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Non-homogeneous Dirichlet boundary value problems in weighted Sobolev spaces

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The paper presents existence and multiplicity results for non-linear boundary value problems on possibly non-smooth and unbounded domains under possibly non-homogeneous Dirichlet boundary conditions. We develop here an appropriate functional setting based on weighted Sobolev spaces. Our results are obtained by using global minimization and a minimax approach using a non-smooth critical point theory.

Keywords: non-linear elliptic problem; non-homogeneous Dirichlet boundary conditions; weighted Sobolev spaces; global optimization; minimax principle

AMS Subject Classifications: 35J92; 35J70; 46E35; 49J40

1. Introduction

Let Ω be a domain in \mathbb{R}^N with $N \geq 1$ such that $|\mathbb{R}^N \setminus \Omega| > 0$, where $|\cdot|$ stands for the Lebesgue measure in \mathbb{R}^N . Denote by $L^1_{\text{loc}}(\Omega)$ (resp. $L^\infty_{\text{loc}}(\Omega)$) the space of locally integrable (resp. locally essentially bounded) functions in Ω . Throughout this paper, a weight on Ω means a function $w \in L^1_{\text{loc}}(\Omega)$ such that $w > 0$ a.e. in Ω . Given $p \in (1, +\infty)$, a weight w on Ω with $w^{\frac{1}{1-p}} \in L^1_{\text{loc}}(\Omega)$ and a weight w_0 on Ω , we consider the following problem

$$\begin{cases} -\Delta_{p,w}u = w_0(x)f(x, u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (\text{P})$$

In the statement of problem (P), the differential part is driven by the weighted p -Laplacian operator $\Delta_{p,w}u = \text{div}(w|\nabla u|^{p-2}\nabla u)$ (see Section 2 for the precise functional setting). The right-hand side of the equation in problem (P) is expressed through a Carathéodory function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ (that is, $f(\cdot, s)$ is measurable for each $s \in \mathbb{R}$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$). The boundary condition is given by a function $g \in L^1_{\text{loc}}(\Omega)$ that is subject to an appropriate trace property. The solutions for problem (P) are understood in the weak sense as formulated in Section 2.

The presence of the weights w and w_0 is aimed to compensate the possible non-regularity and unboundedness of the domain Ω . Another relevant feature of problem (P)

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is that the Dirichlet condition imposed on the boundary is generally non-homogeneous. We emphasize that passing by translation from non-homogeneous boundary condition to homogeneous, one is not possible because the terms of the new problem will not satisfy any more the initial hypotheses. Our approach provides a direct treatment of non-linear elliptic problems with non-homogeneous Dirichlet conditions without needing to make translation for the boundary conditions. The usual classical case is retrieved if the domain is bounded, $g = 0$, and $w = w_0 = 1$. We also mention that degenerate elliptic equations with homogeneous boundary conditions have been studied in [1–3] in the semilinear case, pointing out concentration phenomena. Here we develop an approach based on appropriate function spaces whose properties allow us to handle degenerate quasilinear elliptic equations with possibly non-homogeneous boundary conditions.

The resolution of problem (P) leads naturally to weighted and doubly weighted Sobolev spaces, namely $L^{1,p}(\Omega, w)$ and $W^{1,q,p}(\Omega, w_0, w)$, respectively. Here we present basic results on these spaces that are useful tools in studying problem (P) as well as large classes of other non-linear boundary value problems. Two different approaches are developed for investigating problem (P): global minimization and minimax method. They provide complementary results covering different cases related to the growth condition that should be satisfied by the non-linearity $f(x, s)$ in (P) with respect to the variable $s \in \mathbb{R}$. Namely, we treat the p -sublinear case in (P) through minimization, whereas the p -superlinear case in (P) is handled through the minimax approach. In both situations, we pay a special attention to the case of homogeneous boundary condition, i.e. when $g = 0$. The main original idea in these results is to incorporate the boundary condition in a suitable set of constrains. We are thus led in a natural way to make use of a non-smooth critical point theory for convex, lower semicontinuous perturbations of differentiable functions.

The rest of the paper is organized as follows. Section 2 is devoted to the functional setting of weighted Sobolev spaces. Section 3 contains our results on problem (P) based on the minimization approach. Section 4 contains our results on problem (P) obtained relying on the variational method.

2. Weighted Sobolev spaces

Let $p \in [1, +\infty)$ and introduce

$$\mathcal{W}(\Omega) = \{w \in L^1_{\text{loc}}(\Omega) : w > 0 \text{ a.e. on } \Omega\}.$$

For $w \in \mathcal{W}(\Omega)$, we consider the w -weighted Lebesgue space

$$L^p(\Omega, w) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \int_{\Omega} |u(x)|^p w(x) dx < +\infty \right\},$$

which is a Banach space for the norm

$$\|u\|_{L^p(\Omega, w)} = \left(\int_{\Omega} |u(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

Define

$$\begin{aligned} \mathcal{W}_p(\Omega) &= \left\{ w \in \mathcal{W}(\Omega) : w^{\frac{1}{1-p}} \in L^1_{\text{loc}}(\Omega) \right\} & \text{if } p > 1, \\ \mathcal{W}_1(\Omega) &= \left\{ w \in \mathcal{W}(\Omega) : w^{-1} \in L^{\infty}_{\text{loc}}(\Omega) \right\}. \end{aligned}$$

Remark 1 One has $L^p(\Omega, w) \subset L^1_{\text{loc}}(\Omega)$ provided $w \in \mathcal{W}_p(\Omega)$.

For $p \in [1, +\infty)$ and $w \in \mathcal{W}_p(\Omega)$, we introduce the space

$$L^{1,p}(\Omega, w) = \{u \in L^1_{\text{loc}}(\Omega) : \nabla u \in L^p(\Omega, w)^N\},$$

where ∇u stands for the gradient of u in the distributional sense. The space $L^{1,p}(\Omega, w)$ is equipped with the seminorm

$$\|u\|_{L^{1,p}(\Omega, w)} := \|\nabla u\|_{L^p(\Omega, w)}.$$

We construct the space of zero trace elements in our non-regular setting for domains.

Definition 1 Given $w \in \mathcal{W}_p(\mathbb{R}^N)$, we define the space

$$L_0^{1,p}(\Omega, w) = \left\{ u|_{\Omega} : u \in L^{1,p}(\mathbb{R}^N, w), u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega, \right. \\ \left. u = \lim_{n \rightarrow \infty} u_n \text{ in } L^{1,p}(\mathbb{R}^N, w), \text{ with } (u_n)_{n \geq 1} \subset C_c^\infty(\Omega) \right\}.$$

Here $C_c^\infty(\Omega)$ is the space of C^∞ -functions with compact support in Ω .

Remark 2 The space $L_0^{1,p}(\Omega, w)$ coincides with that introduced in [4, §2.2].

The following lemma summarizes essential properties of the space $L_0^{1,p}(\Omega, w)$. The proof can be found in [4, Proposition 2.7].

LEMMA 1

(a) If $u \in L_0^{1,p}(\Omega, w)$, then the function \bar{u} defined by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega \end{cases}$$

belongs to $L^{1,p}(\mathbb{R}^N, w)$.

(b) $L_0^{1,p}(\Omega, w) \subset L^{1,p}(\Omega, w)$.

(c) $L_0^{1,p}(\Omega, w)$ does not contain the non-zero constants.

(d) $\|\cdot\|_{L^{1,p}(\Omega, w)}$ is a norm on $L_0^{1,p}(\Omega, w)$.

(e) $C_c^\infty(\Omega)$ is dense in $L_0^{1,p}(\Omega, w)$ for the norm $\|\cdot\|_{L^{1,p}(\Omega, w)}$.

Remark 3 The requirement $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$ in Definition 1 cannot be dropped in view of Lemma 1 (c) because $\mathbb{R} \subset \{u|_{\Omega} : u \in L^{1,p}(\mathbb{R}^N, w), u = \lim_{n \rightarrow \infty} u_n \text{ in } L^{1,p}(\mathbb{R}^N, w), \text{ with } (u_n)_{n \geq 1} \subset C_c^\infty(\Omega)\}$.

Now, we focus on doubly weighted Sobolev spaces. Given $p, q \in [1, +\infty)$ and $w \in \mathcal{W}_p(\Omega)$, $w_0 \in \mathcal{W}_q(\Omega)$, we define the (w_0, w) -weighted Sobolev space

$$W^{1,q,p}(\Omega, w_0, w) = \{u \in L^q(\Omega, w_0) : \nabla u \in L^p(\Omega, w)^N\},$$

which is a Banach space for the norm

$$\|u\|_{W^{1,q,p}(\Omega, w_0, w)} := \|u\|_{L^q(\Omega, w_0)} + \|\nabla u\|_{L^p(\Omega, w)}$$

(see [4]).

