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► **To cite this version:**

F Brock, G Croce, O Guibé, A Mercaldo. Symmetry and asymmetry of minimizers of a class of noncoercive functionals. *Advances in Calculus of Variation*, 2017, pp. 15-32. 10.1515/acv-2017-0005 . hal-01262642v3

HAL Id: hal-01262642

<https://hal.science/hal-01262642v3>

Submitted on 23 Aug 2017

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SYMMETRY AND ASYMMETRY OF MINIMIZERS OF A CLASS OF NONCOERCIVE FUNCTIONALS

F. BROCK, G. CROCE, O. GUIBÉ, A. MERCALDO

ABSTRACT. In this paper we prove symmetry results for minimizers of a non coercive functional defined on the class of Sobolev functions with zero mean value. We prove that the minimizers are foliated Schwarz symmetric, i.e. they are axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis. In the two dimensional case we show a symmetry breaking.

1. INTRODUCTION

Consider the functional

$$v \in H_0^1(\Omega) \rightarrow \frac{1}{2} \int_{\Omega} |\nabla v|^2$$

subjected to the constraint $\int_{\Omega} v^2 = 1$, where Ω is the unit ball in the plane. Its critical values are the eigenvalues of the classical fixed membrane problem

$$(1.1) \quad \begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

It is known that the first eigenfunctions are positive and Schwarz symmetric, that is, radial and decreasing in the radial variable. On the contrary, the second eigenfunctions are sign-changing; they are not radial, but they are symmetric with respect to the reflection at some line $\mathbb{R}e$, and they are decreasing in the angle $\arccos\left[\frac{x}{|x|} \cdot e\right] \in (0, \pi)$. These properties can be seen as a spherical version of the Schwarz symmetry along the foliation of the underlying ball Ω by circles. For this reason, this property has been called foliated Schwarz symmetry in the literature.

In the last years much interest has been devoted to the shape of sign changing minimizers of integral functionals, see for example [15], [24], [7] and [21]. In [15], Girao and Weth studied the

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symmetry properties of the minimizers of the problem

$$(1.2) \quad v \rightarrow \frac{\|\nabla v\|_2}{\|v\|_p}, \quad v \in H^1(\Omega), \quad \int_{\Omega} v = 0$$

for $2 \leq p < 2^*$. In view of the zero average constraint, (1.2) is similar to the problem of finding the second eigenfunctions of problem (1.1). They proved that the minimizers are foliated Schwarz symmetric.

In [15] Girao and Weth pointed out another interesting phenomenon related to the shape of the minimizers of (1.2). If p is close to 2, then any minimizer of the above functional is antisymmetric with respect to the reflection at the hyperplane $\{x \cdot e = 0\}$. In contrast to this, the minimizers are not antisymmetric when $N = 2$ and p is sufficiently large. A similar break of symmetry was already observed in [13], [12], [17], [3], [11], [20] for the minimizers of the functional

$$v \rightarrow \frac{\|v'\|_{L^p(0,1)}}{\|v\|_{L^q(0,1)}}, \quad v \in W^{1,p}((0,1)), v(0) = v(1), \quad \int_0^1 v = 0.$$

Indeed, it has been shown that any minimizer is an antisymmetric function, if and only if $q \leq 3p$.

In this paper, we will prove similar symmetry results for the minimizers of a generalized version of the functional studied by Girao and Weth in [15]. We consider

$$(1.3) \quad \lambda^{\theta,p}(\Omega) = \inf \left\{ \int_{\Omega} \frac{|\nabla v|^2}{(1+|v|)^{2\theta}} dx, \quad v \in W^{1,q}(\Omega), v \neq 0, \int_{\Omega} v dx = 0, \|v\|_{L^p(\Omega)} = 1 \right\}$$

where Ω is either a ball or an annulus centered in the origin in \mathbb{R}^N , $N \geq 2$, θ and q satisfy

$$(1.4) \quad 0 < 2\theta < 1,$$

$$(1.5) \quad q = \frac{2N(1-\theta)}{N-2\theta}, \quad \text{if } N \geq 3,$$

$$(1.6) \quad 2(1-\theta) \leq q < 2, \quad \text{if } N = 2,$$

$$(1.7) \quad 1 < p < q^* \quad \text{if } N \geq 3,$$

$$(1.8) \quad 1 < p < +\infty \quad \text{if } N = 2.$$

Observe that, if one defines

$$\Psi(\xi) := \int_0^{\xi} (1+|t|)^{-\theta} dt = \frac{\operatorname{sgn} \xi}{1-\theta} [(1+|\xi|)^{1-\theta} - 1], \quad \xi \in \mathbb{R}$$

then our functional is the integral of $|\nabla \Psi(u)|^2$, that is, (1.3) is equivalent to

$$(1.9) \quad \lambda^{\theta,p}(\Omega) = \inf \left\{ \int_{\Omega} |\nabla \Psi(v)|^2 dx, \quad v \in W^{1,q}(\Omega), v \neq 0, \int_{\Omega} v dx = 0, \|v\|_{L^p(\Omega)} = 1 \right\}$$

The main feature of this functional is that it is not coercive on $H_0^1(\Omega)$, even if it is well defined on this Sobolev space. The lack of coercivity has unpleasant consequences for the minimizers of

$$v \rightarrow \int_{\Omega} \left[\frac{|\nabla v|^2}{(1+|v|)^{2\theta}} - G(x, v) \right] dx$$

for functions G having various growth assumptions. Indeed, it was shown in [6], [2], [19], [14], [22], [5], that the minimizers are less regular than the minimizers of coercive functionals on $H^1(\Omega)$.

After recalling the definition of foliated Schwarz symmetry and proving some new sufficient conditions for this symmetry in Section 3, we will prove the foliated Schwarz symmetry of the minimizers for $N \geq 2$. As already pointed out, the same result has been obtained by Girao and Weth in [15] in the “coercive” case, that is, for $\theta = 0$. We observe that in their proof, Girao and Weth make use of a well-known regularity result of the solutions of the Euler equation. In our case, we have to prove the analogous regularity result for our non coercive functional (see Section 4). Actually we are able to prove the foliated Schwarz symmetry of the minimizers of a more general functional, that is we consider

$$(1.10) \quad \lambda^{\theta,p}(\Omega) = \inf \left\{ \int_{\Omega} \frac{|\nabla v|^2 - F(|x|, v)}{(1 + |v|)^{2\theta}} dx, v \in W^{1,q}(\Omega), v \neq 0, \int_{\Omega} v dx = 0, \|v\|_{L^p(\Omega)} = 1 \right\}$$

where we assume that $F : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function in $r = |x| \in [0, +\infty)$ and continuously differentiable in $t \in \mathbb{R}$, which satisfies

$$(1.11) \quad F(r, 0) = 0,$$

and the growth conditions

$$(1.12) \quad |F(r, t)| \leq c(1 + |t|)^p, \quad c > 0,$$

$$(1.13) \quad |F_t(r, t)| \leq C_1(1 + |t|)^{p-1}, \quad C_1 > 0,$$

for any $r \in [0, +\infty)$, $t \in \mathbb{R}$.

If $p \in (1, 2)$, we add the requirement

$$(1.14) \quad t(1 + |t|)F_t(r, t) - 2\theta|t|F(r, t) \leq 0,$$

for any $r \in [0, +\infty)$, $t \in \mathbb{R}$.

In the last two sections we will focus on the two-dimensional setting, in the case where Ω is a ball. We will prove that there exists a unique minimizer, which is anti-symmetric, for $p = 2$ and sufficiently small θ . On the contrary, the minimizers are not anti-symmetric for p sufficiently large. This shows a symmetry breaking phenomenon, which generalizes the results proved by Girao and Weth in the case $\theta = 0$. Note that because of the difficulty given by the lack of coercivity of our functional, our technique is quite different from that one of [15].

2. EXISTENCE OF A MINIMIZER

In this section we prove the existence of a minimizer for problem (1.10) by adapting the technique of [13]. We will also make use of an estimate proved in [6] (see also [1] and [4]).

Theorem 2.1. *Under the assumptions (1.4)-(1.8), (1.11)-(1.14), there exists a minimizer u which realizes $\lambda^{\theta,p}(\Omega)$, as defined in (1.10).*

Proof. We first observe that the growth assumption (1.12) on F and the condition $\|u\|_{L^p(\Omega)} = 1$ in the functional imply that $\lambda^{\theta,p} \in \mathbb{R}$. For any fixed $n \in \mathbb{N}$, let us define

$$H_n(v) = \int_{\Omega} \frac{|\nabla v|^2 - F(|x|, v)}{(1 + |v|)^{2\theta}} dx - \left(\lambda^{\theta,p}(\Omega) + \frac{1}{n} \right),$$

$$H_{\infty}(v) = \int_{\Omega} \frac{|\nabla v|^2 - F(|x|, v)}{(1 + |v|)^{2\theta}} dx - \lambda^{\theta,p}(\Omega)$$

for any $v \in W^{1,q}(\Omega)$ such that $v \neq 0$, $\|v\|_{L^p(\Omega)} = 1$ and $\int_{\Omega} v = 0$. By the definition of infimum, for any fixed $n \in \mathbb{N}$ there exists $u_n \in W^{1,q}(\Omega)$, $u_n \neq 0$, such that

$$(2.1) \quad \|u_n\|_{L^p(\Omega)} = 1, \quad \int_{\Omega} u_n dx = 0, \quad H_n(u_n) < 0.$$

Now, by the growth assumption (1.12) on F , since the functions u_n have L^p -norm equal to 1, we have

$$(2.2) \quad \int_{\Omega} \frac{|F(|x|, u_n)|}{(1 + |u_n|)^{2\theta}} dx \leq C,$$

where C is a positive constant which does not depend on n .

