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# ON A COUPLING OF SOLUTIONS TO THE INTERFACE SDE ON A STAR GRAPH

HATEM HAJRI <sup>(1)</sup> AND MARC ARNAUDON <sup>(2)</sup>

ABSTRACT. Inspired by Tsirelson proof of the non Brownian character of Walsh Brownian motion filtration on three or more rays, we prove some results on a particular coupling of solutions to the interface SDE on a star graph, recently introduced in [6]. This coupling consists in two solutions which are independent given the driving Brownian motion. As a consequence, we deduce that if the star graph contains 3 or more rays, the argument of the solution at a fixed time is independent of the driving Brownian motion.

## 1. INTRODUCTION AND MAIN RESULTS

A filtration  $(\mathcal{F}_t)_t$  has the Brownian representation property (BRP) if there exists a Brownian motion  $B$  such that every  $(\mathcal{F}_t)_t$ -martingale is a stochastic integral of  $B$ . In 1979 Yor posed the reverse problem, i.e whether a filtration having the BRP is necessarily Brownian [13]. At the end of his paper [12], Walsh suggested the study of a Markov process with state space

$$G = \bigcup_{j=1}^N E_j; \quad E_j = \{r e^{i\theta_j} : r \geq 0\}$$

where  $\theta_j$  are given angles. This process, called since then Walsh Brownian motion (WBM), behaves like a standard Brownian motion on each ray; and at 0 it makes excursions with probability  $p_j$  on  $E_j \setminus \{0\}$ . Later on, a detailed study of WBM was given in [1]. In particular, it was shown that WBM is a strong Markov process with Feller semigroup and that the natural filtration  $(\mathcal{F}_t^Z)_t$  of a WBM  $Z$  has the BRP with respect to the Brownian motion  $B$  given by the martingale part of  $|Z|$ , the geodesic distance between  $Z$  and 0.

After nearly two decades a negative answer to Yor's question was finally given by Tsirelson [11]. The result proved by Tsirelson is the following

**Theorem 1.1.** *If  $(\mathcal{G}_t)_t$  is a Brownian filtration, i.e a filtration generated by a finite or infinite family of independent standard Brownian motions, there does not exist any*

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$(\mathcal{G}_t)_t$ -WBM ( $(\mathcal{G}_t)_t$ -Markov process with semigroup  $P$ , the Feller semigroup of WBM) on a star graph with three or more rays.

To prove Theorem 1.1, Tsirelson performs a beautiful reasoning by contradiction. Suppose there exists a Brownian motion  $B$  such  $Z = F(B)$  is a WBM with  $N \geq 3$  rays. Let  $Z^r = F(B^r)$  where  $B^r = rB + \sqrt{1-r^2}B'$  with  $B'$  an independent copy of  $B$ . Then, it is shown that  $\mathbb{E}[d(Z_t^r, Z_t)]$  converges to 0. However, Tsirelson is able to prove that  $\mathbb{E}[d(Z_t^r, Z_t)] > c > 0$  with  $c$  not depending on  $r$ .

In the present paper we are interested in a simple stochastic differential equation on  $G$  whose solutions are WBMs. This SDE is the interface SDE introduced in [6] and driven by an  $N$  dimensional Brownian motion  $W = (W^1, \dots, W^N)$ . While moving inside  $E_i$ , a solution to this equation follows  $W^i$  so that the origin can be seen as an interface at the intersection of the half lines. For  $N = 2$ , the interface SDE is identified with

$$(1) \quad dX_t = 1_{\{X_t > 0\}} dW_t^1 + 1_{\{X_t \leq 0\}} dW_t^2$$

Equation (1) has a unique strong solution [10, 7]. Not knowing Theorem 1.1, one could have the intuition, that similarly to  $N = 2$ , solutions are also strong ones for  $N \geq 3$ . The Theorem implies this cannot be the case.

The main result proved in [6] was the existence of a stochastic flow of mappings, unique in law and a Wiener stochastic flow [8] which solve the interface SDE. The problem of finding the flows of kernels which “interpolate” between these two particular flows was left open in [6]. The answer to this question needs a complete understanding of weak solutions of this equation.

The purpose of the present paper is to establish new results on weak solutions of the interface SDE in the case  $N \geq 3$ . These results are very different from the case  $N = 2$ . Our proofs are largely inspired by Tsirelson proof of Theorem 1.1.

### 1.1. Notations.

This paragraph contains the main notations and definitions which will be used throughout the paper.

Let  $(G, d)$  be a metric star graph with a finite set of rays  $(E_i)_{1 \leq i \leq N}$  and origin denoted by 0. This means that  $(G, d)$  is a metric space,  $E_i \cap E_j = \{0\}$  for all  $i \neq j$  and for each  $i$ , there is an isometry  $e_i : [0, \infty[ \rightarrow E_i$ . We assume  $d$  is the geodesic distance on  $G$  in the sense that  $d(x, y) = d(x, 0) + d(0, y)$  if  $x$  and  $y$  do not belong to the same  $E_i$ .

For any subset  $A$  of  $G$ , we will use the notation  $A^*$  for  $A \setminus \{0\}$ . Also, we define the function  $\varepsilon : G^* \rightarrow \{1, \dots, N\}$  by  $\varepsilon(x) = i$  if  $x \in E_i^*$ .

Let  $C_b^2(G^*)$  denote the set of all continuous functions  $f : G \rightarrow \mathbb{R}$  such that for all  $i \in [1, N]$ ,  $f \circ e_i$  is  $C^2$  on  $]0, \infty[$  with bounded first and second derivatives both with finite limits at  $0+$ . For  $x = e_i(r) \in G^*$ , set  $f'(x) = (f \circ e_i)'(r)$  and  $f''(x) = (f \circ e_i)''(r)$ .

