ON SARNAK'S CONJECTURE AND VEECH'S QUESTION FOR INTERVAL EXCHANGES

Sébastien Ferenczi, Christian Mauduit

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ABSTRACT. Using a criterion due to Bourgain [10] and the generalization of the self-dual induction defined in [18], for each primitive permutation we build a large family of k-interval exchanges satisfying Sarnak's conjecture, and, for at least one permutation in each Rauzy class, smaller families for which we have weak mixing, which implies a prime number theorem, and simplicity in the sense of Veech.

1. INTRODUCTION

Interval exchanges were originally introduced by Oseledec [44], following an idea of Arnold [3] (see also Katok and Stepin [35]). An exchange of k intervals, denoted throughout this paper by $\mathcal{I}$, is given by a probability vector of k lengths together with a permutation π on k letters. The unit interval is partitioned into k subintervals of lengths $\alpha_1, \ldots, \alpha_k$ which are rearranged by $\mathcal{I}$ according to π.

The history of interval exchanges is made of big questions. The question whether almost every k-interval exchange is (measure-theoretically) weakly mixing, for every given π not satisfying $i \equiv \pi i + j \bmod k$ for some j and all $1 \leq i \leq k$, was open for twenty years before being solved by Avila and Forni [5]. Veech asked whether almost every interval exchange is simple [50], which is again a measure-theoretic property, and this has now been open for more than thirty years. Recently Sarnak [46] stated a conjecture on the orthogonality of the Möbius function with any sequence produced by a dynamical system of zero topological entropy. This class includes interval exchanges, or rather any topological model which makes the transformation continuous: we shall use the associated symbolic systems through the natural coding.

The aim of this note is to build explicitly (i.e. with an explicit algorithm) examples of interval exchanges satisfying these properties. Even for the weak mixing, explicit constructions (as opposed to existence theorems) are scarce: the only weakly mixing k-interval exchanges we have been able to find in the literature are for the symmetric permutation $\pi i = k + 1 - i$ and $k = 3$ [35][22], $k = 4$ [25][47], and $k = 5, 6$ [47], while Theorem 13 of [16] gives a construction for every value of k; for permutations outside the hyperelliptic Rauzy class, the only examples are in Theorem 4.3 of [18] (for all even $k \geq 6$).

The situation is much worse for the very difficult question of simplicity: after a pioneering, but atypical, family of examples by del Junco [32] for $k = 3$, with the stronger property of minimal self-joinings, the only examples which are simple and rigid (rigidity is a typical property for interval exchanges, and excludes minimal self-joinings) appear for $k = 3$ [23] and $k = 4$ [25]. For

Date: March 5, 2015.
2010 *Mathematics Subject Classification.* Primary 37B10, 11N37.

larger values of k not even an existence theorem appears in the literature.

As for Sarnak's conjecture, it came later. For exchanges of two intervals, which are just rotations, it follows from the Prime Number Theorem [30] [49] when the rotation is rational and from a result of Davenport [12] (using a result of Vinogradov [53]) when the rotation is irrational. In the last five years, many other examples of sequences or dynamical systems verifying Sarnak's conjecture have been given: the Thue-Morse sequence [34, 31][1][42], translations on a compact nilmanifold [29], the horocycle flow [11], several classes of distal homogeneous flows [39], Rudin-Shapiro sequences [43], some classes of extensions of rotations on the torus [38].

For exchanges of more than two intervals, Bourgain remarks in [10] that, for uniquely ergodic systems, minimal self-joinings for the invariant measure imply Sarnak's conjecture through a property of spectral disjointness, hence Del Junco's examples do satisfy it. Then Bourgain proves that it is satisfied also by a large class of 3-interval exchanges, including all the examples of [22][23]. Bourgain's results use the *self-dual induction* defined for 3-interval exchanges in [20][21][22]. The examples in [25][16][18] are built by this technique, which was generalized beyond $k = 3$ in [24][18], though it cannot be called self-dual outside the hyperellliptic Rauzy class (the authors of [13] call it the Ferenczi-Zamboni induction). Thus it is not surprising, and easy to prove, that all those examples satisfy the conditions which imply Theorem 3 of [10], and thus Sarnak's conjecture, with a prime numbers theorem whenever they are uniquely ergodic and weakly mixing.

In the present paper we show that a criterion deduced from [10] applies to a large family of constructions of k-interval exchanges built by the self-dual induction and its generalization, for all k, for at least one permutation (called *standard*) in every Rauzy class. Then we build two different families of examples which are uniquely ergodic and satisfy Sarnak's conjecture, using techniques of finite rank in one case, rank one in the other. The three families of examples mentioned so far can be lifted to every primitive permutation by an inverse of the Rauzy induction. Then, for standard permutations, we build smaller subfamilies with weak mixing and thus a prime number theorem by Bourgain's criterion. In the case of rank one, we get also simplicity and rigidity, by isomorphism with a variant of del Junco-Rudolph's map [33] as in [23][25]. Thus not only we build the first explicit weakly mixing examples in every Rauzy class, but also we prove that there are interval exchanges answering positively to Veech's question for more than four intervals, and outside the hyperelliptic Rauzy class. The existence of such examples and *a fortiori* explicit constructions were not known previously.

Acknowledgement: Most of this research was carried out while the first author was in Unité Mixte IMPA-CNRS in Rio de Janeiro and the second author was a temporary visitor of IMPA. The first author was also partially supported by the ANR GeoDyn and the ANR DYna3S, and the second author by the ANR MUNUM, Réseau Franco-Brésilien en Mathématiques (Proc. CNPq 60-0014/01-5 and 69-0140/03-7), Math Am Sud program DYSTIL, BREUDS (IRSES GA318999) and Ciência sem Fronteiras (PVE 407308/2013-0).

2. DEFINITIONS AND NOTATIONS

2.1. **Interval exchanges.** Throughout the paper, *intervals* are semi-open as $[a, b[$. For any question about interval exchanges, we refer the reader to the surveys [52][54][17].

Definition 2.1. *A* k-interval exchange $\mathcal{I}$ *with probability vector* $(\alpha_1, \alpha_2, \dots, \alpha_k)$*, and permutation* π *is defined by*

$$\mathcal{I}x = x + \sum_{\pi^{-1}(j)<\pi^{-1}(i)} \alpha_j - \sum_{j<i} \alpha_j.$$

when x *is in the interval*

$$\Delta_i = \left[\sum_{j<i} \alpha_j, \sum_{j\leq i} \alpha_j \right[.$$

We denote by β_i*,* $1 \leq i \leq k-1$*, the* i*-th discontinuity of* $\mathcal{I}^{-1}$*, namely* $\beta_i = \sum_{\pi^{-1}(j)\leq\pi^{-1}(i)} \alpha_j$*, while* γ_i *is the* i*-th discontinuity of* $\mathcal{I}$*, namely* $\gamma_i = \sum_{j\leq i} \alpha_j$*, we define also* $\gamma_0 = 0$*,* $\gamma_k = 1$*. Then* Δ_i *is the interval* $[\gamma_{i-1}, \gamma_i[$ *if* $1 \leq i \leq k$*.*

Warning: roughly half the texts on interval exchanges re-order the subintervals by π^{-1}; the present definition corresponds to the following ordering of the $\mathcal{I}\Delta_i$: from left to right, $\mathcal{I}\Delta_{\pi(1)}, \dots\mathcal{I}\Delta_{\pi(k)}$.

Definition 2.2. *A* k*-interval exchange* $\mathcal{I}$ *has* alternate discontinuities *if* $\beta_1 < \gamma_1 < \dots, \beta_{k-1} < \gamma_{k-1}$.

Definition 2.3. *A* k*-interval exchange* $\mathcal{I}$ *satisfies the* infinite distinct orbit condition *or* i.d.o.c. *of Keane* [36] *if the* $k-1$ *negative orbits* $\{\mathcal{I}^{-n}\gamma_i, n \geq 0\}$ *,*$1 \leq i \leq k-1$*, of the discontinuities of* $\mathcal{I}$ *are infinite disjoint sets.*

As is proved in [36], the i.d.o.c. condition implies that $\mathcal{I}$ has no periodic orbit and is *minimal*: every orbit is dense. The permutation π is *primitive* (or *irreducible*) if $\pi(\{1, \dots j\}) \neq \{1, \dots j\}$ for every $1 \leq j \leq k-1$; in this case, the i.d.o.c. condition is (strictly) weaker than the *total irrationality*, where the only rational relation satisfied by $\alpha_i, 1 \leq i \leq k$, is $\sum_{i=1}^k \alpha_i = 1$ (see [52]).

Definition 2.4. *For every point* x *in* $[0, 1[$*, its* trajectory *is the infinite sequence* $(x_n)_{n\in\mathbb{N}}$ *defined by* $x_n = i$ *if* $\mathcal{I}^n x$ *falls into* Δ_i*,* $1 \leq i \leq k$*.*

Definition 2.5. *The* induced map *of a map* T *on a set* Y *is the map* $y \to T^{r(y)}y$ *where, for* $y \in Y$*,* $r(y)$ *is the smallest* $r \geq 1$ *such that* $T^r y$ *is in* Y *(in all cases considered in this paper,* $r(y)$ *is finite).*

Definition 2.6. *The* Rauzy induction [45] *associates to a* k *interval exchange* $\mathcal{I}$*, with probability vector* α *and primitive permutation* π*, its induced map on* $[0, \beta_{k-1} \vee \gamma_{k-1}[$*, which is a* k*-interval exchange* $\mathcal{I}'$*, with probability vector* α' *and primitive permutation* π'*. The set of possible* π' *which can be reached from* π *by iterations of the Rauzy induction is the* Rauzy class *of* π*.*

The link between Rauzy classes and connected components of strata in the moduli space of abelian differentials is described in [37][55]. The *hyperelliptic* Rauzy class is the class which contains the *symmetric* permutation $\pi i = k + 1 - i$, $1 \leq i \leq k$. The Rauzy induction will be used in Section 6. Propositions 2.1 and 2.2 below sum up what we need to know about it, and can be found either in the original text [45] or in the surveys [52] [54].

Proposition 2.1. *(Rauzy* [45]*) Each Rauzy class contains a* standard *permutation, that is a permutation such that* $\pi 1 = k$ *and* $\pi k = 1$*.*
If π *and* π' *are two primitive permutations in the same Rauzy class, for any i.d.o.c. interval exchange* $\mathcal{I}$ *with permutation* π*, there exists an i.d.o.c. interval exchange* $\mathcal{I}'$ *with permutation* π' *and an integer* p *smaller than the cardinality of the Rauzy class such that* $\mathcal{I}$ *is reached from* $\mathcal{I}'$ *by applying the Rauzy induction* p *times.*

2.2. **Symbolic dynamics.** We look at finite *words* on a finite alphabet $\mathcal{A} = \{1, \dots k\}$. A word $w_1 \cdots w_t$ has *length* $|w| = t$ (not to be confused with the length of a corresponding interval). The *empty word* is the unique word of length 0. The *concatenation* of two words w and w' is denoted by ww'. A word $w = w_1 \cdots w_t$ *appears at place* i in a word $v = v_1 \cdots v_s$ or an infinite sequence $v = v_1 v_2 \cdots$ if $w_1 = v_i \dots, w_t = v_{i+t-1}$, we say that w is a *factor* of v.