From now on we will denote by C a positive constant which depends on the data and which can vary from line to line.

Since $H_n(u_n) < 0$, estimates (2.1) and (2.2) imply that

$$(2.3) \quad \int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{2\theta}} dx \leq C.$$

Now we prove that $|\nabla u_n|$ is bounded in $L^q(\Omega)$, that is, for any $n \in \mathbb{N}$,

$$(2.4) \quad \|\nabla u_n\|_{L^q(\Omega)} \leq C.$$

We adapt the estimate used in Theorem 2.1 of [6] and we distinguish the case where $N \geq 3$ and the case $N = 2$.

Let $N \geq 3$ with $q = \frac{2N(1-\theta)}{N-2\theta}$. We begin by applying the Hölder inequality since $q < 2$; then we use estimate (2.3) and, since the mean value of u_n is null, by the Sobolev inequality, we get

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^q dx &\leq \left(\int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{2\theta}} dx \right)^{\frac{q}{2}} \left(\int_{\Omega} (1 + |u_n|)^{\frac{2\theta q}{2-q}} dx \right)^{1-\frac{q}{2}} \\ &\leq C \left(\int_{\Omega} \frac{|\nabla u_n|^2}{(1 + |u_n|)^{2\theta}} dx \right)^{\frac{q}{2}} \left(1 + \int_{\Omega} |u_n|^{q^*} dx \right)^{1-\frac{q}{2}} \\ &\leq C \left(1 + \int_{\Omega} |\nabla u_n|^q dx \right)^{\frac{q^*}{q}(1-\frac{q}{2})} \end{aligned}$$

where we have used the equality $\frac{2\theta q}{2-q} = q^*$. Since $N \geq 3$, we deduce that $\frac{q^*}{q}(1 - \frac{q}{2}) < 1$ and (2.4) is proved.

Let $N = 2$. Similarly to above, by using the Hölder inequality, estimate (2.3), the inclusion $L^{\frac{2q}{2-q}}(\Omega) \subset L^{\frac{2q\theta}{2-q}}(\Omega)$ and the Sobolev inequality, we get

$$\begin{aligned} \int_{\Omega} |\nabla u_n|^q dx &\leq \left(\int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\theta}} dx \right)^{\frac{q}{2}} \left(\int_{\Omega} (1+|u_n|)^{\frac{2\theta q}{2-q}} dx \right)^{1-\frac{q}{2}} \\ &\leq C \left(\int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\theta}} dx \right)^{\frac{q}{2}} \left(1 + \int_{\Omega} |u_n|^{\frac{2q}{2-q}} dx \right)^{(1-\frac{q}{2})\theta} \\ &\leq C \left(1 + \int_{\Omega} |\nabla u_n|^q dx \right)^{\theta}. \end{aligned}$$

Since $\theta < 1$, (2.4) follows again.

By the Poincaré-Wirtinger inequality, since the mean value of u_n is zero, we deduce by (2.4) that

$$(2.5) \quad u_n \text{ is bounded in } W^{1,q}(\Omega)$$

and therefore there exists a function $u \in W^{1,q}(\Omega)$ such that, as n goes to ∞ , up to a subsequence,

$$(2.6) \quad u_n \rightharpoonup u \text{ in } W^{1,q}(\Omega) \text{ weakly,}$$

$$(2.7) \quad u_n \rightharpoonup u \text{ in } L^r(\Omega), \quad 1 \leq r < q^*,$$

$$(2.8) \quad u_n \rightarrow u \text{ a.e. in } \Omega.$$

Let $N \geq 3$ with $q = \frac{2N(1-\theta)}{N-2\theta}$. Note that (2.7), since $p < q^*$, implies that

$$\int_{\Omega} u dx = 0, \quad \|u\|_{L^p(\Omega)} = 1.$$

We claim that

$$(2.9) \quad H_{\infty}(u) \leq 0.$$

Let

$$(2.10) \quad \Psi(\xi) := \int_0^{\xi} (1+|t|)^{-\theta} dt = \frac{\operatorname{sgn} \xi}{1-\theta} [(1+|\xi|)^{1-\theta} - 1], \quad \xi \in \mathbb{R}$$

and observe that, by (2.3),

$$\int_{\Omega} |\nabla \Psi(u_n)|^2 dx = \int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\theta}} dx \leq C.$$

Moreover, $\Psi(u_n)$ is bounded in $L^2(\Omega)$, since $2(1-\theta) \leq q$ and u_n is bounded in $L^q(\Omega)$ by (2.5). We infer that $\Psi(u_n)$ is bounded in $W^{1,2}(\Omega)$ and, up to a subsequence, $\Psi(u_n)$ converges weakly in $W^{1,2}(\Omega)$ to a limit which is necessarily $\Psi(u)$, by (2.8). Therefore, by the weak semi-continuity of the norm and inequality in (2.1), as n goes to ∞ , up to a subsequence,

$$\|\Psi(u)\|_{W^{1,2}}^2 \leq \liminf_{n \rightarrow +\infty} \|\Psi(u_n)\|_{W^{1,2}}^2 \leq \lim_{n \rightarrow +\infty} \int_{\Omega} \frac{F(|x|, u_n)}{(1+|u_n|)^{2\theta}} dx + \lim_{n \rightarrow +\infty} \left(\lambda^{\theta,p}(\Omega) + \frac{1}{n} \right).$$

To pass to the limit in the first term of the right hand side, one can use the Lebesgue theorem. Indeed, the pointwise convergence is given by (2.8). The growth assumptions (1.12) on F and (2.7), since $p < q^*$, imply the existence of a function $h \in L^p(\Omega)$ such that

$$\frac{|F(|x|, u_n)|}{(1 + |u_n|)^{2\theta}} \leq h(x) \quad \text{a.e. in } \Omega.$$

Finally we get

$$\int_{\Omega} \frac{|\nabla u|^2}{(1 + |u|)^{2\theta}} dx \leq \int_{\Omega} \frac{F(|x|, u)}{(1 + |u|)^{2\theta}} dx + \lambda^{\theta, p}(\Omega),$$

that is, (2.9) holds. By the definition of $\lambda^{\theta, p}(\Omega)$, necessarily we have $H_{\infty}(u) = 0$. We observe that $\Psi(u) \neq 0$, since $\|u\|_{L^p(\Omega)} = 1$.

It remains to conclude the proof in the case $N = 2$. Indeed when $N = 2$ we have $1 < p < +\infty$ (see (1.8)) and $2(1 - \theta) \leq q < 2$ (see (1.6)), so that the convergences (2.6), (2.7) and (2.8) do not imply, in general, that $\|u\|_{L^p(\Omega)} = 1$. However in view of (2.3) and since $2(1 - \theta) \leq q$ we obtain that $\Psi(u_n)$ is bounded in $W^{1,2}(\Omega)$. From the Sobolev embedding theorem it follows that $\Psi(u_n)$ is bounded in $L^r(\Omega)$ for any $1 \leq r < +\infty$. Since $\Psi(\xi)$ grows like $|\xi|^{1-\theta}$ with $1 - \theta > 0$, we conclude that u_n is bounded in $L^r(\Omega)$ for any $1 \leq r < +\infty$. We obtain that $\|u\|_{L^p(\Omega)} = 1$ and the arguments developed in the case $N = 3$ allow us to conclude that $H_{\infty}(u) = 0$. \square

3. IDENTIFICATION OF SYMMETRY

In this section we generalize some known symmetry criteria (cf. [10]). We first introduce some notation and definitions. Let Ω be a domain that is radially symmetric w.r.t. the origin. In other words, Ω is either an annulus, a ball, or the exterior of a ball in \mathbb{R}^N . If $u : \Omega \rightarrow \mathbb{R}$ is a measurable function, we will for convenience always extend u onto \mathbb{R}^N by setting $u(x) = 0$ for $x \in \mathbb{R}^N \setminus \Omega$.

Definition 3.1. *Let \mathcal{H}_0 be the family of open half-spaces H in \mathbb{R}^N such that $0 \in \partial H$. For any $H \in \mathcal{H}_0$, let σ_H denote the reflection in ∂H . We write*

$$\sigma_H u(x) := u(\sigma_H x), \quad x \in \mathbb{R}^N.$$

The two-point rearrangement w.r.t. H is given by

$$u_H(x) := \begin{cases} \max\{u(x); u(\sigma_H x)\} & \text{if } x \in H, \\ \min\{u(x); u(\sigma_H x)\} & \text{if } x \notin H. \end{cases}$$

The notion of two-point rearrangement was introduced more than fifty years ago as a set transformation in [25], and was applied to variational problems for the first time by Brock and Solynin in [10].