Remark 4 In the definition of the space $W^{1,q,p}(\Omega, w_0, w)$, we utilize the assumption that $w_0 \in \mathcal{W}_q(\Omega)$ in order to ensure that the elements of $L^q(\Omega, w_0)$ belong to $L^1_{\text{loc}}(\Omega)$ (see Remark 1), so that their distributional gradient is well defined.

PROPOSITION 1 *If $p, q \in (1, +\infty)$, the space $W^{1,q,p}(\Omega, w_0, w)$ is reflexive.*

Proof The conclusion readily follows because the map

$$I : W^{1,q,p}(\Omega, w_0, w) \rightarrow L^q(\Omega, w_0) \times (L^p(\Omega, w))^N, \quad u \mapsto (u, \nabla u)$$

is an isometry with closed range in $L^q(\Omega, w_0) \times (L^p(\Omega, w))^N$. \square

For $w \in \mathcal{W}_p(\Omega)$ and $w_0 \in \mathcal{W}_q(\Omega)$, we define $W_0^{1,q,p}(\Omega, w_0, w)$ to be the closure of $C_c^\infty(\Omega)$ in $W^{1,q,p}(\Omega, w_0, w)$. A comparison between the spaces $W_0^{1,q,p}(\Omega, w_0, w)$ and $L_0^{1,p}(\Omega, w)$ is given below. The proof is the same as in [4, Proposition 2.8].

LEMMA 2 *If $p, q \in [1, +\infty)$, $w \in \mathcal{W}_p(\mathbb{R}^N)$, and $w_0 \in \mathcal{W}_q(\mathbb{R}^N)$, then we have $W_0^{1,q,p}(\Omega, w_0, w) \subset L_0^{1,p}(\Omega, w)$ and the inclusion map is continuous.*

We state now the main result of this section.

THEOREM 1 *Let $p, q \in [1, +\infty)$, $w \in \mathcal{W}_p(\mathbb{R}^N)$, $w_0 \in \mathcal{W}_q(\mathbb{R}^N)$. The following conditions are equivalent:*

- (a) *There is a continuous embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$;*
- (b) *The Poincaré inequality holds: there is a constant $c > 0$ such that*

$$\|u\|_{L^q(\Omega, w_0)} \leq c \|\nabla u\|_{L^p(\Omega, w)} \quad \text{for all } u \in W_0^{1,q,p}(\Omega, w_0, w);$$

- (c) *$\|\cdot\|_{W^{1,q,p}(\Omega, w_0, w)}$ and $\|\cdot\|_{L^{1,p}(\Omega, w)}$ are equivalent norms of $W_0^{1,q,p}(\Omega, w_0, w)$.*

Moreover, any of these properties implies

- (d) $L_0^{1,p}(\Omega, w) = W_0^{1,q,p}(\Omega, w_0, w)$.

Proof (a) \Rightarrow (b): By (a), there exists a constant $c > 0$ such that

$$\|u\|_{L^q(\Omega, w_0)} \leq c \|\nabla u\|_{L^p(\Omega, w)} \quad \text{for all } u \in L_0^{1,p}(\Omega, w).$$

Taking Lemma 2 into account, we see that (b) holds true.

(b) \Leftrightarrow (c): This is straightforward.

(c) \Rightarrow (d): By Lemma 2, we know that $C_c^\infty(\Omega) \subset W_0^{1,q,p}(\Omega, w_0, w) \subset L_0^{1,p}(\Omega, w)$. Using (c), it turns out that $W_0^{1,q,p}(\Omega, w_0, w)$ is complete for the norm $\|\cdot\|_{L^{1,p}(\Omega, w)}$, so closed in $L_0^{1,p}(\Omega, w)$. Then the density of $C_c^\infty(\Omega)$ in $L_0^{1,p}(\Omega, w)$ (see Lemma 1 (e)) implies that $L_0^{1,p}(\Omega, w) = W_0^{1,q,p}(\Omega, w_0, w)$.

(c) \Rightarrow (a): It suffices to combine (c) \Rightarrow (d) and (c) \Rightarrow (b). \square

Remark 5 In the classical case where the domain Ω is bounded and $w = w_0 = 1$, the assertions (a)–(d) of Theorem 1 are satisfied for any real number $q \in [1, p^*]$, where p^* stands for the Sobolev critical exponent, that is $p^* = Np/(N - p)$ if $N > p$ and $p^* = +\infty$ if $N \leq p$ (see, e.g. [5, Corollary 9.14]).

We set forth an extension of a fundamental property well known for the usual Sobolev spaces (see, e.g. [6, Lemma 7.6]).

PROPOSITION 2 *Suppose that $p, q \in [1, +\infty)$, $w \in \mathcal{W}_p(\Omega)$, $w_0 \in \mathcal{W}_q(\Omega)$. If $u \in W_0^{1,q,p}(\Omega, w_0, w)$, then $u^+, u^-, |u|$ belong to $W_0^{1,q,p}(\Omega, w_0, w)$ and satisfy*

$$\begin{aligned}\nabla(u^+) &= \begin{cases} \nabla u & \text{a.e. in } \{x \in \Omega : u(x) > 0\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \leq 0\}, \end{cases} \\ \nabla(u^-) &= \begin{cases} 0 & \text{a.e. in } \{x \in \Omega : u(x) \geq 0\}, \\ -\nabla u & \text{a.e. in } \{x \in \Omega : u(x) < 0\}, \end{cases} \\ \nabla|u| &= \begin{cases} \nabla u & \text{a.e. in } \{x \in \Omega : u(x) > 0\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) = 0\}, \\ -\nabla u & \text{a.e. in } \{x \in \Omega : u(x) < 0\}. \end{cases}\end{aligned}$$

Proof Since $u^+ = \frac{1}{2}(|u| + u)$ and $u^- = \frac{1}{2}(|u| - u)$, it is sufficient to prove the result for $|u|$.

Given $\delta > 0$, consider the function

$$f_\delta : \mathbb{R} \rightarrow [0, +\infty), \quad s \mapsto \sqrt{s^2 + \delta^2} - \delta.$$

Note that $f_\delta \in C^\infty(\mathbb{R})$ and $0 \leq f_\delta(s) \leq |s|$ for all $s \in \mathbb{R}$.

Step 1 If $u \in W_0^{1,q,p}(\Omega, w_0, w)$, then we have $f_\delta(u) \in W_0^{1,q,p}(\Omega, w_0, w)$ and $\nabla(f_\delta(u)) = \frac{u}{\sqrt{u^2 + \delta^2}} \nabla u$.

Fix $u \in W_0^{1,q,p}(\Omega, w_0, w)$. The fact that $f_\delta(u) \in L^q(\Omega, w_0)$ follows from $0 \leq f_\delta(u) \leq |u|$. Since f_δ is Lipschitz continuous, it turns out that $f_\delta(u)$ is weakly differentiable and

$$\frac{\partial f_\delta(u)}{\partial x_i} = \frac{u}{\sqrt{u^2 + \delta^2}} \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \dots, N$$

(see, e.g. [6, Theorem 7.8]). The inequality

$$\left| \frac{u}{\sqrt{u^2 + \delta^2}} \frac{\partial u}{\partial x_i} \right| \leq \left| \frac{\partial u}{\partial x_i} \right|$$

implies that $\frac{\partial f_\delta(u)}{\partial x_i} \in L^p(\Omega, w)$ for $i = 1, \dots, N$. This amounts to saying that $f_\delta(u) \in W^{1,q,p}(\Omega, w_0, w)$.

On account of $u \in W_0^{1,q,p}(\Omega, w_0, w)$, there exists a sequence $(u_n)_{n \geq 1} \subset C_c^\infty(\Omega)$ such that $u_n \rightarrow u$ in $W^{1,q,p}(\Omega, w_0, w)$. Recalling that f_δ vanishes at 0, we infer that $f_\delta(u_n) \in C_c^\infty(\Omega)$. From the fact that f_δ is Lipschitz continuous, we easily derive that $f_\delta(u_n) \rightarrow f_\delta(u)$ in $L^q(\Omega, w_0)$. In addition, through Lebesgue's dominated convergence theorem, we derive that $\frac{\partial(f_\delta(u_n))}{\partial x_i} \rightarrow \frac{\partial(f_\delta(u))}{\partial x_i}$ in $L^p(\Omega, w)$ for $i = 1, \dots, N$. We conclude that $f_\delta(u_n) \rightarrow f_\delta(u)$ in $W^{1,q,p}(\Omega, w_0, w)$, whence $f_\delta(u) \in W_0^{1,q,p}(\Omega, w_0, w)$.