Definition 2.7. *The* symbolic dynamical system *defined by an interval exchange $\mathcal{I}$ is the one-sided shift $S(x_0x_1x_2\cdots) = x_1x_2\cdots$ on the subset X_L of $\mathcal{A}^{\mathbb{N}}$ made with the infinite sequences such that for every $t < s$, $x_t \cdots x_s$ is a factor of a trajectory of $\mathcal{I}$.*

Definition 2.8. *A topological dynamical system is* uniquely ergodic *if it admits only one invariant probability measure.*

The Rauzy induction has a nice symbolic translation:

Proposition 2.2. *(Rauzy* [45]*) If $\mathcal{J}$ is the Rauzy induced map of $\mathcal{I}$, each trajectory of $\mathcal{I}$ is the image of a trajectory of $\mathcal{J}$ by an application of the form $x_0x_1x_2 \dots \to \phi(x_0)\phi(x_1)\phi(x_2)\dots$ where, for $1 \le i \le k$, $\phi(i)$ is a word of length 1 or 2.*

2.3. **Veech's question.**

Definition 2.9. (X, T, μ) *is* ergodic *if the only invariant functions for the operator $f \circ T$ in $\mathcal{L}^2(X, \mathbb{R}/\mathbf{Z})$ are the constants.* (X, T, μ) *is* weakly mixing *if it is ergodic and that operator has no nonzero eigenvalue (denoted additively, $f \circ T = f + \zeta$).*

Definition 2.10. *A* self-joining *(of order two) of a system (X, T, μ) is any measure ν on $X \times X$, invariant under $T \times T$, for which both marginals are μ. An ergodic system (X, T, μ) is* simple *(of order two) if any ergodic self-joining of order two ν is either the product measure $\mu \times \mu$ or a measure defined by $\nu(A \times B) = \mu(A \cap U^{-1}B)$ for some measurable transformation U commuting with T.*

Definition 2.11. (X, T, μ) *is* rigid *if there exists a sequence $s_n \to \infty$ such that for any measurable set A $\mu(T^{s_n} A \Delta A) \to 0$.*

In 1982, Veech asked the following question (4.9 of [50]):

Question (Veech): Are almost all interval exchange transformations simple?

Here "almost all" means for Lebesgue-almost every vector α, for every given permutation π. Veech remarked also that almost all interval exchanges are rigid, thus simple non-rigid examples, as those of del Junco mentioned in the introduction, are not relevant to his question.

2.4. **Rokhlin towers.**

Definition 2.12. *In (X, T), a Rokhlin* tower *is a collection of disjoint measurable sets called* levels *F, TF, ..., $T^{h-1}F$. If X is equipped with a partition P such that each level T^rF is contained in one atom $P_{w(r)}$, the* name *of the tower is the word $w(0)\dots w(h-1)$.*

Definition 2.13. *A system (X, T, μ) is of* rank one *if there exists a sequence of Rokhlin towers such that the whole σ-algebra is generated by the partitions $\{F_n, TF_n, \dots, T^{h_n-1}F_n, X \setminus . \cup_{j=0}^{h_n-1} T^j F_n\}$.*

When we have the same result but with R sequences of Rokhlin towers, the system is of *finite rank*; the rank is R if the above is true for R sequences but not for $R-1$ sequences, see [14] for more details.

For topological systems, there is no canonical notion of rank, but the useful notion is that of adic presentation [51], which we translate here from the original vocabulary into the one of Rokhlin towers.

Definition 2.14. *An* adic presentation *of a topological system (X, T) is given by, for each $n \geq 0$, a finite collection $\mathcal{Z}_n$ of Rokhlin towers, such that*

- *the levels of the towers in $\mathcal{Z}_n$ partition X,*
- *each level of a tower in $\mathcal{Z}_n$ is a union of levels of towers in $\mathcal{Z}_{n+1}$,*
- *the levels of the towers in $\cup_{n\geq 0}\mathcal{Z}_n$ form a basis of the topology of X.*

In that case, the towers of $\mathcal{Z}_{n+1}$ are built from the towers of $\mathcal{Z}_n$ by *cutting and stacking*, following recursion rules: a given tower in $\mathcal{Z}_{n+1}$ can be build by taking columns of successive towers in $\mathcal{Z}_n$ and stacking them successively one above another. Thus, for a partition P whose atoms are unions of levels of the towers in $\mathcal{Z}_0$, the names of the towers in $\mathcal{Z}_{n+1}$ are concatenations of names of towers in $\mathcal{Z}_n$, of the form $Z = Z_1 \cdots Z_q$; we always choose to write the unique decomposition corresponding to the construction of the towers. If the $\#\mathcal{Z}_n$ are bounded, then the system (equipped with an invariant measure) is of finite rank, with the additional property that, because of the first condition from Definition 2.14, the towers fill all the space (and not only up to a measure zero set).

2.5. **The self-dual induction and its generalization.** Ferenczi and Zamboni defined in [24] the self-dual induction for the symmetric permutation $\pi i = k+1-i$, and a geometric interpretation was provided in [13]. It was generalized by Ferenczi to all permutations in [18], with geometric interpretation in [19]. Here we sum up the results we shall use; they do not require the knowledge of the whole theory as the examples we shall build are particular cases where the method of [13] applies.

Starting from an interval exchange $\mathcal{I}$, satisfying the i.d.o.c. condition and with alternate discontinuities, the induction builds families of disjoint intervals $E_{i,n} = [\beta_i - l_{i,n}, \beta_i + r_{i,n}[$, $1 \leq i \leq k-1$, such that $E_{i,n}$ contains the discontinuity β_i, and converges to it when n tends to infinity. At a given stage, the induced map S_n of $\mathcal{I}$ is described by either a combinatorial [24][18] or a geometric [13][19] object, or else by *train-track equalities* [18][19] between the parameters $l_{i,n}$ and $r_{i,n}$. These objects constitute the states of the induction; they allow us to know, for all i in a nonempty subset H_n of $\{1, \dots k-1\}$ (determined by the description of the map S_n), the point $\gamma(E_{i,n})$ which is defined to be the first (in the increasing order of m) point $\mathcal{I}^{-m}\gamma_i$ which falls in the interior of $E_{i,n}$; the i.d.o.c. condition implies that $\gamma(E_{i,n}) \neq \beta_i$. Then for all i in H_n we define a *choice* $c_n(i)$ to be $+$ if $\gamma(E_{i,n})$ is left of β_i, $-$ if $\gamma(E_{i,n})$ is right of β_i; $c_n(i)$ can be computed from the $l_{i,n}$ and $r_{i,n}$. For any nonempty subset F_n of H_n, the *induction with decision* F_n creates the new subintervals $E_{i,n+1}$, where

- if i is not in F_n, $E_{i,n+1} = E_{i,n}$,
- if i is in F_n and $c_n(i) = +$, $E_{i,n+1} = E_{i,n} \cap [\gamma(E_{i,n}), 1[$,
- if i is in F_n and $c_n(i) = -$, $E_{i,n+1} = E_{i,n} \cap [0, \gamma(E_{i,n})[$.

These in turn define a new set H_{n+1} (which is not related in a simple way with H_n) and a description of the new S_{n+1}, which allow to iterate the process.

The result which allows us to use this induction to build examples is Theorem 2.28 of [18]: any sequence of choices c_n and decisions F_n satisfying compatibility conditions (c_n must be compatible with the description of S_n determined by the previous choices, the new subintervals created by c_n must have positive length for at least one set of parameters $l_{i,n}$ and $r_{i,n}$ satisfying the train-track equalities at stage n, F_n is included in the H_n determined by the previous choices, $c_{n+1} = c_n$ outside of F_n) and such that for any i in $\{1, \dots k-1\}$, i is in F_n with $c_n(i) = +$ for infinitely many n, i is in F_n with $c_n(i) = -$ for infinitely many n, defines at least one interval exchange which generates it through the induction. This interval exchange satisfies the i.d.o.c. condition. Then it can be studied by Proposition 2.24 of [18], which states that we can build sets $\mathcal{Z}_n$ of $2k-2$ Rokhlin towers, which form an adic presentation of the system in the sense of Definition 2.14 above, and whose names (for the partition $\{\Delta_1, \dots, \Delta_k\}$) have concatenation rules expressed in Proposition 2.21 of [18]; the Rokhlin towers are, up to some technical modification, those created naturally by the induced map S_n.

Here we are interested in some particular cases, namely when the train-track equalities are of the form $l_i + r_i = r_{m(i)} + l_{p(i)}$, $1 \le i \le k-1$, for two bijective maps m and p; in that case we say the induction is in a *binary state*. In a binary state, the set H is $\{1, \dots, k-1\}$, the point $\gamma(E_i)$ used above is the point $\beta_i - l_i + r_{m(i)} = \beta_i + r_i - l_{p(i)}$; the families of names of towers can be denoted by $M_{i,m(i)}$ and $P_{i,p(i)}$, $1 \le i \le k-1$.

Indeed, we want that the induction always stays in binary states; in [24] and [13] it is shown that we can choose decisions to ensure this property, (always under the condition of alternate discontinuities, ultimately in general) for any interval exchange such that the permutation π is in the hyperelliptic Rauzy class, but that it is not true in general outside that class. However, if π is standard, with the condition of alternate discontinuities, the initial state is binary. Then a sufficient (and indeed necessary) condition for the induction with choice c and decision F, starting from a binary state, to lead to a new binary state, is that the induction is made on disjoint unions of *circuits* (equivalently, of *staircases* in the terminology of [13]):

Definition 2.15. *An* induction on the M circuit containing i *is defined by any choice such that* $c(i) = c(m(i)) = \dots = c(m^{s-1}(i)) = -$ *and by the decision* F *made with* $i, m(i), \dots, m^{s-1}(i)$ *for a minimal* s *such that* $m^s(i) = i$.
An induction on the P circuit containing i *is defined by any choice such that* $c(i) = c(p(i)) = \dots = c(p^{s-1}(i)) = +$ *and the decision* F *made with* $i, p(i), \dots, p^{s-1}(i)$*, for a minimal* s *such that* $p^s(i) = i$.
An induction on a disjoint union of circuits is made by successive inductions on each circuit, in any order.

It is a direct consequence of Proposition 2.17 in [18], which uses the cumbersome machinery of the general theory, but it could also be proved by the simpler techniques of Proposition 2.4 of [24], that, when we start from a binary state G, with names M and P, one induction on a circuit creates nonempty subintervals, and leads to another binary state, with names M' and P' built by Proposition 2.21 of [18], which in that case have a simpler form, similar to the names built by Theorem 2.8 of [24], namely

- if i is not in F, $M'_{m^{-1}(i),i} = M_{m^{-1}(i),i}$, $P'_{p^{-1}(i),i} = P_{p^{-1}(i),i}$,

- if i is in F_n and $c_n(i) = +$, $M'_{m^{-1}(i),p(i)} = M_{m^{-1}(i),i}P_{i,p(i)}$, $P'_{p^{-1}(i),i} = P_{p^{-1}(i),i}$,
- if i is in F_n and $c_n(i) = -$, $M'_{m^{-1}(i),i} = M_{m^{-1}(i),i}$, $P'_{p^{-1}(i),m(i)} = P_{p^{-1}(i),i}M_{i,m(i)}$.

