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On the definition of energy for a continuum, its conservation laws, and the energy-momentum tensor

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Abstract

We review the energy concept in the case of a continuum or a system of fields. First, we analyze the emergence of a true local conservation equation for the energy of a continuous medium, taking the example of an isentropic continuum in Newtonian gravity. Next, we consider a continuum or a system of fields in special relativity: we recall that the conservation of the energy-momentum tensor contains two local conservation equations of the same kind as before. We show that both of these equations depend on the reference frame, and that, however, they can be given a rigorous meaning. Then we review the definitions of the canonical and Hilbert energy-momentum tensors from a Lagrangian through the principle of stationary action in a general spacetime. We prove precise results regarding the definition of the Hilbert tensor field, its uniqueness, and its tensoriality. We recall the meaning of its covariant conservation equation. We end with a proof of uniqueness of the energy density and flux, when both depend polynomially of the fields.

Keywords: energy conservation; conservation equation; special relativity; general relativity; Hilbert tensor; variational principle

1 Introduction and summary

In non-relativistic classical physics, the concept of energy emerges when one considers the power done (the scalar product of the force by the velocity) on a

mass point or a volume element. First, in the schematic case of a mass point in a time-independent potential force field V , the energy of the mass point appears from the power equation as a natural conserved scalar quantity (the sum $\frac{1}{2}mv^2 + V$), which is still relevant if the potential depends on time, but is not constant any more. This is well known. In the more realistic case of a continuous medium subjected to internal forces and to an external force field, the energy is a volume density and it still emerges from the power done. However, in general, the local conservation of energy then appears in the form of a balance equation, though it is one in which there is no source term. That is, the energy leaving (or entering) a given domain is exactly identified as a flux going through the boundary surface of the domain. We illustrate this point in **Section 2** on the example of the energy conservation equation for a self-gravitating system of deformable media with isentropic deformation in Newtonian gravity.

In relativistic theories (including relativistic quantum mechanics), on the other hand, the volume energy density is essentially the (0 0) component of the energy-momentum-stress tensor, in what follows “the T-tensor” for brevity. In **Section 3**, we recall why, in the Minkowski spacetime, the conservation equation verified by the T-tensor contains in fact two true local conservation equations of the form found in the non-relativistic example of Section 2; and why one may identify the density and flux in these two equations as those of energy and spatial momentum, respectively. Our contribution here is to note the dependence of the energy density and the energy flux on the reference frame in a precise sense, and to show that these quantities can nevertheless be given a rigorous meaning within a theory of general reference frames and the associated space manifolds.

As is well known, the expression of the T-tensor may be deduced from a Lagrangian, the latter being assumed to govern the relevant system of matter fields via the principle of stationary action. There are two distinct definitions of a T-tensor from a Lagrangian: (i) The so-called “canonical” or “Noether” tensor, say $\boldsymbol{\tau}$, is a by-product of the Euler-Lagrange equations. (ii) The “Hilbert tensor”, say \boldsymbol{T} , is the symmetric tensor obtained as the derivative of the Lagrangian density with respect to variations of the (spacetime) metric. In **Section 4** we review the definitions of the canonical and Hilbert tensors from a Lagrangian through the principle of stationary action in a general spacetime. We recall two important but seemingly not widely known cases where

the “canonical tensor” is, or is not, a tensor. Then we prove precise results regarding the definition of the Hilbert tensor field (Theorem 1). In doing so, we formulate sufficient conditions of regularity for the bounded set in which the action is calculated; we define exact boundary conditions to be verified by the infinitesimal coordinate change; and we give a detailed derivation of the equations. We do not need to use complex notions of fibre bundles. To our knowledge, such a relatively elementary but detailed proof is not available in the literature. Next, we recall the meaning of the standard conservation equation verified by the Hilbert tensor: we argue that one actually needs *local* definitions of the energy and momentum densities and their fluxes, in short a local definition of the T-tensor, and one needs also a *local* conservation equation for the energy. We briefly discuss a recent work that proposes a solution to the latter issue. We end Section 4 by stating and proving precise results regarding the uniqueness and the actual tensoriality of the Hilbert tensor (Theorem 2). In particular, we prove that the same variational equation applies when a *complete* variation of the *sole* metric is applied: Eq. (69), as when the variation of the metric results from a mere *coordinate change* (or diffeomorphism): Eq. (58) — although the meaning of these two equations is totally different, e.g. the l.h.s. of (58) is zero for an invariant Lagrangian. We *prove* that the variational equation (69) characterizes the components of the Hilbert tensor field — whence follows that it is left unchanged by the addition of a four-divergence. We also prove that this is indeed a $(0\ 2)$ tensor, which is just stated in the literature that we consulted. Of course this follows from the invariance of the action but, in our opinion, not in a fully trivial way.