Step 2 $|u|$ is weakly differentiable and the gradient $\nabla|u|$ has the stated expression. For any $\varphi \in C_c^\infty(\Omega)$, Step 1 gives

$$\int_{\Omega} f_{\delta}(u) \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} \frac{u}{\sqrt{u^2 + \delta^2}} \frac{\partial u}{\partial x_i} \varphi dx \quad \text{for all } \delta > 0 \text{ and } i = 1, \dots, N.$$

Letting $\delta \rightarrow 0$, by Lebesgue's dominated convergence theorem we get

$$\int_{\Omega} |u| \frac{\partial \varphi}{\partial x_i} dx = - \int_{\Omega} g_i(x) \varphi dx \quad \text{for } i = 1, \dots, N,$$

where

$$g_i(x) = \begin{cases} \frac{\partial u}{\partial x_i}(x) & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) = 0 \\ -\frac{\partial u}{\partial x_i}(x) & \text{if } u(x) < 0. \end{cases}$$

This proves Step 2.

Step 3 $|u| \in W_0^{1,q,p}(\Omega, w_0, w)$.

Step 2 guarantees that the weak derivatives $\frac{\partial |u|}{\partial x_i}$ exist and they are given by $\frac{\partial |u|}{\partial x_i} = g_i$ for $i = 1, \dots, N$, with g_i as above. As $|u| \in L^q(\Omega, w_0)$ and $g_i \in L^p(\Omega, w)$, we obtain that $|u| \in W^{1,q,p}(\Omega, w_0, w)$.

By applying Lebesgue's dominated convergence theorem, we can show that $f_{\delta}(u) \rightarrow |u|$ in $L^q(\Omega, w_0)$ and $\frac{\partial f_{\delta}(u)}{\partial x_i} \rightarrow \frac{\partial |u|}{\partial x_i}$ in $L^p(\Omega, w)$ for $i = 1, \dots, N$ as $\delta \rightarrow 0$, which results in $f_{\delta}(u) \rightarrow |u|$ in $W^{1,q,p}(\Omega, w_0, w)$ as $\delta \rightarrow 0$. Taking Step 1 into account, the proof is complete. \square

On the basis of the built functional setting, we can make precise the (weak) formulation of problem (P). The differential operator driving the equation in problem (P) is the weighted p -Laplacian $\Delta_{p,w} : L^{1,p}(\Omega, w) \rightarrow L_0^{1,p}(\Omega, w)^*$ given by

$$\langle -\Delta_{p,w} u, v \rangle = \int_{\Omega} w(x) |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \quad (1)$$

for all $u \in L^{1,p}(\Omega, w)$ and $v \in L_0^{1,p}(\Omega, w)$. By a (weak) solution of problem (P), we mean any $u \in L^{1,p}(\Omega, w)$ such that $f(\cdot, u(\cdot))w_0(\cdot) \in L_{\text{loc}}^1(\Omega)$,

$$\langle -\Delta_{p,w} u, v \rangle = \int_{\Omega} f(x, u)v(x)w_0(x) dx \quad \text{for all } v \in C_c^\infty(\Omega), \quad (2)$$

and

$$u - g \in L_0^{1,p}(\Omega, w). \quad (3)$$

3. Minimization approach

For the rest of the paper, we suppose that $q > 1$ and set $q' := \frac{q}{q-1}$. We assume that

(H₁) There exist $a \in L^{q'}(\Omega, w_0)$ and a constant $c > 0$ such that

$$|f(x, s)| \leq a(x) + c|s|^{q-1} \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

(H₂) There exists a compact embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$.

Remark 6 In the case where the domain Ω is bounded and $w = w_0 = 1$, the Rellich–Kondrachov embedding theorem in conjunction with Theorem 1 implies that hypothesis (H₂) is fulfilled for any $q \in [1, p^*)$, and in this case (H₁)–(H₂) hold once f satisfies a subcritical growth condition.

$$\text{Set } F(x, s) = \int_0^s f(x, t) dt.$$

LEMMA 3 *Assume that hypotheses (H₁), (H₂) are fulfilled. Then there hold:*

- (a) *for every $u \in L^{1,p}(\Omega, w)$, we have that $f(\cdot, u(\cdot)) \in L^{q'}(\Omega, w_0)$, so in particular, $f(\cdot, u(\cdot))w_0(\cdot) \in L^1_{\text{loc}}(\Omega)$;*
- (b) *for every $u \in L^{1,p}(\Omega, w)$, we have that $F(\cdot, u(\cdot)) \in L^1(\Omega, w_0)$.*

Proof

- (a) Let $u \in L^{1,p}(\Omega, w)$. By (H₂), we know that $u \in L^q(\Omega, w_0)$. Then (H₁) implies that $f(\cdot, u(\cdot)) \in L^{q'}(\Omega, w_0)$. Thanks to $w_0 \in L^1_{\text{loc}}(\Omega)$ and Hölder inequality, for any compact $K \subset \Omega$ we note that

$$\begin{aligned} & \int_K |f(x, u(x))w_0(x)| dx \\ & \leq \left(\int_K |f(x, u(x))|^{q'} w_0(x) dx \right)^{\frac{1}{q'}} \left(\int_K w_0 dx \right)^{\frac{1}{q}} < +\infty, \end{aligned}$$

whence $f(\cdot, u(\cdot))w_0(\cdot) \in L^1_{\text{loc}}(\Omega)$.

- (b) Hypothesis (H₁) and Young inequality entail

$$|F(x, s)| \leq a(x)|s| + c \frac{|s|^q}{q} \leq \frac{1}{q'} a(x)^{q'} + \frac{1}{q} (c+1)|s|^q,$$

which leads to the desired conclusion. □

Related to problem (P) we also assume:

$$(H_3) \ w \in \mathcal{W}_p(\mathbb{R}^N), w_0 \in \mathcal{W}_q(\mathbb{R}^N), \text{ and } g \in L^{1,p}(\Omega, w) \cap L^q(\Omega, w_0).$$

This enables us to characterize the solutions to problem (P).

LEMMA 4 *Assume that hypotheses (H₁), (H₂), (H₃) are satisfied. Then $u \in g + L^{1,p}_0(\Omega, w)$ is a solution of problem (P) if and only if*

$$\langle -\Delta_{p,w} u, v \rangle = \int_{\Omega} f(x, u)v(x)w_0(x) dx \quad \text{for all } v \in L^{1,p}_0(\Omega, w). \quad (4)$$

Proof Let $u \in g + L^{1,p}_0(\Omega, w)$ be a solution of problem (P). Given $v \in L^{1,p}_0(\Omega, w)$, there is a sequence $(v_n)_{n \geq 1} \subset C_c^\infty(\Omega)$ such that $\lim_{n \rightarrow \infty} \|v - v_n\|_{L^{1,p}(\Omega, w)} = 0$ (see Lemma 1(e)). In view of (2), we have

$$\int_{\Omega} w(x) |\nabla u|^{p-2} \nabla u \cdot \nabla v_n \, dx = \int_{\Omega} f(x, u) v_n(x) w_0(x) \, dx \quad \text{for all } n.$$

Passing to the limit, the Hölder inequality ensures that the left-hand side converges to $\langle -\Delta_{p,w} u, v \rangle$. Assumption (H₂) yields $\lim_{n \rightarrow \infty} \|v_n - v\|_{L^q(\Omega, w_0)} = 0$, which implies that the right-hand side converges to $\int_{\Omega} f(x, u) v(x) w_0(x) \, dx$. Therefore, (4) holds true.

Conversely, any element $u \in g + L_0^{1,p}(\Omega, w)$ fulfilling (4) satisfies

$$\int_K |f(x, u)| w_0(x) \, dx \leq \left(\int_{\Omega} |f(x, u)|^{q'} w_0(x) \, dx \right)^{\frac{1}{q'}} \left(\int_K w_0(x) \, dx \right)^{\frac{1}{q}} < +\infty$$

for all compact $K \subset \Omega$ (by Lemma 3(a)), so $f(\cdot, u(\cdot)) w_0(\cdot) \in L_{\text{loc}}^1(\Omega)$, it also satisfies relation (2) (by (4) and the fact that $C_c^\infty(\Omega) \subset L_0^{1,p}(\Omega, w)$) and relation (3), hence is a solution of problem (P). \square

The Euler functional $\Phi : L^{1,p}(\Omega, w) \rightarrow \mathbb{R}$ associated to problem (P) is given by

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p w(x) \, dx - \int_{\Omega} F(x, u(x)) w_0(x) \, dx \quad \text{for all } u \in L^{1,p}(\Omega, w), \quad (5)$$

which is well defined due to Lemma 3(b).

