = o(\text{Var} Z)$ . Notice that

$$\begin{aligned} \mathbb{E}[Z_0^2(\delta_1) \mathbf{1}(A_\lambda)] &\leq \lambda^2 \int_{\mathcal{A}(s, T^*, K, \delta_1)} \int_{\mathcal{A}(s, T^*, K, \delta_1)} \mathbb{E}[\tilde{\xi}(x, \mathcal{P}_\lambda^y) \tilde{\xi}(y, \mathcal{P}_\lambda^x)] dy dx \\ &= O(\lambda^2 (\text{Vol}(\mathcal{A}(s, T^*, K, \delta_1)))^2 (\log \log \lambda)^{(6(d-1)+1)d}) \\ &= O((\log \lambda)^{2(d-2)+2/d} (\log \log \lambda)^{(6(d-1)+1)d+1}), \end{aligned}$$

where we use (7.13). Thus by (3.6) we get  $\mathbb{E}[Z_0^2(\delta_1) \mathbf{1}(A_\lambda)] P[A_\lambda^c] = o(1)$ . The desired bound  $\text{Var}[Z_0(\delta_1)|A_\lambda] = o(\text{Var}[Z])$  follows from this estimate and the identity  $\text{Var}[Z_0(\delta_1)|A_\lambda] = P[A_\lambda]^{-2} (\text{Var}[Z_0(\delta_1) \mathbf{1}(A_\lambda)] - \mathbb{E}[Z_0^2(\delta_1) \mathbf{1}(A_\lambda)] (1 - P[A_\lambda]))$ .

Now we show  $\text{Var}[Z_0(\delta_1) \mathbf{1}(A_\lambda)] = o(\text{Var}[Z])$  when  $\xi$  is the volume score. Recall from (2.3) that  $\mathcal{F}_{d-1}(x)$  is the collection of facets in  $K_\lambda$  which contain  $x$ . Regardless of whether we use (2.2) and (2.3), we have on  $A_\lambda$  that

$$\begin{aligned} \xi_V(x, \mathcal{P}_\lambda) &\leq \lambda \text{card}(\mathcal{F}_{d-1}(x)) \\ &\quad \times \sup\{\text{Vol}(\text{Cap of } K): \text{Cap of } K \text{ tangent to } K(v = s') \text{ with } s \leq s' \leq T^*\}. \end{aligned}$$

Indeed, for any face  $F$  and facet  $F'$  containing  $F$ , let  $H$  be the hyperplane containing  $F'$ . Then  $C_F(K_\lambda) \cap K$  is included in the cap of  $K$  bounded by  $H$ . Since  $H$  meets  $\mathcal{A}(s, T^*, K)$  but not  $K(v \geq T^*)$ , it is tangent to some  $K(v = s')$  with  $s \leq s' \leq T^*$ . Consequently, Lemma 2.4 in [4] yields

$$\sup\{\text{Vol}(\text{Cap of } K): \text{Cap of } K \text{ tangent to } K(v = s') \text{ with } s \leq s' \leq T^*\} = O\left(\frac{\log \log \lambda}{\lambda}\right). \quad (7.20)$$

Moreover, (7.15) implies that  $\text{card}(\mathcal{F}_{d-1}(x)) = O((\log \log \lambda)^{(6(d-1)+1)d/2})$ . Thus on  $A_\lambda$  we have

$$\sup_{x \in Q_0^c \cap \mathcal{A}(s, T^*, K)} |\xi_V(x, \mathcal{P}_\lambda)| = O\left((\log \log \lambda)^{(6(d-1)+1)\frac{d}{2}+1}\right).$$

Using this bound in place of the bound (7.15) and following the method for the  $k$ -face functional verbatim, we obtain  $\text{Var}[Z_0(\delta_1)\mathbf{1}(A_\lambda)] = o((\log \lambda)^{(d-1)}) = o(\text{Var}[Z])$ . Following the discussion for the  $k$ -face functional, we also have  $\mathbb{E}[Z_0^2(\delta_1)\mathbf{1}(A_\lambda)] = o(1)$ . This gives  $\text{Var}[Z_0(\delta_1)|A_\lambda] = o(\text{Var}[Z])$  as explained for the  $k$ -face score.

To show part (b) of Lemma 3.7, we recall Figure 2 and notice that

$$\begin{aligned} 0 &\leq \text{Vol}(K \setminus K_\lambda) - \frac{1}{\lambda} \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda) \\ &\leq \text{card}(\{\text{facets intersecting one of the } p_d(\mathcal{V}_i, \delta_0) \text{ and its complement}\}) \\ &\quad \times \sup\{\text{Vol}(\text{Cap of } K): \text{Cap of } K \text{ tangent to } K(v = s') \text{ with } s \leq s' \leq T^*\} \\ &\leq c \sum_{x \in \mathcal{P}_\lambda(s, T^*, K, \delta_1)} \frac{\log \log \lambda}{\lambda}, \end{aligned}$$

where the last inequality uses the estimate (7.20). We now apply the exact same method as for the proof of part (a) in order to bound the variance of the difference  $\text{Vol}(K \setminus K_\lambda) - \frac{1}{\lambda} \sum_{x \in \mathcal{P}_\lambda} \xi_V(x, \mathcal{P}_\lambda)$  and show the required statement (b). This concludes the proof of Lemma 3.7.  $\square$