These formulas give us also the m' and p'; as a consequence, the same cycle appears again in the new state. Thus we can make several consecutive inductions on a given circuit; moreover, if from G we make s successive inductions on an M circuit where s is the length of the cycle, then we arrive at G (with the same train-track equalities and bijections m, p), and at the end the new names have the form $M'_{i,j} = M_{i,j}$, $P'_{i,j} = P_{i,j}M_{j,m(j)} \cdots M_{m^{s-1}(j),j}$, and *mutatis mutandis* for a P circuit.

For a standard permutation π, with the condition of alternate discontinuities, in the initial state of the constructions we are in a binary state G_0. The bijections p and m are retrieved from the knowledge of the initial names, namely $M_{\pi i,i-1,0} = i-1$, $P_{\pi i,i,0} = i$, $2 \le i \le k-1$, $M_{1,k-1,0} = k-1$, $P_{1,1,0} = k1$ (note that $k1$ is a word of two symbols, not a typographical error).

3. Sarnak's conjecture and Bourgain's criterion

We denote by μ the Möbius function, defined by $\mu(1) = 1, \mu(p_1 \dots p_s) = (-1)^s$ if $p_1, \dots, p_s$ are distinct prime numbers and $\mu(n) = 0$ if n is divisible by the square of a prime number.

Definition 3.1. *A sequence $\mathbf{u} = (u(n))_{n\in\mathbb{N}}$ of complex numbers of modulus less than 1 is (asymptotically) orthogonal to the Möbius function if $\sum_{n\le x} \mu(n)u(n) = o(x)$.*

Definition 3.2. *Let X be a compact metric space, T be a continuous transformation of X and (X,T) the associated dynamical system. A sequence $\mathbf{u} = (u(n))_{n\in\mathbb{N}}$ of elements of X is produced by (X,T) if there exist $x_0 \in X$ and f continuous on X such that for any n in $\mathbb{N}$, we have $u(n) = f(T^n(x_0))$.*

Conjecture (Sarnak): Any bounded sequence of complex numbers produced by a zero topological entropy dynamical system is orthogonal to the Möbius function.

We want to test this conjecture for an interval exchange $\mathcal{I}$, where the topological model we choose is the symbolic dynamical system of Definition 2.7 above. All the results quoted in the introduction are valid for this model; note that this model is not always uniquely ergodic, but Sarnak's conjecture does not require that property.

3.1. **Bourgain's criterion.** The following result is not completely explicit in [10], as it is stated only in a particular case, as Theorem 3, and its proof is understated; here we give a general result and explain how to deduce it from [10]:

Theorem 3.1. *For every positive integer K there exists a constant $C(K)$ such that Sarnak's conjecture is satisfied by any topological dynamical system (X,T) admitting an adic presentation as in Defintion 2.14 such that, if the names (for some partition whose atoms are union of levels of the towers in $\mathcal{Z}_0$) of the towers in $\mathcal{Z}_n$ form sets of words $\mathcal{W}_n$, every $W \in \mathcal{W}_n$ has a canonical decomposition (deduced from the construction of the towers) of the form*

$$W = W_1^{k_1} \cdots W_r^{k_r},$$

for $r \le K$ words W_i, $1 \le i \le r$ in $\mathcal{W}_{n-1}$, integers $k_1, \dots, k_r$, all depending on W, and for any W in $\mathcal{W}_n$ and any $s \le n$, if we decompose W into words W_l in $\mathcal{W}_{n-s}$ by iteration of the above formula, then for all l we have $|W| > \beta(s)|W_l|$, for s large enough and some function $\beta(s) > C(K)^s$.

If such a system is uniquely ergodic, and weakly mixing for its invariant probability, it satisfies also a prime number theorem *for any word $W = w_1 \dots w_N$ which is a factor of a word in any $\mathcal{W}_n$, namely*

$$\sum_{i=1}^{N} \Lambda(i) w_i = \sum_{i=1}^{N} w_i + o(N),$$

where Λ is the von Mangoldt function defined by $\Lambda(n) = \ln p$ if $n = p^k$, p prime and $k \geq 1$, $\Lambda(n) = 0$ otherwise.

Proof

Theorem 2 of [10], together with the Remark at the beginning of Section 2, page 119 in that paper, gives, for any word $w_1 \cdots w_N$ in some $\mathcal{W}_m$ and N large enough, an estimate for

$$\int |\sum_{1}^{N} w_n e(n\theta)||\sum_{1}^{N} \mu(n) e(n\theta)| d\theta,$$

and this, through the relation 1.62 on p. 118, implies that $\sum_1^N w_n \mu(n) = o(N)$, Note that the assumption in [10] that the words W_n are on the alphabet $\{0, 1\}$ is not used in the proof, which works for any finite alphabet, while the condition $\beta(s) > C_0 s$ written in p. 119 of [10] is a misprint for $\beta(s) > C_0^s$, corrected in Theorem 3.1 above.

Now, if we replace w_n by $u(n) = f(T^n(x_0))$, because of Definition 2.14 above we can first assume that f is constant on all levels of the towers of some stage m, and then conclude by approximation. Such an f is also constant on all levels of all towers at stages $q > m$; fixing x_0 and N, except for some initial values $u(1)$ to $u(N_0)$ where N_0 is much smaller than N, we can replace $u(n)$ by w'_n, where w'_n is the value of f on the n-th level of some tower with name W in some $\mathcal{W}_q$ for $q \geq m$. Then the $w'_1 \cdots w'_N$ are built by the same induction rules as the $w_1 \cdots w_N$, and the estimates using the w'_n are computed as those using the w_n in the proof of Theorem 2 of [10], thus we get the same result.

The prime number theorem is in (3.4), (3.7), (3.14) of [10] ((3.14) is proved for the particular case of 3-interval exchanges but holds in the same way for the more general case). □

The orthogonality of an infinite sequence $\mathbf{w} = (w_i)_{i \in \mathbb{N}}$ with the Möbius function, namely $\sum_{i=1}^{N} \mu(i) w_i = o(N)$, does not imply the existence of a prime number theorem for the sequence $\mathbf{w}$, that is to say a control of the sums $\sum_{i=1}^{N} \Lambda(i) w_i$ (or $\sum_{i<N, i \text{ prime}} w_i$). Much better qualitative estimates (e.g. for type II sums in the Vinogradov method) are usually needed to obtain a prime number theorem (see [48] for interesting comments on this question). Such examples of prime number theorems can be found in [42][29][28][9][39][40, 41][43]. Bourgain succeeded in obtaining such estimates, by using the Hardy-Littlewood method for all systems satisfying all the hypotheses of Theorem 3.1 above.

We remark that the paper [10] is apparently focused on rank one systems (see also [2] for some improvement on the rank one results). But indeed the hidden Theorem 3.1 above applies to a much wider class of systems, generally but not necessarily of finite rank, and even for some famous rank one systems this criterion works while the supposedly main Theorem 1 of [10] does not apply; this will also be the case for the rank one systems we shall build in Section 5.

To show what Bourgain's criterion may mean, we look at the *Sturmian* symbolic systems defined by i.d.o.c. 2-interval exchanges, aka irrational rotations. Then it can be deduced from [4], or reproved using Section 2.5, that the system admits an adic presentation with sets $\mathcal{Z}_n$ consisting of two Rokhlin towers, with names (for the partition defined by the value of the first coordinate) form the set $\mathcal{W}_n = \{A_{n-1}, A_n\}$ with the recursion formula $A_{n+1} = A_n^{a_{n+1}} A_{n-1}$ if the angle of the rotation has a continued fraction expansion $[0, a_1 + 1, a_2, a_3, \ldots]$. As $|A_n| > |A_{n-1}|$ we get that Bourgain's criterion is satisfied if all the a_n, for n large enough, are greater than $C(2)$; thus it applies to a class of rotations but falls short of proving the conjecture for all irrational rotations, by which it is known to be satisfied, as mentioned in the introduction.

3.2. **Constructions.** We shall now build examples using successive inductions on circuits as in Section 2.5. We fix a few notations.

A *circuit* is given by a cycle of m, resp. p, in a given state; we describe it formally as a succession of *vertices* $i, m(i), \ldots, m^{s-1}(i)$, resp. $i, p(i), \ldots, p^{s-1}(i)$ and *edges* $\mathcal{M}_{i,m(i)}, \ldots, \mathcal{M}_{m^{s-1}(i),i}$, resp. $\mathcal{P}_{i,p(i)}, \ldots, \mathcal{P}_{p^{s-1}(i),i}$ (these are indeed vertices and edges in some graphs defined in [18]). A given edge $\mathcal{P}_{i,j}$ or $\mathcal{M}_{i,j}$ may appear several times in a construction; the varying names $P_{i,j,n}$, resp. $M_{i,j,n}$, build recursively as in Section 2.5, are called the *n-labels* of the edges $\mathcal{P}_{i,j}$, resp. $\mathcal{M}_{i,j}$.

An M circuit is denoted as $\mathcal{M}_v = \mathcal{M}_{i,m(i)}, \ldots, \mathcal{M}_{m^{s-1}(i),i}$, where i is any of its vertices, and the numbering v is partly arbitrary and may change according to our needs. Any concatenated name $M_{j,m(j),n} \cdots M_{m^{s-1}(j),j,n}$, for any vertex j, is called an n–label of $\mathcal{M}_v$, and all these n-labels have a common length denoted by $|M_{v,n}|$; if $M_{v,n}$ and $M'_{v,n}$ are two different n-labels of the same circuit, any cycle $(M'_{v,n})^q$ contains $M_{v,n}^{q-1}$. We define similar notions for P circuits. The P circuit made with the single edge $\mathcal{P}_{1,1}$ is always numbered $\mathcal{P}_{1,1}$, and the M circuit containing the vertex 1 is $\mathcal{M}_1$. The number of vertices of $\mathcal{M}_v$, resp. $\mathcal{P}_u$, is denoted by s_v, resp. s'_u.

In state G_0 described in Section 2.5 for a given standard permutation π, all the edges $\mathcal{M}_{i,j}$, resp. $\mathcal{P}_{i,j}$, are partitioned in r disjoint M circuits, resp. r' disjoint P circuits, denoted by $\mathcal{M}_{i_t,m(i_t)}, \ldots, \mathcal{M}_{m^{s_t-1}(i_t),i_t}$, $1 \leq t \leq r$, $\mathcal{P}_{j_t,p(j_t)}, \ldots, \mathcal{P}_{p^{s'_t-1}(j_t),j_t}$, $1 \leq t \leq r'$; r, r' and the circuits depend on π; there are several possible choices for $i_1, \ldots, i_r$, $j_1, \ldots, j_{r'}$, we fix one. There is a circuit $\mathcal{P}_{1,1}$; in Theorems 4.3 and 5.2 we shall use the fact that $|P_{1,1,0}| = 2$ while $|M_{v,0}| = s_v$ and $|P_{u,0}| = s'_u$ for the other circuits.