Finally, in **Section 5** we investigate if the energy equation is unique for a given system of fields, i.e., if the energy density and fluxes can be considered to be uniquely defined. We show that, if the energy density and its flux depend on the fields (both the matter fields and the “long-distance” fields) in a polynomial way, then they are determined uniquely. We show this by considering separately the contributions of matter (including its potential energy in the long-distance fields) and the long-distance fields.

2 Local energy conservation for an isentropically deformable medium in Newtonian gravity

2.1 Local energy balance for the matter fields

Let us consider a deformable continuous medium, of mass density field ρ , having a general motion (including deformation and rotation), with velocity field \mathbf{v} , with respect to some inertial frame F . I.e., $\mathbf{v} := \frac{d\mathbf{x}}{dt}$, where $\mathbf{x}(X) := (x^i)_{i=1,2,3}$ is the spatial position associated with an event X in the frame F , and where $t \mapsto X(t)$ is the world line of a given “particle” of the medium, parameterized by the Newtonian time t . The internal force field in that medium is assumed to be described by the Cauchy stress tensor field $\boldsymbol{\sigma}$. We assume that this motion takes place in a gravitational field, with Newtonian gravity potential U . Newton’s second law for a volume element of the medium writes then:

$$\rho \frac{d\mathbf{v}}{dt} = \rho \nabla U + \mathbf{div} \boldsymbol{\sigma}, \quad (1)$$

where $\frac{d}{dt}$ means the “material” (or “total”) derivative: for a vector, ¹

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{grad} \mathbf{v})(\mathbf{v}). \quad (2)$$

The power (per unit volume) is got by taking the scalar product of (1) with the velocity \mathbf{v} . On the new l.h.s., we have $\mathbf{h}(\mathbf{v}, \frac{d\mathbf{v}}{dt}) = \frac{d}{dt}(\frac{\mathbf{v}^2}{2})$, while on the r.h.s. we note that

$$\mathbf{v} \cdot \mathbf{div} \boldsymbol{\sigma} := \mathbf{h}(\mathbf{v}, \mathbf{div} \boldsymbol{\sigma}) = \mathbf{div} \boldsymbol{\sigma}(\mathbf{v}) - \boldsymbol{\sigma} : \mathbf{D}, \quad (3)$$

where

$$\mathbf{D} := \frac{1}{2} (\mathbf{grad} \mathbf{v} + (\mathbf{grad} \mathbf{v})^T) \quad (4)$$

¹ Of course, ∇U is the spatial vector having components $(\nabla U)^i = U_{,i}$ ($i = 1, 2, 3$) in any Cartesian coordinate system (“Cartesian components”) — i.e., in any coordinate system for which we have the identity $h_{ij} \equiv \delta_{ij}$, where \mathbf{h} is the (Euclidean) spatial metric. Also, $\mathbf{div} \boldsymbol{\sigma}$ is the spatial vector with Cartesian components $(\mathbf{div} \boldsymbol{\sigma})^i = \sigma^{ij}_{,j}$. And $\mathbf{grad} \mathbf{v}$ is the mixed spatial tensor having Cartesian components $(\mathbf{grad} \mathbf{v})^i_j = v^i_{,j}$. Mixed tensors are identified with linear mappings: $(\mathbf{grad} \mathbf{v})(\mathbf{w})$ is the spatial vector having components $(\mathbf{grad} \mathbf{v})^i_j w^j$, which is the same vector independently of the coordinate system. We use the spatial metric \mathbf{h} to raise or lower the indices, e.g. $v_i := h_{ij} v^j$. The equations in this section are valid in any spatial coordinates, whether Cartesian or curvilinear.