Note that one has $u = u_H$ iff $u(x) \geq u(\sigma_H x)$ for all $x \in H$. Similarly, $\sigma_H u = u_H$ iff $u(x) \leq u(\sigma_H x)$ for all $x \in H$.

We will make use of the following properties of the two-point rearrangement (see [10]).

Lemma 3.1. *Let $H \in \mathcal{H}_0$.*

(1) If $A \in C([0, +\infty), \mathbb{R})$, $u : \Omega \rightarrow \mathbb{R}$ is measurable and $A(|x|, u) \in L^1(\Omega)$, then $A(|x|, u_H) \in L^1(\Omega)$ and

$$(3.1) \quad \int_{\Omega} A(|x|, u) dx = \int_{\Omega} A(|x|, u_H) dx.$$

(2) If $B \in L^\infty(\mathbb{R})$, $u \in W^{1,p}(\Omega)$ for some $p \in [1, +\infty)$, then

$$(3.2) \quad \int_{\Omega} B(u) |\nabla u|^p dx = \int_{\Omega} B(u_H) |\nabla u_H|^p dx$$

Proof. Since $|\sigma_H x| = |x|$, we have for a.e. $x \in H \cap \Omega$,

$$A(|x|, u(x)) + A(|\sigma_H x|, u(\sigma_H x)) = A(|x|, u_H(x)) + A(|\sigma_H x|, u_H(\sigma_H x)),$$

and

$$B(u(x)) |\nabla u(x)|^p + B(u(\sigma_H x)) |\nabla u(\sigma_H x)|^p = B(u_H(x)) |\nabla u_H(x)|^p + B(u_H(\sigma_H x)) |\nabla u_H(\sigma_H x)|^p.$$

Now (3.1) and (3.2) follow from this by integration over $H \cap \Omega$. \square

In order to study the symmetry of minimizers of (1.10) we introduce the notion of foliated Schwarz symmetrization of a function, a function which is axially symmetric with respect to an axis passing through the origin and nonincreasing in the polar angle from this axis.

Definition 3.2. If $u : \Omega \rightarrow \mathbb{R}$ is measurable, the foliated Schwarz symmetrization u^* of u is defined as the (unique) function satisfying the following properties:

(1) there is a function $w : [0, +\infty) \times [0, \pi) \rightarrow \mathbb{R}$, $w = w(r, \theta)$, which is nonincreasing in θ , and

$$u^*(x) = w(|x|, \arccos(x_1/|x|)), \quad (x \in \Omega);$$

(2) $\mathcal{L}^{N-1}\{x : a < u(x) \leq b, |x| = r\} = \mathcal{L}^{N-1}\{x : a < u^*(x) \leq b, |x| = r\}$ for all $a, b \in \mathbb{R}$ with $a < b$, and $r \geq 0$.

Definition 3.3. Let P_N denote the point $(1, 0, \dots, 0)$, the 'north pole' of the unit sphere \mathcal{S}^{N-1} . We say that u is foliated Schwarz symmetric w.r.t. P_N if $u = u^*$ - that is, u depends solely on r and on θ - the 'geographical width' -, and is nonincreasing in θ .

We also say that u is foliated Schwarz symmetric w.r.t. a point $P \in \mathcal{S}^{N-1}$ if there is a rotation about the origin ρ such that $\rho(P_N) = P$, and $u(\rho(\cdot)) = u^*(\cdot)$.

In other words, a function $u : \Omega \rightarrow \mathbb{R}$ is foliated Schwarz symmetric with respect to P if, for every $r > 0$ and $c \in \mathbb{R}$, the restricted superlevel set $\{x : |x| = r, u(x) \geq c\}$ is equal to $\{x : |x| = r\}$ or a geodesic ball in the sphere $\{x : |x| = r\}$ centered at rP . In particular u is axially symmetric with respect to the axis $\mathbb{R}P$.

Moreover a measurable function $u : \Omega \rightarrow \mathbb{R}$ is foliated Schwarz symmetric w.r.t. $P \in \mathcal{S}^{N-1}$ iff $u = u_H$ for all $H \in \mathcal{H}_0$ with $P \in H$.

The main result of this section is the following result which gives a tool to establish if a measurable function is foliated Schwarz symmetric with respect to some point P .

Theorem 3.1. Let $u \in L^p(\Omega)$ for some $p \in [1, +\infty)$, and assume that for every $H \in \mathcal{H}_0$ one has either $u = u_H$, or $\sigma_H u = u_H$. Then u is foliated Schwarz symmetric w.r.t. some point $P \in \mathcal{S}^{N-1}$.

Note that the above result has been shown for continuous functions by Weth in [24].

Theorem 3.2. *Let $u \in C(\mathbb{R}^N)$ and assume that for every $H \in \mathcal{H}_0$ one has either $u = u_H$, or $\sigma_H u = u_H$. Then u is foliated Schwarz symmetric w.r.t. some point $P \in \mathcal{S}^{N-1}$.*

The idea in our proof is to use an approximation argument. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \geq 0$, with $\int_{\mathbb{R}^N} \varphi(x) dx = 1$. Moreover, assume that φ is radial and radially non increasing, that is, there is a nonincreasing function $h : [0, +\infty) \rightarrow [0, +\infty)$ such that $\varphi(x) = h(|x|)$ for all $x \in \mathbb{R}^N$. For any function $\varepsilon > 0$, define φ_ε by $\varphi_\varepsilon(x) := \varepsilon^{-N} \varphi(\varepsilon^{-1}x)$, ($x \in \mathbb{R}^N$). For any $u \in L^1_{loc}(\mathbb{R}^N)$ let u_ε be the standard mollifier of u , given by

$$u_\varepsilon(x) := (u * \varphi_\varepsilon)(x) \equiv \int_{\mathbb{R}^N} u(y) \varphi_\varepsilon(x - y) dy, \quad (x \in \mathbb{R}^N).$$

The following property is crucial. It allows a reduction to C^∞ -functions.

Lemma 3.2. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty)$, and let $H \in \mathcal{H}_0$ such that $u = u_H$. Then $u_\varepsilon = (u_\varepsilon)_H$ for every $\varepsilon > 0$.*

Proof. It is easy to see that

$$|x - y| = |\sigma_H x - \sigma_H y| \leq |\sigma_H x - y| = |x - \sigma_H y|,$$

whenever $x, y \in H$. Since $u(y) \geq u(\sigma_H y)$ and since φ_ε is radial and radially nonincreasing, we have for every $x \in H$,

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(\sigma_H x) &= \int_{\mathbb{R}^N} u(y) [\varphi_\varepsilon(x - y) - \varphi_\varepsilon(\sigma_H x - y)] dx \\ &= \int_H \{u(y) [\varphi_\varepsilon(x - y) - \varphi_\varepsilon(\sigma_H x - y)] + u(\sigma_H y) [\varphi_\varepsilon(x - \sigma_H y) - \varphi_\varepsilon(\sigma_H x - \sigma_H y)]\} dx \\ &= \int_H (u(y) - u(\sigma_H y)) [\varphi_\varepsilon(x - y) - \varphi_\varepsilon(\sigma_H x - y)] dx \geq 0. \end{aligned}$$

The Lemma is proved. \square

Corollary 3.1. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty)$, and let $H \in \mathcal{H}_0$ such that $\sigma_H u = u_H$. Then $\sigma_H(u_\varepsilon) = (u_\varepsilon)_H$ for every $\varepsilon > 0$.*

We are now able to prove Theorem 3.1.

Proof of Theorem 3.1. Since for every $H \in \mathcal{H}_0$ one has either $u = u_H$, or $\sigma_H u = u_H$, Lemma 3.2 and Corollary 3.1 apply. Then either $u_\varepsilon = (u_\varepsilon)_H$, or $\sigma_H u_\varepsilon = (u_\varepsilon)_H$ for every $\varepsilon > 0$. Since $u_\varepsilon \in C^\infty(\mathbb{R}^N)$, Theorem 3.2 tells us that u_ε is foliated Schwarz symmetric w.r.t. some point $P_\varepsilon \in \mathcal{S}^{N-1}$, for every $\varepsilon > 0$. Since \mathcal{S}^{N-1} is compact, there is a sequence of positive numbers $\{\varepsilon_n\}$ and a point $P \in \mathcal{S}^{N-1}$ such that u_{ε_n} is foliated Schwarz symmetric w.r.t. a point $P_n \in \mathcal{S}^{N-1}$ and $\varepsilon_n \rightarrow 0$, $P_n \rightarrow P$ as $n \rightarrow +\infty$. Let ρ_n and ρ be rotations such that $\rho_n(N) = P_n$, ($n \in \mathbb{N}$), and $\rho(N) = P$. Writing $u_n := u_{\varepsilon_n}$ we have that

$$(3.3) \quad u_n(\rho_n(\cdot)) = (u_n)^*(\cdot), \quad (n \in \mathbb{N}).$$

Since $u_n \rightarrow u$, it follows that $(u_n)^* \rightarrow u^*$ in $L^p(\mathbb{R}^N)$, and since $P_n \rightarrow P$ we also have that $u_n(\rho_n(\cdot)) \rightarrow u(\rho(\cdot))$ in $L^p(\mathbb{R}^N)$, as $n \rightarrow \infty$. This, together with (3.3) implies that $u(\rho(\cdot)) = u^*(\cdot)$. The Theorem is proved. \square