Consider the constrained minimization problem

$$\begin{aligned} & \text{Minimize} && \Phi(u). \\ & u \in g + L_0^{1,p}(\Omega, w) \end{aligned} \quad (Q)$$

The following result discusses the connection between problems (P) and (Q).

PROPOSITION 3

- (a) *If $u \in g + L_0^{1,p}(\Omega, w)$ is a solution of (Q), then u is a solution of (P).*
- (b) *Moreover, if the function $F(x, \cdot)$ is concave for a.e. $x \in \Omega$, then u is a solution of problem (P) if and only if u is a solution of problem (Q).*

Proof

- (a) Assume that $u \in g + L_0^{1,p}(\Omega, w)$ is a solution of problem (Q). Then 0 is a global minimizer for the function $J_v : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$J_v(t) = \Phi(u + tv) \quad \text{for all } t \in \mathbb{R}$$

whenever $v \in C_c^\infty(\Omega)$. Note that J_v is of class C^1 with the derivative

$$J'_v(t) = \int_{\Omega} w |\nabla(u+tv)|^{p-2} \nabla(u+tv) \cdot \nabla v \, dx - \int_{\Omega} f(x, u+tv) v w_0 \, dx \quad \text{for all } t \in \mathbb{R}.$$

This can be shown by utilizing Lebesgue's dominated convergence theorem, (H₁) and the continuous inclusion $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$. Then the equality $J'_v(0) = 0$ expresses that u is a solution of problem (P).

(b) Let u be a solution of problem (P). Then we have $u \in g + L_0^{1,p}(\Omega, w)$ and, owing to Lemma 4,

$$\int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f(x, u) v(x) w_0(x) \, dx \quad \text{for all } v \in L_0^{1,p}(\Omega, w).$$

Since the functions $\xi \in \mathbb{R}^N \mapsto \frac{1}{p} |\xi|^p$ and $t \in \mathbb{R} \mapsto -F(x, t)$ (with $x \in \Omega$) are convex, we obtain

$$\begin{aligned} \Phi(u) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p w \, dx + \int_{\Omega} w |\nabla u|^{p-2} \nabla u \cdot \nabla(v - u) \, dx \\ &\quad - \int_{\Omega} F(x, u) w_0 \, dx - \int_{\Omega} f(x, u) w_0 (v - u) \, dx \\ &\leq \Phi(v) \quad \text{for all } v \in u + L_0^{1,p}(\Omega, w), \end{aligned}$$

hence u is a solution of problem (Q). □

The main result of this section is the following existence theorem.

THEOREM 2 *Under hypotheses (H₁) with $q < p$, (H₂), (H₃), there exists at least a solution to problem (P). If the function $F(x, \cdot)$ is concave for a.e. $x \in \Omega$, then the solution is unique.*

Proof First we check that problem (Q) has at least a solution. In view of the continuous embedding in (H₂) and of Theorem 1, we have

$$L_0^{1,p}(\Omega, w) = W_0^{1,q,p}(\Omega, w_0, w).$$

From Proposition 1 we know that $W^{1,q,p}(\Omega, w_0, w)$ is a reflexive Banach space. Thanks to (H₃), the set $g + L_0^{1,p}(\Omega, w)$ is closed and convex in $W^{1,q,p}(\Omega, w_0, w)$. Therefore, in order to get the existence of a solution to (Q), it is sufficient to show that $\Phi : g + L_0^{1,p}(\Omega, w) \rightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous and coercive on $g + L_0^{1,p}(\Omega, w)$ with respect to the norm in $W^{1,q,p}(\Omega, w_0, w)$.

The first term in Φ is continuous as seen from the inequality

$$\left| \left(\int_{\Omega} |\nabla u|^p w \, dx \right)^{\frac{1}{p}} - \left(\int_{\Omega} |\nabla v|^p w \, dx \right)^{\frac{1}{p}} \right| \leq \|u - v\|_{L^{1,p}(\Omega, w)},$$

which is valid for all $u, v \in g + L_0^{1,p}(\Omega, w)$. Consequently, it is weakly lower semicontinuous due to its convexity. To justify the weak lower semicontinuity of the second term in the expression of Φ , let $(u_n)_{n \geq 1} \subset g + L_0^{1,p}(\Omega, w)$ be a sequence such that $u_n \rightharpoonup u$ (weakly) in $W^{1,q,p}(\Omega, w_0, w)$. Hence we have $u_n - g \rightharpoonup u - g$ in $L_0^{1,p}(\Omega, w)$ (since $L_0^{1,p}(\Omega, w)$ is a closed subspace of $W^{1,q,p}(\Omega, w_0, w)$, by virtue of (H₂) and Theorem 1). Furthermore, by (H₂), the embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ is compact, so up to a subsequence there holds $u_n \rightarrow u$ in $L^q(\Omega, w_0)$, which reads as $u_n w_0^{\frac{1}{q}} \rightarrow u w_0^{\frac{1}{q}}$ in $L^q(\Omega)$. Therefore, along a relabelled subsequence, it is true that $u_n \rightarrow u$ a.e. in Ω and $|u_n(x)| w_0(x)^{\frac{1}{q}} \leq h(x)$ for a.a.

$x \in \Omega$, with some $h \in L^q(\Omega)$. On the other hand, by hypothesis (H₁) and Young inequality, we have the estimate

$$\begin{aligned} |F(x, u_n(x))|w_0(x) &\leq a(x)|u_n(x)|w_0(x) + \frac{c}{q}|u_n(x)|^q w_0(x) \\ &\leq \frac{1}{q'} a(x)^{q'} w_0(x) + \frac{1+c}{q} |h(x)|^q \quad \text{for a.a. } x \in \Omega. \end{aligned}$$

Observe that the right-hand side of the last inequality belongs to $L^1(\Omega)$ and that $F(x, u_n(x)) \rightarrow F(x, u(x))$ a.e. in Ω . Then Lebesgue's dominated convergence theorem yields $\int_{\Omega} F(x, u_n)w_0 dx \rightarrow \int_{\Omega} F(x, u)w_0 dx$ as $n \rightarrow \infty$, thus Φ is sequentially weakly lower semicontinuous.

Next, we show that Φ is coercive on $g + L_0^{1,p}(\Omega, w)$, that is, $\Phi(u) \rightarrow +\infty$ as $\|u\|_{W^{1,q,p}(\Omega, w_0, w)} \rightarrow +\infty$ with $u \in g + L_0^{1,p}(\Omega, w)$. Theorem 1 ensures that if $\|u\|_{W^{1,q,p}(\Omega, w_0, w)} \rightarrow +\infty$ holds, then $\|u - g\|_{L^{1,p}(\Omega, w)} \rightarrow +\infty$ occurs. We notice that

$$\int_{\Omega} |\nabla u|^p w dx \geq \frac{1}{2^{p-1}} \int_{\Omega} |\nabla(u - g)|^p w dx - \int_{\Omega} |\nabla g|^p w dx \quad (6)$$

for all $u \in L^{1,p}(\Omega, w)$. As before, from hypothesis (H₁) we find that

$$\begin{aligned} \left| \int_{\Omega} F(x, u(x))w_0(x) dx \right| &\leq \frac{1}{q'} \|a\|_{L^{q'}(\Omega, w_0)}^q + \frac{1+c}{q} \|u\|_{L^q(\Omega, w_0)}^q \\ &\leq c_1 + c_2 \|u - g\|_{L^{1,p}(\Omega, w)}^q, \end{aligned}$$

with constants $c_1, c_2 > 0$. Combining with (6), we arrive at

$$\Phi(u) \geq \frac{1}{2^{p-1}} \|u - g\|_{L^{1,p}(\Omega, w)}^p - c_2 \|u - g\|_{L^{1,p}(\Omega, w)}^q - c_1 - \|g\|_{L^{1,p}(\Omega, w)}.$$

Since $q < p$, Φ is coercive on $g + L_0^{1,p}(\Omega, w)$. Therefore problem (Q) has a solution. Now applying Proposition 3(a) completes the existence part of the proof.

For the uniqueness of solution of problem (P), through Proposition 3(b), we have to establish the uniqueness of solution for problem (Q). To do this, it suffices to check that the functional Φ is strictly convex on $g + L_0^{1,p}(\Omega, w)$. To this end, it is seen from Theorem 1 that if $u, v \in g + L_0^{1,p}(\Omega, w)$ with $u \neq v$, then we have $\nabla u \neq \nabla v$. We note that the function $u \mapsto \int_{\Omega} |\nabla u|^p w dx$ is strictly convex on $g + L_0^{1,p}(\Omega, w)$ due to the corresponding property of $\|\cdot\|_{L^p(\Omega, w; \mathbb{R}^N)}^p$, where $L^p(\Omega, w; \mathbb{R}^N) = \{\xi : \Omega \rightarrow \mathbb{R}^N : |\xi| \in L^p(\Omega, w)\}$. Taking into account that, by our assumption, the function $u \mapsto -\int_{\Omega} F(x, u)w_0 dx$ is convex, the solution for problem (Q) is unique. \square

Our next goal is to get through minimization existence of non-trivial solutions for problem (P). This problem statement is meaningful provided $g = 0$ and $f(x, 0) = 0$ for a.e. $x \in \Omega$, which we suppose in the following result. Moreover, we establish existence of constant sign solutions of (P).

