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# On a coupling of solutions to the interface SDE on a star graph

Marc Arnaudon \*      Hatem Hajri †

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## Abstract

We prove some results on a particular coupling of solutions to the interface SDE on a star graph, recently introduced in [6]. This coupling consists in two solutions which are independent given the driving Brownian motion. As a consequence, we deduce that if the star graph contains 3 or more rays, the argument of the solution at a fixed time is independent of the driving Brownian motion.

## 1 Introduction and main results

Our purpose in this paper is to prove some results on the interface SDE defined on a metric star graph  $G$  consisting of  $N$  half lines  $(E_i)_{1 \leq i \leq N}$  sharing the same origin [6]. This SDE is driven by an  $N$  dimensional Brownian motion  $W = (W^1, \dots, W^N)$  and its solution  $X$  is a Walsh Brownian motion on  $G$  [1]. While it moves inside  $E_i$ ,  $X$  follows  $W^i$  so that the origin can be thought of as an interface at the intersection of the half lines. The case  $N = 2$  is a version of the perturbed Tanaka's equation which has a unique strong solution [11, 8].

The interface SDE exhibits an interesting "phase transition" when passing from two to three or more rays. In fact, for  $N \geq 3$ , solutions are only weak [12]. The main result proved in [6] was the construction of a stochastic flow of mappings, unique in law and a Wiener stochastic flow, unique up to a modification, which solve the interface SDE. The problem of finding the flows of kernels which "interpolate" between these two particular flows was left open in [6]. This question can not be answered without a complete understanding of

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the extra randomness that should be added to  $W$  to construct  $X$ . The present paper aims to bring some light on this problem. Our main result states that for fixed  $t$ , the argument of  $X_t$  is independent of  $W$  in the case  $N \geq 3$ .

## 1.1 Notations

This paragraph contains the main notations and definitions which will be used throughout the paper.

Let  $(G, d)$  be a metric star graph with  $N$  ( $1 \leq N < \infty$ ) rays  $(E_i)_{1 \leq i \leq N}$  and origin denoted by 0. This means that  $(G, d)$  is a metric space,  $E_i \cap E_j = \{0\}$  for all  $i \neq j$  and for each  $i$ , there is an isometry  $e_i : [0, \infty[ \rightarrow E_i$ . We assume that  $d$  is the geodesic distance on  $G$  in the sense that  $d(x, y) = d(x, 0) + d(0, y)$  if  $x$  and  $y$  do not belong to the same  $E_i$ .

For any subset  $A$  of  $G$ , we will use the notation  $A^*$  for  $A \setminus \{0\}$ . Also, we define the function  $\varepsilon : G^* \rightarrow \{1, \dots, N\}$  by  $\varepsilon(x) = i$  if  $x \in E_i^*$ .

Let  $C_b^2(G^*)$  denote the set of all continuous functions  $f : G \rightarrow \mathbb{R}$  such that for all  $i \in [1, N]$ ,  $f \circ e_i$  is  $C^2$  on  $]0, \infty[$  with bounded first and second derivatives both with finite limits at  $0+$ . For  $x = e_i(r) \in G^*$ , set  $f'(x) = (f \circ e_i)'(r)$  and  $f''(x) = (f \circ e_i)''(r)$ .

Let  $p_1, \dots, p_N \in (0, 1)$  such that  $\sum_{i=1}^N p_i = 1$  and define

$$\mathcal{D} = \left\{ f \in C_b^2(G^*) : \sum_{i=1}^N p_i (f \circ e_i)'(0+) = 0 \right\}.$$

For  $f \in C_b^2(G^*)$ , we will take the convention  $f'(0) = \sum_{i=1}^N p_i (f \circ e_i)'(0+)$  and  $f''(0) = \sum_{i=1}^N p_i (f \circ e_i)''(0+)$  so that  $\mathcal{D}$  can be written as  $\mathcal{D} = \{f \in C_b^2(G^*) : f'(0) = 0\}$ . We are now in position to recall the following

**Definition 1.1.** *A solution of the interface SDE (I) on  $G$  is a pair  $(X, W)$  of processes defined on a filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$  such that*

- (i)  $W = (W^1, \dots, W^N)$  is a standard  $(\mathcal{F}_t)$ -Brownian motion in  $\mathbb{R}^N$ ;
- (ii)  $X$  is an  $(\mathcal{F}_t)$ -adapted continuous process on  $G$ ;
- (iii) For all  $f \in \mathcal{D}$ ,

$$f(X_t) = f(X_0) + \sum_{i=1}^N \int_0^t f'(X_s) 1_{\{X_s \in E_i\}} dW_s^i + \frac{1}{2} \int_0^t f''(X_s) ds \quad (1)$$

It has been proved in [6] (Theorem 2.3) that for all  $x \in G$ , (I) admits a solution  $(X, W)$  with  $X_0 = x$  and moreover the law of  $(X, W)$  is unique. Theorem 2.3 in [6] also states that  $X$  is a Walsh Brownian motion on  $G$  (which enters  $E_i$  with probability  $p_i$ ) and  $X$  is  $\sigma(W)$ -measurable if and only if  $N \leq 2$ .