Let  $p_1, \dots, p_N \in (0, 1)$  such that  $\sum_{i=1}^N p_i = 1$  and define

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^N p_i (f \circ e_i)'(0+) = 0 \right\}.$$

For  $f \in C_b^2(G^*)$ , we will take the convention  $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$  and  $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$  so that  $\mathcal{D}$  can be written as  $\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}$ . We are now in position to recall the following

**Definition 1.2.** *A solution of the interface SDE (I) on  $G$  with initial condition  $X_0 = x$  is a pair of processes  $(X, W)$  defined on a filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$  such that*

- (i)  $W = (W^1, \dots, W^N)$  is a standard  $(\mathcal{F}_t)$ -Brownian motion in  $\mathbb{R}^N$ ;
- (ii)  $X$  is an  $(\mathcal{F}_t)$ -adapted continuous process on  $G$ ;
- (iii) For all  $f \in \mathcal{D}$ ,

$$(2) \quad f(X_t) = f(x) + \sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds$$

To emphasize on the filtration  $(\mathcal{F}_t)_t$ , we will sometimes say  $(X, W)$  is an  $(\mathcal{F}_t)_t$ -solution. It has been proved in [6] (Theorem 2.3) that for all  $x \in G$ , (I) admits a solution  $(X, W)$  with  $X_0 = x$ , the law of  $(X, W)$  is unique and  $X$  is an  $(\mathcal{F}_t)$ -WBM on  $G$ . We will denote by  $Q_x$  the law of a solution  $(X, W)$  with  $X_0 = x$ .

Tsirelson theorem 1.1 combined with Theorem 2.3 in [6] show that  $X$  is  $\sigma(W)$ -measurable if and only if  $N \leq 2$ .

Let us give an intuitive description of solutions to the previous equation. Given a WBM  $X$  started from  $x$ , we will denote from now on by  $B^X$  the martingale part of  $|X| - |x|$ . Freidlin-Sheu formula [4] says that for all  $f \in C_b^2(G^*)$

$$(3) \quad f(X_t) = f(x) + \int_0^t f'(X_s) dB_s^X + \frac{1}{2} \int_0^t f''(X_s) ds + \sum_{i=1}^N p_i (f \circ e_i)'(0+) L_t(|X|)$$

with  $L_t(|X|)$  denoting the local time of  $|X|$ . Comparing (2) with (3), one gets

$$(4) \quad B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW_s^i$$

Thus, while it moves inside  $E_i$ ,  $X$  follows the Brownian motion  $W^i$  which shows that (2) extends (1) in a natural way.

Let us now introduce the following

**Definition 1.3.** We say that  $(X, Y, W)$  is a coupling of solutions to (I) if  $(X, W)$  and  $(Y, W)$  satisfy Definition 1.2 on the same filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$ .

A trivial coupling of solutions to (I) is given by  $(X, X, W)$  where  $(X, W)$  solves (I). This is also the law unique coupling of solutions to (I) if  $N \leq 2$  as  $\sigma(X) \subset \sigma(W)$  in this case. Let us now introduce another interesting coupling.

**Definition 1.4.** A coupling  $(X, Y, W)$  of solutions to (I) is called the Wiener coupling if  $X$  and  $Y$  are independent given  $W$ .

The existence of the Wiener coupling is easy to check. For this note there exists a law unique triplet  $(X, Y, W)$  such that  $(X, W)$  and  $(Y, W)$  are distributed respectively as  $Q_x$  and  $Q_y$  and moreover  $X$  and  $Y$  are independent given  $W$ . It remains to check that  $W$  is a standard  $(\mathcal{F}_t)$ -Brownian motion in  $\mathbb{R}^N$  where  $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, u \leq t)$ . This holds from the conditional independence between  $X$  and  $Y$  given  $W$  and the fact that  $W$  is a Brownian motion with respect to the natural filtrations of  $(X, W)$  and  $(Y, W)$ . The reason for choosing the name Wiener for this coupling will be justified in Section 3 in connection with stochastic flows.

## 1.2. Main results.

Given a WBM  $X$  on  $G$ , we define the process  $\bar{X}$  by

$$\bar{X}_t = 1_{\{X_t \neq 0\}} \sum_{i=1}^N 1_{\{\varepsilon(X_t)=i\}} \times e_i \left( \frac{e_i^{-1}(X_t)}{Np_i} \right)$$

Note that  $\bar{X} = X$  if  $p_i = \frac{1}{N}$  for all  $1 \leq i \leq N$ . Following the terminology used in [2], the process  $\bar{X}$  is a spidermartingale (“martingale-araignée”). In fact, for all  $1 \leq i \leq N$ , define

$$(5) \quad \bar{X}_t^i = |\bar{X}_t| \text{ if } \bar{X}_t \in E_i \text{ and } \bar{X}_t^i = 0 \text{ if not}$$

Note that  $\bar{X}_t^i = f^i(X_t)$ , where  $f^i(x) = \frac{|x|}{Np_i} 1_{\{x \in E_i\}}$ . Applying Freidlin-Sheu formula (3) for  $X$  and the function  $f^i$  shows that

$$(6) \quad \bar{X}_t^i = \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \frac{1}{N} L_t(|X|)$$

In particular,  $\bar{X}_t^i - \bar{X}_t^j$  is a martingale for all  $i, j \in [1, N]$ . Proposition 5 in [2] shows that  $\bar{X}$  is a spidermartingale.

Our main result in this paper is the following.

**Theorem 1.5.** Assume  $N \geq 3$ . Let  $(X, Y, W)$  be the Wiener coupling of solutions to (I) with  $X_0 = Y_0 = 0$ . Then

(i)  $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$  is a martingale. In particular,

$$\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)] = 2 \frac{N-2}{N} \sqrt{\frac{2t}{\pi}}.$$

- (ii) Call  $g_t^X$  and  $g_t^Y$  the last zeroes before  $t$  of  $X$  and  $Y$ , then for all  $t > 0$ ,  $\mathbb{P}(g_t^X = g_t^Y) = 0$  and  $\mathbb{P}(X_t = Y_t) = 0$ .
- (iii)  $\varepsilon(X_t)$  and  $\varepsilon(Y_t)$  are independent for all  $t > 0$ .

Another important fact about  $(X, Y, W)$  proved in [6], also true for  $N = 2$ , says that  $(X, Y, W)$  is a strong Markov process associated with a Feller semigroup. This result will be sketched in Section 3 below.

The claim (ii) says that common zeros of  $X$  and  $Y$  are rare. It has been proved in [6], that couplings  $(X, Y)$  to  $(I)$  have the same law before  $T = \inf\{t \geq 0 : X_t = Y_t\}$  and that  $T < \infty$  with probability one. The strong Markov property shows then that the set of common zeros of  $X$  and  $Y$  is infinite.