is the strain rate tensor.² We note also that $\mathbf{v} \cdot \nabla U = \frac{dU}{dt} - \frac{\partial U}{\partial t}$. Still, we assume that the deformation of the continuum is isentropic, which means that the power of the internal forces is stored as the rate of elastic energy:

$$\boldsymbol{\sigma} : \mathbf{D} = \rho \frac{d\Pi}{dt}, \quad (5)$$

with Π the mass density of internal (elastic) energy in the continuous medium. This assumption applies, in particular, to any *elastic medium*, and also [1] to any *barotropic perfect fluid*, as is commonly assumed in astrophysics — although a barotropic fluid is not plainly an elastic medium in the sense that it does not have a reference configuration. We thus get:

$$\rho \frac{d}{dt} \left(\frac{\mathbf{v}^2}{2} \right) = \rho \left(\frac{dU}{dt} - \frac{\partial U}{\partial t} \right) + \operatorname{div} \boldsymbol{\sigma}(\mathbf{v}) - \rho \frac{d\Pi}{dt}. \quad (6)$$

It suggests itself to put together the terms containing an exact total derivative:

$$\rho \frac{d}{dt} \left(\frac{\mathbf{v}^2}{2} + \Pi - U \right) = -\rho \frac{\partial U}{\partial t} + \operatorname{div} \boldsymbol{\sigma}(\mathbf{v}). \quad (7)$$

On the r.h.s., we have a source term due to the external force field, plus a flux term. On the l.h.s., we have

$$e_m := \frac{\mathbf{v}^2}{2} + \Pi - U. \quad (8)$$

Using the continuity equation that expresses the mass conservation:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (9)$$

we get easily the well known fact that (for whatever scalar function e_m , actually):

$$\rho \frac{d e_m}{d t} := \rho \left(\frac{\partial e_m}{\partial t} + \mathbf{v} \cdot \nabla e_m \right) = \frac{\partial(\rho e_m)}{\partial t} + \operatorname{div}(\rho e_m \mathbf{v}). \quad (10)$$

From (8) and (10), we may rewrite (7) as

$$\frac{\partial w_m}{\partial t} + \operatorname{div} \boldsymbol{\Phi}_m = -\rho \frac{\partial U}{\partial t}, \quad (11)$$

² $\boldsymbol{\sigma}(\mathbf{v})$ is the vector transformed of \mathbf{v} by the linear mapping associated with the mixed tensor form of $\boldsymbol{\sigma}$, thus $[\boldsymbol{\sigma}(\mathbf{v})]^i := \sigma^i_j v^j = \sigma^{ij} v_j$. And $\boldsymbol{\sigma} : \mathbf{D} := \sigma^i_j D^j_i$ is the scalar product of mixed tensors, defined by double contraction.

with

$$w_m := \rho e_m = \rho \left(\frac{\mathbf{v}^2}{2} + \Pi - U \right) \quad (12)$$

and

$$\mathbf{\Phi}_m := w_m \mathbf{v} - \boldsymbol{\sigma}(\mathbf{v}). \quad (13)$$

That is, we got a balance equation with an external source term on the r.h.s. The scalar field w_m is thus the *volume energy density of matter*, including its potential energy in the gravitational field, and the spatial vector field $\mathbf{\Phi}_m$ is thus (the surface density of) the *matter energy flux*. Equation (11) can be found in the literature, see Eq. (66.11) in Fock [1], but we find that its detailed derivation is instructive.