4. SYMMETRY OF MINIMIZERS

In this section we study the properties of symmetry of minimizers of (1.10). The main result is the following

Theorem 4.1. *Assume (1.12), (1.13), and (1.14) if $p \in (1, 2)$. Then every minimizer of (1.10) is foliated Schwarz symmetric w.r.t. some point $P \in \mathcal{S}^{N-1}$.*

Remark 4.1. Condition (1.14) is equivalent to

$$(4.1) \quad t \frac{\partial}{\partial t} \left(\frac{F(r, t)}{(1 + |t|)^{2\theta}} \right) \leq 0 \quad \forall (r, t) \in [0, +\infty) \times \mathbb{R}.$$

It is satisfied, for instance, if $F(r, t) = F(r, -t)$ and if

$$(4.2) \quad (1 + t)F_t(r, t) - 2\theta F(r, t) \leq 0 \quad \text{for } t \geq 0.$$

An example is

$$F(r, t) = -c_0 |t|^\alpha, \quad \alpha \geq 2\theta, \quad c_0 \geq 0.$$

Observe that F satisfies the growth condition (1.12) with suitable c_0 and α such that $2\theta \leq \alpha \leq p$.

Proof. We divide the proof into four steps.

Step 1 Let $H \in \mathcal{H}_0$, and let u be a minimizer of (1.10). The Euler equation satisfied by u is

$$(4.3) \quad -\nabla \left(\frac{\nabla u}{(1 + |u|)^{2\theta}} \right) - \theta \frac{|\nabla u|^2 \text{sgn } u}{(1 + |u|)^{2\theta+1}} + c + d|u|^{p-2}u = g(|x|, u) \quad \text{in } \Omega,$$

$$(4.4) \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $c, d \in \mathbb{R}$,

$$(4.5) \quad g(r, t) := \frac{\partial}{\partial t} \left(\frac{F(r, t)}{2(1 + |t|)^{2\theta}} \right), \quad \forall (r, t) \in [0, +\infty) \times \mathbb{R}$$

and ν denotes the exterior unit normal to Ω . Setting

$$I(v) := \int_{\Omega} \frac{|\nabla v|^2 - F(|x|, v)}{(1 + |v|)^{2\theta}} dx,$$

we have, by Lemma 3.1,

$$u_H \neq 0, \quad u_H \in W^{1,q}(\Omega), \quad \int_{\Omega} u_H dx = 0, \quad \|u_H\|_{L^p} = 1, \quad I(u) = I(u_H).$$

Hence, u_H is a minimizer, too, so that it satisfies

$$(4.6) \quad -\nabla \left(\frac{\nabla u_H}{(1 + |u_H|)^{2\theta}} \right) - \theta \frac{|\nabla u_H|^2 \text{sgn } u_H}{(1 + |u_H|)^{2\theta+1}} + c' + d'|u_H|^{p-2}u_H = g(|x|, u_H) \quad \text{in } \Omega,$$

$$(4.7) \quad \frac{\partial u_H}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $c', d' \in \mathbb{R}$.

Step 2 We claim that $u, u_H \in W^{1,q}(\Omega) \cap W^{2,2}(\Omega) \cap C^1(\overline{\Omega})$. Set

$$\Phi(\eta) := \Psi^{-1}(\eta)$$

where Ψ has been defined in (2.10). Let $U := \Psi(u)$. Note that $u = \Phi(U)$, $u_H = \Phi(U_H)$,

$$\Phi(\eta) = \left([1 + (1 - \theta)|\eta|]^{1/(1-\theta)} - 1 \right) \operatorname{sgn} \eta,$$

and Φ is locally Lipschitz continuous. Rewriting (4.3) and (4.6) in terms of U and U_H we find

$$(4.8) \quad -\Delta U + dM(U) = N(|x|, U),$$

$$(4.9) \quad -\Delta U_H + dM(U_H) = N(|x|, U_H)$$

in Ω , where

$$\begin{aligned} M(t) &:= |\Phi(t)|^{p-2} \Phi(t) (1 + |\Phi(t)|)^\theta, \\ N(r, t) &:= (g(r, t) - c) (1 + |\Phi(t)|)^\theta, \end{aligned}$$

for any $r \in [0, +\infty)$, $t \in \mathbb{R}$.

Observe that, by the growth conditions (1.12), (1.13) and definition of $\Phi(t)$, we have

$$\begin{aligned} |g(r, t)| &\leq c_\theta (1 + |t|)^{p-1-2\theta}, \\ |M(t)| &\leq c'_\theta (1 + |t|)^{\frac{p-1+\theta}{1-\theta}}, \\ |N(r, t)| &\leq c''_\theta (1 + |t|)^{p-1-2\theta + \frac{\theta}{1-\theta}}. \end{aligned}$$

Now, the growths of M and N allow us to apply classical techniques for Neumann problems (see p. 272 of [18] and p. 271 of [23]) to state that $U \in H^1(\Omega)$ is in fact $C^{1,\beta}(\overline{\Omega})$, with $\beta \in (0, 1)$. Therefore u has the same regularity.

Step 3 Integrating (4.3) and (4.6) give

$$(4.10) \quad -\theta \int_{\Omega} \frac{|\nabla u|^2 \operatorname{sgn} u}{(1 + |u|)^{2\theta}} dx + c \int_{\Omega} dx + d \int_{\Omega} |u|^{p-2} u dx = \int_{\Omega} g(|x|, u) dx,$$

$$(4.11) \quad -\theta \int_{\Omega} \frac{|\nabla u_H|^2 \operatorname{sgn} u_H}{(1 + |u_H|)^{2\theta}} dx + c' \int_{\Omega} dx + d' \int_{\Omega} |u_H|^{p-2} u_H dx = \int_{\Omega} g(|x|, u_H) dx.$$

Further, multiplying (4.3) and (4.6) with u and u_H respectively, then integrating and using the constraints, yield

$$(4.12) \quad \int_{\Omega} \frac{|\nabla u|^2 [1 + (1 - \theta)|u|]}{(1 + |u|)^{2\theta+1}} dx + d = \int_{\Omega} u g(|x|, u) dx,$$

$$(4.13) \quad \int_{\Omega} \frac{|\nabla u_H|^2 [1 + (1 - \theta)|u_H|]}{(1 + |u_H|)^{2\theta+1}} dx + d' = \int_{\Omega} u_H g(|x|, u_H) dx.$$

Now (4.10)–(4.13) together with Lemma 3.1 show that necessarily

$$c = c', \quad d = d'.$$

Moreover, if $p \in (1, 2)$, then (4.1) holds, so that (4.12) yields $d \leq 0$.

Step 4

Note that $t \rightarrow M(t)$ is nondecreasing. Set $h := U_H - U$, and note that $h \geq 0$ in $\Omega \cap H$. Subtracting (4.9) from (4.8), we split into two cases.

(1) Let $p \geq 2$. Then we find that

$$-\Delta h = L(x)h \quad \text{in } \Omega \cap H,$$

where

$$L(x) := \begin{cases} \frac{N(|x|, U_H(x)) - N(|x|, U(x)) - d[M(U_H(x)) - M(U(x))]}{h(x)} & \text{if } h(x) > 0, \\ 0 & \text{if } h(x) = 0 \end{cases}$$

is a bounded function.

(2) Let $p \in (1, 2)$. Then $d \leq 0$, so that

$$-\Delta h \geq P(x)h \quad \text{in } \Omega \cap H,$$

where

$$P(x) := \begin{cases} \frac{N(|x|, U_H(x)) - N(|x|, U(x))}{h(x)} & \text{if } h(x) > 0, \\ 0 & \text{if } h(x) = 0 \end{cases}$$

is a bounded function.

Thus, in both cases the Strong Maximum Principle tells us that either $h(x) \equiv 0$, or $h(x) > 0$ throughout $\Omega \cap H$. This implies that we have either $u = u_H$, or $\sigma_H u = u_H$ in Ω . By Theorem 3.1 we deduce that u is foliated Schwarz symmetric. \square

5. ANTI-SYMMETRY FOR $p = 2$ IN DIMENSION 2

In this section we study symmetry properties of the solutions to (1.10) in the case $p = 2$, $\Omega = B$, where B is a ball in \mathbb{R}^2 , and $F \equiv 0$. We will show that for small parameter values θ , there exists a unique minimizer of

$$v \rightarrow \int_B \frac{|\nabla v|^2}{(1 + |v|)^{2\theta}} dx, \quad v \in W^{1,q}(B), v \neq 0, \int_B v dx = 0, \|v\|_{L^2(B)} = 1,$$

which is anti-symmetric. Recall that θ satisfies (1.4) and q satisfies (1.6). With abuse of notations, we will denote the infimum of the above functional by $\lambda^{\theta,2}(B)$.