To explain the meaning of the previous equation, note that comparing (1) with Freidlin-Sheu formula [4] (see also [7]), shows that if  $X_0 = 0$ , the martingale part  $B^X$  of  $|X|$  should be given by

$$B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}} dW_s^i \quad (2)$$

Thus, while it evolves  $E_i$ ,  $X$  follows the Brownian motion  $W^i$ .

Let us now introduce the following

**Definition 1.2.** We say that  $(X, Y, W)$  is a coupling of solutions to (I) if  $(X, W)$  and  $(Y, W)$  satisfy Definition 1.1 on the same filtered probability space  $(\Omega, \mathcal{A}, (\mathcal{F}_t)_t, \mathbb{P})$ .

A trivial coupling of solutions to (I) is given by  $(X, X, W)$  where  $(X, W)$  solves (I). This is also the law unique coupling of solutions to (I) if  $N \leq 2$  as  $\sigma(X) \subset \sigma(W)$  in this case. Let us now introduce another interesting coupling.

**Definition 1.3.** A coupling  $(X, Y, W)$  of solutions to (I) is called the Wiener coupling if  $X$  and  $Y$  are independent given  $W$ .

It is easy to see that there exists a Wiener coupling (since the condition  $X$  and  $Y$  are independent given  $W$  uniquely determine the law of  $(X, Y, W)$ ). Moreover the law of this coupling is unique. The reason for designating this coupling by the Wiener coupling will be justified in Section 2.2 in connection with stochastic flows of kernels.

## 1.2 Main results

Given a Walsh Brownian motion  $X$  on  $G$ , we define the process  $\overline{X}$  by

$$\overline{X}_t = 1_{\{X_t \neq 0\}} \sum_{i=1}^N 1_{\{\varepsilon(X_t)=i\}} \times e_i \left( \frac{e_i^{-1}(X_t)}{Np_i} \right)$$

Following the terminology used in [2], the process  $\overline{X}$  is a spidermartingale (“martingale-araignée”). In fact, for all  $1 \leq i \leq N$ , define

$$\overline{X}_t^i = |\overline{X}_t| \text{ if } \overline{X}_t \in E_i \text{ and } \overline{X}_t^i = 0 \text{ if not}$$

Note that  $\overline{X}_t^i = f^i(X_t)$ , where  $f^i(x) = \frac{|x|}{Np_i} 1_{\{x \in E_i\}}$ . Applying Freidlin-Sheu formula for  $X$  and the function  $f^i$ , it comes that

$$\overline{X}_t^i = \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB_s^X + \frac{1}{N} L_t(|X|) \quad (3)$$

where  $L_t(|X|)$  is the local at zero of the reflected Brownian motion  $|X|$ . In particular,  $\overline{X}_t^i - \overline{X}_t^j$  is a martingale for all  $i, j \in [1, N]$ . Proposition 5 in [2] shows that  $\overline{X}$  is a spider-martingale.

Our main result in this paper is the following

**Theorem 1.4.** *Suppose  $N \geq 3$ . Let  $(X, Y, W)$  be the Wiener coupling of solutions to (I) with  $X_0 = Y_0 = 0$ . Then*

(i)  *$d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$  is a martingale. In particular,*

$$E[d(\overline{X}_t, \overline{Y}_t)] = 2 \frac{N-2}{N} \sqrt{\frac{2t}{\pi}}$$

(ii) *Call  $g_t^X$  and  $g_t^Y$  the last zeroes before  $t$  of  $X$  and  $Y$ , then for all  $t > 0$ ,  $\mathbb{P}(g_t^X = g_t^Y) = 0$  and  $\mathbb{P}(X_t = Y_t) = 0$ .*

(iii)  *$\varepsilon(X_t)$  and  $\varepsilon(Y_t)$  are independent for all  $t > 0$ .*

This theorem yields the following important

**Corollary 1.5.** *Suppose  $N \geq 3$ . Let  $(X, W)$  be a solution of (I) with  $X_0 = 0$ . Then for each  $t > 0$ ,  $\varepsilon(X_t)$  is independent of  $W$ .*

This corollary seems to us quite remarkable. In fact, it is known since the work of Tsirelson [12], that  $X$  can not be  $\sigma(W)$ -measurable. As a consequence of this result and (2), it can be deduced that  $\varepsilon(X_t)$  is not  $\sigma(W)$ -measurable (actually neither  $|X|$  nor  $\varepsilon(X_t)$  are  $\sigma(W)$ -measurable). However, Corollary (1.5) gives a much stronger result than the non-measurability of  $\varepsilon(X_t)$  with respect to  $\sigma(W)$ . Comparing this with the case  $N = 2$  in which  $\varepsilon(X_t)$  is  $\sigma(W)$ -measurable, shows that stochastic differential equations on star graphs with  $N \geq 3$  rays involve interesting phenomena.