**The case  $N = 2$ .** Point (i) in Theorem 1.5 is also true for  $N = 2$  since  $X = Y$  in this case [6]. This can also be deduced from the proofs below. In fact, Proposition 2.1 claims that  $\Lambda_t$  defined as the local time of the semimartingale  $d(\overline{X}_t, \overline{Y}_t)$  is zero for all  $N \geq 2$ . By the usual Tanaka formula (see also Proposition 2.6),

$$d(\overline{X}_t, \overline{Y}_t) = M_t + \frac{1}{2}\Lambda_t$$

where  $M$  is a martingale. Taking the expectation shows that  $\overline{X} = \overline{Y}$  and so  $X = Y$ . The same reasoning applies to any coupling  $(X, Y)$  and in particular pathwise uniqueness holds for (2) in the case  $N = 2$ . Since weak uniqueness is also satisfied, this yields the strong solvability of (2) when  $N = 2$  (or equivalently (1)).

Theorem 1.5 yields the following important

**Corollary 1.6.** *Assume  $N \geq 3$ . Let  $(X, W)$  be a solution of (I) with  $X_0 = 0$ . Then for each  $t > 0$ ,  $\varepsilon(X_t)$  is independent of  $W$ .*

This corollary seems to us quite remarkable. In fact, admitting Tsirelson theorem 1.1 and using (4), it can be deduced that  $\varepsilon(X_t)$  is not  $\sigma(W)$ -measurable (actually neither  $\varepsilon(X_t)$  nor  $|X_t|$  are  $\sigma(W)$ -measurable). However, Corollary (1.6) gives a much stronger result than this non-measurability. Comparing this with the case  $N = 2$ , in which  $\varepsilon(X_t)$  is  $\sigma(W)$ -measurable, shows that stochastic differential equations on star graphs with  $N \geq 3$  rays involve interesting “phase transitions”.

Corollary 1.6 is easy to deduce from Theorem 1.5. For this, define  $C_t = \mathbb{P}(\varepsilon(X_t) = i | W)$ . Since  $X$  and  $Y$  are independent given  $W$  and  $(X, W)$ ,  $(Y, W)$  have the same law,

$$\mathbb{P}(\varepsilon(X_t) = i)^2 = \mathbb{P}(\varepsilon(X_t) = i, \varepsilon(Y_t) = i) = \mathbb{E}[C_t^2]$$

Thus  $\mathbb{E}[C_t] = \mathbb{E}[C_t^2]^{\frac{1}{2}}$  and so there exists a constant  $c_t$  such that  $C_t = c_t$  a.s. Taking the expectation shows that  $c_t = p_i$ .

Let us now explain our arguments to prove Theorem 1.5.

In Section 2.1, we prove that for any coupling  $(X, Y, W)$  of solutions to (I) such that  $X_0 = Y_0 = 0$ , we have  $L_t(D) = 0$  where  $D_t = d(\overline{X}_t, \overline{Y}_t)$ .

Next, inspired by Tsirelson arguments [11], we consider a perturbation  $W^r = rW + \sqrt{1-r^2}\hat{W}$ ,  $r < 1$  of  $W$ ,  $\hat{W}$  is an independent copy of  $W$ , and  $X^r, Y^r$  such that

- $(X^r, W^r)$  and  $(Y^r, W)$  are solutions to (I).
- $X^r$  and  $Y^r$  are independent given  $(W, \hat{W})$ .

The coupling  $(X^r, Y^r)$  satisfies

$$(7) \quad \frac{d}{dt} \langle |X^r|, |Y^r| \rangle_t \leq r < 1$$

A crucial result proved in [11] (see also [2, 3]) which will be used below says that, since (7) holds,  $L_t(|X^r|)$  and  $L_t(|Y^r|)$  have rare common points of increase (see (ii) in Proposition 2.4 for more precision). The process  $(X^r, Y^r, W)$  is shown to converge in law as  $r \rightarrow 1$  to the Wiener coupling  $(X, Y, W)$  described above. The passage to the limit  $r \rightarrow 1$  allows to deduce the properties mentioned in Theorem 1.5.

Section 3 is a complement based on stochastic flows to the previous results. We consider the Wiener stochastic flow of kernels  $K$  constructed in [6] which is a strong solution to the flows of kernels version of (2) driven by a real white noise  $(W_{s,t})_{s \leq t}$ . The Wiener coupling  $(X, Y, W)$  is shown to be the strong Markov process associated to a Feller semigroup  $Q$  obtained from  $K$ .

## 2. PROOFS

### 2.1. The local time of the distance.

The subject of this paragraph is to prove the following result which, in the case  $N = 2$ , is proved in [6].

**Proposition 2.1.** *Assume  $N \geq 2$ . Let  $(X, Y, W)$  be a coupling of two solutions to (I) with  $X_0 = Y_0 = 0$  and let  $D_t = d(\overline{X}_t, \overline{Y}_t)$ . Then  $D$  is a semimartingale with  $L_t(D) = 0$ .*

*Proof.* The fact that  $D$  is a semimartingale is shown in [11] (see [2] and Proposition 2.6 below for more details). We follow the proof of Proposition 4.5 in [6] and first prove that a.s.

$$(8) \quad \int_{]0, +\infty[} L_t^a(D) \frac{da}{a} < \infty$$

where  $L_t^a(D)$  is the local time of  $D$  at level  $a$  and time  $t$ . Recall that by the occupation formula

$$\int_{]0,+\infty[} L_t^a(D) \frac{da}{a} = \int_0^t 1_{\{D_s > 0\}} \frac{d\langle D \rangle_s}{D_s}$$

By (6),

$$\begin{aligned} |\bar{X}_t| &= \sum_{i=1}^N \bar{X}_t^i = M_t^1 + L_t(|X|) \\ |\bar{Y}_t| &= \sum_{i=1}^N \bar{Y}_t^i = M_t^2 + L_t(|Y|) \end{aligned}$$

with

$$M_t^1 = \sum_{i=1}^N \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB_s^X, \quad M_t^2 = \sum_{i=1}^N \frac{1}{Np_i} \int_0^t 1_{\{Y_s \in E_i\}} dB_s^Y$$