In the following we will use the notations of the proof of Theorem 4.1. More in details, let u_θ be a minimizer for $\lambda^{\theta,2}(B)$, with corresponding constants $c = c_\theta$ and $d = d_\theta$, see equation (4.10). By (4.12) we have,

$$(5.1) \quad d_\theta = - \int_B \frac{|\nabla u_\theta|^2 [1 + (1 - \theta)|u_\theta|]}{(1 + |u_\theta|)^{2\theta+1}} dx.$$

We will also frequently work with the functions

$$U_\theta := \Psi_\theta(u_\theta)$$

where

$$\Psi_\theta(\xi) = \frac{\text{sgn}(\xi)}{1 - \theta} [(1 + |\xi|)^{1-\theta} - 1]$$

(see (2.10)), and

$$\Phi_\theta(\eta) = \Psi_\theta^{-1}(\eta) = \left([1 + (1 - \theta)|\eta|]^{1/(1-\theta)} - 1 \right) \text{sgn}(\eta).$$

Our calculations will often contain a generic constant C that may vary from line to line, but will be *independent of θ* .

Furthermore, as a consequence of Theorem 4.1, we will assume that u_θ is foliated Schwarz symmetric w.r.t. the positive x_1 -half axis, that is,

$$(5.2) \quad u_\theta(x_1, x_2) = u_\theta(x_1, -x_2).$$

The anti-symmetry of u_θ then reads as $u_\theta(x_1, x_2) := -u_\theta(-x_1, x_2)$, if θ is small.

Lemma 5.1. *Under assumptions (1.4), (1.6), the function $\theta \rightarrow \lambda^{\theta,2}(B)$ is decreasing. Moreover $\lambda^{\theta,2}(B) \leq \lambda^2(B)$, where*

$$\lambda^2(B) = \inf \left\{ \int_B |\nabla u|^2 dx, u \in H^1(B), \int_B u dx = 0, \|u\|_{L^2(B)} = 1 \right\}.$$

Proof. Let $\theta_1 < \theta_2$, let u_{θ_1} be a minimizer for $\lambda^{\theta_1,2}(B)$, and let $2(1 - \theta_1) \leq q < 2$. We obtain

$$\lambda^{\theta_1,2}(B) = \int_B \frac{|\nabla u_1|^2}{(1 + |u_1|)^{2\theta_1}} dx \geq \int_B \frac{|\nabla u_1|^2}{(1 + |u_1|)^{2\theta_2}} dx \geq \lambda^{\theta_2,2}(B).$$

Next, let u be a minimizer for $\lambda^2(B)$. Then

$$\lambda^2(B) \geq \int_B |\nabla u|^2 dx \geq \int_B \frac{|\nabla u|^2}{(1 + |u|)^{2\theta}} dx \geq \lambda^{\theta,2}(B).$$

□

Lemma 5.2. *Under assumptions (1.4), (1.6), let u_θ be a minimizer for $\lambda^{\theta,2}(B)$. Let d_θ be defined by (5.1). There holds $\lim_{\theta \rightarrow 0} d_\theta = -\lambda^2(B)$.*

Proof. First we observe that

$$(5.3) \quad -d_\theta = \int_B \frac{|\nabla u_\theta|^2(1 + (1 - \theta)|u_\theta|)}{(1 + |u_\theta|)^{2\theta+1}} dx \leq \int_B \frac{|\nabla u_\theta|^2}{(1 + |u_\theta|)^{2\theta}} dx \leq \lambda^{\theta,2}(B) \leq \lambda^2(B)$$

by Lemma 5.1. On the other hand,

$$(5.4) \quad -d_\theta \geq \int_B \frac{|\nabla u_\theta|^2(1 - \theta)}{(1 + |u_\theta|)^{2\theta}} dx = (1 - \theta) \int_B |\nabla U_\theta|^2 dx.$$

Moreover, it is easy to see that $\|U_\theta\|_{L^2(B)}$ is uniformly bounded, since $\|u_\theta\|_{L^2(B)} = 1$ and $\theta \leq \frac{1}{2}$. Therefore also $\|U_\theta\|_{H^1(B)}$ is uniformly bounded. By compactness, as $\theta \rightarrow 0$, U_θ converges weakly to some function $V \in H^1(B)$ and strongly in $L^2(B)$. By the lower semi-continuity of the norm we get from (5.4)

$$(5.5) \quad \liminf_{\theta \rightarrow 0} (-d_\theta) \geq \|\nabla V\|_{L^2(B)}^2.$$

Further, the a.e. limit of U_θ is the limit of u_θ , say u . By the uniqueness of the limit, $u = V$ a.e. in B . We recall that $\|u_\theta\|_{L^2(B)} = 1$ and $\int_B u_\theta dx = 0$. Since $\|\Psi_\theta(u_\theta)\|_{H^1(B)}$, by the growth of Ψ_θ , we

deduce that $u_\theta \rightarrow u$ in $L^2(B)$ and that $\|V\|_{L^2(B)} = 1$ and $\int_B V dx = 0$. Together with (5.5), this implies that

$$(5.6) \quad \liminf_{\theta \rightarrow 0} (-d_\theta) \geq \lambda^2(B).$$

Now the Lemma follows from inequalities (5.3) and (5.6). \square

Proposition 5.1. *Let u_θ be a minimizer for $\lambda^{\theta,2}(B)$. Under assumptions (1.4), (1.6), we have*

(1) $\|u_\theta\|_{W^{1,\infty}(B)} \leq C$, where C does not depend on θ .

(2) Let u be the limit of u_θ , as $\theta \rightarrow 0$, in $W^{1,r}(\Omega)$, for every $r \in (1, +\infty)$. Then $\|u\|_{L^2(B)} = 1$

$$\text{and } \int_B u dx = 0.$$

Proof. The $H^1(B)$ -norm of U_θ is uniformly bounded by Lemma 5.1. By multiplying equation (4.8) by U_θ and integrating on B , one has that the right hand side of the equality is uniformly bounded, due to Lemma 5.1. The growth of M and Lemma 5.2 imply that $\|U_\theta\|_{L^{\frac{p-1+\theta}{1-\theta}}(B)} \leq C$. This allows us to use the bootstrap argument described at p. 271 of [23]. \square

Let $v_\theta(x_1, x_2) := -u_\theta(-x_1, x_2)$. Then we obtain from (4.10),

$$(5.7) \quad c_\theta = \frac{\theta}{|B|} \int_B \frac{|\nabla u_\theta|^2 \text{sgn}(u_\theta)}{(1 + |u_\theta|)^{2\theta+1}} dx = -\frac{\theta}{|B|} \int_B \frac{|\nabla v_\theta|^2 \text{sgn}(v_\theta)}{(1 + |v_\theta|)^{2\theta+1}} dx.$$

Lemma 5.3. *Under assumptions (1.4), (1.6), let u_θ be a minimizer for $\lambda^{\theta,2}(B)$. Let c_θ be defined by (5.7). There holds $|c_\theta| \leq C\theta\|u_\theta - v_\theta\|_{L^2(B)}$ for a positive constant C independent on θ .*

Proof. By multiplying equation (4.3), (with $p = 2$ and $g = 0$), by $(1 + |u_\theta|)^\theta$, we have

$$(5.8) \quad -\nabla((1 + |u_\theta|)^{-\theta} \nabla u_\theta) + c_\theta(1 + |u_\theta|)^\theta + d_\theta u_\theta(1 + |u_\theta|)^\theta = 0.$$

Integrating this gives

$$c_\theta \int_B (1 + |u_\theta|)^\theta dx + d_\theta \int_B u_\theta(1 + |u_\theta|)^\theta dx = 0,$$

since $\frac{\partial u_\theta}{\partial \nu} = 0$ on ∂B . The first integral in this identity is bounded from below, and $|d_\theta|$ is bounded

by Lemma 5.2. If J denotes $\int_B u_\theta(1 + |u_\theta|)^\theta dx$, we deduce that

$$(5.9) \quad |c_\theta| \leq C|J|$$

for a constant C independent on θ . A change of variables gives

$$J = \frac{1}{2} \int_B [u_\theta(1 + |u_\theta|)^\theta - v_\theta(1 + |v_\theta|)^\theta] dx.$$

Since $\int_B u_\theta dx = \int_B v_\theta dx = 0$, we get

$$J = \frac{1}{2} \int_B (u_\theta - v_\theta)[(1 + |v_\theta|)^\theta - 1] dx + \frac{1}{2} \int_B u_\theta[(1 + |u_\theta|)^\theta - (1 + |v_\theta|)^\theta] dx.$$

Let J_1 denote the first term and J_2 the second one in this identity. A short computation shows that

$$|J_1| \leq \frac{\theta}{2} \int_B |u_\theta - v_\theta| |v_\theta| dx, \quad |J_2| \leq \frac{\theta}{2} \int_B |u_\theta - v_\theta| |u_\theta| dx.$$

Since u_θ and v_θ are uniformly bounded by Proposition 5.1, this gives

$$|J| \leq C\theta \|u_\theta - v_\theta\|_{L^2(B)}.$$

The conclusion follows from estimate (5.9). \square

Now we can prove the main result of the section

Theorem 5.1. *There is a number $\theta_0 > 0$, such that every minimizer u_θ of $\lambda^{\theta,2}(B)$ satisfying (5.2) is unique and anti-symmetric w.r.t. x_1 , that is,*

$$(5.10) \quad u_\theta(x_1, x_2) := -u_\theta(-x_1, x_2),$$

for any $0 < \theta < \theta_0$.