Corollary 1.5 is easy to deduce from Theorem 1.4. For this, define  $C_t = \mathbb{P}(\varepsilon(X_t) = i | W)$ . Since  $X$  and  $Y$  are independent given  $W$  and  $(X, W)$ ,  $(Y, W)$  have the same law,

$$\mathbb{P}(\varepsilon(X_t) = i)^2 = \mathbb{P}(\varepsilon(X_t) = i, \varepsilon(Y_t) = i) = E[C_t^2]$$

Thus  $E[C_t] = E[C_t^2]^{\frac{1}{2}}$  and so there exists a constant  $c_t$  such that  $C_t = c_t$  a.s. Taking the expectation, we see that  $c_t = p_i$ .

Let us now explain our arguments to prove Theorem 1.4. In Section 2.1, we prove that for any coupling  $(X, Y, W)$  of solutions to (I),  $L_t(D) = 0$  where  $D_t = d(\bar{X}_t, \bar{Y}_t)$ . Next, we consider the Wiener stochastic flow of kernels  $K$  constructed in [6] which is a strong solution to the flows of kernels version of (1). Inspired by Tsirelson arguments [12], we then consider a perturbation  $K^r$  of  $K$ ,  $r \rightarrow 1$ . The semigroup

$$Q_t^r(f \otimes g)(x, y) = E[K_{0,t}f(x)K_{0,t}^rg(y)]$$

on  $G^2$  is Feller. Interesting results about the Markov process  $(X^r, Y^r)$  associated to  $Q^r$  are known [2, 3, 12]. The process  $(X^r, Y^r)$  converges in law as  $r \rightarrow 1$  to the Wiener coupling  $(X, Y)$  described above. Letting  $r$  go to 1 and using the previous results, we deduce Theorem 1.4.

## 2 Proofs

### 2.1 The local time of the distance

The subject of this paragraph is to prove the following

**Proposition 2.1.** *Let  $(X, Y, W)$  be a coupling of two solutions to (I) with  $X_0 = Y_0 = 0$  and let  $D_t = d(\bar{X}_t, \bar{Y}_t)$ . Then  $L_t(D) = 0$ .*

*Proof.* We follow the proof of Proposition 4.5 in [6] and first prove that a.s.

$$\int_{]0, +\infty[} L_t^a(D) \frac{da}{a} = \int_0^t 1_{\{D_s > 0\}} \frac{d\langle D \rangle_s}{D_s} < \infty. \quad (4)$$

By (3),  $|\bar{X}_t| = \sum_{i=1}^N \bar{X}_t^i = M_t^1 + L_t(|X|)$ ,  $|\bar{Y}_t| = \sum_{i=1}^N \bar{Y}_t^i = M_t^2 + L_t(|Y|)$ , with

$$M_t^1 = \sum_{i=1}^N \frac{1}{Np_i} \int_0^t 1_{\{X_s \in E_i\}} dB_s^X, \quad M_t^2 = \sum_{i=1}^N \frac{1}{Np_i} \int_0^t 1_{\{Y_s \in E_i\}} dB_s^Y$$

In particular,

$$d\langle M^1 \rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i\}} ds, \quad d\langle M^2 \rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{Y_s \in E_i\}} ds$$

and

$$d\langle M^1, M^2 \rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2} \int_0^t 1_{\{X_s \in E_i, Y_s \in E_i\}} ds.$$

Proposition 7 [2] tells us that

$$D_t - \int_0^t 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (dM_s^1 + dM_s^2) - \int_0^t 1_{\{\varepsilon(X_s) = \varepsilon(Y_s)\}} \operatorname{sgn}(M_s^1 - M_s^2) (dM_s^1 - dM_s^2)$$

is a continuous increasing process. Consequently,

$$d\langle D \rangle_s = \sum_{i=1}^N \frac{1}{(Np_i)^2} 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} (1_{\{X_s \in E_i\}} + 1_{\{Y_s \in E_i\}}) ds \leq C 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} ds$$

where  $C$  is a positive constant. Notice also that there exists  $C' > 0$  such that  $D_s \geq C'(|X_s| + |Y_s|)$  for all  $s$  such that  $\varepsilon(X_s) \neq \varepsilon(Y_s)$ . Thus, to get (4), it is sufficient to prove that

$$\int_0^t 1_{\{X_s \neq 0, Y_s \neq 0\}} 1_{\{\varepsilon(X_s) \neq \varepsilon(Y_s)\}} \frac{ds}{|X_s| + |Y_s|} < \infty$$