2.2 Balance for the gravitational field and local energy conservation equation

Now we assume that all of the matter that produces the gravitational field is indeed in the form of isentropically deformable continuous media. (Of course, the characteristics of the media may vary in space.) Thus the point-dependent mass density ρ is just the source of the gravitational field. It therefore obeys the gravitational field equation, i.e. the Poisson equation:

$$\Delta U = -4\pi G\rho. \quad (14)$$

By using Cartesian coordinates, for which we have $\Delta U = U_{,i,i}$, one checks easily that Eq. (14) implies the following:

$$\frac{\partial w_g}{\partial t} + \operatorname{div} \mathbf{\Phi}_g = \rho \frac{\partial U}{\partial t}, \quad (15)$$

where

$$w_g := \frac{(\nabla U)^2}{8\pi G} \quad (16)$$

is the *volume energy density of the gravitational field*, and

$$\mathbf{\Phi}_g := -\frac{\partial U}{\partial t} \frac{\nabla U}{4\pi G} \quad (17)$$

is the *gravitational energy flux*. Equation (15) may be termed the energy balance equation of the gravitational field. Like (11), this also is a balance equation with a source term. The source term in Eq. (15) is just the opposite

of the source term in (11). Therefore, combining (11) with (15), we get the *local energy conservation equation in Newtonian gravity* [2]:

$$\frac{\partial w}{\partial t} + \operatorname{div} \mathbf{\Phi} = 0, \quad (18)$$

with the total energy density $w := w_m + w_g$ and the total energy flux $\mathbf{\Phi} := \mathbf{\Phi}_m + \mathbf{\Phi}_g$. Note that (18) has essentially the same form as the continuity equation (9). There is also a local conservation equation for momentum in Newtonian gravity, and global (integral) conservation laws can also be derived; see e.g. Refs. [2, 3]. Strangely enough, however, we did not see the local Eq. (18) [with the definitions (12)–(13) and (16)–(17)] in the literature, e.g. it does not seem to be there in Fock’s dense treatise [1], and definitely it is not there in Chandrasekhar’s paper [3]. The $i = 0$ component of Eq. (13) in Ref. [3] is just the continuity equation (9), with ρ being indeed the (Newtonian) density of mass; thus, it is not the conservation equation for the Newtonian energy, but the one for the mass.

3 Local conservation equations and the energy-momentum tensor in Minkowski spacetime

Recall that the energy-momentum tensor of a continuum or a system of fields is a second-order spacetime tensor field \mathbf{T} , preferably symmetric. In the Minkowski spacetime, \mathbf{T} verifies [4] the *local conservation equation*

$$T^{\mu\nu}{}_{;\nu} = 0 \quad (\text{in Cartesian coordinates}). \quad (19)$$

[Here, Cartesian coordinates are now ones such that the *spacetime* metric has components $g_{\mu\nu} = \eta_{\mu\nu}$, where the matrix $(\eta_{\mu\nu}) := \operatorname{diag}(1, -1, -1, -1)$.] Actually, (19) is the conjunction of two conservation equations having the form (18). One is precisely the scalar conservation equation (18):

$$\frac{\partial w}{\partial t} + \operatorname{div} \mathbf{\Phi} = 0 \quad (t := \frac{x^0}{c}), \quad (20)$$

in which now (the tensor \mathbf{T} being taken in mass units as in Fock [1])

$$w := c^2 T^{00}, \quad \mathbf{\Phi} := c^3 T^{0i} \partial_i \quad (\text{sum over } i = 1, 2, 3). \quad (21)$$

The other conservation equation involved in (19) and having the form (18) is a (spatial) vector equation:

$$\frac{\partial \mathbf{P}}{\partial t} + \mathbf{div} \boldsymbol{\Sigma} = 0, \quad (22)$$

where

$$\mathbf{P} := cT^{i0}\partial_i, \quad \boldsymbol{\Sigma} := c^2T^{ij}\partial_i \otimes \partial_j. \quad (23)$$

We may integrate either of the two conservation equations (20) and (22) in *any* bounded spatial domain Ω (the integrability in an unbounded domain being not guaranteed). This gives us two integral conservation equations:

$$\frac{d}{dt} \left(\int_{\Omega} w \, dV \right) = - \int_{\partial\Omega} \boldsymbol{\Phi} \cdot \mathbf{n} \, dS, \quad (24)$$

and

$$\frac{d}{dt} \left(\int_{\Omega} \mathbf{P} \, dV \right) = - \int_{\partial\Omega} \boldsymbol{\Sigma} \cdot \mathbf{n} \, dS. \quad (25)$$