Proof. We first prove that any minimizer is anti-symmetric. Let $U_\theta := \Psi_\theta(u_\theta)$ and $V_\theta := \Psi_\theta(v_\theta)$, where $v_\theta(x_1, x_2) = -u_\theta(-x_1, x_2)$. Writing equation (5.8) in terms of U_θ we have

$$-\Delta U_\theta + c_\theta(1 + |u_\theta|)^\theta + d_\theta u_\theta(1 + |u_\theta|)^\theta = 0.$$

Similarly

$$-\Delta V_\theta - c_\theta(1 + |v_\theta|)^\theta + d_\theta v_\theta(1 + |v_\theta|)^\theta = 0.$$

Subtract both equations from each other. Assuming that $U_\theta - V_\theta \neq 0$ along a sequence $\theta \rightarrow 0$, we multiply by $\frac{U_\theta - V_\theta}{\|U_\theta - V_\theta\|_{L^2(B)}^2}$ and integrate. Then we obtain

$$\begin{aligned} & \frac{\|\nabla(U_\theta - V_\theta)\|_{L^2(B)}^2}{\|U_\theta - V_\theta\|_{L^2(B)}^2} + \frac{c_\theta}{\|U_\theta - V_\theta\|_{L^2(B)}} \int_B [(1 + |u_\theta|)^\theta + (1 + |v_\theta|)^\theta] \frac{U_\theta - V_\theta}{\|U_\theta - V_\theta\|_{L^2(B)}} dx = \\ & = -d_\theta \int_B [u_\theta(1 + |u_\theta|)^\theta - v_\theta(1 + |v_\theta|)^\theta] \frac{U_\theta - V_\theta}{\|U_\theta - V_\theta\|_{L^2(B)}} dx. \end{aligned}$$

The second term of the left-hand side tends to zero, by Lemma 5.3, and since $(1 + |u_\theta|)^\theta + (1 + |v_\theta|)^\theta$ is uniformly bounded by Proposition 5.1. To estimate the right-hand side, we first observe that $-d_\theta \rightarrow \lambda^2(B)$, by Lemma 5.2. Moreover, it is not difficult to prove the following estimate:

$$|u_\theta(1 + |u_\theta|)^\theta - v_\theta(1 + |v_\theta|)^\theta| \leq (1 + \theta)[1 + (1 - \theta)|\xi_\theta|]^{\frac{2\theta}{1-\theta}} |U_\theta - V_\theta|,$$

where ξ_θ is between $U_\theta = \Psi_\theta(u_\theta)$ and $V_\theta = \Psi_\theta(v_\theta)$. By Proposition 5.1, we deduce that $(1 + \theta)[1 + (1 - \theta)|\xi_\theta|]^{\frac{2\theta}{1-\theta}} \rightarrow 1$, as $\theta \rightarrow 0$, uniformly in B .

Now, set $W_\theta := \frac{U_\theta - V_\theta}{\|U_\theta - V_\theta\|_{L^2(B)}}$. By the above identity, the norms $\|\nabla W_\theta\|_{L^2(B)}$ are uniformly bounded. Hence there is a function $\tilde{W} \in H^1(B)$, such that along a subsequence, $\nabla W_\theta \rightarrow \nabla \tilde{W}$ weakly in $L^2(B)$ and $W_\theta \rightarrow \tilde{W}$ in $L^2(B)$, so that $\|\tilde{W}\|_{L^2(B)} = 1$. This also implies

$$\|\nabla \tilde{W}\|_{L^2(B)}^2 \leq \lambda^2(B).$$

Next we claim that $\left| \int_B W_\theta dx \right| \leq C\theta$. Indeed, since $\int_B u_\theta dx = \int_B v_\theta dx = 0$, we have

$$\begin{aligned} \left| \int_B (U_\theta - V_\theta) dx \right| &= \int_B (\Psi_\theta(u_\theta) - u_\theta - \Psi_\theta(v_\theta) + v_\theta) dx \\ &\leq \int_B \left| \int_{v_\theta}^{u_\theta} (\Psi'_\theta(t) - 1)(u_\theta - v_\theta) dt \right| dx \leq \theta \int_B |u_\theta - v_\theta| dx. \end{aligned}$$

Now, Φ_θ is locally Lipschitz continuous, uniformly in θ . By Proposition 5.1, we obtain

$$\left| \int_B (U_\theta - V_\theta) dx \right| \leq \theta \int_B |u_\theta - v_\theta| dx \leq C\theta \int_B |U_\theta - V_\theta| dx \leq C\theta \|U_\theta - V_\theta\|_{L^2(B)}$$

and the claim follows. This and the fact that $W_\theta \rightarrow \tilde{W}$ in $L^2(B)$ prove that $\int_B \tilde{W} dx = 0$. Then, by definition of $\lambda^2(B)$, we have that

$$\|\nabla \tilde{W}\|_{L^2(B)}^2 \geq \lambda^2(B).$$

Hence \tilde{W} is a (nonzero) eigenfunction for the Neumann Laplacian in B . By the properties of U_θ and V_θ and by (5.2), one has $W_\theta(x_1, x_2) = W_\theta(-x_1, x_2) = W_\theta(x_1, -x_2)$. Thus the same symmetry properties hold for \tilde{W} . But this is in contradiction with the shape of the eigenfunction in a ball, which is given by

$$J_n(\alpha_{nk}|(x_1, x_2)|/R) \cdot \begin{cases} \cos(n\varphi), & l = 1 \\ \sin(n\varphi), & l = 2(n \neq 0) \end{cases},$$

where we have used the polar coordinates, R is the radius of the ball, and α_{nk} are the positive roots of the derivative of the Bessel function J_n , (see for example [16]). We thus have proved that any minimizer is anti-symmetric.

Note that the anti-symmetry also implies that $c_\theta = 0$, which can be seen by integrating (4.3).

It remains to prove that the minimizer is unique for small θ . Assume this is not the case. Therefore along a sequence $\theta \rightarrow 0$ along which there exist two distinct minimizers u_θ and u'_θ . Let the corresponding constants d of (4.3) be denoted by d_θ and d'_θ . Multiplying (4.3) for u_θ with u_θ , and integrating by parts gives

$$\begin{aligned} (5.11) \quad d_\theta &= - \int_B \frac{|\nabla u_\theta|^2}{(1 + |u_\theta|)^{2\theta}} dx + \theta \int_B \frac{|\nabla u_\theta|^2 |u_\theta|}{(1 + |u_\theta|)^{2\theta+1}} dx \\ &= -\lambda^{2,\theta}(B) + \theta \int_B \frac{|\nabla u_\theta|^2 |u_\theta|}{(1 + |u_\theta|)^{2\theta+1}} dx. \end{aligned}$$

Similarly

$$(5.12) \quad d'_\theta = -\lambda^{2,\theta}(B) + \theta \int_B \frac{|\nabla u'_\theta|^2 |u'_\theta|}{(1 + |u'_\theta|)^{2\theta+1}} dx.$$

We define

$$g_\theta(\xi) := \frac{|\xi|}{(1 + |\xi|)^{2\theta+1}}.$$

Since the functions g_θ are locally Lipschitz continuous, uniformly in θ , we can estimate, using Proposition 1,

$$\begin{aligned}
(5.13) \quad & \left| \frac{|\nabla u_\theta|^2 |u_\theta|}{(1+|u_\theta|)^{2\theta+1}} - \frac{|\nabla u'_\theta|^2 |u'_\theta|}{(1+|u'_\theta|)^{2\theta+1}} \right| \\
&= \left| \frac{|u_\theta|}{(1+|u_\theta|)^{2\theta+1}} (|\nabla u_\theta|^2 - |\nabla u'_\theta|^2) + |\nabla u'_\theta|^2 (g_\theta(u_\theta) - g_\theta(u'_\theta)) \right| \\
&\leq C (|\nabla u_\theta - \nabla u'_\theta| + |u_\theta - u'_\theta|).
\end{aligned}$$

Subtracting (5.12) from (5.11) and taking into account (5.13), we obtain,

$$\begin{aligned}
(5.14) \quad |d_\theta - d'_\theta| &\leq \theta \int_B \left| \frac{|\nabla u_\theta|^2 |u_\theta|}{(1+|u_\theta|)^{2\theta+1}} - \frac{|\nabla u'_\theta|^2 |u'_\theta|}{(1+|u'_\theta|)^{2\theta+1}} \right| dx \\
&\leq C\theta (\|u_\theta - u'_\theta\|_{L^2(B)} + \|\nabla(u_\theta - u'_\theta)\|_{L^2(B)}).
\end{aligned}$$

Now we claim that

$$(5.15) \quad |d_\theta - d'_\theta| \leq C\theta \|U_\theta - U'_\theta\|_{L^2(B)}.$$