Let us prove for example that

$$(1) = \int_0^t \frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} ds < \infty.$$

Let  $f(z) = |z|$  if  $z \in E_1$  and  $f(z) = -|z|$  if not and set  $x_t = f(X_t)$ ,  $y_t = f(Y_t)$ . Clearly

$$\frac{1}{|X_s| + |Y_s|} 1_{\{X_s \in E_1^*, Y_s \notin E_1\}} = \frac{1}{2} \frac{|\operatorname{sgn}(x_s) - \operatorname{sgn}(y_s)|}{|x_s - y_s|} 1_{\{y_s < 0 < x_s\}}.$$

Now as in [6], let  $(f_n)_n \subset C^1(\mathbb{R})$  such that  $f_n \rightarrow \operatorname{sgn}$  pointwise and  $(f_n)_n$  is uniformly bounded in total variation. Then defining  $z_s^u = (1-u)x_s + uy_s$ , we have

$$\begin{aligned} (1) &\leq \liminf_n \int_0^t 1_{\{y_s < 0 < x_s\}} \frac{|f_n(x_s) - f_n(y_s)|}{|x_s - y_s|} \frac{ds}{2} \\ &\leq \liminf_n \int_0^t 1_{\{y_s < 0 < x_s\}} \int_0^1 |f_n'(z_s^u)| du \frac{ds}{2} \end{aligned}$$

Writing Freidlin-Sheu formula for the function  $f$  applied to  $X$  and  $Y$  shows that on  $\{y_s < 0 < x_s\}$ ,

$$\frac{d}{ds} \langle z^u \rangle_s = u^2 + (1-u)^2 \geq \frac{1}{2}.$$

Thus

$$\begin{aligned}
(1) &\leq \liminf_n \int_0^1 \int_0^t 1_{\{y_s < 0 < x_s\}} |f'_n(z_s^u)| d\langle z^u \rangle_s du \\
&\leq \liminf_n \int_0^1 \int_{\mathbb{R}} |f'_n(a)| L_t^a(z^u) da du.
\end{aligned}$$

So a sufficient condition for (1) to be finite is

$$\sup_{a \in \mathbb{R}, u \in [0,1]} \mathbb{E}[L_t^a(z^u)] < \infty.$$

By Tanaka's formula

$$\begin{aligned}
\mathbb{E}[L_t^a(z^u)] &= \mathbb{E}[|z_t^u - a|] - \mathbb{E}[|z_0^u - a|] - \mathbb{E}\left[\int_0^t \text{sgn}(z_s^u - a) dz_s^u\right] \\
&\leq \mathbb{E}[|z_t^u - z_0^u|] - \mathbb{E}\left[\int_0^t \text{sgn}(z_s^u - a) dz_s^u\right].
\end{aligned}$$

Since  $x$  and  $y$  are two skew Brownian motions, it is easily seen that  $\sup_{u \in [0,1]} \mathbb{E}[|z_t^u - z_0^u|] < \infty$ . The same argument shows that

$$\mathbb{E}\left[\int_0^t \text{sgn}(z_s^u - a) dz_s^u\right]$$

is uniformly bounded with respect to  $(u, a)$  and this shows that (1) is finite. Finally  $\int_{]0, +\infty[} L_t^a(D) \frac{da}{a}$  is finite a.s. Since  $\lim_{a \downarrow 0} L^a(D) = L^0(D)$ , we deduce  $L_t^0(D) = 0$ .  $\square$

## 2.2 The Wiener flow

In this paragraph, we are interested in stochastic flows of kernels [9] which play a crucial role in our proofs. We start this paragraph by recalling the following definition.

**Definition 2.2.** *Let  $K$  be a stochastic flow of kernels on  $G$  and  $\mathcal{W} = (W^i, 1 \leq i \leq N)$  be a family of independent real white noises. We say that  $(K, \mathcal{W})$  solves (I) if for all  $s \leq t$ ,  $f \in \mathcal{D}$  and  $x \in G$ , a.s.*

$$K_{s,t}f(x) = f(x) + \sum_{i=1}^N \int_s^t K_{s,u}(1_{E_i}f')(x) dW_u^i + \frac{1}{2} \int_s^t K_{s,u}f''(x) du.$$

We will say it is a Wiener solution if for all  $s \leq t$ ,  $\mathcal{F}_{s,t}^K \subset \mathcal{F}_{s,t}^{\mathcal{W}}$ .