THEOREM 3 *Assume that hypotheses (H₁) with $q < p$, (H₂), (H₃) with $g = 0$ and $f(x, 0) = 0$ for a.e. $x \in \Omega$ are satisfied. If, in addition, one has*

(H₄⁺) There are constants $\theta \in [1, p)$ and $\varepsilon_0 > 0$ such that

$$\liminf_{t \rightarrow 0^+} \frac{F(x, t)}{t^\theta} \geq \varepsilon_0 \quad \text{uniformly for a.a. } x \in \Omega,$$

then problem (P) possesses at least one non-trivial non-negative solution. Symmetrically, if, in addition, one has

(H₄⁻) There are constants $\theta \in [1, p)$ and $\varepsilon_0 > 0$ such that

$$\liminf_{t \rightarrow 0^-} \frac{F(x, t)}{|t|^\theta} \geq \varepsilon_0 \quad \text{uniformly for a.a. } x \in \Omega,$$

then problem (P) possesses at least one non-trivial non-positive solution.

Proof We suppose that (H₄⁺) holds. Consider the truncation

$$\hat{f}(x, s) = \begin{cases} f(x, s) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0, \end{cases}$$

which is a Carathéodory function, and the associated problem

$$\begin{cases} -\Delta_{p,w} u = w_0(x) \hat{f}(x, u) & \text{in } \Omega \\ u \in L_0^{1,p}(\Omega, w). \end{cases} \quad (\hat{P})$$

From Theorem 2 we know that problem (\hat{P}) admits a solution u , which (in view of the proof of Theorem 2) is actually a solution for the problem

$$\begin{aligned} & \text{Minimize } \hat{\Phi}(u), \\ & u \in L_0^{1,p}(\Omega, w) \end{aligned} \quad (\hat{Q})$$

where $\hat{\Phi}$ is given by (5) with $\hat{F}(x, s) := \int_0^s \hat{f}(x, t) dt$ instead of $F(x, s)$ (that is, (\hat{Q}) is problem (Q) with $g = 0$ and \hat{f} in place of f). Let us note from (5) that $\hat{\Phi}$ coincides with Φ on the non-negative functions.

Let us prove that $u \not\equiv 0$. Fix $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq 0$ in Ω , $\varphi \not\equiv 0$. By (H₄⁺), there exist $\delta, \varepsilon > 0$ such that $F(x, t) \geq \varepsilon t^\theta$ for all $t \in (0, \delta)$ and a.a. $x \in \Omega$. Then, since $\theta < p$, we have

$$\hat{\Phi}(s\varphi) = \Phi(s\varphi) \leq s^p \|\varphi\|_{L^{1,p}(\Omega, w)}^p - s^\theta \int_\Omega \varepsilon \varphi(x)^\theta w_0(x) dx < 0 = \hat{\Phi}(0)$$

provided $s \in (0, \frac{\delta}{\|\varphi\|_\infty})$ is small enough. So, 0 is not a minimizer of $\hat{\Phi}$. Recalling that u solves problem (\hat{Q}), it turns out that $u \not\equiv 0$.

We claim that $u \geq 0$ a.e. in Ω . Since u is a solution of problem (\hat{P}), using Lemma 4 we may write

$$\langle -\Delta_{p,w} u, v \rangle = \int_\Omega \hat{f}(x, u) v(x) w_0(x) dx \quad \text{for all } v \in L_0^{1,p}(\Omega, w). \quad (7)$$

By (H₂) and Theorem 1, we have $L_0^{1,p}(\Omega, w) = W_0^{1,q,p}(\Omega, w_0, w)$, so the fact that $u \in L_0^{1,p}(\Omega, w)$ and Proposition 2 imply that $u^- \in L_0^{1,p}(\Omega, w)$. Hence, setting $v = -u^-$ is

admissible in (7), which yields

$$\int_{\{u < 0\}} |\nabla u|^p w(x) dx = \int_{\{u < 0\}} \hat{f}(x, u(x)) u(x) w_0(x) dx = 0.$$

This results in $\nabla u = 0$ a.e. in $\{x \in \Omega : u(x) < 0\}$. Then Proposition 2 leads to $\nabla u^- = 0$ a.e. in Ω , thereby u^- is constant a.e. in Ω . Since u^- belongs to $L_0^{1,p}(\Omega, w)$ and $L_0^{1,p}(\Omega, w)$ does not contain the non-zero constants (cf. Lemma 1(c)), it turns out that $u^- = 0$ a.e. in Ω . Therefore, the claim that $u \geq 0$ a.e. in Ω is proven. Taking this into account, (7) becomes

$$\langle -\Delta_{p,w} u, v \rangle = \int_{\Omega} f(x, u) v(x) w_0(x) dx \quad \text{for all } v \in L_0^{1,p}(\Omega, w),$$

which completes the proof. \square

From Theorem 3 we deduce the following multiplicity result.

COROLLARY 1 *Assume that hypotheses (H₁) with $q < p$, (H₂), (H₃), (H₄⁺), (H₄⁻) hold. Then problem (P) has at least two non-trivial solutions of opposite constant sign.*

Example 1 Let $p > q > 1$. Let $w \in \mathcal{W}_p(\mathbb{R}^N)$ and $w_0 \in \mathcal{W}_q(\mathbb{R}^N)$. Assume that the embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ is compact and that Ω is bounded. The hypotheses of Corollary 1 are fulfilled by the non-linearity

$$f(x, s) = f(s) = \lambda |s|^{q-2} s + \mu |s|^{\theta-2} s$$

(for the sake of simplicity we dropped the dependence with respect to $x \in \Omega$), with some $\theta \in [1, q)$, $\lambda \in \mathbb{R}$, and $\mu \geq 0$, with $\mu > 0$ if $\lambda \leq 0$.

Remark 7 Large classes of examples of domains and related weights satisfying hypotheses (H₂), (H₃) are constructed in [4]. Namely, we mention weights behaving like $|\cdot|^\alpha$ for certain $\alpha \in \mathbb{R}$ and weights expressed by means of Jacobians for conformal diffeomorphisms.

4. Minimax approach

In this section, we focus on the situation $q > p$ in (H₁), which is complementary to the one handled in Section 3. Under this condition, the functional Φ introduced in (5) is no more coercive and the approach in Section 3 relying on global minimization does not work. Instead, we will use minimax techniques arguing with the restriction of the functional Φ to the Banach space $W^{1,q,p}(\Omega, w_0, w)$ with w, w_0 as in (H₃), that is, $\Phi : W^{1,q,p}(\Omega, w_0, w) \rightarrow \mathbb{R}$ given by

$$\Phi(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p w(x) dx - \int_{\Omega} F(x, u(x)) w_0(x) dx \quad \text{for all } u \in W^{1,q,p}(\Omega, w_0, w). \quad (8)$$

LEMMA 5 Under assumption (H₁), the functional Φ in (8) is of class C^1 and its differential is expressed by

$$\Phi'(u)(v) = \int_{\Omega} |\nabla u|^{p-2} (\nabla u \cdot \nabla v) w(x) dx - \int_{\Omega} f(x, u(x)) v(x) w_0(x) dx \quad (9)$$

for all $u, v \in W^{1,q,p}(\Omega, w_0, w)$.