As in the proof of Lemma 5.3, by multiplying equation (4.3), (with $p = 2$ and $g = 0$), by $(1+|u_\theta|)^\theta$, we have

$$(5.16) \quad -\nabla((1+|u_\theta|)^{-\theta} \nabla u_\theta) + c_\theta(1+|u_\theta|)^\theta + d_\theta u_\theta(1+|u_\theta|)^\theta = 0.$$

Moreover the analogous of this equality holds true for u'_θ . Moreover by multiplying this equation by $u_\theta - u'_\theta$, we get

$$\begin{aligned}
(5.17) \quad & \int_B \left(\frac{\nabla u_\theta}{(1+|u_\theta|)^\theta} - \frac{\nabla u'_\theta}{(1+|u'_\theta|)^\theta} \right) \nabla(u_\theta - u'_\theta) dx \\
&+ d_\theta \int_B [u_\theta(1+|u_\theta|)^\theta - u'_\theta(1+|u'_\theta|)^\theta] (u_\theta - u'_\theta) dx \\
&+ (d_\theta - d'_\theta) \int_B [u'_\theta(1+|u_\theta|)^\theta] (u_\theta - u'_\theta) dx = 0.
\end{aligned}$$

Now we evaluate the three integrals on the left-hand side. For value of θ small enough, we have

$$\begin{aligned}
(5.18) \quad & \int_B \left(\frac{\nabla u_\theta}{(1+|u_\theta|)^\theta} - \frac{\nabla u'_\theta}{(1+|u'_\theta|)^\theta} \right) \nabla(u_\theta - u'_\theta) dx \\
&= \int_B \frac{1}{(1+|u_\theta|)^\theta} |\nabla(u_\theta - u'_\theta)|^2 dx + \\
&+ \int_B \nabla u'_\theta \cdot \nabla(u_\theta - u'_\theta) \left(\frac{1}{(1+|u_\theta|)^\theta} - \frac{1}{(1+|u'_\theta|)^\theta} \right) dx \\
&\geq \frac{1}{2} \|\nabla(u_\theta - u'_\theta)\|_{L^2(B)}^2 - C \|\nabla(u_\theta - u'_\theta)\|_{L^2(B)} \|u_\theta - u'_\theta\|_{L^2(B)}.
\end{aligned}$$

Moreover, as in the calculation of J in the previous arguments (see after (5.9)), we get

$$(5.19) \quad \left| d_\theta \int_B [u_\theta(1+|u_\theta|)^\theta - u'_\theta(1+|u'_\theta|)^\theta] (u_\theta - u'_\theta) dx \right| \leq C \|u_\theta - u'_\theta\|_{L^2(B)}^2$$

(5.20)

$$\int_B \nabla u'_\theta \cdot \nabla (u_\theta - u'_\theta) \left(\frac{1}{(1 + |u_\theta|)^\theta} - \frac{1}{(1 + |u'_\theta|)^\theta} \right) dx \leq C\theta \|u_\theta - u'_\theta\|_{L^2(B)} (\|u_\theta - u'_\theta\|_{L^2(B)} + \|\nabla(u_\theta - u'_\theta)\|_{L^2(B)}).$$

Combining (5.18)- (5.20), via Young inequality, we get

$$\|\nabla(u_\theta - u'_\theta)\|_{L^2(B)}^2 \leq C\|u_\theta - u'_\theta\|_{L^2(B)}^2.$$

Now as in the previous calculation we get

$$\|u_\theta - u'_\theta\|_{L^2(B)} \leq \|U_\theta - U'_\theta\|_{L^2(B)}$$

This gives (5.15).

Next we define

$$h_\theta(\xi) := \xi(1 + |\xi|)^\theta - \Psi_\theta(\xi) = \xi(1 + |\xi|)^\theta - \frac{\operatorname{sgn} \xi}{1 - \theta} \left[(1 + |\xi|)^{1-\theta} - 1 \right].$$

It is easy to see that h_θ is locally Lipschitz continuous with

$$|h'_\theta(\xi)| \leq C\theta, \quad (|\xi| \leq M),$$

where the constant C depends only on M , ($M > 0$). Using Proposition 1 we obtain from this

$$(5.21) \quad |h_\theta(u_\theta) - h_\theta(u'_\theta)| \leq C\theta |u_\theta - u'_\theta|.$$

Now let $U_\theta := \Psi_\theta(u_\theta)$ and $U'_\theta := \Psi_\theta(u'_\theta)$. Arguing similarly as before, we first observe

$$(5.22) \quad |u_\theta - u'_\theta| \leq C|\Psi_\theta(u_\theta) - \Psi_\theta(u'_\theta)| = C|U_\theta - U'_\theta|.$$

We have by (4.3),

$$(5.23) \quad -\Delta U_\theta + d_\theta u_\theta (1 + |u_\theta|)^\theta = 0,$$

$$(5.24) \quad -\Delta U'_\theta + d'_\theta u'_\theta (1 + |u'_\theta|)^\theta = 0.$$

Subtracting (5.24) from (5.23), multiplying with $(U_\theta - U'_\theta)$ and integrating by parts gives,

$$(5.25) \quad 0 = \|\nabla(U_\theta - U'_\theta)\|_2^2 + d_\theta \int_B \left(u_\theta (1 + |u_\theta|)^\theta - u'_\theta (1 + |u'_\theta|)^\theta \right) (U_\theta - U'_\theta) dx \\ + (d_\theta - d'_\theta) \int_B u'_\theta (1 + |u'_\theta|)^\theta (U_\theta - U'_\theta) dx.$$

Now define $W_\theta := (U_\theta - U'_\theta)\|U_\theta - U'_\theta\|_{L^2(B)}^{-1}$. Then we obtain from (5.25),

$$(5.26) \quad 0 = \|\nabla W_\theta\|_{L^2(B)}^2 + d_\theta \int_B \left(u_\theta (1 + |u_\theta|)^\theta - u'_\theta (1 + |u'_\theta|)^\theta \right) W_\theta \|U_\theta - U'_\theta\|_{L^2(B)}^{-1} dx \\ + (d_\theta - d'_\theta) \int_B u'_\theta (1 + |u'_\theta|)^\theta W_\theta \|U_\theta - U'_\theta\|_{L^2(B)}^{-1} dx. \\ = \|\nabla W_\theta\|_{L^2(B)}^2 + d_\theta \|W_\theta\|_{L^2(B)}^2 \\ + d_\theta \int_B (h_\theta(u_\theta) - h_\theta(u'_\theta)) W_\theta \|U_\theta - U'_\theta\|_{L^2(B)}^{-1} dx \\ + (d_\theta - d'_\theta) \int_B u'_\theta (1 + |u'_\theta|)^\theta W_\theta \|U_\theta - U'_\theta\|_{L^2(B)}^{-1} dx.$$

Since $\|W_\theta\|_{L^2(B)} = 1$, and in view of the estimates (5.14) and (5.22), we see that the last two terms in (5.26) tend to zero as $\theta \rightarrow 0$. Hence the functions W_θ are uniformly bounded in $H^1(B)$. By passing to another subsequence if necessary, we find a function $\overline{W} \in H^1(B)$ such that $W_\theta \rightarrow \overline{W}$ weakly in $H^1(B)$ and $W_\theta \rightarrow \overline{W}$ in $L^2(B)$. Then, passing to the limit in (5.26) we obtain, since $\liminf_{\theta \rightarrow 0} \|\nabla W_\theta\|_{L^2(B)} \geq \|\nabla \overline{W}\|_{L^2(B)}$,

$$(5.27) \quad \|\nabla \overline{W}\|_{L^2(B)}^2 \leq \lambda^2(B) \|\overline{W}\|_{L^2(B)}^2.$$

Since also $\int_B \overline{W} dx = 0$ and $\int_B \overline{W}^2 dx = 1$, we must have equality in (5.27), and \overline{W} is an anti-symmetric eigenfunction for the Neumann Laplacian in B , that is, $\overline{W} = u$. In other words, we have

$$\int_B W_\theta U_\theta dx \rightarrow \int_B u^2 dx = 1, \quad \text{as } \theta \rightarrow 0.$$

On the other hand, we calculate

$$\begin{aligned} \int_B W_\theta U_\theta dx &= \frac{\int_B U_\theta^2 dx - \int_B U'_\theta U_\theta dx}{\|U_\theta - U'_\theta\|_{L^2(B)}} \\ &= \frac{1 - \int_B U'_\theta U_\theta dx}{\sqrt{2 - 2 \int_B U'_\theta U_\theta dx}} \\ &= \frac{1}{\sqrt{2}} \sqrt{1 - \int_B U'_\theta U_\theta dx} \\ &\rightarrow 0, \quad \text{as } \theta \rightarrow 0, \end{aligned}$$

which gives a contradiction. The proof is complete. \square

6. SYMMETRY BREAKING IN DIMENSION 2

In this section we continue studying the two dimensional case, assuming again that $F \equiv 0$. We show that for p sufficiently large the minimizers of $\lambda^{\theta,p}$ do not verify the properties of anti-symmetry described in the previous section; therefore a phenomenon of symmetry breaking occurs.