It has been proved in [6], that given  $\mathcal{W}$  a real white noise, there exists a Wiener stochastic flow of kernels  $K$ , unique up to modification, such that  $(K, \mathcal{W})$  solves (I). Let  $Q$  be the Feller semigroup on  $G^2 \times \mathbb{R}^N$  defined by

$$Q_t(f \otimes g \otimes h)(x, y, w) = E[K_{0,t}f(x)K_{0,t}g(y)h(w + \mathcal{W}_{0,t})].$$

Denote by  $(X, Y, W)$  the Markov process associated to  $(Q_t)_t$  and started from  $(x, y, 0)$ .

**Proposition 2.3.**  $(X, Y, W)$  is the Wiener coupling solution of (I) with  $X_0 = x$  and  $Y_0 = y$ .

*Proof.* We follow the proof of Lemma 6.2 in [6]. Let  $\mathcal{F}_t = \sigma(X_s, Y_s, W_s; s \leq t)$ . Clearly  $X$  and  $Y$  are two  $(\mathcal{F}_t)$ -Walsh Brownian motions. We first prove that  $(X, W)$  (and similarly  $(Y, W)$ ) solves (I). By Freidlin-Sheu's formula, for all  $f \in \mathcal{D}$ ,

$$f(X_t) = f(x) + \int_0^t f'(X_s)dB_s^X + \frac{1}{2} \int_0^t f''(X_s)ds. \quad (5)$$

Thus for  $(X, W)$  to solve (I), it will suffice to show  $B_t^X = \sum_{i=1}^N \int_0^t 1_{\{X_s \in E_i\}}dW_s^i$ . Set

$$\mathcal{D}_1 = \{f \in \mathcal{D} : f, f', f'' \in C_0(G)\}. \quad (6)$$

By the condition  $f', f'' \in C_0(G)$ , it is meant here that for all  $i \in [1, N]$ , the limits  $\lim_{x \in E_i, x \neq 0} f'(x)$  are equal and the same holds for  $f''$  (not to be confused with the conventions taken for  $f'(0), f''(0)$ ). As  $f \in \mathcal{D}$ , it is clear that this yields  $\lim_{x \in E_i, x \neq 0} f'(x) = 0$ . Denote by  $A$  the generator of  $\tilde{Q}_t(f \otimes g) = Q_t(f \otimes I \otimes g)$  and  $\mathcal{D}(A)$  its domain, then  $\mathcal{D}_1 \otimes C_0^2(\mathbb{R}^N) \subset \mathcal{D}(A)$  and for all  $f \in \mathcal{D}_1$  and  $g \in C_0^2(\mathbb{R}^N)$ ,

$$A(f \otimes g)(x, w) = \frac{1}{2}f(x)\Delta g(w) + \frac{1}{2}f''(x)g(w) + \sum_{i=1}^N (f'1_{E_i})(x) \frac{\partial g}{\partial w^i}(w).$$

In particular, for all  $f \in \mathcal{D}_1$  and  $g \in C_0^2(\mathbb{R}^N)$ ,

$$f(X_t)g(W_t) - \int_0^t A(f \otimes g)(X_s, W_s)ds \text{ is a martingale.} \quad (7)$$

On another hand, using (5), it is possible to write Itô's formula for  $f(X_t)g(W_t)$ . Combining this Itô's formula with (7), it comes that

$$\sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s)ds = \sum_{i=1}^N \int_0^t (f'1_{E_i})(X_s) \frac{\partial g}{\partial w^i}(W_s) d\langle B^X, W^i \rangle_s.$$

Since this holds for all  $f \in \mathcal{D}_1$  and  $g \in C_0^2(\mathbb{R}^N)$ , by an approximation argument, we deduce  $\langle B^X, W^i \rangle_t = \int_0^t 1_{\{X_s \in E_i\}} ds$  for all  $i$ .

Now it remains to prove that  $X$  and  $Y$  are independent given  $W$ . We will check that

$$E\left[\prod_{i=1}^n f_i(X_{t_i})g_i(Y_{t_i})h_i(W_{t_i})\right] = E\left[\prod_{i=1}^n E[f_i(X_{t_i})|W]E[g_i(Y_{t_i})|W]h_i(W_{t_i})\right]$$

for all measurable and bounded test functions  $(f_i, g_i, h_i)_i$ . Since  $K$  is a measurable function of  $\mathcal{W}$ , we may assume  $K$  (and so  $\mathcal{W}$ ) is defined on the same space as  $X$  and  $Y$  and that  $W_t = \mathcal{W}_{0,t}$ . By an easy induction (see the proof of Proposition 4.1 in [5]),

$$E\left[\prod_{i=1}^n f_i(X_{t_i})g_i(Y_{t_i})h_i(W_{t_i})\right] = E\left[\prod_{i=1}^n K_{0,t_i}f_i(x)K_{0,t_i}g_i(y)h_i(W_{t_i})\right] \quad (8)$$

From (8), we also have  $K_{0,t_i}f_i(x) = E[f_i(X_{t_i})|\mathcal{F}_{0,t_i}^W]$  and  $K_{0,t_i}g_i(y) = E[g_i(Y_{t_i})|\mathcal{F}_{0,t_i}^W]$ . This completes the proof.  $\square$

Since  $(Q_t)_t$  is Feller, the proposition shows in particular that  $(X, Y)$  is a Feller process on  $G^2$ .