We can now build a large family of interval exchanges which satisfy Sarnak's conjecture (in a forthcoming paper, we shall show that it corresponds to a set of parameters of measure zero but positive Hausdorff dimension):

Theorem 3.2. *Let π be a standard permutation on $\{1, \ldots, k\}$. Any k-interval-exchange with permutation π, defined from state G_0 by a finite preliminary sequence of inductions leading to G_0, and then successively for each $n \geq 1$,*

- $s_t q_{t,n} > 0$ *consecutive inductions on the M circuit containing i_t, for each $1 \leq t \leq r$, in any order on t,*
- $s'_t q'_{t,n} > 0$ *consecutive inductions on the P circuit containing j_t, for each $1 \leq t \leq r'$, in any order on t,*

satisfies Bourgain's criterion and thus Sarnak's conjecture if there exists N_0 *such that*

$$\min_{\substack{1\leq t\leq r\\ 1\leq t'\leq r'\\ n>N_0}} \{q_{t,n}, q'_{t',n}\} > C(2k-2).$$

Proof

At the beginning of the runs of $s_t q_{t,n}$ inductions on M circuits, we are in state G_0; the circuits have edges $\mathcal{P}_{i,j}$ whose labels we denote by $P_{i,j,n}$, $n\geq 1$, $\mathcal{M}_{i,j}$ whose labels we denote by $M_{i,j,n}$, $n\geq 1$. At the end of all these runs, we have G_0 again with the same M edge labels, while the P labels become

$$P_{j,p(j),n+1} = P_{j,p(j),n}(M_{p(j),mp(j),n}\dots M_{m^{s_t-1}p(j),p(j),n})^{q_{t,n}}$$

$$\text{whenever}\quad p(j)=m^l(i_t)\quad\text{for some}\quad 0\leq l\leq s_t-1, 1\leq t\leq r.$$

Similarly, the runs of $s'_t q'_{t,n}$ inductions on P circuits keep the P labels, while the M labels become

$$M_{i,m(i),n+1} = M_{i,m(i),n}(P_{m(i),pm(i),n+1}\dots P_{p^{s'_t-1}m(i),m(i),n+1})^{q'_{t,n}}$$

$$\text{whenever}\quad m(i)=p^l(j_t)\quad\text{for some}\quad 0\leq l\leq s'_t-1, 1\leq t\leq r'.$$

By the remarks in Section 2.5, the system admits an adic presentation, where the set of names of the towers in $\mathcal{Z}_{2n}$ is the set of all labels $M_{i,j,n-1}$ and $P_{i',j',n}$, while for $\mathcal{Z}_{2n+1}$ it is the set of all labels $M_{i,j,n}$ and $P_{i',j',n}$. We split the towers further: for each tower of name $P_{i',j',n}$ in $\mathcal{Z}_{2n}$ with $i'\neq j'$, we keep in the tower of name $P_{i',j',n}$ only the part corresponding to this $P_{i',j',n}$ which in the next stage of concatenation is isolated at the beginning of $P_{i',j',n+1}$, and stack all the other parts to form new towers whose names are all the n-labels of the P circuits with at least two edges; for each tower of name $M_{i,j,n}$ in $\mathcal{Z}_{2n+1}$ with $i\neq j$, we keep in the tower of name $M_{i,j,n}$ only the part corresponding to this $M_{i,j,n}$ which in the next stage of concatenation is isolated at the beginning of $M_{i,j,n+1}$, and stack all the other parts to form new towers whose names are all the n-labels of the M circuits with at least two edges. We still have an adic presentation, but now we use the sets of names $\mathcal{W}_{2n}$, made of all labels $M_{i,j,n-1}$, all labels $P_{i',j',n}$, and all the n-labels of all the P circuits with at least two edges, while $\mathcal{W}_{2n+1}$ is made of all labels $M_{i,j,n}$, all the n-labels of all the M circuits with at least two edges, and all labels $P_{i',j',n}$. The concatenation formulas giving the decomposition of $M_{i,j,n}$ in $\mathcal{W}_{2n+1}$ into words of $\mathcal{W}_{2n}$ are the formulas computed above under the form $M_{i,j,n} = M_{i,j,n-1}P_{v,n}^{q'_{t,n}}$, for some label of some circuit P_v, which is an edge label if the circuit has only one edge, while the new n-labels of the circuits are then written by concatenating these formulas; for $P_{i,j,n}$ in $\mathcal{W}_{2n+1}$, we write just that it is equal to $P_{i,j,n}$ in $\mathcal{W}_{2n}$; and *mutatis mutandis* for the decomposition of words in $\mathcal{W}_{2n}$. Thus we generate the $\mathcal{W}_n$ by formulas as in Theorem 3.1 with $K\leq 2k-2$ (the number r is 2 for the new labels of edges, and up to $k-1$ of them are concatenated to make the new labels of circuits).

We check now the condition on lengths. Let $Q=\min_{\substack{1\leq t\leq r\\ 1\leq t'\leq r'\\ n>N_0}} \{q_{t,n}, q'_{t',n}\}$; we look first at the M words of $\mathcal{W}_{2n+1}$ and suppose $2n+1$ is larger than $2N_0+1+s$. The word $M_{i,j,n}$ gets the decomposition $M_{i,j,n-1}P_{v,n}^{q'_{t,n}}$, thus its length is at least $Q|P_{v,n}|$. The circuit $\mathcal{P}_v$ has j as one of its vertices, thus $P_{v,n}$ contains some label $P_{i',j,n} = P_{i',j,n-1}M_{u,n-1}^{q'_{t,n-1}}$, and the circuit $\mathcal{M}_u$ contains the edge $\mathcal{M}_{i,j}$, thus $P_{v,n}$ is longer than $M_{i,j,n-1}$, $|M_{i,j,n}| > Q|M_{i,j,n-1}|$, and $|M_{i,j,n}| > Q|W'|$ for every W' in its decomposition into words of $\mathcal{W}_{2n}$. Then we make the next step of the decomposition of $M_{i,j,n}$;

$M_{i,j,n-1}$ stays, and each edge label $P_{i',j',n}$ in the labels $P_{v,n}$ gets a decomposition $P_{i',j',n-1}M_{u,n-1}^{q_{t',n-1}}$, for some M circuit. Each of these $M_{u,n-1}$ appears at least Q^2 times in the decomposition of $M_{i,j,n}$; we have just seen that $M_{i,j,n-1}$ is the label of an edge in one of these circuits, and thus shorter than some $M_{u,n-1}$. As for the $P_{i',j',n-1}$, the same reasoning as just above proves that each one is shorter than some $M_{u,n-1}$; thus $|M_{i,j,n}| > Q^2|W'|$ for every W' in its decomposition into words of $\mathcal{W}_{2n-1}$. Iterations of these computations give $|M_{i,j,n}| > Q^s|W'|$ for every W' in its decomposition into words of $\mathcal{W}_{2n+1-s}$. The same happens when we start from the n-labels of M circuits. When we start from labels of P edges in $\mathcal{W}_{2n+1}$, then we begin by an equality with the same name in $\mathcal{W}_{2n}$, and then the computation proceeds similarly at each stage, thus we get the same estimate with Q^s replaced by Q^{s-1}; and *mutatis mutandis* if we start from names in $\mathcal{W}_{2n}$.

Thus, if we take the set $\mathcal{W}_{2N_0}$ as initial words instead of $\mathcal{W}_0$, we get always $\beta(s) > Q^{s-1}$, and we have the required condition for large s. □

Note that we have used a finite rank property (see after Definition 2.13 above), with $2k-2$ towers at each stage (the further splitting is just an auxiliary step), but this bound for the rank is not optimal. In contrast, the examples on $k=3$ intervals in [10] and $k=4$ intervals in [25] used a further induction on one interval, which reduces the number of towers, giving the optimal bound k on the rank, but seems very difficult to generalize to a larger number of intervals.

4. Examples with comparable towers

To go further towards a prime number theorem we need unique ergodicity. The standard way to get it would be to ensure all the towers have comparable measures at each stage. However, in our constructions, the M towers and the P towers will not have comparable measures in general; we shall show that comparability inside each family is enough, but this requires a new nontrivial proof.

We need first the following lemma, which comes from the minimality of the system in any of our constructions.

Lemma 4.1. *For any $1 \le j \le k-1$, let $\Xi_0(j) = j$, then $\Xi_{2i+1}(j)$ is the set of all vertices of all M circuits which have at least one vertex in $\Xi_{2i}(j)$, $i \ge 0$, $\Xi_{2i+2}(j)$ is the set of all vertices of all P circuits which have at least one vertex in $\Xi_{2i+1}(j)$, $i \ge 0$. Similarly we define the $\Xi'_i(j)$ by $\Xi'_0(j) = j$, then $\Xi'_{2i+1}(j)$ is the set of all vertices of all P circuits which have at least one vertex in $\Xi'_{2i}(j)$, $i \ge 0$, $\Xi'_{2i+2}(j)$ is the set of all vertices of all M circuits which have at least one vertex in $\Xi_{2i+1}(j)$, $i \ge 0$.*

Then there exists L such that for all $i \ge L$ and every $1 \le j \le k-1$, $\Xi'_i(j) = \Xi_i(j) = \{1, \dots, k-1\}$.

Proof

For all t large enough $\Xi_{t+1}(j) = \Xi_t(j)$; for such a t, $\Xi_t(j)$ is such that for all i in $\Xi_t(j)$, $p(i)$ and $m(i)$ are also in $\Xi_t(j)$, thus $\Xi_t(j) = \{1, \dots, k-1\}$ (this can be checked on the definition of G_0, or comes from the fact that there exist minimal interval exchanges with initial state G_0), and similarly for the Ξ'. □

Here is now a still large family of uniquely ergodic interval exchanges satisfying Sarnak's conjecture. Note that here we use Rokhlin towers and finite rank but, in contrast with Section 5 below, not rank-one techniques; ergodicians who are not afraid of technicalities can verify that the following examples are of rank $k-1$.