In particular,

$$\langle M^1 \rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i\}} ds, \quad \langle M^2 \rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{Y_s \in E_i\}} ds$$

and

$$\langle M^1, M^2 \rangle_t = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i, Y_s \in E_i\}} ds.$$

Proposition 7 [2] tells us that

$$\begin{aligned} D_t &- \int_0^t 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (dM_s^1 + dM_s^2) \\ &- \int_0^t 1_{\{\varepsilon(X_s) = \varepsilon(Y_s)\}} \operatorname{sgn}(M_s^1 - M_s^2) (dM_s^1 - dM_s^2) \end{aligned}$$

is a continuous increasing process. Consequently,

$$d\langle D \rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2} 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (1_{\{X_s \in E_i\}} + 1_{\{Y_s \in E_i\}}) ds \leq C 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} ds$$

where  $C$  is a positive constant. Note there exists  $C' > 0$  such that  $D_s \geq C'(|X_s| + |Y_s|)$  for all  $s$  such that  $\varepsilon(X_s) \neq \varepsilon(Y_s)$ . Thus, to get (8), it is sufficient to prove

$$\int_0^t 1_{\{X_s \neq 0, Y_s \neq 0\}} 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} \frac{ds}{|X_s| + |Y_s|} < \infty$$

Let us prove for instance that

$$(1) = \int_0^t \frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} ds < \infty$$

Define  $f(z) = |z|$  if  $z \in E_1$  and  $f(z) = -|z|$  if not and set  $x_t = f(X_t), y_t = f(Y_t)$ . Clearly

$$\frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} = \frac{1}{2} \frac{|\operatorname{sgn}(x_s) - \operatorname{sgn}(y_s)|}{|x_s - y_s|} 1_{\{y_s < 0 < x_s\}}.$$

As in [6], let  $(f_n)_n \subset C^1(\mathbb{R})$  such that  $f_n \rightarrow \operatorname{sgn}$  pointwise and  $(f_n)_n$  is uniformly bounded in total variation. Defining  $z_s^u = (1-u)x_s + uy_s$ , we have by Fatou's Lemma

$$\begin{aligned} (1) &\leq \liminf_n \int_0^t 1_{\{y_s < 0 < x_s\}} \frac{|f_n(x_s) - f_n(y_s)|}{|x_s - y_s|} \frac{ds}{2} \\ &\leq \liminf_n \int_0^t 1_{\{y_s < 0 < x_s\}} \int_0^1 |f'_n(z_s^u)| du \frac{ds}{2} \end{aligned}$$

Writing Freidlin-Sheu formula for the function  $f$  applied to  $X$  and  $Y$  shows that on  $\{y_s < 0 < x_s\}$ ,

$$\frac{d}{ds} \langle z^u \rangle_s = u^2 + (1-u)^2 \geq \frac{1}{2}$$

Thus

$$\begin{aligned} (1) &\leq \liminf_n \int_0^1 \int_0^t 1_{\{y_s < 0 < x_s\}} |f'_n(z_s^u)| d\langle z^u \rangle_s du \\ &\leq \liminf_n \int_0^1 \int_{\mathbb{R}} |f'_n(a)| L_t^a(z^u) da du \end{aligned}$$

So a sufficient condition for (1) to be finite is

$$\sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}[L_t^a(z^u)] < \infty$$

By Tanaka's formula

$$\begin{aligned} \mathbb{E}[L_t^a(z^u)] &= \mathbb{E}[|z_t^u - a|] - \mathbb{E}[|z_0^u - a|] - \mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a) dz_s^u\right] \\ &\leq \mathbb{E}[|z_t^u - z_0^u|] - \mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a) dz_s^u\right] \end{aligned}$$

Since  $x$  and  $y$  are two skew Brownian motions, it is easily seen that  $\sup_{u \in [0,1]} \mathbb{E}[|z_t^u - z_0^u|] < \infty$ . The same argument shows that

$$\mathbb{E}\left[\int_0^t \operatorname{sgn}(z_s^u - a) dz_s^u\right]$$

is uniformly bounded with respect to  $(u, a)$  and consequently (1) is finite. Finally  $\int_{]0, +\infty[} L_t^a(D) \frac{da}{a}$  is finite a.s. Since  $\lim_{a \downarrow 0} L^a(D) = L^0(D)$ , we deduce  $L_t^0(D) = 0$ .  $\square$

## 2.2. Proof of Theorem 1.5.

This section gives the proof of our main result. First, we define the perturbation of the Wiener coupling as described in the introduction and then perform a passage to the limit.

**Lemma 2.2.** *For all  $r \in [0, 1]$ , there exists a law unique process  $(X, W, \hat{W})$  such that, denoting  $\mathcal{F}_t = \sigma(X_u, W_u, \hat{W}_u, u \leq t)$ ,*

- $W$  and  $\hat{W}$  are two independent  $(\mathcal{F}_t)_t$ -Brownian motions in  $\mathbb{R}^N$ .
- $(X, W^r)$  is an  $(\mathcal{F}_t)_t$ -solution to (I) with  $X_0 = 0$  and where  $W^r = rW + \sqrt{1 - r^2}\hat{W}$ .

*Proof.* The proof of this lemma is similar to that of Theorem 2.3 in [6]. For the existence part, take independent processes  $X, V^1, \dots, V^N, \dots, V^{2N}$  where  $X$  is a WBM started from 0 and each  $V^i$  is a standard Brownian motion. Denote by  $(\mathcal{G}_t)_t$  the natural filtration of  $(X, V^1, \dots, V^N, \dots, V^{2N})$  and for  $1 \leq i \leq N$ , define

$$d\Gamma_t^i = 1_{\{X_t \in E_i\}} dB_t^X + 1_{\{X_t \notin E_i\}} dV_t^i, \quad dW_t^i = r d\Gamma_t^i + \sqrt{1 - r^2} dV_t^{i+N}$$

and

$$d\hat{W}_t^i = \sqrt{1 - r^2} d\Gamma_t^i - r dV_t^{i+N}.$$

Then  $W = (W^1, \dots, W^N)$ ,  $\hat{W} = (\hat{W}^1, \dots, \hat{W}^N)$  are two independent  $(\mathcal{G}_t)_t$ -Brownian motions in  $\mathbb{R}^N$ ,  $(X, \Gamma^1, \dots, \Gamma^N)$  is a  $(\mathcal{G}_t)_t$ -solution to (I) and since  $d\Gamma_t^i = r dW_t^i + \sqrt{1 - r^2} d\hat{W}_t^i$ , existence holds.