Proof Using the growth condition required in (H₁) and the continuous embedding $W^{1,q,p}(\Omega, w_0, w) \subset L^q(\Omega, w_0)$, we can proceed as in [7, p.91–93]. \square

On the basis of (H₂) and (H₃), we have that $g + L_0^{1,p}(\Omega, w)$ is a closed, convex subset of $W^{1,q,p}(\Omega, w_0, w)$ (see Theorem 1). Consequently, its indicator function $I_{g+L_0^{1,p}(\Omega, w)} : W^{1,q,p}(\Omega, w_0, w) \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$I_{g+L_0^{1,p}(\Omega, w)}(u) = \begin{cases} 0 & \text{if } u \in g + L_0^{1,p}(\Omega, w) \\ +\infty & \text{if } u \in W^{1,q,p}(\Omega, w_0, w) \setminus (g + L_0^{1,p}(\Omega, w)) \end{cases}$$

is lower semicontinuous, convex, and not identically $+\infty$. We recall from [8] (see also [9, Chapter 3] for an extension) that a critical point of $\Phi + I_{g+L_0^{1,p}(\Omega, w)} : W^{1,q,p}(\Omega, w_0, w) \rightarrow \mathbb{R} \cup \{+\infty\}$ means a point $u \in W^{1,q,p}(\Omega, w_0, w)$ such that

$$\Phi'(u)(v - u) + I_{g+L_0^{1,p}(\Omega, w)}(v) - I_{g+L_0^{1,p}(\Omega, w)}(u) \geq 0 \quad \text{for all } v \in W^{1,q,p}(\Omega, w_0, w).$$

Equivalently, this is expressed as

$$u \in g + L_0^{1,p}(\Omega, w) \quad \text{and} \quad \Phi'(u)(v - u) \geq 0 \quad \text{for all } v \in g + L_0^{1,p}(\Omega, w).$$

Since $L_0^{1,p}(\Omega, w)$ is a vector subspace of $W^{1,q,p}(\Omega, w_0, w)$, it reads as

$$u \in g + L_0^{1,p}(\Omega, w) \quad \text{and} \quad \Phi'(u)(v) = 0 \quad \text{for all } v \in L_0^{1,p}(\Omega, w). \quad (10)$$

LEMMA 6 Assume that (H₁), (H₂), (H₃) hold. Then $u \in W^{1,q,p}(\Omega, w_0, w)$ is a solution of problem (P) if and only if u is a critical point of $\Phi + I_{g+L_0^{1,p}(\Omega, w)}$.

Proof The proof follows directly from (10) and Lemma 4. \square

In order to compensate the lack of coercivity, we impose some additional hypotheses:

(H₄) $w_0 \in \mathcal{W}_p(\mathbb{R}^N) (\subset \mathcal{W}_q(\mathbb{R}^N))$, $g \in L^p(\Omega, w_0)$, and there exists a compact embedding $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$, so a constant $c_p > 0$ can be found such that

$$\|u\|_{L^p(\Omega, w_0)} \leq c_p \|u\|_{L^{1,p}(\Omega, w)} \quad \text{for all } u \in L_0^{1,p}(\Omega, w).$$

(H₅) There exist constants $r \in (p, q]$, $c_1 > 0$, $\alpha \in [0, (pc_p^p)^{-1}]$, $\beta > 0$ and a function $\varphi \in L^1(\Omega, w_0)$ such that

$$c_1 |s|^r - \varphi(x) \leq F(x, s) \leq \alpha |s|^p + \beta |s|^q \quad \text{for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R}.$$

(H₆) $\int_{\Omega} F(x, g(x)) w_0(x) dx \geq \frac{1}{p} \|g\|_{L^{1,p}(\Omega, w)}^p$.

Remark 8

- (a) The assumptions $g \in L^q(\Omega, w_0)$ in (H₃) and $g \in L^p(\Omega, w_0)$ in (H₄) imply that $g \in L^r(\Omega, w_0)$ because we have $|g|^r \leq |g|^p + |g|^q$. Similarly, the continuous embeddings $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ in (H₂) and $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$ in (H₄) determine a continuous embedding $L_0^{1,p}(\Omega, w) \subset L^r(\Omega, w_0)$.
- (b) If we have $w_0 \in L^1(\Omega)$ (and $w_0 \in \mathcal{W}_p(\mathbb{R}^N)$), then hypothesis (H₄) is fulfilled. Indeed, according to (H₂), we have the compact embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$, so there is a constant $c_q > 0$ such that

$$\|u\|_{L^q(\Omega, w_0)} \leq c_q \|u\|_{L^{1,p}(\Omega, w)} \quad \text{for all } u \in L_0^{1,p}(\Omega, w). \quad (11)$$

Because $q > p$, we have the continuous embedding $L^q(\Omega, w_0) \subset L^p(\Omega, w_0)$ and Hölder inequality yields

$$\|u\|_{L^p(\Omega, w_0)} \leq \|w_0\|_{L^1(\Omega)}^{\frac{q-p}{q}} \|u\|_{L^q(\Omega, w_0)} \quad \text{for all } u \in L_0^{1,p}(\Omega, w).$$

- (c) If $pF(x, g(x))w_0(x) - |\nabla g(x)|^p w(x) \geq 0$ for a.a. $x \in \Omega$, then (H₆) is satisfied.

Two more hypotheses are needed to develop our variational approach:

(H₇) $\|g\|_{L^{1,p}(\Omega, w)} < \left(\frac{1-\alpha c_p^p}{\beta c_q^q}\right)^{\frac{1}{q-p}}$, with c_p as in (H₄) and $c_q > 0$ standing for the best constant for the embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ (see (11)).

(H₈) There are constants $\mu > p$, $m \in (1, p)$, $\tilde{c} > 0$, and a function $\psi \in L^1(\Omega, w_0)$ such that a continuous embedding $L_0^{1,p}(\Omega, w) \subset L^m(\Omega, w_0)$ holds and

$$f(x, s)(s - g(x)) - \mu F(x, s) \geq -\tilde{c}|s|^m - \psi(x) \quad \text{for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}.$$

Remark 9

- (a) In the case where g is a constant function, (H₇) is satisfied.
- (b) Hypothesis (H₈) extends the classical Ambrosetti–Rabinowitz condition by considering the boundary condition g , relaxing the growth, and no requirement about the sign of F .

Our main result of this section is the following multiplicity theorem.

THEOREM 4 *Assume that hypotheses (H₁)–(H₈) hold and that $q > p$. Then problem (P) admits at least two solutions: one is g , and it is a local minimizer of the functional $\Phi + I_{g+L_0^{1,p}(\Omega, w)}$, whereas the second one is of mountain pass type.*

Proof We are arguing on the Banach space $W^{1,q,p}(\Omega, w_0, w)$. Assumptions (H₂) and (H₃) guarantee that $g + L_0^{1,p}(\Omega, w) \subset W^{1,q,p}(\Omega, w_0, w)$.

Claim 1 Given $u_0 \in L_0^{1,p}(\Omega, w)$, $u_0 \neq 0$, we have

$$\lim_{t \rightarrow +\infty} \Phi(g + tu_0) = -\infty.$$

Using the first inequality in (H₅), we find that

$$\begin{aligned}
\Phi(g + tu_0) &= \frac{1}{p} \int_{\Omega} |\nabla(g + tu_0)|^p w \, dx - \int_{\Omega} F(x, g + tu_0) w_0 \, dx \\
&\leq \frac{1}{p} \int_{\Omega} |\nabla(g + tu_0)|^p w \, dx - c_1 \int_{\Omega} |g + tu_0|^r w_0 \, dx + \int_{\Omega} \varphi w_0 \, dx \\
&\leq \frac{2^{p-1}}{p} \|g\|_{L^{1,p}(\Omega, w)}^p + \frac{2^{p-1} t^p}{p} \|u_0\|_{L^{1,p}(\Omega, w)}^p - \frac{c_1}{2^{r-1}} t^r \|u_0\|_{L^r(\Omega, w_0)}^r \\
&\quad + c_1 \|g\|_{L^r(\Omega, w_0)}^r + \|\varphi\|_{L^1(\Omega, w_0)}.
\end{aligned}$$

This implies Claim 1 because $r > p$.

Claim 2 g is a local minimizer of $\Phi + I_{g+L_0^{1,p}(\Omega, w)}$, so a solution of problem (P) (by Lemma 6), and for every $\rho > 0$ small enough there holds

$$\inf_{\substack{u \in g + L_0^{1,p}(\Omega, w) \\ \|u - g\|_{L^{1,p}(\Omega, w)} = \rho}} \Phi(u) > \Phi(g).$$

Using the second inequality in (H₅) in conjunction with (H₄), (11), and (H₆), for all $u \in g + L_0^{1,p}(\Omega, w)$ we have

$$\begin{aligned}
\Phi(u) - \Phi(g) &= \frac{1}{p} \int_{\Omega} |\nabla u|^p w \, dx - \int_{\Omega} F(x, u) w_0 \, dx - \frac{1}{p} \int_{\Omega} |\nabla g|^p w \, dx + \int_{\Omega} F(x, g) w_0 \, dx \\
&\geq \frac{1}{p} \|u\|_{L^{1,p}(\Omega, w)}^p - \alpha \|u\|_{L^p(\Omega, w_0)}^p - \beta \|u\|_{L^q(\Omega, w_0)}^q \\
&\quad + \int_{\Omega} F(x, g) w_0 \, dx - \frac{1}{p} \|g\|_{L^{1,p}(\Omega, w)}^p \\
&\geq \|u\|_{L^{1,p}(\Omega, w)}^p \left(\frac{1}{p} - \alpha c_p^p - \beta c_q^q (\|g\|_{L^{1,p}(\Omega, w)} + \|u - g\|_{L^{1,p}(\Omega, w)})^{q-p} \right).
\end{aligned}$$

Since $C := \frac{1}{p} - \alpha c_p^p > \beta c_q^q \|g\|_{L^{1,p}(\Omega, w)}^{q-p}$ (see (H₇)), for $\rho > 0$ sufficiently small we obtain $\Phi(u) - \Phi(g) > c(\rho) > 0$ whenever $\|u - g\|_{L^{1,p}(\Omega, w)} = \rho$, with some constant $c(\rho) > 0$. This proves Claim 2.