Proposition 4.2. *A k-interval-exchange defined as in Theorem 3.2 is uniquely ergodic if there exist a constant z and two sequences $\bar{q}_n$ and $\bar{q}'_n$ such that, for all n and t, $1 \leq \bar{q}_n \leq q_{t,n} \leq z\bar{q}_n$, $1 \leq \bar{q}'_n \leq q'_{t,n} \leq z\bar{q}'_n$. Then it satisfies Sarnak's conjecture if $\min_{n>N_0}\{\bar{q}_n, \bar{q}'_n\} > C(2k-2)$ for some N_0.*

Proof

At each stage the space is filled by $2k-2$ Rokhlin towers, whose names are the labels of the $\mathcal{M}_{i,j}$ and $\mathcal{P}_{i,j}$ (the auxilary towers whose names form the $\mathcal{W}_n$ are used only for Bourgain's criterion, not for other properties). The towers are made by cutting and stacking following the recursion rules above; thus, for any invariant measure μ, the vector made of the measures of a level in the s-th tower, $1 \leq s \leq 2k-2$ at stage t_1 is given by applying to the corresponding vector at stage $t_2 > t_1$ the matrix A_{t_1,t_2} which has on its line α and column β the number of times the name of the β-th tower at stage t_1 appears in the decomposition of the name of the α-th tower at stage t_2 into names of towers at stage t_1. Unique ergodicity is equivalent to the fact that $\cap_{t>0}A_{1,t}\mathcal{C}$ is one-dimensional, where $\mathcal{C}$ is the positive cone. The matrix $A_{1,t}$ is a product of matrices corresponding to transitions between each stage and the next one. We shall first get some estimates on the matrix B_n corresponding to the transitions between the stage situated just before the runs of $s_tq_{t,n}$ inductions on M circuits and the stage just before the runs of $s_tq_{t,n+L+1}$ inductions on M circuits (for an L which will come from Lemma 4.1). The coefficients of B_n are the number of times we see the $M_{i,j,n}$, and $P_{i,j,n}$ in the decompositions of the $M_{i',j',n+L+1}$, and $P_{i',j',n+L+1}$. Fix first some $M_{i,j,n}$; this appears one time in one M circuit whose vertices form $\Xi_1(i)$, thus in any of its n-labels M_n; the labels M_n appears in the decomposition of labels P_{n+1} of all the circuits with vertices in $\Xi_2(i)$, and in each of those P_{n+1} the labels M_n appears at least $\bar{q}_n$ times and at most $zk^2\bar{q}_n$ times, as each of these P_{n+1} is made with one label P_n and between 1 and k labels M_n, each one being iterated between $\bar{q}_n$ and $zk\bar{q}_n$ times. Similarly, the labels P_{n+1} appear in all labels M_{n+1} of the circuits with vertices in $\Xi_3(i)$, and in each of those M_{n+1} the labels P_{n+1} appear at least $\bar{q}'_n$ times and at most $zk^2\bar{q}'_n$ times. We continue until we reach labels M_{n+L} of circuits whose vertices form $\Xi_{2L+1}(i) = \{1, \dots, k-1\}$ by Lemma 4.1. Each label $P_{c,d,n+L+1}$ contains at least $\bar{q}_{n+L}$ and at most $zk\bar{q}_{n+L}$ occurrences of these labels M_{n+L}. Now the occurrences of our $M_{i,j,n}$ in $P_{c,d,n+L+1}$ are those who are inside successive labels M_n, P_{n+1}, M_{n+2}, …plus some which come from an initial $P_{\alpha,\beta,n+j}$ in $P_{\alpha,\beta,n+j+1}$, or $M_{\alpha,\beta,n+j}$ in $M_{\alpha,\beta,n+j+1}$, which at worst will multiply the number of occurrences by a fixed factor, because all the $\bar{q}_n$ and $\bar{q}'_n$ are at least one (to take these into account we replace z by a larger z'). Thus the total number of these occurrences is at least $\Pi_{j=n}^{n+L-1}(\bar{q}_j\bar{q}'_j)\bar{q}_{n+L}$ and at most $2k^{4L+2}(z')^{2L+1}\Pi_{j=n}^{n+L-1}(\bar{q}_j\bar{q}'_j)\bar{q}_{n+L}$ times. By continuing to circuits P_{n+L+1} we reach the labels $M_{a,b,n+L+1}$, each of whom contains our initial $M_{i,j,n}$ between $\Pi_{j=n}^{n+L}(\bar{q}_j\bar{q}'_j)$ and $2k^{4L+4}(z')^{2L+2}\Pi_{j=n}^{n+L}(\bar{q}_j\bar{q}'_j)$ times.

If we start from a fixed $P_{i,j,n}$; it appears one time in one label $P_{i,j,n+1}$, from which we go similarly to labels P_{n+1} for one circuit whose vertices are in $\Xi'_1(i)$, labels M_{n+2} of circuits with vertices in $\Xi'_2(i)$, and so on. At the end, as $\Xi'_{2L}(i)$ is $\{1, \dots, k-1\}$, we get that the number of occurrences of that $P_{i,j,n}$ in any $P_{c,d,n+L+1}$ is at least $\bar{q}'_n\Pi_{j=n+1}^{n+L-1}(\bar{q}_j\bar{q}'_j)\bar{q}_{n+L}$ and at most $2k^{4L+1}(z')^{2L}q'_n\Pi_{j=n+1}^{n+L-1}(\bar{q}_j\bar{q}'_j)\bar{q}_{n+L}$, and in any $M_{a,b,n+L+1}$ is at least $\bar{q}'_n\Pi_{j=n+1}^{n+L}(\bar{q}_j\bar{q}'_j)$ and at most $2k^{4L+1}(z')^{2L+1}\bar{q}'_n\Pi_{j=n+1}^{n+L}(\bar{q}_j\bar{q}'_j)$ (the main difference with the case starting from an M circuit at stage n is the absence of $\bar{q}_n$ in the formulas).

Now, we use a theory of Garrett Birkhoff [6][7] as described in [26], see also [27]. For some projective metric d on the convex cone, by Theorem 7.8 of [26] a positive matrix Λ contracts the distances by a factor $\frac{e^{\frac{\theta}{4}}-e^{\frac{-\theta}{4}}}{e^{\frac{\theta}{4}}+e^{\frac{-\theta}{4}}}$, where by Proposition 7.13 of [26] $\theta(\Lambda) = \max_{i,j,i',j'} \log \frac{\Lambda_{i,i'}\Lambda_{j,j'}}{\Lambda_{j,i'}\Lambda_{i,j'}}$. By the above estimates, if $\Lambda = B_n$, all the $\frac{\Lambda_{i,i'}\Lambda_{j,j'}}{\Lambda_{j,i'}\Lambda_{i,j'}}$ are bounded by a constant C independent of n: when both i and j correspond to M (resp. P) coordinates, $\frac{\Lambda_{i,i'}}{\Lambda_{j,i'}}$ and $\frac{\Lambda_{j,j'}}{\Lambda_{i,j'}}$ are bounded, while if i corresponds to an M coordinate and j to a P coordinate $\frac{\Lambda_{i,i'}}{\Lambda_{j,i'}}$ will be bounded by $C'\bar{q}'_{n+L}$, and $\frac{\Lambda_{j,j'}}{\Lambda_{i,j'}}$ by $C''(\bar{q}'_{n+L})^{-1}$, and similarly in the opposite case. Thus B_n contracts the distances by a fixed factor. As the matrix $A_{1,t}$ is indeed a product of infinitely many B_n, we get the required characterization of $\cap_{t>0} A_{1,t}\mathcal{C}$. And Sarnak's conjecture comes from Theorem 3.2. □

To get weak mixing and a prime number theorem, we need more work: the following examples generalize to every standard permutation the weakly mixing families built for particular cases in Theorem 13 of [16] and Theorem 4.3 of [18].

Theorem 4.3. *Let $k \geq 3$, and π a standard permutation on $\{1,\dots,k\}$. Let q be the least common multiple of $s_1,\dots,s_r$, q' the least common multiple of $s'_1,\dots,s'_{r'}$. One can construct recursively two sequences $(q_n)_{n\geq 1}$ and $(q'_n)_{n\geq 1}$, such that the k-interval-exchange built as in Theorem 3.2, with the preliminary sequence of inductions defined in 1.1 to 2.4 below and $q_{t,n} = q_n\frac{q}{s_t}$, $q'_{t,n} = q'_n\frac{q'}{s'_t}$ is uniquely ergodic, weakly mixing, and satisfies Sarnak's conjecture with a prime number theorem.*

Proof

The words $P_{i,j,n}$ and $M_{i,j,n}$ are built as in the proof of Theorem 3.2; again, we use the $2k-2$ Rokhlin towers whose names are the labels of the $\mathcal{M}_{i,j}$ and $\mathcal{P}_{i,j}$, and not the auxilary towers of Theorem 3.1. Note that $s_t q_{t,n} = qq_n$, $s'_t q'_{t,n} = q'q'_n$, and the q_n and q'_n differ by a constant from $\bar{q}_n$ and $\bar{q}'_n$ of Proposition 4.2.

We choose some circuit $\mathcal{P}_u$ to be precised later, with $\mathcal{P}_u \neq \mathcal{P}_{1,1}$. Then $|P_{1,1,n+1}| = |P_{1,1,n}| + q_n X_n$, $|P_{u,n+1}| = |P_{u,n}| + q_n Y_n$, with $X_n = \frac{q}{s_1}|M_{1,n}|$, and $Y_n = \sum_{i=1}^{r''} c_i \frac{q}{s_i}|M_{i,n}|$, where the vertices of $\mathcal{P}_u$ are partitioned among the M circuits $\mathcal{M}_i$, $2 \leq i \leq r''$, the circuit $\mathcal{M}_i$ having c_i common vertices with $\mathcal{P}_u$; the $\mathcal{M}_i$ and c_i do not depend on n; for a given $\mathcal{P}_u$ the $\mathcal{M}_i$, $2 \leq i \leq r''$, are all different but one of them may be $\mathcal{M}_1$.

Let $D_n = Y_n|P_{1,1,n}| - X_n|P_{u,n}|$ for $n \geq 1$. We make the recursion hypothesis that $|P_{1,1,n}|$ and $|P_{u,n}|$ are coprime and $D_n \neq 0$. We show first how, by a suitable choice of $\mathcal{P}_u$ and initial inductions we can ensure it is satisfied at stage 1. Let Y_1 (determined by the choice of $\mathcal{P}_u$) and X_1 be as above, we define an auxiliary $D_0 = Y_1|P_{1,1,0}| - X_1|P_{u,0}|$.

- If there exists at least one circuit $\mathcal{P}_u \neq \mathcal{P}_{1,1}$ with an odd number of vertices,
  - 1.1 if $D_0 \neq 0$ with one of these choices of $\mathcal{P}_u$, $|P_{u,0}|$ is odd, and the hypothesis is satisfied without any initial induction, as then $|P_{1,1,1}| = |P_{1,1,0}|$ and $|P_{u,1}| = |P_{u,0}|$ are coprime, $D_1 = D_0 \neq 0$; this is true in particular if for such a $\mathcal{P}_u$, $r'' = 2$ and $\mathcal{M}_1 = \mathcal{M}_2$, as then $X_1 = \frac{q}{s_1}|M_{1,1}|$, $Y_1 = c\frac{q}{s_1}|M_{1,1}|$, $D_0 = (2c - |P_{u,0}|)\frac{q}{s_1}|M_{1,1}|$ and $|P_{u,0}|$ is odd;
  - 1.2 if for all such $\mathcal{P}_u$, we have $D_0 = 0$, then there exists some $\mathcal{M}_j \neq \mathcal{M}_1$, and we choose one such $\mathcal{P}_u$ and a corresponding $\mathcal{M}_j$; at the beginning we make $q_0 s_j$ inductions on $\mathcal{M}_j$. After that, we get new values $|P_{1,1,1}| = |P_{1,1,0}|$, $|P_{u,1}| = |P_{u,0}| + q_0 m$ for some $m \neq 0$; we take $q_0 \neq 0$ and even, thus $D_1 \neq D_0$ and our hypothesis is satisfied at this stage.
- If every circuit $\mathcal{P}_v \neq \mathcal{P}_{1,1}$ has an even number of vertices, then $k-1$ is odd and there exists at least one circuit $\mathcal{M}$ in G_0 with an odd number s of vertices;