### 2.3 Perturbation of the Wiener flow

We now fix a measurable function  $F$  such that if  $\mathcal{V}$  is a real white noise on  $\mathbb{R}^N$ , then  $K = F(\mathcal{V})$  solves (I). Let  $\mathcal{W}$  and  $\mathcal{W}'$  be two independent real white noises on  $\mathbb{R}^N$  and for  $r > 0$ , define the real white noise on  $\mathbb{R}^N$ :

$$\mathcal{W}^r = r\mathcal{W} + \sqrt{1-r^2}\mathcal{W}'.$$

Let  $K = F(\mathcal{W})$  and  $K^r = F(\mathcal{W}^r)$  and note that  $(K, K^r)$  and  $(K^r, K)$  have the same law since this is also true for  $(\mathcal{W}, \mathcal{W}^r)$  and  $(\mathcal{W}^r, \mathcal{W})$ . Set now

$$Q_t^r(f \otimes g \otimes h)(x, y, w) = E[K_{0,t}f(x)K_{0,t}^rg(y)h(w + \mathcal{W}_{0,t})].$$

Then  $Q^r$  is a Feller semigroup on  $G^2 \times \mathbb{R}^N$ . Call  $(X^r, Y^r, W^r)$  the associated Markov process started from  $(0, 0, 0)$ , then again  $(X^r, Y^r)$  and  $(Y^r, X^r)$  have the same law.

**Proposition 2.4.** *The following assertions hold*

(i)  $d\langle B^{X^r}, B^{Y^r} \rangle_t = r1_{\{\varepsilon(X_t^r) = \varepsilon(Y_t^r)\}} dt.$

(ii)  $\int_0^t 1_{\{Y_s^r \neq 0\}} dL_s(|X^r|) = L_t(|X^r|)$  and  $\int_0^t 1_{\{X_s^r \neq 0\}} dL_s(|Y^r|) = L_t(|Y^r|)$ .

*Proof.* The proof of (i) follows similar arguments as in the proof of Proposition 2.3. Denote by  $A^r$ , the generator of  $Q_t^r(f \otimes g \otimes \text{Id})$ , then  $\mathcal{D}_1 \otimes \mathcal{D}_1 \subset \mathcal{D}(A^r)$  where  $\mathcal{D}_1$  is given by (6). Moreover for all  $f, g \in \mathcal{D}_1$ ,

$$A^r(f \otimes g)(x, y) = \frac{1}{2}f(x)g''(y) + \frac{1}{2}f''(x)g(y) + rf'(x)g'(y)1_{\{\varepsilon(x)=\varepsilon(y)\}}$$

Writing Itô's formula for  $f(X^r)$  and  $g(Y^r)$  and using martingales as in the proof of Proposition 2.3, we easily deduce (i). (ii) is Lemma 4.12 in [12] (see also [2, 3]). □

## 2.4 Proof of Theorem 1.4

Let  $(X, Y, W)$  be a Wiener coupling solutions of  $(I)$  with  $X_0 = Y_0 = 0$ .

**Lemma 2.5.** *As  $r \rightarrow 1$ ,  $(X^r, Y^r, W^r)$  converges in law to  $(X, Y, W)$ .*

*Proof.* Let  $r_n$  be a sequence in  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} r_n = 1$ . Slutsky lemma (see Theorem 1 in [2]) shows that for all  $g : G \rightarrow \mathbb{R}$  measurable bounded,  $y \in G$  and  $t > 0$ ,  $K_{0,t}^{r_n}g(y)$  converges in probability to  $K_{0,t}g(y)$  and for any measurable function  $L$ ,  $L(\mathcal{W}^{r_n})$  converges in probability to  $L(\mathcal{W})$ . Now for any  $p \geq 1$ ,  $(f_i, g_i, h_i)_{1 \leq i \leq p}$  bounded,  $(t_i)_{1 \leq i \leq p}$

$$E\left[\prod_{i=1}^p f_i(X_{t_i}^{r_n})g_i(Y_{t_i}^{r_n})h_i(W_{t_i}^{r_n})\right] = E\left[\prod_{i=1}^p K_{0,t_i}^{r_n}f_i(0)K_{0,t_i}^{r_n}g_i(0)h_i(\mathcal{W}_{0,t_i}^{r_n})\right]$$

which converges as  $n$  goes to  $\infty$  to

$$E\left[\prod_{i=1}^p K_{0,t_i}f_i(0)K_{0,t_i}g_i(0)h_i(\mathcal{W}_{0,t_i})\right]$$

This gives the desired result. □

Let us now recall Proposition 7 in [2].