Thus in Eqs. (24) and (25), the change on the l.h.s. is due to the flux through the boundary $\partial\Omega$ on the r.h.s.. The scalar w is interpreted as the volume density of energy, and the spatial vector \mathbf{P} is interpreted as the volume density of momentum [4]. Therefore, in view of (24) and (25), the spatial vector $\boldsymbol{\Phi}$ is interpreted as the surface density of the energy flux, and the spatial tensor $\boldsymbol{\Sigma}$ is interpreted as the surface density of the momentum flux. This interpretation may be justified in several ways, notably the following two [1]:

— First, by examining the non-relativistic limit for a barotropic perfect fluid or an elastic solid. In the second approximation, w , $\boldsymbol{\Phi}$, \mathbf{P} , and $\boldsymbol{\Sigma}$ in Eqs. (21) and (23) have then the following expressions ([1], Sect. 32):

$$w := c^2 T^{00} \simeq c^2 \rho + \frac{1}{2} \rho \mathbf{v}^2 + \rho \Pi, \quad (26)$$

$$\boldsymbol{\Phi} := c^3 T^{0i} \partial_i = c^3 T^{i0} \partial_i = c^2 \mathbf{P} \simeq c^2 \rho \mathbf{v} + \left(\frac{1}{2} \rho \mathbf{v}^2 + \rho \Pi \right) \mathbf{v} - \boldsymbol{\sigma}(\mathbf{v}), \quad (27)$$

$$\boldsymbol{\Sigma} := c^2 T^{ij} \partial_i \otimes \partial_j \simeq \rho \mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}. \quad (28)$$

Therefore, at the first approximation, the special-relativistic local energy conservation (20) reduces to the continuity equation (9), and at the second approximation it expresses the conservation of that rest-mass energy corrected

by adding the conserved Newtonian energy [Eq. (18) with $U \equiv 0$]. Also, at the first approximation, the special-relativistic momentum conservation (22) reduces to the Newtonian momentum conservation equation in the absence of external field:

$$\frac{\partial(\rho\mathbf{v})}{\partial t} + \mathbf{div}(\rho\mathbf{v} \otimes \mathbf{v} - \boldsymbol{\sigma}) = 0. \quad (29)$$

— Second, by recognizing in w and \mathbf{P} , for the electromagnetic field, the usual definition of the electromagnetic energy density and the Poynting vector from the relevant expression of \mathbf{T} ([1], Sect. 33).

One does not use the symmetry of the tensor \mathbf{T} to derive Eqs. (20)–(22) and (24)–(25). If that symmetry is true, it implies that $\boldsymbol{\Phi} = c^2\mathbf{P}$ is true generally: the density of energy flux is equal to c^2 times the density of momentum. The same Eqs. (24) and (25) apply also to Newtonian gravity, as follows from Eqs. (18) and the gravitational extension [2, 3] of (29).

As is easy to check, under a purely spatial change of the chart (coordinate system):

$$x'^j = f^j((x^k)) \quad (j, k = 1, 2, 3) \quad (\text{or } \mathbf{x}' = \mathbf{f}(\mathbf{x})), \quad \text{and} \quad x'^0 = x^0, \quad (30)$$

the energy density w [Eq. (21)₁] is an invariant scalar, while $\boldsymbol{\Phi}$ [Eq. (21)₂] and \mathbf{P} [Eq. (23)₁] transform indeed as spatial vectors; and $\boldsymbol{\Sigma}$ [Eq. (23)₂] transforms indeed as a (2 0) spatial tensor. One may give a rigorous geometric meaning to such “spatial” objects by defining a relevant *space manifold* M_F , as follows [5]. In a *general* spacetime, one can formally define a *reference frame* F as being an equivalence class of charts having the same domain of definition U (an open subset of the spacetime manifold V) and exchanging by a coordinate change (“transition map”) having the form (30). Let $P_S : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $\mathbf{X} := (x^\mu) \mapsto \mathbf{x} := (x^j)$, be the “spatial projection”. The elements (points) of the space manifold M_F are the *world lines*, each of which is the set of events that have a given spatial projection \mathbf{x} in some chart $\chi : U \rightarrow \mathbb{R}^4$, $X \mapsto \mathbf{X}$, belonging to the class F . I.e., a world line l is an element of M_F , iff there is a chart $\chi \in F$ and a triplet $\mathbf{x} \in P_S(\chi(U))$, such that l is the set of all events X in the domain U , whose spatial coordinates are \mathbf{x} :

$$l := \{ X \in U; P_S(\chi(X)) = \mathbf{x} \}. \quad (31)$$

It results easily from (30) that (31) holds true then in any chart $\chi' \in F$, of course with the transformed spatial projection triplet $\mathbf{x}' = \mathbf{f}(\mathbf{x}) := (f^j(\mathbf{x}))$