2.1 if there is such an $\mathcal{M} \neq \mathcal{M}_1$; there is at least one circuit $\mathcal{P}_u \neq \mathcal{P}_{1,1}$ with an odd number c of common vertices with $\mathcal{M}$; we choose such a circuit $\mathcal{P}_u$ and make first sq_0 inductions on $\mathcal{M}$; after these inductions $|P_{1,1,1}| = |P_{1,1,0}| = 2$, $|P_{u,1}| = |P_{u,0}| + q_0 m$, where m is c times the length of the 0-label of $\mathcal{M}$ and thus is odd also, $D_1 = 2Y_1 - (|P_{u,0}| + q_0 m)X_1$. By taking an odd q_0, we get $|P_{u,1}|$ odd; at most one of the possible such q_0 gives a $D_1 = 0$, and by avoiding it we get our hypothesis;

2.2 if the only possible $\mathcal{M}$ is $\mathcal{M} = \mathcal{M}_1$ and there is at least one circuit $\mathcal{P}_u \neq \mathcal{P}_{1,1}$ with no common vertex with $\mathcal{M}_1$; we choose such a circuit $\mathcal{P}_u$ and make first $s_1 q_0$ inductions on $\mathcal{M}_1$; after these inductions $|P_{1,1,1}| = 2 + q_0 m$ where $m = |M_{1,0}|$ is odd, $|P_{u,1}| = |P_{u,0}|$, $D_1 = (2 + q_0 m)Y_1 - |P_{u,0}|X_1$.. Let $2, d_1, \dots, d_z$ be the prime factors of $|P_{u,0}|$; if $d_i \neq 2$ divides m, it cannot divide $2 + q_0 m$. Then if we choose a q_0 not congruent to 0 modulo 2, and with $2 + q_0 m$ not congruent to 0 modulo any of the d_i which do not divide m, which is possible by the Chinese reminder theorem, we get that $|P_{1,1,1}|$ and $|P_{u,1}|$ are coprime; at most one of the possible such q_0 gives a $D_1 = 0$, and by avoiding it we get our hypothesis;

2.3 if $\mathcal{M} = \mathcal{M}_1$, every circuit $\mathcal{P}_v$ has common vertices with $\mathcal{M}_1$, at least one circuit $\mathcal{P}_u \neq \mathcal{P}_{1,1}$ has c common vertices with $\mathcal{M}_1$ and $2e \neq 2c$ vertices; we choose such a circuit $\mathcal{P}_u$ and make first $s_1 q_0$ inductions on $\mathcal{M}_1$; after these inductions $|P_{1,1,1}| = 2 + q_0 m$ for some odd $m = |M_{1,0}|$, $|P_{u,1}| = 2e + cq_0 m$, $X_1 = \frac{q}{s_1} m$, $Y_1 = c\frac{q}{s_1} m + m'$ for some m' corresponding to the (possible) other M circuits which have common vertices with P_u. Then any common divisor of $|P_{1,1,1}|$ and $|P_{u,1}|$ must divide $2 + q_0 m$ and $|2c - 2e| \neq 0$, and by the same reasoning as in 2.2 we find q_0 such $|P_{1,1,n}|$ and $|P_{u,1}|$ are coprime. Then $D_1 = (2c - 2e)\frac{q}{s_1} m + 2m' + q_0 m m'$, thus if $m' = 0$ we have always $D_1 \neq 0$, and if $m' \neq 0$ at most one of the possible q_0 gives a $D_1 = 0$, and by avoiding it we get our hypothesis;

2.4 if $\mathcal{M} = \mathcal{M}_1$ and every circuit $\mathcal{P}_u \neq \mathcal{P}_{1,1}$ has $2c$ vertices and c common vertices with $\mathcal{M}_1$ for some c; we choose any one of these $\mathcal{P}_u$; then $\mathcal{P}_u$ has $2c$ vertices, c of which are common with $\mathcal{M}_1$; we choose a circuit $\mathcal{M}_2 \neq \mathcal{M}_1$ which has $c' > 0$ common vertices with $\mathcal{P}_u$. By making s_2 preliminary inductions on $\mathcal{M}_2$, we change y_0 to something larger than $2c$ while the other parameters are unchanged, and we get to case 2.3, from which we proceed as above.

We suppose now that the recursion hypothesis is satisfied for n, and shall choose q_n and q'_n such that they will be satisfied for $n+1$. Namely, $|P_{1,1,n+1}| = |P_{1,1,n}| + q_n X_n$, $|P_{u,n+1}| = |P_{u,n}| + q_n Y_n$. Any common factor of $|P_{1,1,n+1}|$ and $|P_{u,n+1}|$ has to divide $Y_n|P_{1,1,n+1}| - X_n|P_{u,n+1}| = Y_n|P_{1,1,n}| - X_n|P_{u,n}| = D_n \neq 0$, which is independent of q_n. Let $\mathcal{D}$ be the set of all prime factors of D_n, $\mathcal{D}_1$ the set of those factors which divide also $|P_{1,1,n}|$, $\mathcal{D}_2$ the set of the other factors. If d is in $\mathcal{D}_2$ and divides X_n, any choice of q_n ensures that d does not divide $|P_{1,1,n+1}|$; if d is in $\mathcal{D}_2$ and does not divide X_n, d does not divide $|P_{1,1,n+1}|$ for any q_n such that $q_n \equiv X_n^{-1}(z - |P_{1,1,n}|)$ modulo d, for any $z \not\equiv 0$ modulo d. Similarly if d is in $\mathcal{D}_1$, and therefore does not divide $|P_{u,n}|$, either d does not divide $|P_{u,n+1}|$ for any value of q_n, or this can be ensured by a congruence condition modulo d. Thus, by the Chinese remainder theorem, we can find infinitely many values of q_n such that no prime number d divides the three numbers D_n, $|P_{1,1,n+1}|$ and $|P_{u,n+1}|$, and this ensures that $|P_{1,1,n+1}|$ and $|P_{u,n+1}|$ are coprime. Note that q_n depends only on the parameters q_1, ... q_{n-1}, q'_1, ... q'_{n-1}.

Thus for any n there exist positive integers U_n and V_n such that $|U_n|P_{1,1,n}| - V_n|P_{u,n}|| = 1$. As the value of $|P_{1,1,n+1}|$ and $|P_{u,n+1}|$ depend only on the parameters q_1, ... q_n, q'_1, ... q'_{n-1}, we can then choose q'_n larger than $U_{n+1} \vee V_{n+1}$. We also ask that q'_n is large enough to ensure that $D_{n+1} \neq 0$: indeed, by construction $D_{n+1} = D_n + q_n\theta_n$ for a θ_n independent of q'_n, thus if $\theta_n = 0$ any choice of q'_n is good, and otherwise $q'_n|\theta_n| > |D_n|$ is sufficient.

We shall now prove that, with the choices of the q_n and q'_n made above, $\mathcal{I}$ is weakly mixing by using a standard technique known as the *Chacon trick*, see [17] [8]. At the beginning of the run of qq_n inductions, the name $(P_{1,1,n})^{q'_{n-1}}$ appears in every $M_{i,j,n}$ such that i and 1 are in the same M circuit; similarly, if $P_{u,n}$ is one fixed n-label of the circuit $\mathcal{P}_u$, the name $(P_{u,n})^{q'_{n-1}-1}$ appears in one $M_{i,j,n}$. Thus in some M tower we see $(P_{1,1,n})^{q'_{n-1}}$, when we read the name along some sequence of levels. Let τ_n be the union of all these levels. For any point ω in τ_n, $\mathcal{I}^{|P_{1,1,n}|}\omega$, $\mathcal{I}^{2|P_{1,1,n}|}\omega$,.... $\mathcal{I}^{U_n|P_{1,1,n}|}\omega$ are in the same level of the tower with name $P_{1,1,n}$ as ω. Similarly, in the name of some other M tower we see $P_{u,n}^{q'_{n-1}-1}$ along a sequence of levels; let τ'_n be the union of all these levels. For any point ω in τ'_n, $\mathcal{I}^{|P_{u,n}|}\omega$, $\mathcal{I}^{2|P_{u,n}|}\omega$,.... $\mathcal{I}^{V_n|P_{u,n}|}\omega$ are in the same level of the tower with name $P_{u,n}$ as ω.

Let μ be an invariant probability for $\mathcal{I}$, f be an eigenfunction for the eigenvalue ζ. For each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ there exists f_n, which satisfies $\int ||f - f_n||d\mu < \varepsilon$ and is constant on each level of each tower at the beginning of the run of qq_n negative choices (where $||x||$ denotes its distance to the nearest integer). Thus for μ-almost every ω in τ_n, $f_n(\mathcal{I}^{U_n|P_{1,1,n}|}\omega) = f_n(\omega)$ while $f(\mathcal{I}^{U_n|P_{1,1,n}|}\omega) = \zeta U_n|P_{1,1,n}| + f(\omega)$; we have

$$\int_{\tau_n} ||f_n \circ \mathcal{I}^{U_n|P_{1,1,n}|} - \zeta U_n|P_{1,1,n}| - f_n||d\mu = \int_{\tau_n} ||\zeta U_n|P_{1,1,n}|||d\mu =$$

$$||\zeta U_n|P_{1,1,n}|||\mu(\tau_n) \quad \text{and}$$

$$\int_{\tau_n} ||f_n \circ \mathcal{I}^{U_n|P_{1,1,n}|} - \zeta U_n|P_{1,1,n}| - f_n||d\mu \leq \int_{\tau_n} ||f_n \circ \mathcal{I}^{U_n|P_{1,1,n}|} - f \circ \mathcal{I}^{U_n|P_{1,1,n}|}||d\mu + \int_{\tau_n} ||f_n - f||d\mu < 2\varepsilon.$$

Thus $\mu(\tau_n)||\zeta U_n|P_{1,1,n}||| < 2\varepsilon$, and similarly $\mu(\tau'_n)||\zeta V_n|P_{u,n}||| < 2\varepsilon$; as $U_n|P_{1,1,n}| - V_n|P_{u,n}| = \pm 1$, we shall conclude that $\zeta = 0$, and thus get the weak mixing, if we can prove that $\mu(\tau_n)$ and $\mu(\tau'_n)$ are bounded away from 0.

For this, we need first to check that all the lengths of the $M_{i,j,n}$, resp. $P_{i,j,n}$, are comparable. But each recursion formula giving $P_{i,j,n+1}$ has exactly $qq_n + 1$ terms, and each recursion formula giving $M_{i,j,n+1}$ has exactly $q'q'_n + 1$ terms. Thus, starting from the lengths at stage 1 which are fixed, we get that $c_1|P_{i,j,n}| \leq |P_{i',j',n}| \leq c_2|P_{i,j,n}|$, $c_1|M_{i,j,n}| \leq |M_{i',j',n}| \leq c_2|M_{i,j,n}|$ for any given n and all i, j, i', j'.