Let us denote by $W_{as}^{1,q}(B)$ the subset of the Sobolev space $W^{1,q}(B)$ of the functions which are anti-symmetric with respect to the plane $P \equiv \{x \in \mathbb{R}^{N+1} : x_N = 0\}$, that is,

$$W_{as}^{1,q}(B) := \{v \in W^{1,q}(B) : u(x', -x_N) = -u(x', x_N)\}.$$

Let

$$\mathcal{F}(v) = \int_B \frac{|\nabla v|^2}{(1 + |v|)^{2\theta}} dx, \quad v \in W^{1,q}(B), v \neq 0, \int_B v dx = 0, \|v\|_{L^2(B)} = 1.$$

Recall that θ satisfies (1.4) and q satisfies (1.6). Let

$$\lambda^{\theta,p}(B) := \inf \left\{ \mathcal{F}(v), v \in W^{1,q}(B), v \neq 0, \int_B v dx = 0, \|v\|_{L^p(B)} = 1 \right\}$$

and

$$\lambda_{as}^{\theta,p}(B) := \inf \left\{ \mathcal{F}(v), v \in W_{as}^{1,q}(B), v \neq 0, \int_B v dx = 0, \|v\|_{L^p(B)} = 1 \right\}.$$

Observe that the existence of a function realizing $\lambda_{as}^{\theta,p}(B)$ can be proved analogously as in Theorem 2.1.

Let us also recall a well-known result. For any bounded smooth domain $\Omega \subset \mathbb{R}^2$, let

$$\Lambda_{as}^p(\Omega) = \inf \left\{ \|\nabla v\|_{L^2(\Omega)}^2, v \in W_{as}^{1,2}(\Omega), v \neq 0, \int_{\Omega} v \, dx = 0, \|v\|_{L^p(\Omega)} = 1 \right\}.$$

In [15] the behaviour of $\Lambda_{as}^p(\Omega)$ is studied and it is proved that

$$(6.1) \quad \Lambda_{as}^p(\Omega) \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

It is easy to prove the same result for $\lambda_{as}^{\theta,p}(B)$:

Proposition 6.1. *We have $\lambda_{as}^{\theta,p}(B) \rightarrow 0$, as $p \rightarrow \infty$.*

Proof. Since $\int_B |\nabla v|^2 \, dx \geq \mathcal{F}(v)$, one has

$$\begin{aligned} \Lambda_{as}^p(B) &\geq \inf \left\{ \mathcal{F}(v), v \in W_{as}^{1,2}(B), v \neq 0, \int_B v \, dx = 0, \|v\|_{L^p(B)} = 1 \right\} \\ &\geq \inf \left\{ \mathcal{F}(v), v \in W_{as}^{1,q}(B), v \neq 0, \int_B v \, dx = 0, \|v\|_{L^p(B)} = 1 \right\} \\ &= \lambda_{as}^{\theta,p}(B) \end{aligned}$$

By (6.1) the conclusion follows. \square

Now we can prove the main result of the section.

Theorem 6.1. *For p sufficiently large, $\lambda^{\theta,p}(B) < \lambda_{as}^{\theta,p}(B)$. Therefore the minimizers of \mathcal{F} are not anti-symmetric for p sufficiently large.*

Proof. Let v_p be an eigenfunction for $\lambda_{as}^{\theta,p}(B)$. Hence $\|v_p\|_{L^p(B)} = 1$. Let $B_+ = \{(x_1, x_2) \in B : x_2 > 0\}$, and let \bar{u}_p be defined by

$$\bar{u}_p(x) = \begin{cases} v_p(x), & x \in B_+, \\ 0, & x \in B \setminus B_+. \end{cases}$$

Then

$$(6.2) \quad \int_B \frac{|\nabla \bar{u}_p|^2}{(1 + |\bar{u}_p|)^{2\theta}} \, dx = \frac{\lambda_{as}^{\theta,p}(B)}{2}, \quad \|\bar{u}_p\|_{L^p(B)}^p = \frac{1}{2}.$$

We claim that

$$(6.3) \quad \int_B \bar{u}_p \, dx \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

By Proposition 6.1, we deduce that

$$\lambda_{as}^{\theta,p}(\Omega) = 2\|\nabla \Psi(\bar{u}_p)\|_{L^2(B)} \rightarrow 0, \quad \text{as } p \rightarrow \infty,$$

where Ψ has been defined in (2.10). Since $\bar{u}_p = 0$ in $B \setminus B_+$, we can use Poincaré-Wirtinger inequality which implies

$$\|\Psi(\bar{u}_p)\|_{L^2(B)} \rightarrow 0, \quad \text{as } p \rightarrow \infty.$$

Therefore, up to subsequence, $\Psi(\bar{u}_p) \rightarrow 0$ and $u_p \rightarrow 0$ a.e. in B . On the other hand, there exists a function $h \in L^2(B)$ such that $|\Psi(\bar{u}_p)| \leq h$ a.e. in B . By definition of $\Psi(t)$, we deduce the existence of a function $k \in L^{2(1-\theta)}(B)$ such that $|\bar{u}_p| \leq k$ a.e. in B . Hence Lebesgue's theorem applies and we get $\int_B |\bar{u}_p| dx \rightarrow 0$. This proves (6.3).

Next we define

$$\tilde{u}_p := \frac{\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx}{\left\| \bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx \right\|_{L^p(B)}}.$$

Therefore

$$\lambda^{\theta,p}(B) \leq \int_B \frac{|\nabla \tilde{u}_p|^2}{(1 + |\tilde{u}_p|)^{2\theta}} dx = \frac{1}{\left\| \bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx \right\|_{L^p(B)}^2} \int_B \frac{|\nabla \bar{u}_p|^2}{\left(1 + \frac{|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx\|_{L^p(B)}}\right)^{2\theta}} dx.$$

Let $\varepsilon > 0$ be sufficiently small. For a suitable $p(\varepsilon) > 0$ and for any $p > p(\varepsilon)$, one has, by (6.2)

$$(6.4) \quad \lambda^{\theta,p}(B) \leq \frac{1}{\left[\left(\frac{1}{2}\right)^{\frac{1}{p}} - \varepsilon\right]^2} \int_B \frac{|\nabla \bar{u}_p|^2}{\left[1 + \frac{|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}}\right]^{2\theta}} dx.$$

Let us set $M_\varepsilon = \frac{1 + \varepsilon}{1 - \varepsilon}$.

We claim that

$$(6.5) \quad G(\bar{u}_p) \equiv \frac{1 + |\bar{u}_p|}{1 + \frac{|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}}} \leq M_\varepsilon.$$

First of all, it is easy to verify that, for any $p > p(\varepsilon)$,

$$(6.6) \quad 1 + \left| \bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx \right| \geq 1 + \left| \bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx \right| \geq 1 + \|\bar{u}_p\| - \varepsilon \geq (1 - \varepsilon)(1 + |\bar{u}_p|).$$

Now we distinguish two cases.

(1) If $\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon} \leq 1$, then $G(\bar{u}_p) \leq \frac{1 + |\bar{u}_p|}{1 + |\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}$. By (6.6) one has $G(\bar{u}_p) \leq \frac{1}{1 - \varepsilon}$.

(2) If $\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon} > 1$, then by (6.6),

$$1 + \frac{|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}} \geq \frac{1 + |\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}} \geq \frac{1 - \varepsilon}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}} (1 + |\bar{u}_p|).$$

Therefore (6.5) is proved, that is,

$$(6.7) \quad \frac{1}{1 + \frac{|\bar{u}_p - \frac{1}{|B|} \int_B \bar{u}_p dx|}{\left(\frac{1}{2}\right)^{\frac{1}{p} + \varepsilon}}} \leq \frac{M_\varepsilon}{1 + |\bar{u}_p|}.$$

Combining estimates (6.4) and (6.7) we get

$$\lambda^{\theta,p}(B) \leq \frac{M_\varepsilon^{2\theta}}{\left[\left(\frac{1}{2}\right)^{\frac{1}{p}} - \varepsilon\right]^2} \int_B \frac{|\nabla \bar{u}_p|^2}{(1 + |\bar{u}_p|)^{2\theta}} dx.$$

It is clear that $\frac{M_\varepsilon^{2\theta}}{\left[\left(\frac{1}{2}\right)^{\frac{1}{p}} - \varepsilon\right]^2} = \left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{2\theta} \frac{1}{\left[\left(\frac{1}{2}\right)^{\frac{1}{p}} - \varepsilon\right]^2} < 2$ for $p > p(\varepsilon)$. Therefore, for p sufficiently large, one has

$$\lambda^{\theta,p}(B) < 2 \int_B \frac{|\nabla \bar{u}_p|^2}{(1 + |\bar{u}_p|)^{2\theta}} dx = \lambda_{as}^{\theta,p}(B)$$

by (6.2). □

Acknowledgements. This work started during a visit of A. Mercaldo to Université de Rouen which was financed by the Fédération Normandie Mathématiques. The last author is a member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) which supported visitings of F. Brock to Università degli Studi di Napoli Federico II. All these institutions are gratefully acknowledged.

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