[5]. For any chart $\chi \in \mathbb{F}$, one defines the “associated chart” as the mapping which associates, with a world line $l \in \mathbb{M}_{\mathbb{F}}$, the constant triplet of the spatial coordinates of the events $X \in l$:

$$\tilde{\chi} : \mathbb{M} \rightarrow \mathbb{R}^3, \quad l \mapsto \mathbf{x} \text{ such that } \forall X \in l, P_S(\chi(X)) = \mathbf{x}. \quad (32)$$

The set $\mathbb{M}_{\mathbb{F}}$ is endowed with a natural structure of three-dimensional *differentiable manifold*, of which the basic atlas is made of the associated charts $\tilde{\chi}$, where χ is any chart belonging to the reference frame \mathbb{F} [5]. The “spatial” objects defined above: the scalar w , the vectors Φ , \mathbf{P} , and the tensor Σ , are simply and rigorously tensor fields on the manifold $\mathbb{M}_{\mathbb{F}}$. (Of course they have in general, in addition, a dependence on the time coordinate x^0 , thus they are, strictly speaking, one-parameter families of tensor fields on $\mathbb{M}_{\mathbb{F}}$.) Fixing a reference frame in this sense can be done, for instance, by choosing one local coordinate system (chart χ) on the spacetime, with its domain of definition U : the corresponding reference frame \mathbb{F} is then the equivalence class of this chart. As soon as one has fixed a reference frame, then Eqs. (20) and (22), as well as Eqs. (24) and (25), are coordinate-free equations on the space manifold $\mathbb{M}_{\mathbb{F}}$. In particular, the bounded spatial domain Ω is an open subset of the manifold $\mathbb{M}_{\mathbb{F}}$, having a regular boundary $\partial\Omega$, so that the divergence theorem applies. (See Appendix A for a precise definition of the needed regularity.)

On the other hand, if one makes a general coordinate change, for which the change in the spatial coordinates depends on the time coordinate (already if one makes a Lorentz transformation transforming the Cartesian system into another one, but with a non-zero “boost”), then (21) defines completely different quantities w' and Φ' , as compared with the initial ones. The same is true for \mathbf{P} and Σ as defined by (23). This means that *there is one definition of the energy and momentum (and their fluxes) per reference frame*. It is not specific to special relativity. Indeed the energy depends on the reference frame. This is true in non-relativistic physics — as may be checked here on the fact that w_m and Φ_m defined in Eqs. (12)–(13) involve the velocity \mathbf{v} that depends on the inertial frame, whereas ρ, Π, U and σ are Galilean invariants. It is also true in relativistic physics, and also in a general spacetime, and be it for the classical or the quantum-mechanical energy [6].

4 Definition of the energy-momentum tensor from a Lagrangian

4.1 Lagrangian and stationary action principle

We assume that the equations of motion for some “matter fields” ϕ^A ($A = 1, \dots, n$) derive from a Lagrangian L through the *principle of stationary action* in a general spacetime:

$$\text{For any variation field } \delta\phi^A = \delta\phi^A(X) \text{ with } \delta\phi^A|_{\partial U} = 0, \text{ we have } \delta S = 0. \quad (33)$$