Let the Ξ_i and L be as in Lemma 4.1. For a given n the strings $P_{1,1,n}^{q'_{n-1}}$ appear in all the n-labels of the M circuits whose vertices form $\Xi_1(1)$ where they fill at least a fixed proportion $\frac{c_1}{q'}$ of the length; then they appear in all the $n+1$-labels of the M circuits whose vertices form $\Xi_2(1)$, and where they fill at least a fixed proportion $\frac{c_1}{q'}$ of the length, and so on, thus eventually they fill a fixed proportion κ of the lengths of all the $M_{i,j,n+L+1}$ and $P_{,i,j,n+L+1}$, and these are the names of Rokhlin towers filling all the space. This implies that $\mu(\tau_n) \geq \kappa$, and a similar reasoning works for $\mu(\tau'_n)$.

On the other hand, it is immediate that these systems satisfy Proposition 4.2 (the final condition in that proposition may be easily satisfied as we were able to choose the values q_n and q'_n in this proof to be as large as desired), which completes the proof of Theorem 4.3. □

5. Examples with rank one and simplicity for every standard permutation

A different way to ensure unique ergodicity is to have, at each stage, one tower much larger than all the others. This means we shall build rank one examples, and these are the first ones for which we are able to use directly the $2k-2$ Rokhlin towers given by the induction (see the remark after Theorem 3.2), and thus can work with all k.

Proposition 5.1. *A k-interval-exchange defined as in Theorem 3.2 is uniquely ergodic, of rank one, and satisfies Sarnak's conjecture if there exists a sequence $(a_n)_{n\in\mathbb{N}}$, tending to infinity with n, such that, for all n and $t \neq 1$, $q_{1,n} \geq a_n q_{t,n}$, $q'_{1,n} \geq a_n q'_{t,n}$.*

Proof

We use again the $2k-2$ Rokhlin towers whose names are the labels of the $\mathcal{M}_{i,j}$ and $\mathcal{P}_{i,j}$. For $n \geq 2$, we consider the towers just before the runs of $s'_t q'_{t,n}$ inductions, which have names $P_{i,j,n}$ and $M_{i',j',n-1}$. Like in Proposition 4.2, we compute how many times these appear in the labels $M_{a,b,n+L+1}$ and $P_{c,d,n+L+1}$.The number of occurrences of $P_{i,j,n}$ into $M_{a,b,n+L+1}$ is split into at most $(2k)^{2L}$ terms, each one corresponding to those who are inside successive labels M_n or P_n, P_{n+1} or M_{n+1}, M_{n+2} or $P_{n+2}, \ldots$, or come from an initial $P_{\alpha,\beta,n+j}$ in $P_{\alpha,\beta,n+j+1}$, or $M_{\alpha,\beta,n+j}$ in $M_{\alpha,\beta,n+j+1}$. Each of this terms is a product involving some bounded quantities, and some or all of the the $q_{1,n+1}$ to $q_{1,n+L}$, $q'_{1,n}$ to $q'_{1,n+L}$. For all m, 1 is the only vertex of the circuit with label $P_{1,1,m}$, and is a vertex of one M circuit, the one with label $M_{1,m}$. Thus, if $(i,j) = (1,1)$, at least one of these terms does involve all the $q_{1,m}$ and $q'_{1,m}$, $n \leq m \leq n+L$, while if $(i,j) \neq (1,1)$, in every one of these terms at least one $q'_{1,m}$ is replaced by either a $q'_{t,m}$, $t > 1$, or nothing, or else at least one $q_{1,m}$ is replaced by either a $q_{t,m}$, $t > 1$, or nothing. Thus in each given $M_{a,b,n+L+1}$, we see at least $a_n C'$ more times labels $P_{1,1,n}$ than any other $P_{i,j,n}$, for a constant C'. The same is true if we compare $P_{1,1,n}$ with any $M_{i',j',n-1}$ inside any $M_{a,b,n+L+1}$, as each $M_{i',j',n-1}$ appears only in the labels of one M circuit, and the same is true, *mutatis mutandis*, if we compare $P_{1,1,n}$ with any other $P_{i,j,n}$, or any $M_{i',j',n-1}$, inside any $P_{c,d,n+L+1}$. Thus, for any invariant measure, the measure of any level of the tower with name $P_{1,1,n}$ is much larger than the measure of any level of any other tower just after the runs of $s_t q_{t,n-1}$ inductions.

As for the lengths of these names, the $M_{1,n-1}$ are $C' a_{n-1}$ times longer than all the labels of the other M circuits at that stage, and thus the name $P_{1,1,n}$ is $C' a_{n-1}$ times longer than any name $P_{i,j,n}$ where j is not a vertex of the circuit with label $M_{1,n-1}$, and of a comparable length with any name $P_{i,j,n}$ where j is a vertex of the circuit with label $M_{1,n-1}$ (namely, at most such a name could be k times longer than $P_{1,1,n}$) ; $P_{1,1,n}$ is also $C' a_n$ times longer than the name $M_{1,n-1}$ and thus than any name $M_{i,j,n-1}$. Thus the measure of the whole tower with name $P_{1,1,n}$ is also much larger than the measure of any other tower just before the runs of $s'_t q'_{t,n}$ inductions, thus is close to one, and the system is of rank one.

Because our system is of rank one for any invariant measure, every invariant measure is ergodic, thus there is only one, and we have unique ergodicity.

Note that if we consider the towers just before the runs of $s_t q_{t,n}$ inductions, they have names $P_{i,j,n}$ and $M_{i',j',n}$; the new tower with name $P_{i,j,n}$ is much thinner than the previous tower with name $P_{i,j,n}$, and the new tower with name$P_{1,1,n}$ is not of large measure anymore: at this stage, the tower with largest measure is the one with name $M_{m^{-1}(1),1,n}$, though it will lose its pre-eminence to the one with name $P_{1,1,n+1}$ after the runs of $s_t q_{t,n}$ inductions.

Sarnak's conjecture is ensured by a weaker condition than the one in Theorem 3.2 because we can identify the longest names (for the auxiliary towers with names in $\mathcal{W}_n$) at each stage. Let $Q = \min_{n>N_0}\{q_{1,n}, q'_{1,n}\}$; for some N'_0 and $n > N'_0$, the maximal length of a word in $\mathcal{W}_{2n+1}$ is reached by the n-labels of the circuit $\mathcal{M}_1$, containing $M_{1,m(1),n}$ and is at most twice the length of $M_{1,m(1),n}$ (the other edge labels are much shorter); the minimal length of a word in $\mathcal{W}_{2n+1}$ is the length of some $P_{i,j,n}$; the maximal length of a word in $\mathcal{W}_{2n}$ is reached by the n-labels of some fixed P circuit $\mathcal{P}_v$ and is at most k^2 times the length of $P_{1,1,n}$ (we have seen that there might be longer $P_{i,j,n}$, and they might form a circuit, thus the factor k^2). The minimal length of a word in $\mathcal{W}_{2n}$ is the length of some $M_{i,j,n-1}$. If $n > N'_0$, by Lemma 4.1 the minimal length of an M word in $\mathcal{W}_{2n+L+1}$ is at least $|M_{1,n}|$; then this length is at least $Q|P_{1,1,n}| > Q^2|M_{1,n-1}| > Q^3|P_{1,1,n-1}| \ldots$ as long as we are above N_0. Thus, if $2n+1-s$ is at least $2N_1+1$, where $N_1 = N_0 \vee N'_0$, the minimal length of an M word in $\mathcal{W}_{2n+L+1}$ is at least $\frac{1}{k^2}Q^s$ times the maximal length of a word in $\mathcal{W}_{2n+1-s}$, and similarly for P words. Thus, if take as initial words by the $\mathcal{W}_{N_1}$, we get $\beta(s+L+1) > \frac{1}{k^2}Q^s$. Hence, if we choose N_0 such that $a_n > C(2k-2)$ for all $n > N_0$, the hypothesis is enough to satisfy Bourgain's criterion. $\square$

In this class, we can have not only weak mixing, but also simplicity.

Theorem 5.2. *Let $k \geq 3$, and π a standard permutation on $\{1, \ldots, k\}$: one can construct recursively sequences $q'_{t,n}$, $1 \leq t \leq r'$, and $q_{t,n}$, $1 \leq t \leq r$, $n \geq 1$, such that the k-interval-exchange defined as in Theorem 3.2, with the preliminary sequence of inductions defined in* 3.1 *to* 3.m *below, is uniquely ergodic, rank one, weakly mixing, simple, rigid, and satisfies Sarnak's conjecture with a prime number theorem.*

Proof

We fix first the preliminary inductions so that $|P_{1,1,1}|$ and $|M_{1,1}|$ are coprime.

3.1 If $\mathcal{M}_1$ has an odd number of vertices, its label has odd length, and we have what we require without any preliminary induction.

3.2 If the label of $\mathcal{M}_1$ has even length, $\mathcal{M}_1$ has an even number of vertices; then there exists a P circuit $\mathcal{P}_2 \neq \mathcal{P}_{1,1}$ with an odd number of common vertices with $\mathcal{M}_1$; if $\mathcal{P}_2$ has an odd number s'_2 of vertices, we make s'_2 inductions on $\mathcal{P}_2$; after that the length of the label of $\mathcal{M}_1$ becomes odd, the label of $\mathcal{P}_{1,1}$ is unchanged and we have what we require.

3.3 If the label of $\mathcal{P}_2$ has even length, there exists an M circuit $\mathcal{M}_2 \neq \mathcal{M}_1$ with an odd number of common vertices with $\mathcal{P}_2$; if $\mathcal{M}_2$ has an odd number s_2 of vertices, we make s_2 inductions on $\mathcal{M}_2$; after that the length of the label of $\mathcal{P}_2$ becomes odd, the labels of $\mathcal{P}_{1,1}$ and $\mathcal{M}_1$ are unchanged, we are as in 3.2, and can continue to get what we require.

3.4 If the label of $\mathcal{M}_2$ has even length, there exists a P circuit $\mathcal{P}_3 \neq \mathcal{P}_2$, $\mathcal{P}_3 \neq \mathcal{P}_{1,1}$ with an odd number of common vertices with $\mathcal{M}_2$; if $\mathcal{P}_3$ has an odd number s'_3 of vertices, we make s'_3 inductions on $\mathcal{P}_3$; after that the length of the label of $\mathcal{M}_2$ becomes odd, the labels of $\mathcal{P}_{1,1}$, $\mathcal{P}_2$ and $\mathcal{M}_1$ are unchanged, we are as in 3.3, and can continue to get what we require.

3.m We continue in the same way, possibly with $\mathcal{P}_{l+2} = \mathcal{P}_l$ or $\mathcal{M}_{l+2} = \mathcal{M}_l$ for some l, until we cannot find any new circuits P or M; if we can stop somewhere with a circuit of odd length, we go back through the steps to get what we require.