**Proposition 2.6.** *Let  $Z^1$  and  $Z^2$  be two Walsh Brownian motion with respect to the same filtration. Denote by  $\Lambda$  the local time of  $D_t = d(\overline{Z}_t^1, \overline{Z}_t^2)$ . Then*

$$D_t = M_t + \frac{1}{2}\Lambda_t + (N-2) \left( \int_0^t 1_{\{\overline{Z}_s^1 \neq 0\}} dL_s^2 + \int_0^t 1_{\{\overline{Z}_s^2 \neq 0\}} dL_s^1 \right)$$

with  $M$  a martingale and  $L^1, L^2$  are (see Proposition 5 in [2]) the bounded variation parts of  $\overline{X}_t^i$  and  $\overline{Y}_t^i$ .

Note that  $L_t^1 = \frac{1}{N}L_t(|Z^1|)$  and  $L_t^2 = \frac{1}{N}L_t(|Z^2|)$  by (3).

Applying the previous proposition to  $(X^r, Y^r)$  and using Proposition 2.4 (ii), we get

$$d(\overline{X}_t^r, \overline{Y}_t^r) = M_t^r + \frac{1}{2}\Lambda_t^r + \frac{(N-2)}{N}(L_t(|X^r|) + L_t(|Y^r|))$$

with  $M^r$  a martingale and  $\Lambda^r$  the local time of  $d(\overline{X}_t^r, \overline{Y}_t^r)$ . In particular,

$$E[d(\overline{X}_t^r, \overline{Y}_t^r)] \geq 2\frac{(N-2)}{N}E[R_t] \quad (9)$$

with  $R$  a reflected Brownian motion started from 0.

The previous proposition applied to the Wiener coupling  $(X, Y)$  and the result of Section 2.1 show that

$$d(\overline{X}_t, \overline{Y}_t) = M_t + \frac{(N-2)}{N}\left(\int_0^t 1_{\{X_s \neq 0\}}dL_s(|Y|) + \int_0^t 1_{\{Y_s \neq 0\}}dL_s(|X|)\right) \quad (10)$$

with  $M$  a martingale. By the Balayage formula (see [10] on page 111 or the proof of Proposition 8 in [2]) and the fact that  $L_t(D) = 0$ ,

$$d(\overline{X}_t, \overline{Y}_t) = \text{Martingale} + \frac{N-2}{N}\left(1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y}_t| + 1_{\{\overline{Y}_{g^1} \neq 0\}}|\overline{X}_t|\right) \quad (11)$$

Here  $g^1 := g_t^{\overline{X}}$  and  $g^2 := g_t^{\overline{Y}}$ . Admit for a moment that  $E[d(\overline{X}_t^r, \overline{Y}_t^r)]$  converges to  $E[d(\overline{X}_t, \overline{Y}_t)]$ . It comes from (9), (11),  $(X, Y)$  has the same law as  $(Y, X)$ , that

$$2\frac{N-2}{N}E\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y}_t|\right] \geq 2\frac{(N-2)}{N}E[R_t]$$

So

$$E[|\overline{Y}_t|] \geq E\left[1_{\{\overline{X}_{g^2} \neq 0\}}|\overline{Y}_t|\right] \geq E[R_t]$$

But  $E[|\overline{Y}_t|] = E[R_t]$  and so  $\overline{X}_{g^2} \neq 0$  a.s. By symmetry  $\overline{Y}_{g^1} \neq 0$ . Returning back to (11), we deduce that  $d(\overline{X}_t, \overline{Y}_t) - \frac{N-2}{N}(|\overline{X}_t| + |\overline{Y}_t|)$  is a martingale which proves Theorem 1.4 (i).

Note that  $g^1 = g_t^X$ ,  $g^2 = g_t^Y$  and  $Z$  has the same set of zeros as  $\overline{Z}$  for  $Z$  a Walsh Brownian motion. This shows that  $X_{g_t^Y} \neq 0$  and  $Y_{g_t^X} \neq 0$  a.s. In particular  $g_t^X \neq g_t^Y$  a.s and since  $\{X_t = Y_t\} \subset \{g_t^X = g_t^Y\}$  (as  $X, Y$  follow the same Brownian motion on the same ray), Theorem 1.4 (ii) is also proved.

**Remark 2.7.** Using the convergence of  $E[d(\overline{X}_t^r, \overline{Y}_t^r)]$  to  $E[d(\overline{X}_t, \overline{Y}_t)]$ , (9) and (10), we easily deduce that

$$\int_0^t 1_{\{X_s \neq 0\}}dL_s(|Y|) = L_t(|Y|); \int_0^t 1_{\{Y_s \neq 0\}}dL_s(|X|) = L_t(|X|)$$

which is similar to Proposition 2.4 (ii).