Here, ∂U is the boundary, assumed smooth, of some bounded open set U in the spacetime, and S is the action: in some chart χ whose domain of definition W contains U , it writes

$$S = S_U := \int_{\chi(U)} L(\phi^A(\mathbf{X}), \phi^A_{,\mu}(\mathbf{X}), \mathbf{X}) \sqrt{-g(\mathbf{X})} \, d^4 \mathbf{X}, \quad (34)$$

where $\phi^A : \mathbf{X} \mapsto \phi^A(\mathbf{X})$, $\chi(W) \rightarrow \mathbb{R}^{n_A}$, is the local expression of the field ϕ^A in the chart χ , and $g := \det(g_{\mu\nu})$, the $g_{\mu\nu}$ ’s being the components of the metric tensor in the chart χ ; note that $\chi(W)$ is an open subset of \mathbb{R}^4 . Thus, the field ϕ^A has n_A real components (or n_A complex components for a complex field, with \mathbb{R}^{n_A} replaced by \mathbb{C}^{n_A}). At this stage we do not need to know the exact geometric nature of the fields: whether they are scalars, vectors, more general tensors, or otherwise. We just assume that, on changing the chart: $\chi \hookrightarrow \chi'$, the local expression of each of them has some definite transformation law, say $\phi^A(\mathbf{X}) \hookrightarrow \phi'^A(\mathbf{X}')$,³ and that the Lagrangian is then invariant under the coordinate change:

$$L(\phi'^A(\mathbf{X}'), \phi'^A_{,\mu}(\mathbf{X}'), \mathbf{X}') = L(\phi^A(\mathbf{X}), \phi^A_{,\mu}(\mathbf{X}), \mathbf{X}), \quad \mathbf{X}' = \chi'(\chi^{-1}(\mathbf{X})) := F(\mathbf{X}). \quad (35)$$

³ If ϕ^A is a section of a vector bundle with base V , say E , to write its local expression needs that not only a chart on V but also a frame field (e_a) on E is given. However, in the case of a tensor field, a relevant frame field is determined uniquely by the data of the chart χ with the associated natural basis (∂_μ) and the dual basis (dx^μ). In a very general case, E has the form $E = T \otimes N$ with T an usual tensor bundle on V and N a vector bundle of a different kind. Then it is natural to take a frame field of the form $(T_b \otimes N_c)$, with (T_b) a frame field on T , determined by the chart χ , and with (N_c) a frame field on N , which is left unchanged when changing the chart. Thus the transformation law on changing the chart is determined.

That invariance has to be true at least when the chart belongs to some well-defined class and implies that the same invariance is valid for the action (34). In this section, we shall consider the usual case that *all* charts (in the atlas of the spacetime manifold) are allowed, i.e., we shall discuss generally-covariant theories. However, it also makes sense to consider instead the class associated with a particular (“privileged”) reference frame. Thus, the Lagrangian is a smooth real function $L = L(\mathbf{q}^A, \mathbf{q}_\mu^A, \mathbf{X})$, where $\mathbf{X} \in \mathbb{R}^4$ is the coordinate vector specifying the spacetime position, $\mathbf{q}^A \in \mathbb{R}^{n_A}$, and also $\mathbf{q}_\mu^A \in \mathbb{R}^{n_A}$ for $\mu = 0, \dots, 3$. These five vectors of \mathbb{R}^{n_A} specify the values that may be taken at \mathbf{X} by the local expression of the field ϕ^A and its partial derivatives. This means that, in the expression (34) of the action, one makes the substitution

$$\mathbf{q}^A = \phi^A(\mathbf{X}), \quad \mathbf{q}_\mu^A = \phi_{,\mu}^A(\mathbf{X}) \quad (A = 1, \dots, n; \mu = 0, \dots, 3). \quad (36)$$

Note also that $dV_4 := \sqrt{-g} d^4 \mathbf{X}$, with $g := \det(g_{\mu\nu})$, is the invariant four-volume element on the spacetime [thus $g < 0$ for a Lorentzian metric on the four-dimensional spacetime].

The stationarity (33) is equivalent to the Euler-Lagrange equations {see e.g. [4, 7]; δS is defined from a Gateaux derivative, as with Eq. (41) below}. In a general spacetime, the latter equations write [8]:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}_\mu^A} \right) = \frac{\partial \mathcal{L}}{\partial \mathbf{q}^A} \quad (A = 1, \dots, n), \quad \mathcal{L} := L\sqrt{-g} \quad (37)$$

in U , with the implicit assignment (36). The domain of definition W of the coordinate system now has to contain not only U but also the boundary ∂U , because the derivation of (37) needs to use the divergence theorem.