Now we consider the set

$$U = \{u \in W^{1,q,p}(\Omega, w_0, w) : \|u - g\|_{L^{1,p}(\Omega, w)} < \rho\}$$

with $\rho > 0$ given in Claim 2. Note that U is an open neighbourhood of g in $W^{1,q,p}(\Omega, w_0, w)$. By virtue of Claim 1, we can find $g_0 \in g + L_0^{1,p}(\Omega, w)$ such that $g_0 \notin \bar{U}$ and $\Phi(g_0) < \Phi(g)$. Set $Q = \{tg + (1-t)g_0 : t \in [0, 1]\}$, $\partial Q = \{g, g_0\}$, and $S = \partial U = \{u \in W^{1,q,p}(\Omega, w_0, w) : \|u - g\|_{L^{1,p}(\Omega, w)} = \rho\}$. The sets $(Q, \partial Q, S)$ link, that is

$$S \cap \partial Q = \emptyset \tag{12}$$

and

$$\gamma(Q) \cap S \neq \emptyset \text{ for all } \gamma \in \Gamma := \{\gamma \in C(Q, W^{1,q,p}(\Omega, w_0, w)) : \gamma|_{\partial Q} = \text{id}_{\partial Q}\}. \tag{13}$$

Claims 1 and 2 entail

$$\inf_S (\Phi + I_{g+L_0^{1,p}(\Omega,w)}) > \max\{\Phi(g), \Phi(g_0)\} = \max_{\partial Q} (\Phi + I_{g+L_0^{1,p}(\Omega,w)}). \quad (14)$$

For proving the existence of a critical point of the functional $\Phi + I_{g+L_0^{1,p}(\Omega,w)}$ distinct from g , it suffices to apply the minimax principle in [9, Theorem 3.2]. This is possible on the basis of (12), (13), (14), once we know that $\Phi + I_{g+L_0^{1,p}(\Omega,w)}$ verifies the compactness property called the Palais–Smale condition.

Claim 3 The functional $\Phi + I_{g+L_0^{1,p}(\Omega,w)} : W^{1,q,p}(\Omega, w_0, w) \rightarrow \mathbb{R}$ satisfies the Palais–Smale condition, that is, every sequence $(u_n)_{n \geq 1} \subset g + L_0^{1,p}(\Omega, w)$ such that

$$|\Phi(u_n)| \leq M \quad \text{for all } n \geq 1, \quad (15)$$

with some $M > 0$, and

$$|\Phi'(u_n)(v)| \leq \varepsilon_n \|v\|_{W^{1,q,p}(\Omega, w_0, w)} \quad \text{for all } v \in L_0^{1,p}(\Omega, w), \text{ all } n \geq 1, \quad (16)$$

with $(\varepsilon_n)_{n \geq 1} \subset (0, +\infty)$, $\varepsilon_n \rightarrow 0$, has a convergent subsequence in $W^{1,q,p}(\Omega, w_0, w)$.

Let a sequence $(u_n)_{n \geq 1} \subset g + L_0^{1,p}(\Omega, w)$ satisfy (15) and (16). Then, for all $n \geq 1$, by (15) we have

$$\frac{1}{p} \int_{\Omega} |\nabla u_n|^p w \, dx - \int_{\Omega} F(x, u_n) w_0 \, dx \leq M$$

and, from (16) with $v = u_n - g$,

$$\begin{aligned} & - \int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n \cdot \nabla (u_n - g)) w \, dx + \int_{\Omega} f(x, u_n) (u_n - g) w_0 \, dx \\ & \leq \varepsilon_n \|u_n - g\|_{W^{1,q,p}(\Omega, w_0, w)}. \end{aligned}$$

Multiplying the former inequality by $\mu > p$ (see (H₈)), adding it to the latter one, and using (H₈), we obtain

$$\begin{aligned} & \left(\frac{\mu}{p} - 1\right) \|u_n\|_{L^{1,p}(\Omega, w)}^p \\ & \leq \int_{\Omega} (\mu F(x, u_n) - f(x, u_n)(u_n - g)) w_0 \, dx + \mu M + \varepsilon_n \|u_n - g\|_{W^{1,q,p}(\Omega, w_0, w)} \\ & \quad - \int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n \cdot \nabla g) w \, dx \\ & \leq \tilde{c} \|u_n\|_{L^m(\Omega, w_0)}^m + \|\psi\|_{L^1(\Omega, w_0)} + \mu M + \varepsilon_n \|u_n\|_{W^{1,q,p}(\Omega, w_0, w)} \\ & \quad + \varepsilon_n \|g\|_{W^{1,q,p}(\Omega, w_0, w)} + \|u_n\|_{L^{1,p}(\Omega, w)}^{p-1} \|g\|_{L^{1,p}(\Omega, w)}. \end{aligned}$$

Due to the continuous embeddings $L_0^{1,p}(\Omega, w) \subset L^m(\Omega, w_0)$ and $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ (see (H₂) and (H₈)), this implies that

$$(\|u_n\|_{L^{1,p}(\Omega, w)})_{n \geq 1} \text{ is bounded.} \quad (17)$$

Writing $u_n = g + v_n$ with $v_n \in L_0^{1,p}(\Omega, w)$, from (17) we know that the sequence $(v_n)_{n \geq 1}$ is bounded in $L_0^{1,p}(\Omega, w)$, hence in $L^q(\Omega, w_0)$ as postulated by (H₂). Therefore we have

$$(\|u_n\|_{L^q(\Omega, w_0)})_{n \geq 1} \text{ is bounded.} \quad (18)$$

Combining (17) and (18), we infer that

$$(u_n)_{n \geq 1} \text{ is bounded in } W^{1,q,p}(\Omega, w_0, w).$$

So, along a relabelled subsequence, we obtain that

$$u_n \rightharpoonup u \quad \text{in } W^{1,q,p}(\Omega, w_0, w). \quad (19)$$

We claim that expressing $u = g + v$, with $v \in L_0^{1,p}(\Omega, w)$, there holds

$$v_n \rightharpoonup v \quad \text{in } L_0^{1,p}(\Omega, w). \quad (20)$$

To this end, we note that (19) implies $v_n \rightharpoonup v$ in $W^{1,q,p}(\Omega, w_0, w)$. In order to check (20), let $\varphi \in L_0^{1,p}(\Omega, w)^*$. Since $L_0^{1,p}(\Omega, w)$ is a closed subspace of $W^{1,q,p}(\Omega, w_0, w)$, the Hahn–Banach theorem provides $\psi \in W^{1,q,p}(\Omega, w_0, w)^*$ such that $\psi|_{L_0^{1,p}(\Omega, w)} = \varphi$. It follows that $\varphi(v_n) = \psi(v_n) \rightarrow \psi(v) = \varphi(v)$ as $n \rightarrow \infty$, which establishes (20).

Now, (20), the compactness of the embeddings $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$ and $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ (see (H₂), (H₄)), and as $g \in L^p(\Omega, w_0) \cap L^q(\Omega, w_0)$ (see (H₃), (H₄)) lead to

$$u_n \rightarrow u \quad \text{in } L^p(\Omega, w_0) \text{ and } L^q(\Omega, w_0) \quad \text{as } n \rightarrow \infty, \quad (21)$$

and so

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^p(\Omega, w_0)} = \|u\|_{L^p(\Omega, w_0)}. \quad (22)$$

Next, we show that

$$\lim_{n \rightarrow \infty} \|u_n\|_{L^{1,p}(\Omega, w)} = \|u\|_{L^{1,p}(\Omega, w)}. \quad (23)$$

Indeed, the weak lower semicontinuity of the seminorm $\|\cdot\|_{L^{1,p}(\Omega, w)}$ yields

$$\liminf_{n \rightarrow \infty} \|u_n\|_{L^{1,p}(\Omega, w)} \geq \|u\|_{L^{1,p}(\Omega, w)}. \quad (24)$$

Next, inserting in (16) the test function $v = u_n - u \in L_0^{1,p}(\Omega, w)$ and taking into account (19), we obtain

$$\lim_{n \rightarrow \infty} \Phi'(u_n)(u_n - u) = 0,$$

that is

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n \cdot \nabla (u_n - u)) w \, dx - \int_{\Omega} f(x, u_n) (u_n - u) w_0 \, dx \right) = 0.$$

Since, by (H₁) and (21), the limit of the second integral is equal to zero, we derive

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|u_n\|_{L^{1,p}(\Omega, w)}^p &= \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n|^{p-2} (\nabla u_n \cdot \nabla u) w \, dx \\ &\leq \limsup_{n \rightarrow \infty} \|u_n\|_{L^{1,p}(\Omega, w)}^{p-1} \|u\|_{L^{1,p}(\Omega, w)}, \end{aligned}$$

thereby

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L^{1,p}(\Omega, w)} \leq \|u\|_{L^{1,p}(\Omega, w)}.$$

Combining this with (24), the convergence in (23) ensues.