Suppose we cannot stop anywhere with a circuit of odd length; having defined our $\mathcal{M}_i$ and $\mathcal{P}_i$ as above, we denote the remaining circuits by $\bar{\mathcal{M}}_j$ and $\bar{\mathcal{P}}_j$. We have partitioned all the vertices $\{1, \dots k-1\}$ into circuits $\mathcal{M}_i$ with an even number of vertices, $\bar{\mathcal{M}}_j$, $\mathcal{P}_{1,1}$, $\mathcal{P}_i$ with an even number of vertices, $\bar{\mathcal{P}}_j$; because we have added everything we can to the $\mathcal{M}_i$ and $\mathcal{P}_i$, each $\mathcal{M}_i$ and $\bar{\mathcal{P}}_j$, resp. $\mathcal{P}_i$ and $\bar{\mathcal{M}}_j$, have an even (possibly zero) number of common vertices. But then the number of vertices of $\bar{\mathcal{M}} = \cup \bar{\mathcal{M}}_j$ and $\bar{\mathcal{P}} = \cup \bar{\mathcal{P}}_j$ have the same parity (namely, the parity of the number of vertices of $\bar{\mathcal{P}} \cap \bar{\mathcal{M}}$), thus because of $\mathcal{P}_{1,1}$, the size of the set of vertices, that is the number $k-1$, has two different parities, which gives a contradiction.

Then the words $P_{i,j,n}$ and $M_{i,j,n}$ are built as in the proof of Theorem 3.2. We fix some P circuit different from $\mathcal{P}_{1,1}$, but with $b_1 \neq 0$ common vertices with the circuit $\mathcal{M}_1$ (this is possible as $\mathcal{M}_1$ is not reduced to the vertex 1, by minimality) and denote it by $\mathcal{P}_2$. We shall now choose our $q_{t,n}$ and $q'_{t,n}$ so that for each $n \geq 2$ the lengths $|P_{1,1,n}|$ and $b_1|P_{2,n}|$ are coprime, and $|M_{1,n}|$ is congruent to 1 modulo $|P_{1,1,n}|$. We make the induction hypothesis that $|P_{1,1,n}|$ and $|M_{1,n}|$ are coprime: it is satisfied for $n=1$.

Knowing the $P_{i,j,n}$ and $M_{i',j',n}$, we choose first the $q_{i,n}$ so that $|P_{1,1,n+1}|$ and $b_1|P_{2,n+1}|$ are coprime. We have

$$|P_{1,1,n+1}| = |P_{1,1,n}| + q_{1,n}|M_{1,n}|,$$

$$|P_{2,n+1}| = |P_{2,n}| + b_1 q_{1,n}|M_{1,n}| + \sum_{v=2}^{V} b_v q_{v,n}|M_{v,n}|,$$

where the vertices of $\mathcal{P}_2$ belong to circuits $\mathcal{M}_v$, $1 \leq v \leq V_u$, with b_v common vertices between $\mathcal{M}_v$ and $\mathcal{P}_2$. Every common divisor to $|P_{1,1,n+1}|$ and $b_1|P_{2,n+1}|$ divides also

$$D_n = \left| -b_1^2|P_{1,1,n}| + b_1|P_{2,n}| + b_1 \sum_{v=2}^{V} b_v q_{v,n}|M_{v,n}| \right|.$$

The integer D_n does not depend on $q_{1,n}$. If there exists $v \neq 1$ such that $b_v \neq 0$ then we can choose the $q_{t,n}$, $2 \leq t \leq r$, such that $D_n \neq 0$. Otherwise, for all v we have $b_v = 0$; then all P circuits which have a common vertex with $\mathcal{M}_1$ have all their vertices in $\mathcal{M}_1$. Again by minimality this implies that $\mathcal{M}_1$ is the only M circuit, and $b_1 = s'_2$, $D_n = |-b_1^2|P_{1,1,n}| + b_1|P_{2,n}||$. The recursion formulas give $|P_{1,1,n}| = |P_{1,1,n-1}| + q_{1,n-1}|M_{1,n-1}|$ and $|P_{2,n}| = |P_{2,n-1}| + s'_2 q_{1,n-1}|M_{1,n-1}|$, thus $D_n = D_{n-1} = \ldots = D_1$. In this situation, either $\mathcal{M}_1$ has an odd number of vertices, or we can choose the circuit $\mathcal{P}_2$ such that it has an odd number of vertices; thus in the preliminary inductions, we stop at 3.1 or 3.2: either we change nothing or we change only the M edges, thus $D_1 = D_0 = s'_2 \neq 0$.

We know that $|P_{1,1,n+1}| = |P_{1,1,n}| + q_{1,n}|M_{1,n}|$ and that $|P_{1,1,n}|$ and $|M_{1,n}|$ are coprime. By the Chinese remainder theorem, we can find infinitely many values of $q_{1,n}$ such that no prime number d divides the numbers D_n and $P_{1,1,n+1}$. This is done by choosing the congruences of $q_{1,n}$ modulo those prime factors of D_n which do not divide $|M_{1,n}|$, while any congruence is permitted modulo the divisors of D_n which divide $|M_{1,n}|$. Thus we get that $|P_{1,1,n+1}|$ and $b_1|P_{2,n+1}|$ are coprime.

We choose now the $q'_{i,n}$ such that $|M_{1,n+1}|$ is congruent to 1 modulo $|P_{1,1,n+1}|$, which in particular implies the induction hypothesis at stage $n+1$.. We have

$$|M_{1,n+1}| = |M_{1,n}| + q'_{1,n}|P_{1,1,n+1}| + \sum_{u=1}^{U} c_u q'_{u,n}|P_{u,n+1}|,$$

where the vertices of M_1 belong to $P_{1,1}$ or to circuits P_u, $1 \le u \le U$, with c_u common vertices between M_1 and P_u, and $c_2 = b_1$. Now we choose first $q'_{u,n} = |P_{1,1,n}|$ for the $u \ge 3$ such that $c_u \ne 0$ (if they exist); then $q'_{2,n}$ such that $|M_{1,n}| + b_1 q'_{2,n}|P_{2,n+1}|$ is congruent to 1 modulo $|P_{1,1,n+1}|$, which is possible as $b_1|P_{2,n+1}|$ is invertible modulo $|P_{1,1,n+1}|$; then we fix $q'_{1,n}$ (which will be required later to be much larger than the $q'_{i,n}$, $i > 1$), and we have what we require.

We add the conditions that for all $n \ge 1$, $q_{1,n} > 2^n q_{t,n}$ and $q'_{1,n} > 2^n q'_{t,n}$, for every $2 \le t \le k-1$, which can be ensured in the previous construction. Then our system satisfies the conditions of Proposition 5.1.

We look at the actual expressions of the names; if the vertex $m^l(1)$ is in the circuit $\mathcal{P}_{u_l}$, $1 \le l \le s_1 - 1$, $P_{u_i,n}$ denotes the n-label of this circuit which begins with $M_{m^l(1),pm^l(1),n}$, while $M_{1,n}$ denotes the n-label of the circuit $\mathcal{M}_1$ which begins with $M_{1,m(1),n}$. Then we have

$$M_{1,n} = M_{1,m(1),n-1}(P_{u_1,n})^{q'_{u_1,n-1}} M_{m(1),m^2(1),n-1}(P_{u_2,n})^{q'_{u_2,n-1}} \dots$$

$$M_{m^{s_1-2}(1),m^{s_1-1}(1),n-1}(P_{u_{s_1-2},n})^{q'_{u_{s_1-2},n-1}} M_{m^{s_1-1}(1),1,n-1}(P_{1,1,n})^{q'_{1,n-1}},$$

$$P_{1,1,n+1} = P_{1,1,n} M_{1,n}^{q_{1,n}},$$

hence

$$P_{1,1,n+1} = P_{1,1,n}(R_n P_{1,1,n}^{q'_{1,n-1}})^{q_{1,n}},$$

where $|R_n| = \bar{q}_{n-1}|P_{1,1,n}| + 1$, for some $\bar{q}_{n-1}$ large enough if $q'_{2,n-1}$ is large, but much smaller than $q'_{1,n-1}$.

Thus our system is measure-theoretically isomorphic to the rank one system built by towers of name $H_{n+1} = H_n(H_n^{\bar{q}_{n-1}} s H_n^{q'_{1,n-1}})^{q_{1,n}}$, the isomorphism being made by turning the strings R_n on one side, the strings $H_n^{\bar{q}_{n-1}}$ on the other side, into spacers, see Definition 1.8 and Theorem 4.8 of [25]. If $\bar{q}_n$ and $q'_{1,n}$ are large enough, this last system is weakly mixing, rigid and simple exactly in the same way as the rank one map of del Junco - Rudolph [33], which is the rank one system build by towers of names $H_0 = 0$, $H_{n+1} = H_n^{2^n} 1 H_n^{2^n}$; the main (and quite involved) argument in Theorem 1 of [33] uses only the fact that there are isolated spacers between long concatenations of the same tower. □

6. Other permutations

We have used twice the standardness of the permutation: to get a binary initial state (under the condition of alternate discontinuities), and to have enough knowledge on the lengths of the initial circuits to begin the control of lengths which is crucial in Theorems 4.3 and 5.2. These two theorems seem difficult to generalize to a larger class of permutations, and indeed would be false for circular permutations, for which weak mixing cannot occur (though it can, and does occur, in their Rauzy class, except for $k = 2$).

As for Theorem 3.2, Propositions 4.2 and 5.1, they are valid, by the same method, for any permutation and order of the discontinuities for which the initial state is binary, and these include all permutations with $\pi 1 = k$ under the condition of alternate discontinuities, and all permutations with $\pi k = 1$ under the dual condition $\gamma_1 < \beta_1 < \gamma_2 < \ldots$ Indeed, they are also valid if a binary state can be reached after a finite sequence of initial inductions; this requires the description and knowledge of our induction in every case: it is true for every permutation in the hyperelliptic Rauzy class [24], whatever the order of the discontinuities, and we have checked it experimentally for every permutation we have tried; we conjecture that, with the new induction, we can make similar families of examples for any permutation.

But if we want to build examples for every primitive permutation, the Rauzy induction does the trick, as, when it is applied backwards, it preserves Bourgain's criterion:

Proposition 6.1. *Let $\mathcal{I}$ be an interval exchange satisfying the hypotheses of Theorem 3.2, resp. Proposition 4.2, resp. Proposition 5.1. Then any pre-image of $\mathcal{I}$ for a finite iterate of the Rauzy induction satifies its conclusions.*

Proof

The system $\mathcal{I}$ has an adic presentation with towers of names W in $\mathcal{W}_n$ described in the proof of Theorem 3.2; by applying Proposition 2.2 p times, we get that a pre-image $\mathcal{I}'$ for p iterates of the Rauzy induction has an adic presentation with towers of names ψW, where the map ψ is defined by applying the composition of the p (possibly different) maps ϕ of Proposition 2.2 letter by letter. The words ψW, $W \in \mathcal{W}_n$, are built by the same recursion formulas as the W, with different initial words, and all the results depend only on these recursion rules. □

This does not apply to weak mixing or simplicity, which rely on the lengths of the initial words. But, by Proposition 2.1, the examples satisfying Sarnak's conjecture, unique ergodicity or rank one built in Theorem 3.2, Propositions 4.2 and 5.1 for every standard permutation yield examples for every primitive permutation.

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INSTITUT DE MATHÉMATIQUES DE MARSEILLE, CNRS - UMR 7373, CASE 907 - 163 AV. DE LUMINY, F13288 MARSEILLE CEDEX 9, FRANCE.

E-mail address: ssferenczi@gmail.com, mauduit@iml.univ-mrs.fr