Now it remains to prove the following

**Lemma 2.8.** *We have*

$$\lim_{r \rightarrow 1} E[d(\overline{X}_t^r, \overline{Y}_t^r)] = E[d(\overline{X}_t, \overline{Y}_t)].$$

*Proof.* From the convergence in law given in Lemma 2.5, it is easily seen that  $(\overline{X}^r, \overline{Y}^r)$  converges in law to  $(\overline{X}, \overline{Y})$ . This is because  $\overline{Z}$  is a continuous of  $Z$ . Let  $r_n$  be a sequence converging to 1. Skorokhod representation theorem says that it is possible to construct on some probability space  $(\Omega', \mathcal{A}', \mathbb{P}')$ , random variables  $(X^n, Y^n)_{n \geq 1}$  and  $(X^\infty, Y^\infty)$  such that for each  $n$ ,  $(X^n, Y^n)$  has the same law as  $(\overline{X}^{r_n}, \overline{Y}^{r_n})$  and  $(X^\infty, Y^\infty)$  has the same law as  $(\overline{X}, \overline{Y})$  and moreover  $(X^n, Y^n)$  converges a.s. to  $(X^\infty, Y^\infty)$ . The lemma holds as soon as we prove that

$$\lim_{n \rightarrow \infty} E[d(X_t^n, Y_t^n)] = E[d(X_t^\infty, Y_t^\infty)].$$

For each  $\epsilon > 0$ ,

$$\begin{aligned} E[d(X_t^n, X_t^\infty)] &\leq \epsilon + E[d(X_t^n, X_t^\infty)1_{\{d(X_t^n, X_t^\infty) > \epsilon\}}] \\ &\leq \epsilon + E[d(X_t^n, X_t^\infty)^2]^{1/2} P[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \\ &\leq \epsilon + CP[d(X_t^n, X_t^\infty) > \epsilon]^{1/2} \end{aligned}$$

for some finite constant  $C$ . Thus,  $\limsup_n E[d(X_t^n, X_t^\infty)] = 0$  and similarly  $\limsup_n E[d(Y_t^n, Y_t^\infty)] = 0$ . Using the triangle inequality, the lemma easily follows.  $\square$

Let us now prove Theorem 1.4 (iii). Denote by  $\mathcal{G}$  the natural filtration of the Wiener coupling  $(X, Y)$ . For a random time  $R$ , let us recall the following  $\sigma$ -fields (see [2] on page 286)

$$\mathcal{G}_R = \sigma(X_R : X \text{ is } \mathcal{G} - \text{ optional process})$$

$$\mathcal{G}_{R+} = \sigma(X_R : X \text{ is } \mathcal{G} - \text{ progressive process})$$

It is known (see for example Proposition 19 in [2]), that  $\varepsilon(X_t)$  is independent of  $\mathcal{G}_{g_t^X}$  and that  $\varepsilon(X_t)$  is  $\mathcal{G}_{g_t^X+}$  measurable (the same holds for  $Y$ ). Let  $g^1 = g_t^X, g^2 = g_t^Y$ . The event  $\{g^1 < g^2\} \in \mathcal{G}_{g^2}$  (see Proposition 13 in [2]). On this event, we have

$$\varepsilon(X_t) = \limsup_{\epsilon \rightarrow 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$$

Since  $(g^1+\epsilon) \wedge g^2 \leq g^2$ , by Proposition 13 in [2] again,  $\mathcal{G}_{(g^1+\epsilon) \wedge g^2} \subset \mathcal{G}_{g^2}$ . So  $\limsup_{\epsilon \rightarrow 0+} \varepsilon(X_{(g^1+\epsilon) \wedge g^2})$  is  $\mathcal{G}_{g^2}$ -measurable. Conditioning on  $\mathcal{G}_{g^2}$ , we deduce that

$$E[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^1 < g^2\}}] = E[f(\varepsilon(Y_t))]E[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}]$$

Similarly

$$E[f(\varepsilon(X_t))f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}] = E[f(\varepsilon(X_t))E[f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}]]$$

Summing, we get

$$E[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = E[f(\varepsilon(X_t)) (E[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}] + E[f(\varepsilon(Y_t))1_{\{g^2 < g^1\}}])]$$

But  $\{g^1 < g^2\} = \{g^2 < g^1\}^c$  also belongs to  $\mathcal{G}_{g^1}$  which is independent of  $\varepsilon(X_t)$  so that

$$E[f(\varepsilon(X_t))1_{\{g^1 < g^2\}}] = \frac{1}{2}E[f(\varepsilon(X_t))]$$

By symmetry, we conclude that  $E[f(\varepsilon(X_t))f(\varepsilon(Y_t))] = E[f(\varepsilon(X_t))]E[f(\varepsilon(Y_t))]$ .

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