We consider the reflexive Banach space $W^{1,p,p}(\Omega, w_0, w)$ endowed with the norm

$$\|z\| := \left(\|z\|_{L^p(\Omega, w_0)}^p + \|z\|_{L^{1,p}(\Omega, w)}^p \right)^{\frac{1}{p}}$$

(recall that $w, w_0 \in \mathcal{W}_p(\mathbb{R}^N)$, see (H₃), (H₄)). Equipped with this norm, the map $z \mapsto (zw_0^{\frac{1}{p}}, w^{\frac{1}{p}}\nabla z)$ is an isometrical embedding of $W^{1,p,p}(\Omega, w_0, w)$ into the space $L^p(\Omega) \times L^p(\Omega)^N$ endowed with its usual norm. The last norm is uniformly convex, thereby $W^{1,p,p}(\Omega, w_0, w)$ is uniformly convex (see [10, Proposition 5.2.7]), so that it satisfies the Kadec–Klee property (see [10, p.453]).

We now claim that

$$u_n \rightarrow u \text{ in } W^{1,p,p}(\Omega, w_0, w) \quad \text{as } n \rightarrow \infty. \quad (25)$$

To show this, using that

$$\|u_n\| \rightarrow \|u\| \quad \text{as } n \rightarrow \infty \quad (26)$$

(see (22) and (23)), and because $\|\cdot\|$ satisfies the Kadec–Klee property, it suffices to check that

$$u_n \rightharpoonup u \text{ in } W^{1,p,p}(\Omega, w_0, w) \quad \text{as } n \rightarrow \infty. \quad (27)$$

From (26) it follows that $(u_n)_{n \geq 1}$ is bounded in $W^{1,p,p}(\Omega, w_0, w)$, hence there is $\hat{u} \in W^{1,p,p}(\Omega, w_0, w)$ such that along a subsequence

$$u_n \rightharpoonup \hat{u} \text{ in } W^{1,p,p}(\Omega, w_0, w) \text{ as } n \rightarrow \infty.$$

Since u_n is of the form $u_n = v_n + g$ with $v_n \in L_0^{1,p}(\Omega, w)$, there is $\hat{v} \in L_0^{1,p}(\Omega, w)$ such that $\hat{u} = \hat{v} + g$ and we have $v_n \rightharpoonup \hat{v}$ in $W^{1,p,p}(\Omega, w_0, w)$ (where we use that $g + L_0^{1,p}(\Omega, w)$ is convex, closed and so weakly closed). Taking into account that $L_0^{1,p}(\Omega, w) = W_0^{1,p,p}(\Omega, w_0, w)$ (due to (H₄) and Theorem 1), arguing as for (20), we obtain that $v_n \rightharpoonup \hat{v}$ in $L_0^{1,p}(\Omega, w)$, hence (by (H₄)), $u_n \rightarrow \hat{u}$ in $L^p(\Omega, w_0)$. On the other hand, from (21) we know that $u_n \rightarrow u$ in $L^p(\Omega, w_0)$. We infer that $\hat{u} = u$, whence (27). Therefore, (25) holds.

From (25), we derive that

$$\nabla u_n \rightarrow \nabla u \text{ in } L^p(\Omega, w)^N.$$

Combining this with (21), we conclude that

$$u_n \rightarrow u \text{ in } W^{1,q,p}(\Omega, w_0, w).$$

This proves Claim 3 and completes the proof of the theorem. \square

Example 2 Let $q > p > 1$, with $q < p + 1$. Let $w \in \mathcal{W}_p(\mathbb{R}^N)$ and $w_0 \in \mathcal{W}_p(\mathbb{R}^N) \cap L^1(\Omega)$, so that $L^\infty(\Omega) \subset L^q(\Omega, w_0) \subset L^p(\Omega, w_0)$. Assume that the embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ is compact and that Ω is bounded. The hypotheses of Theorem 4 are satisfied by the non-linearity

$$f(x, s) = f(s) = |s|^{q-2}s + \lambda|s|^{\theta-2}s + \nu|s|^{p-2}s$$

(for the sake of simplicity we dropped the dependence with respect to $x \in \Omega$), with some $\theta \in (p, q)$, $\lambda \geq 0$, $\nu \in [0, c_p^{-p}]$, where c_p is the best constant for the continuous embedding $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$, and for any $g \in L^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ with

$$\frac{1}{p} \|g\|_{L^{1,p}(\Omega, w)}^p \leq \frac{1}{q} \|g\|_{L^q(\Omega, w_0)}^q + \frac{\lambda}{\theta} \|g\|_{L^\theta(\Omega, w_0)}^\theta + \frac{\nu}{p} \|g\|_{L^p(\Omega, w_0)}^p$$

and

$$\|g\|_{L^{1,p}(\Omega, w)} < \max_{\varepsilon \in [0, (\frac{\theta}{\lambda p} (c_p^{-p} - \nu))^{\frac{1}{\theta-p}}]} \left(\frac{\frac{1}{p} - \left(\frac{\nu}{p} - \frac{\lambda \varepsilon^{\theta-p}}{\theta}\right) c_p^p}{\left(\frac{1}{q} + \frac{\lambda}{\theta \varepsilon^{q-\theta}}\right) c_q^q} \right)^{\frac{1}{q-p}},$$

where c_q is the best constant for the continuous embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$. In particular, any constant function g fulfills these properties.

The assumptions of Theorem 4 simplify in the case where $g \equiv 0$, namely instead of (H₄), (H₆), (H₇), (H₈), we assume:

(H'₄) $w, w_0 \in \mathcal{W}_p(\mathbb{R}^N)$ and there is a compact embedding $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$, so a constant $c_p > 0$ can be found such that

$$\|u\|_{L^p(\Omega, w_0)} \leq c_p \|u\|_{L^{1,p}(\Omega, w)} \quad \text{for all } u \in L_0^{1,p}(\Omega, w).$$

(H'₈) There are constants $\mu > p$, $m \in (1, p)$, $\tilde{c} > 0$, and a function $\psi \in L^1(\Omega, w_0)$ such that there is a continuous embedding $L_0^{1,p}(\Omega, w) \subset L^m(\Omega, w_0)$ and

$$f(x, s)s - \mu F(x, s) \geq -\tilde{c}|s|^m - \psi(x) \quad \text{for a.a. } x \in \Omega, \text{ all } s \in \mathbb{R}.$$

COROLLARY 2 *Assume (H₁), (H₂), (H'₄), (H₅), (H'₈) and $q > p$. Then the homogeneous Dirichlet problem*

$$\begin{cases} -\Delta_{p,w} u = w_0(x)f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits at least two solutions: $u = 0$ (which is a local minimizer of the corresponding energy functional on $L_0^{1,p}(\Omega, w)$) and a non-zero solution (which is a critical point of mountain pass type of the energy functional).

Example 3 Let $q > p > 1$. Let $w \in \mathcal{W}_p(\mathbb{R}^N)$ and $w_0 \in \mathcal{W}_p(\mathbb{R}^N) \cap L^1(\Omega)$, so that $L^\infty(\Omega) \subset L^q(\Omega, w_0) \subset L^p(\Omega, w_0)$. Assume that the embedding $L_0^{1,p}(\Omega, w) \subset L^q(\Omega, w_0)$ is compact and that Ω is bounded. The hypotheses of Corollary 2 are fulfilled by the non-linearity

$$f(x, s) = f(s) = |s|^{q-2}s + \lambda|s|^{\theta-2}s + \nu|s|^{p-2}s$$

(for the sake of simplicity we dropped the dependence with respect to $x \in \Omega$), with some $\theta \in (p, q)$, $\lambda \geq 0$, and $\nu \in [0, c_p^{-p}]$, where c_p is the best constant for the continuous embedding $L_0^{1,p}(\Omega, w) \subset L^p(\Omega, w_0)$.

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