Now let  $(X, W, \hat{W})$  and  $(\mathcal{F}_t)_t$  be as in the lemma. Introduce a Brownian motion  $B$  independent of  $(X, W, \hat{W})$  and define  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_u, u \leq t)$ . Write  $\hat{W} = (\hat{W}^1, \dots, \hat{W}^N)$  and for  $1 \leq i \leq N$ , define

$$d\Gamma_t^i = r dW_t^i + \sqrt{1 - r^2} d\hat{W}_t^i, \quad dV_t^{i+N} = \sqrt{1 - r^2} dW_t^i - r d\hat{W}_t^i$$

and

$$dV_t^i = 1_{\{X_t \notin E_i\}} d\Gamma_t^i + 1_{\{X_t \in E_i\}} dB_t.$$

Note that  $V^1, \dots, V^N, \dots, V^{2N}$  are independent  $(\mathcal{G}_t)_t$ -Brownian motions. Using  $1_{\{X_t \in E_i\}} dB_t^X = 1_{\{X_t \in E_i\}} d\Gamma_t^i$ , simple calculations show that  $(V^1, \dots, V^N, \dots, V^{2N})$  is independent of  $B^X$ . Since  $X$  is a  $(\mathcal{G}_t)_t$ -WBM, Lemma 4.3 in [6] claims that  $X, V^1, \dots, V^N, \dots, V^{2N}$  are independent. Finally  $(X^+, W^+, \hat{W}^+)$  constructed from  $X, V^1, \dots, V^N, \dots, V^{2N}$  as in the existence part coincides with  $(X, W, \hat{W})$ . This finishes the proof.  $\square$

An immediate consequence of the previous lemma is the following

**Lemma 2.3.** *For all  $r \in [0, 1]$ , there exists a law unique process  $(X, Y, W, \hat{W})$  such that, denoting  $\mathcal{F}_t = \sigma(X_u, Y_u, W_u, \hat{W}_u, u \leq t)$ ,*

- *$W$  and  $\hat{W}$  are two independent  $(\mathcal{F}_t)_t$ -Brownian motions in  $\mathbb{R}^N$ .*
- *$(X, W^r)$  and  $(Y, W)$  are two  $(\mathcal{F}_t)_t$ -solutions to (I) with  $X_0 = Y_0 = 0$  and where  $W^r = rW + \sqrt{1-r^2}\hat{W}$ .*
- *$X$  and  $Y$  are independent given  $(W, \hat{W})$ .*

The proof of this lemma is similar to the existence and law uniqueness of the Wiener coupling and is left as an exercise.

In the sequel, we will denote  $(X, Y, W, \hat{W})$  by  $(X^r, Y^r, W, \hat{W})$  and use the notation  $W^r$  to denote  $rW + \sqrt{1-r^2}\hat{W}$ .

**Proposition 2.4.** *The following assertions hold*

- (i)  $d\langle B^{X^r}, B^{Y^r} \rangle_t = r \mathbf{1}_{\{\varepsilon(X_t^r) = \varepsilon(Y_t^r)\}} dt$ .
- (ii)  $\int_0^t \mathbf{1}_{\{Y_s^r \neq 0\}} dL_s(|X^r|) = L_t(|X^r|)$  and  $\int_0^t \mathbf{1}_{\{X_s^r \neq 0\}} dL_s(|Y^r|) = L_t(|Y^r|)$ .

*Proof.* Write  $W^r = (W^{r,1}, \dots, W^{r,N})$ . By the previous lemma

$$dB_t^{X^r} = \sum_{i=1}^N \mathbf{1}_{\{X_t^r \in E_i\}} dW_t^{r,i} \text{ and } dB_t^{Y^r} = \sum_{i=1}^N \mathbf{1}_{\{Y_t^r \in E_i\}} dW_t^i$$

which yields (i). (ii) is Lemma 4.12 in [11] (see also [2, 3]).  $\square$

The next lemma establishes the convergence in law of  $(X^r, Y^r, W)$  to the Wiener coupling  $(X, Y, W)$ .

**Lemma 2.5.** *As  $r \rightarrow 1$ ,  $(X^r, Y^r, W)$  converges in law to  $(X, Y, W)$ , the Wiener coupling of solutions to (I) with  $X_0 = Y_0 = 0$ .*

*Proof.* Let  $(r_n)_n$  be a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} r_n = 1$ . For any  $p \geq 1$ ,  $(f_i, g_i, h_i)_{1 \leq i \leq p}$  bounded,  $(t_i)_{1 \leq i \leq p}$

$$\mathbb{E} \left[ \prod_{i=1}^p f_i(X_{t_i}^{r_n}) g_i(Y_{t_i}^{r_n}) h_i(W_{t_i}) \right] = \mathbb{E} \left[ \prod_{i=1}^p \mathbb{E}[f_i(X_{t_i}^{r_n}) | W, \hat{W}] \mathbb{E}[g_i(Y_{t_i}^{r_n}) | W, \hat{W}] h_i(W_{t_i}) \right]$$

Note that  $(Y^{r_n}, W)$  is independent of  $\hat{W}$ . This can be deduced from the uniqueness part in Lemma 2.2 by taking  $r = 1$ . Consequently  $\mathbb{E}[g_i(Y_{t_i}^{r_n}) | W, \hat{W}] = \mathbb{E}[g_i(Y_{t_i}^{r_n}) | W]$  a.s. Now  $\sigma(W, \hat{W}) = \sigma(W^{r_n}, \overline{W}^{r_n})$  where  $\overline{W}^{r_n}$  is the independent complement to  $W^{r_n}$  given by

$$\overline{W}^{r_n} = \sqrt{1-r_n^2}W - r_n\hat{W}.$$

By the proof of Lemma 2.2,  $(X^{r_n}, W^{r_n}, \overline{W}^{r_n})$  and  $(Y^{r_n}, W, \hat{W})$  have the same law. Consequently  $(X^{r_n}, W^{r_n})$  is also independent of  $\overline{W}^{r_n}$  and so

$$\mathbb{E}[f_i(X_{t_i}^{r_n})|W, \hat{W}] = \mathbb{E}[f_i(X_{t_i}^{r_n})|W^{r_n}, \overline{W}^{r_n}] = \mathbb{E}[f_i(X_{t_i}^{r_n})|W^{r_n}].$$