*Proof of Lemma 3.8.* Recall the definition of  $Z$  at (3.13). For  $1 \leq i \leq |\mathcal{V}_K|$  the assertions

$$\max\{|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]|, |\mathbb{E}[Z_i] - \mathbb{E}[Z_i|A_\lambda]|\} = o(\mathbb{E}[Z])$$

and

$$\max\{|\text{Var}[Z] - \text{Var}[Z|A_\lambda]|, |\text{Var}[Z_i] - \text{Var}[Z_i|A_\lambda]|\} = o(\text{Var}[Z])$$

follow from Lemmas 8.2 and 8.3 of [4]. We now show

$$|\text{Var}[Z] - \text{Var}[Z\mathbf{1}(A_\lambda)]| = o(\text{Var}[Z]); \quad |\text{Var}[Z_i] - \text{Var}[Z_i\mathbf{1}(A_\lambda)]| = o(\text{Var}[Z]). \quad (7.21)$$

We only prove the first assertion, as the second follows from identical methods. We prove the first assertion when  $Z$  is the number of  $k$ -dimensional faces of  $K_\lambda$  and then treat the case when  $Z$  is the defect volume of  $K_\lambda$ . We have

$$\begin{aligned} \text{Var}[Z] &= \text{Var}[Z\mathbf{1}(A_\lambda) + Z\mathbf{1}(A_\lambda^c)] \\ &= \text{Var}[Z\mathbf{1}(A_\lambda)] + \text{Var}[Z\mathbf{1}(A_\lambda^c)] + 2\text{Cov}(Z\mathbf{1}(A_\lambda), Z\mathbf{1}(A_\lambda^c)). \end{aligned}$$

Consequently,

$$|\text{Var}[Z] - \text{Var}[Z\mathbf{1}(A_\lambda)]| \leq \mathbb{E}[Z^2\mathbf{1}(A_\lambda^c)] + 2\sqrt{\text{Var}[Z\mathbf{1}(A_\lambda)]}\sqrt{\mathbb{E}[Z^2\mathbf{1}(A_\lambda^c)]}. \quad (7.22)$$

We first estimate  $\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c)]$  as follows. Using the event  $B_\lambda$  provided by [4] (and denoted by  $B$  there, see p. 1519 of [4]), we write

$$\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c)] = \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda)] + \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c)]. \quad (7.23)$$

We treat separately each term on the right hand side of (7.23). Let us start with the first one: On the event  $B_\lambda$ , we know that only the points inside  $K(v \leq d6^d \lambda^{-1} \log \lambda)$  are needed to construct  $K_\lambda$  and that their cardinality is  $O((\log \lambda)^d)$ . Consequently, by McMullen's bound, we have  $Z = O((\log \lambda)^{d^2/2})$ . It follows from (3.6) that

$$\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda)] = O((\log \lambda)^{d^2} P[A_\lambda^c]) = O((\log \lambda)^{-3d^2}). \quad (7.24)$$

To estimate the second term of (7.23), we proceed as in the proof of Lemma 8.2 in [4]:

$$\begin{aligned} \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c)] &= \sum_{m=0}^{\infty} \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c) \mid \text{card}(\mathcal{P}_\lambda) = m] P[\text{card}(\mathcal{P}_\lambda) = m] \\ &= \sum_{m=0}^{\lfloor 3\text{Vol}(K)\lambda \rfloor} \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c) \mid \text{card}(\mathcal{P}_\lambda) = m] P[\text{card}(\mathcal{P}_\lambda) = m] \\ &\quad + \sum_{m=\lfloor 3\text{Vol}(K)\lambda \rfloor + 1}^{\infty} \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c) \mid \text{card}(\mathcal{P}_\lambda) = m] P[\text{card}(\mathcal{P}_\lambda) = m]. \end{aligned} \quad (7.25)$$

When  $\text{card}(\mathcal{P}_\lambda) = m$ , we have  $Z = O(m^{d/2})$ . In particular, when  $m \in \{0, 1, \dots, \lfloor 3\text{Vol}(K)\lambda \rfloor\}$ , we bound  $Z^2$  by  $O(\lambda^d)$ . Consequently, the identity (7.25) gives

$$\begin{aligned} \mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c)] &= O\left(\lambda^d P[A_\lambda^c \cap B_\lambda^c] + \sum_{m=\lfloor 3\text{Vol}(K)\lambda \rfloor + 1}^{\infty} m^d P[\text{card}(\mathcal{P}_\lambda) = m]\right) \\ &= O(\lambda^d P[B_\lambda^c | A_\lambda^c] P[A_\lambda^c] + \mathbb{E}[(\text{card}(\mathcal{P}_\lambda))^d \mathbf{1}(\text{card}(\mathcal{P}_\lambda) \geq 3\text{Vol}(K)\lambda)]). \end{aligned}$$

In view of the bound (8.2) in [4], the first term is  $O(\lambda^{-2d+1}(\log \lambda)^{-4d^2})$  whereas the second one is decreasing exponentially fast in  $\lambda$ . Consequently, we have

$$\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda^c)] = o(\lambda^{-2d+1}). \quad (7.26)$$

Inserting (7.24) and (7.26) into (7.23), we get  $\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c)] = o((\log \lambda)^{-3d^2})$ . This fact combined with (7.22) and Theorem 1.3 from [5] implies (7.21).

