

Appendix A: Chebyshev system

The study of Bézier curves is intimately linked to the theory of Chebyshev systems (Gasca and Micchelli, 2013), (Schumaker, 2007), (Karlín and Studden, 1966). In the following our definitions are restricted to the interval $[0, 1]$, but it is not mandatory: any interval between an initial point a and final point b , open or closed would work.

The following determinant is traditionally denoted

$$M \begin{pmatrix} x_0, & \dots, & x_j \\ f_0, & \dots, & f_j \end{pmatrix} := \begin{vmatrix} f_0(x_0) & \dots & f_j(x_0) \\ f_0^{(1)}(x_0) & \dots & f_j^{(1)}(x_0) \\ \dots & \dots & \dots \\ f_0^{(j-1)}(x_0) & \dots & f_j^{(j-1)}(x_0) \end{vmatrix}.$$

Definition A1 *The functions f_0, f_1, \dots, f_J are called an extended Chebyshev system of class C^{J-1} on $[0, 1]$, or ET-system if they are J times differentiable on $[0, 1]$ and if*

$$\forall x_0 \leq \dots \leq x_J \in [0, 1], M \begin{pmatrix} x_0, & \dots, & x_J \\ f_0, & \dots, & f_J \end{pmatrix} > 0.$$

Definition A2 *The functions f_0, \dots, f_j are called an extended complete Chebyshev system on $[0, 1]$, or ECT-system if*

$$\forall 1 \leq j \leq J, \forall x_0 \leq \dots \leq x_j \in [0, 1], M \begin{pmatrix} x_0, & \dots, & x_j \\ f_0, & \dots, & f_j \end{pmatrix} > 0.$$

In Definition A2, when passing from an ET to an ECT, J is replaced by j and x_0 (resp. $\dots x_j$) can be repeated at most j times.

A consequence of definitions A1 and A2 which will be useful for the proofs in Appendix B is that all the columns vectors of the determinant $M \begin{pmatrix} x_1, & \dots, & x_J \\ f_1, & \dots, & f_J \end{pmatrix} > 0$ are linearly independent.

Two important results on ET and ECT-systems are the following. The first one is based on the notion of multiplicity of the intersection of a curve and a hyperplane at x .

Definition A3 *Let H be a hyperplane in dimension J containing the point (a_1, \dots, a_J) , and spanned by $J - 1$ vectors ${}^t(\alpha_{j,1}, \dots, \alpha_{j,J})_{j=1, J-1}$. Denoting*

$$E(t_1, \dots, t_J) := \begin{vmatrix} t_1 - a_1 & \dots & t_J - a_J \\ \alpha_{1,1} & \dots & \alpha_{1,J} \\ \dots & \dots & \dots \\ \alpha_{J-1,1} & \dots & \alpha_{J-1,J} \end{vmatrix},$$

the equation of H is given by

$$E(t_1, \dots, t_J) = 0.$$

The multiplicity of the intersection at x of the hyperplane H and the curve $C_J = (f_1(x), \dots, f_J(x))$ is defined as the order of the first non vanishing derivative of the determinant $E(f_1(x), \dots, f_J(x))$.

If the curve C_J intersects the hyperplane H at x and if its tangent at x is not contained in H then the multiplicity is 1.

Theorem A1 (extracted from Karlin and Studden (1966), chap I first part of Theorem 4.3) Let $Z(F_\alpha)$ be the number of zeros of F_α on $[0, 1]$ counting multiplicities. If one at least of the coordinates of α is different from zero, then $Z(F_\alpha) \leq J$.

The following specialization is well known in GCAD.

Corollary A1 Let $f_0(x) = 1$ for all $x \in [0, 1]$. If f_0, f_1, \dots, f_J is an ET-system on $[0, 1]$, then any hyperplane in \mathbb{T} intersects the curve C_J on $[0, 1]$ at most J times counting multiplicities.

The second theorem (see Karlin and Studden (1966), Theorem 1.1 Chap XI) gives a characterization of an ECT-system in terms of Wronskians. The Wronskian for $1 \leq j \leq J$ is defined as

$$W_{f_0, \dots, f_j}(x) := \begin{vmatrix} f_0(x) & \cdots & f_j(x) \\ f_0^{(1)}(x) & \cdots & f_j^{(1)}(x) \\ \vdots & \cdots & \vdots \\ f_0^{(j-1)}(x) & \cdots & f_j^{(j-1)}(x) \end{vmatrix}.$$

This notation is a shortcut for $M \begin{pmatrix} x, & \cdots, & x \\ f_0, & \cdots, & f_j \end{pmatrix}$.

Theorem A2 (extracted from Karlin and Studden (1966)) Let f_0, f_1, \dots, f_J be of class C^J on $[0, 1]$. Then f_0, \dots, f_J is an ECT-system on $[0, 1]$ if and only if for $j = 0, \dots, J$ we have $W_{f_0, \dots, f_j}(x) > 0$ on $[0, 1]$.

Examples. Theorem A2 gives a means to easily check that a set of functions is an ECT.

Since in our notation $f_0(x) = 1$, a direct consequence of $\{f_j(x)\}_{j=0}^J$ being an ECT, is that the functions $f_1^{(1)}(x), \dots, f_J^{(1)}(x)$ form also an ECT on $[0, 1]$.

With a sequence of increasing positive real d_j verifying $d_1 = 1$ and $d_1 < \dots < d_J$, the following $W_{f_1^{(1)}, \dots, f_J^{(1)}}(x)$ never vanish on the interval $[0, 1]$:

1. $f_j(x) = x^{d_j}$,

$$W_{f_1^{(1)}, \dots, f_J^{(1)}}(x) = \prod_{j=1}^J d_j \prod_{1 \leq j < k} (d_k - d_j) x^{\sum_{j=1}^J d_j - J(J+1)/2}.$$

2. $f_j(x) = x^j$, i.e. the functions f_j form a sequence of monomials, the previous formula simplifies to

$$W_{f_1^{(1)}, \dots, f_J^{(1)}}(x) = \prod_{j=1}^J j!$$

3. $f_j(x) = \exp(d_j x)$,

$$W_{f_1^{(1)}, \dots, f_J^{(1)}}(x) = \prod_{j=1}^J d_j \prod_{1 \leq j < k} (d_k - d_j) \exp\left(\sum_{j=1}^J d_j x\right).$$

References

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- Karlin, S. and W. Studden (1966). *Tchebycheff systems: with applications in analysis and statistics*. Pure and applied mathematics. Interscience Publishers.
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