Slutsky lemma (see Theorem 1 in [2]) shows that for all  $f : G \rightarrow \mathbb{R}$  measurable bounded and  $t > 0$ , as  $n \rightarrow \infty$ ,

$$\mathbb{E}[f(X_{t_i}^{r_n})|W^{r_n}] \longrightarrow Q_t f(W)$$

in probability where  $Q_t f(W) = \int f(y)Q_t(W, dy)$  and  $Q_t(W, dy)$  is a regular conditional expectation of  $X_t$  given  $W$ . Finally

$$\begin{aligned} \lim_n \mathbb{E} \left[ \prod_{i=1}^p f_i(X_{t_i}^{r_n}) g_i(Y_{t_i}^{r_n}) h_i(W_{t_i}) \right] &= \mathbb{E} \left[ \prod_{i=1}^p Q_{t_i} f_i(W) Q_{t_i} g_i(W) h_i(W_{t_i}) \right] \\ &= \mathbb{E} \left[ \prod_{i=1}^p f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i}) \right] \end{aligned}$$

and the lemma is proved. □

Let us now recall Proposition 7 in [2].

**Proposition 2.6.** *Let  $Z^1$  and  $Z^2$  be two WBMs with respect to the same filtration such that  $Z_0^1 = Z_0^2 = 0$ . Denote by  $\Lambda$  the local time of  $D_t = d(\overline{Z}_t^1, \overline{Z}_t^2)$ . Then*

$$D_t = M_t + \frac{1}{2}\Lambda_t + (N-2) \left( \int_0^t 1_{\{\overline{Z}_s^1 \neq 0\}} dL_s^2 + \int_0^t 1_{\{\overline{Z}_s^2 \neq 0\}} dL_s^1 \right)$$

with  $M$  a martingale,  $M_0 = 0$  and  $L^1, L^2$  are (see Proposition 5 in [2]) the bounded variation parts of  $\overline{X}_t^i$  (defined by (5)) and  $\overline{Y}_t^i$ .

Note that  $L_t^1 = \frac{1}{N}L_t(|Z^1|)$  and  $L_t^2 = \frac{1}{N}L_t(|Z^2|)$  by (6).

Applying the previous proposition to  $(Z^1, Z^2) = (X^r, Y^r)$  and using Proposition 2.4 (ii), we get

$$d(\overline{X}_t^r, \overline{Y}_t^r) = M_t^r + \frac{1}{2}\Lambda_t^r + \frac{(N-2)}{N} (L_t(|X^r|) + L_t(|Y^r|))$$

with  $M^r$  a martingale and  $\Lambda^r$  the local time of  $d(\overline{X}_t^r, \overline{Y}_t^r)$ . In particular,

$$(9) \quad \mathbb{E}[d(\overline{X}_t^r, \overline{Y}_t^r)] \geq 2 \frac{(N-2)}{N} \mathbb{E}[R_t]$$

with  $R$  a reflected Brownian motion started from 0.

Proposition 2.6 applied to the Wiener coupling  $(Z^1, Z^2) = (X, Y)$  and the result of Section 2.1 show that

$$(10) \quad d(\overline{X}_t, \overline{Y}_t) = M_t + \frac{(N-2)}{N} \left( \int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) + \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|) \right)$$

with  $M$  a martingale. By the Balayage formula (see [9] on page 111 or the proof of Proposition 8 in [2]) and the fact that  $L_t(D) = 0$ ,

$$(11) \quad d(\overline{X}_t, \overline{Y}_t) = \text{Martingale} + \frac{N-2}{N} \left( 1_{\{\overline{X}_{g^2} \neq 0\}} |\overline{Y}_t| + 1_{\{\overline{Y}_{g^1} \neq 0\}} |\overline{X}_t| \right)$$

where  $g^1 := g_t^{\overline{X}}$  and  $g^2 := g_t^{\overline{Y}}$ . Admit for a moment that  $\mathbb{E}[d(\overline{X}_t^r, \overline{Y}_t^r)]$  converges to  $\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)]$ . It comes from (9), (11),  $(X, Y)$  has the same law as  $(Y, X)$ , that

$$2 \frac{N-2}{N} \mathbb{E} \left[ 1_{\{\overline{X}_{g^2} \neq 0\}} |\overline{Y}_t| \right] \geq 2 \frac{(N-2)}{N} \mathbb{E}[R_t]$$

Consequently

$$\mathbb{E} [|\overline{Y}_t|] \geq \mathbb{E} \left[ 1_{\{\overline{X}_{g^2} \neq 0\}} |\overline{Y}_t| \right] \geq \mathbb{E}[R_t]$$

Note that this consequence is true only if  $N \geq 3$ . But  $\mathbb{E} [|\overline{Y}_t|] = \mathbb{E}[R_t]$  and so  $\overline{X}_{g^2} \neq 0$  a.s. By symmetry  $\overline{Y}_{g^2} \neq 0$ . Returning back to (11), we deduce that  $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N} (|\overline{X}_t| + |\overline{Y}_t|)$  is a martingale which proves Theorem 1.5 (i).

Note that  $g^1 = g_t^X$ ,  $g^2 = g_t^Y$  and for  $Z$  a WBM, the sets of zeros of  $Z$  and  $\overline{Z}$  are equal. Consequently  $X_{g_t^Y} \neq 0$  and  $Y_{g_t^X} \neq 0$  a.s. In particular  $g_t^X \neq g_t^Y$  a.s and since  $\{X_t = Y_t\} \subset \{g_t^X = g_t^Y\}$  (as  $X, Y$  follow the same Brownian motion on the same ray), Theorem 1.5 (ii) is also proved.

**Remark 2.7.** Using the convergence of  $\mathbb{E}[d(\overline{X}_t^r, \overline{Y}_t^r)]$  to  $\mathbb{E}[d(\overline{X}_t, \overline{Y}_t)]$ , (9) and (10), we easily deduce that

$$\int_0^t 1_{\{X_s \neq 0\}} dL_s(|Y|) = L_t(|Y|); \int_0^t 1_{\{Y_s \neq 0\}} dL_s(|X|) = L_t(|X|)$$

which is similar to Proposition 2.4 (ii).