We now adapt the above proof when  $Z$  is the defect volume. Regarding the first term of (7.23), Theorem 2.7 in [4] implies that on  $B_\lambda$ , we have  $Z \leq \text{Vol}(K(v \leq d6^d \log \lambda / \lambda)) = O((\log \lambda / \lambda)(\log \lambda)^{d-1})$ . Consequently, (7.24) is replaced by

$$\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c \cap B_\lambda)] = O\left(\frac{(\log \lambda)^{2d}}{\lambda^2} P[A_\lambda^c]\right) = O\left(\frac{(\log \lambda)^{-4d^2+2d}}{\lambda^2}\right). \quad (7.27)$$

For the second term of (7.23), we simply bound  $Z^2$  by a constant and we get from (7.25) that (7.26) also holds for the defect volume. Now, inserting (7.27) and (7.26) into (7.23), we find  $\mathbb{E}[Z^2 \mathbf{1}(A_\lambda^c)] = o((\log \lambda)^{-4d^2+2d} \lambda^{-2})$  and so (7.21) also holds when  $Z$  is the defect volume. This completes the proof of Lemma 3.8.  $\square$

*Acknowledgements.* P. Calka is grateful to the Department of Mathematics at Lehigh University for its support and great help during his several stays. J. Yukich likewise thanks the Department of Mathematics at the Université de Rouen for its kind hospitality and support.

## References

- [1] I. Bárány (1989), Intrinsic volumes and  $f$ -vectors of random polytopes, *Math. Ann.*, **285**, 671-699.
- [2] I. Bárány and C. Buchta (1993), Random polytopes in a convex polytope, independence of shape, and concentration of vertices, *Math. Ann.*, **297**, 467-497.
- [3] I. Bárány and D. G. Larman (1989), Convex bodies, economic cap coverings, random polytopes, *Mathematika*, **35**, 274-291.
- [4] I. Bárány and M. Reitzner (2010), Poisson polytopes, *Ann. Probab.*, **38**, 1507-1531.
- [5] I. Bárány and M. Reitzner (2010), The variance of random polytopes, *Advances in Mathematics*, **225**, 1986-2001.
- [6] Yu. Baryshnikov (2000), Supporting-points processes and some of their applications, *Probab. Theory Relat. Fields*, **117**, 163-182.
- [7] P. Calka and J. E. Yukich (2014), Variance asymptotics for random polytopes in smooth convex bodies, *Prob. Theory and Related Fields*, **158**, Issue 1, 435-463.
- [8] P. Calka and J. E. Yukich (2015), Variance asymptotics and scaling limits for Gaussian polytopes, *Prob. Theory and Related Fields*, **163**, Issue 1, 259-301.
- [9] P. Calka, T. Schreiber, and J. E. Yukich (2013), Brownian limits, local limits, and variance asymptotics for convex hulls in the ball, *Ann. Probab.*, **41**, 50-108.
- [10] P. McMullen (1970), The maximum number of faces of a convex polytope, *Mathematika*, **17**, 179-184.

- [11] M. Reitzner (2005), Central limit theorems for random polytopes, *Prob. Theory Related Fields*, **133**, 488-507.
- [12] M. Reitzner (2005), The combinatorial structure of random polytopes, *Adv. Mathematics*, **191**, 178-208.
- [13] M. Reitzner (2010), Random Polytopes, *New Perspectives in Stochastic Geometry*, Oxford University Press, 45-77.
- [14] A. Rényi and R. Sulanke (1963), Über die konvexe Hülle von  $n$  zufällig gewählten Punkten. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **2**, 75-84.
- [15] A. Rényi and R. Sulanke (1964), Über die konvexe Hülle von  $n$  zufällig gewählten Punkten II. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **3**, 138-147.
- [16] R. Schneider and W. Weil (2008), *Stochastic and Integral Geometry*, Springer.
- [17] T. Schreiber and J. E. Yukich (2008), Variance asymptotics and central limit theorems for generalized growth processes with applications to convex hulls and maximal points, *Ann. Probab.*, **36**, 363-396.
- [18] W. Weil and J. A. Wieacker (1993), Stochastic geometry, in *Handbook of Convex Geometry* (P. M. Gruber and J. M. Wills, eds.), vol. B, 1391-1438, North-Holland/Elsevier, Amsterdam.

Pierre Calka, Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, Avenue de l'Université, BP.12, Technopôle du Madrillet, F76801 Saint-Etienne-du-Rouvray France; pierre.calka@univ-rouen.fr

J. E. Yukich, Department of Mathematics, Lehigh University, Bethlehem PA 18015; joseph.yukich@lehigh.edu