Now it remains to prove the following

**Lemma 2.8.** *We have*

$$\lim_{r \rightarrow 1} \mathbb{E}[d(\overline{X}_t^r, \overline{Y}_t^r)] = \mathbb{E}[d(\overline{X}_t, \overline{Y}_t)].$$

*Proof.* From the convergence in law given in Lemma 2.5, it is easily seen that  $(\overline{X}^r, \overline{Y}^r)$  converges in law to  $(\overline{X}, \overline{Y})$ . This is because  $\overline{Z}$  is a continuous function of  $Z$ . Let  $r_n$  be a sequence converging to 1. Skorokhod representation theorem says that it is possible to construct on some probability space  $(\Omega', \mathcal{A}', \mathbb{P}')$ , random variables  $(X^n, Y^n)_{n \geq 1}$

and  $(X^\infty, Y^\infty)$  such that for each  $n$ ,  $(X^n, Y^n)$  has the same law as  $(\overline{X^{r_n}}, \overline{Y^{r_n}})$  and  $(X^\infty, Y^\infty)$  has the same law as  $(\overline{X}, \overline{Y})$  and moreover  $(X^n, Y^n)$  converges a.s. to  $(X^\infty, Y^\infty)$ . The lemma holds as soon as we prove

$$\lim_{n \rightarrow \infty} \mathbb{E}[d(X_t^n, Y_t^n)] = \mathbb{E}[d(X_t^\infty, Y_t^\infty)].$$

For each  $\epsilon > 0$ ,

$$\begin{aligned} \mathbb{E}[d(X_t^n, X_t^\infty)] &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)1_{\{d(X_t^n, X_t^\infty) > \epsilon\}}] \\ &\leq \epsilon + \mathbb{E}[d(X_t^n, X_t^\infty)^2]^{1/2} \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \\ &\leq \epsilon + C \times \mathbb{P}[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \end{aligned}$$

for some finite constant  $C$ . Thus,  $\limsup_n \mathbb{E}[d(X_t^n, X_t^\infty)] = 0$  and similarly  $\limsup_n \mathbb{E}[d(Y_t^n, Y_t^\infty)] = 0$ . The lemma follows now using the triangle inequality.  $\square$

Let us now prove Theorem 1.5 (iii).

Denote by  $\mathcal{G}$  the natural filtration of the Wiener coupling  $(X, Y)$ . For a random time  $R$ , let us recall the following  $\sigma$ -fields (see [2] on page 286)

$$\begin{aligned} \mathcal{G}_R &= \sigma(U_R : U \text{ is a } \mathcal{G} - \text{ optional process}), \\ \mathcal{G}_{R+} &= \sigma(U_R : U \text{ is a } \mathcal{G} - \text{ progressive process}). \end{aligned}$$

In the sequel, we will always consider the completions of these sigma-fields by null sets. Let  $g^1 = g_t^X, g^2 = g_t^Y$ . It is known (see for example Proposition 19 in [2]), that  $\varepsilon(X_t)$  is independent of  $\mathcal{G}_{g^1}$  and  $\varepsilon(X_t)$  is  $\mathcal{G}_{g^1+}$  measurable (the same holds for  $Y$ ). The event  $\{g^1 < g^2\} \in \mathcal{G}_{g^2}$  (see Proposition 13 in [2]) and on this event,  $\varepsilon(X_t) = \limsup_{\epsilon \rightarrow 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$ . Since  $(g^1 + \epsilon) \wedge g^2 \leq g^2$ , by Proposition 13 in [2] again,  $\mathcal{G}_{(g^1+\epsilon) \wedge g^2} \subset \mathcal{G}_{g^2}$  and so  $\limsup_{\epsilon \rightarrow 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$  is  $\mathcal{G}_{g^2}$ -measurable. Take  $f$  an indicator function on a subset of  $\{1, \dots, N\}$ . By conditioning with respect to  $\mathcal{G}_{g^2}$ , we deduce

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^1 < g^2\}}] = \mathbb{E}[f(\varepsilon(Y_t))\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}]]$$

and

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}] = \mathbb{E}[f(\varepsilon(X_t))\mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}]]$$

Summing and using  $\mathbb{P}(g^1 = g^2) = 0$ , we get

$$\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t))(\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}] + \mathbb{E}[f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}])]$$

But  $\{g^1 < g^2\} = \{g^2 < g^1\}^c$  a.s. Since  $\mathcal{G}_{g^1}$  is complete,  $\{g^1 < g^2\} \in \mathcal{G}_{g^1}$  which is independent of  $\varepsilon(X_t)$  so that

$$\mathbb{E}[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}] = \frac{1}{2}\mathbb{E}[f(\varepsilon(X_t))].$$

Using the symmetry, we arrive at  $\mathbb{E}[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = \mathbb{E}[f(\varepsilon(X_t))]\mathbb{E}[f(\varepsilon(Y_t))]$ .

### 3. INTERPRETATION USING STOCHASTIC FLOWS

This section gives an interpretation of the Wiener coupling using the Wiener stochastic flow of kernels solving the generalized interface equation considered in [6]. For basic definitions of stochastic flows of mappings, kernels and real white noises, the reader is referred to [8].

For a family of doubly indexed random variables  $Z = (Z_{s,t})_{s \leq t}$ , define  $\mathcal{F}_{s,t}^Z = \sigma(Z_{u,v}, s \leq u \leq v \leq t)$  for all  $s \leq t$ . The extension to flows of kernels of the interface SDE is the following.

**Definition 3.1.** *Let  $K$  be a stochastic flow of kernels on  $G$  and  $\mathcal{W} = (\mathcal{W}^i, 1 \leq i \leq N)$  be a family of independent real white noises. We say that  $(K, \mathcal{W})$  solves (I) if for all  $s \leq t$ ,  $f \in \mathcal{D}$  and  $x \in G$ , a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^N \int_s^t K_{s,u}(1_{E_i}f')(x)d\mathcal{W}_{s,u}^i + \frac{1}{2} \int_s^t K_{s,u}f''(x)du.$$

We say  $K$  is a Wiener solution if for all  $s \leq t$ ,  $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^{\mathcal{W}}$ . When  $K$  is induced by a stochastic flow of mappings  $\varphi$  ( $K = \delta_\varphi$ ), we say  $(\varphi, \mathcal{W})$  is a solution of (I).

Note that when  $K = \delta_\varphi$ , the flow  $\varphi$  defines a system of solutions to the interface SDE (1.2) for all possible time and position initial conditions.

If  $(K, \mathcal{W})$  solves (I), then  $\mathcal{F}_{s,t}^{\mathcal{W}} \subset \mathcal{F}_{s,t}^K$  for all  $s \leq t$  [6]. Therefore Wiener solutions are characterized by  $\mathcal{F}_{s,t}^{\mathcal{W}} = \mathcal{F}_{s,t}^K$  for all  $s \leq t$ .

It has been proved in [6] that there exists a law unique stochastic flow of mappings  $\varphi$  and a real white noise  $\mathcal{W}$  such that  $(\varphi, \mathcal{W})$  solves (I). Filtering this flow with respect to  $\mathcal{W}$  gives rise to a Wiener stochastic flow of kernels  $K_{s,t}(x) = \mathbb{E}[\delta_{\varphi_{s,t}(x)} | \mathcal{F}_{s,t}^{\mathcal{W}}]$  solution of (I) which is unique up to modification.

In the case  $N = 2$  the Wiener flow and the flow of mappings coincide ( $K = \delta_\varphi$ ) while  $K \neq \delta_\varphi$  if  $N \geq 3$  and other flows solving (I) may exist [6].

Let  $(K, \mathcal{W})$  be the Wiener stochastic flow which solves (I). Then

$$Q_t(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t}f(x)K_{0,t}g(y)h(w + \mathcal{W}_{0,t})]$$

defines a Feller semigroup on  $G^2 \times \mathbb{R}^N$ . Denote by  $(X, Y, W)$  the Markov process associated to  $(Q_t)_t$  and started from  $(x, y, 0)$ .

**Proposition 3.2.**  *$(X, Y, W)$  is the Wiener coupling solution of (I) with  $X_0 = x$  and  $Y_0 = y$ .*

*Proof.* Note that

$$\tilde{Q}_t(f \otimes h)(x, w) := Q_t(f \otimes I \otimes h)(x, w) = \mathbb{E}[f(\varphi_{0,t}(x))h(w + \mathcal{W}_{0,t})]$$

In particular  $(X, W)$  has the same law as  $(\varphi_{0,t}(x), \mathcal{W}_{0,t})_{t \geq 0}$  and so it is a solution to  $(I)$ . The same holds for  $(Y, W)$ . Now it remains to prove that  $X$  and  $Y$  are independent given  $W$ . We will check that

$$\mathbb{E} \left[ \prod_{i=1}^n f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i}) \right] = \mathbb{E} \left[ \prod_{i=1}^n \mathbb{E}[f_i(X_{t_i}) | W] \mathbb{E}[g_i(Y_{t_i}) | W] h_i(W_{t_i}) \right]$$

for all measurable and bounded test functions  $(f_i, g_i, h_i)_i$ . Since  $K$  is a measurable function of  $\mathcal{W}$ , we may assume  $K$  (and so  $\mathcal{W}$ ) is defined on the same space as  $X$  and  $Y$  and that  $W_t = \mathcal{W}_{0,t}$ . By an easy induction (see the proof of Proposition 4.1 in [5]),

$$(12) \quad \mathbb{E} \left[ \prod_{i=1}^n f_i(X_{t_i}) g_i(Y_{t_i}) h_i(W_{t_i}) \right] = \mathbb{E} \left[ \prod_{i=1}^n K_{0,t_i} f_i(x) K_{0,t_i} g_i(y) h_i(W_{t_i}) \right]$$

From (12), we also deduce  $K_{0,t_i} f_i(x) = \mathbb{E}[f_i(X_{t_i}) | \mathcal{F}_{0,t_i}^W]$  and  $K_{0,t_i} g_i(y) = \mathbb{E}[g_i(Y_{t_i}) | \mathcal{F}_{0,t_i}^W]$ . This completes the proof.  $\square$

Let  $(\mathcal{W}_{s,t})_{s \leq t}$  and  $(\hat{\mathcal{W}}_{s,t})_{s \leq t}$  be two independent real white noises and set  $\mathcal{W}_{s,t}^r = r\mathcal{W}_{s,t} + \sqrt{1-r^2}\hat{\mathcal{W}}_{s,t}$ . Denote by  $K$  and  $K^r$  the Wiener flows solutions of  $(I)$  respectively driven by  $\mathcal{W}$  and  $\mathcal{W}^r$  and define

$$(13) \quad Q_t^r(f \otimes g \otimes h)(x, y, w) = \mathbb{E}[K_{0,t}^r f(x) K_{0,t} g(y) h(w + \mathcal{W}_{0,t})]$$

Then  $Q^r$  is a Feller semigroup. Following the proof of Proposition 3.2, one can prove that  $(X^r, Y^r, W)$  given in Lemma 2.5 is the Markov process associated to  $Q^r$  and starting from  $(0, 0, 0)$ . In particular this is also a Feller process.

### Final remarks and open problems.

There are interesting open problems related to the interface SDE. Let us mention some of them.

- What is the conditional law of  $|X_t|$  (and more generally of  $X_t$ ) given  $W$ ?
- What are the couplings which “interpolate” between the coalescing coupling and the Wiener one?
- What are the stochastic flows which “interpolate” between the coalescing flow and the Wiener one? (see [6] for more details).

Let us finish with the following remark regarding the first question. Let  $W$  be a standard Brownian motion and let  $X^1, X^2, \dots$  be WBMs started from 0 such that  $(X^i, W)$  is solution to  $(I)$  with  $X_0^i = 0$  for all  $i$  and  $X^1, X^2, \dots$  are independent given  $W$ . Then by the law of the large numbers for all  $f \in C_0(G)$ , a.s.  $\mathbb{E}[f(X_t^1) | W] = \lim_n \frac{1}{n} \sum_{i=1}^n f(X_t^i)$  (see Section 2.6 in [8]).

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