4.2 The “canonical” (or “Noether”) T-tensor

This object has the following expression in a given chart:

$$\tau_\mu{}^\nu(\mathbf{X}) = \phi_{,\mu}^A(\mathbf{X}) \left(\frac{\partial L}{\partial \mathbf{q}_\nu^A} \right)_{\mathbf{q}^B = \phi^B(\mathbf{X}), \mathbf{q}_\rho^B = \phi_{,\rho}^B(\mathbf{X})} - \delta_\mu^\nu L(\phi^A(\mathbf{X}), \phi_{,\rho}^A(\mathbf{X}), \mathbf{X}). \quad (38)$$

When $-g = 1$ and $\mathcal{L} = L = L(\mathbf{q}^A, \mathbf{q}_\mu^A)$ does not depend explicitly on the spacetime position, this object occurs naturally from the derivation of the Euler-Lagrange equations (37), which imply that it verifies the desired local conservation equation $\tau_{\mu,\nu}{}^\nu = 0$ [4]. However, such an independence happens in

practice only in a flat spacetime. Moreover, in fact, this object is not necessarily a tensor — even in a flat spacetime, cf. the case of the electromagnetic field [8]: \mathbf{F} being the field tensor and \mathbf{A} the 4-potential, we have [4]

$$4\pi\tau_\mu{}^\nu = -A_{\rho,\mu}F^{\nu\rho} + \frac{1}{4}(F_{\rho\sigma}F^{\rho\sigma})\delta_\mu^\nu. \quad (39)$$

(Henceforth, indices are raised or lowered with the spacetime metric.) On the r.h.s., everything, *but* $A_{\rho,\mu}$, is tensorial, hence $\tau_\mu{}^\nu$ is not a tensor. This is not noted by Landau & Lifshitz, probably because (39) does behave as a tensor for linear coordinate changes, as are the Lorentz transformations to which one often restricts oneself in special relativity. In a general spacetime, $\boldsymbol{\tau}$ is a tensor for a scalar field [8] — and also for the Dirac field [9].

4.3 Hilbert’s variational definition of the T-tensor

While following the line of the classic derivation by Landau & Lifshitz [4], we will include many mathematical details which appear necessary and that, for the most part, we did not find in the literature that we consulted. By this, we do not mean the geometric formulation of the physical fields as sections of appropriate fibre bundles, which has been implemented rather extensively in Ref. [10], and which we will not need. [See the remarks following the definition (34) of the action.] Instead, we mean the precise definition of “the variation of the action under an infinitesimal diffeomorphism”, the regularity of the boundary and the exact boundary conditions, and a clear derivation of the main formulas.

One considers a given chart $\chi : X \mapsto \mathbf{X} = (x^\mu)$ and one imposes a small change to it: $x^\mu \mapsto x^\mu + \delta x^\mu$. As we shall see, the domain of definition W of χ must include the closure \bar{U} of the bounded open set U in which one computes the action (34), and we must assume that $\delta x^\mu = 0$ at the events $X \in W$ which do not belong to U . (Alternatively, one may regard the mapping defined in coordinates by $x^\mu \mapsto x^\mu + \delta x^\mu$ as a diffeomorphism of the spacetime manifold V , which coincides with the identity map for $X \notin U$.) Thus, $\delta x^\mu = \epsilon\xi^\mu$ with ξ any smooth vector field that vanishes if $X \notin U$, and $\epsilon \ll 1$. That is, we change the chart χ for a new chart χ_ϵ given by

$$\chi_\epsilon(X) := \mathbf{X} + \epsilon\xi(\mathbf{X}) \quad X \in W, \quad \mathbf{X} := \chi(X). \quad (40)$$

(The vector field $\mathbf{X} \mapsto \xi(\mathbf{X}) = (\xi^\mu(\mathbf{X}))$ is the local expression of ξ in the chart χ .) After such a coordinate change, the local expressions of the fields

change, each according to its specific transformation behaviour, and the domain $\chi(U) \subset \mathbb{R}^4$ also changes, so both the integrand and the integration domain change in the action (34), which thus takes *a priori* a different value. One seeks to calculate the first-order term, as $\epsilon \rightarrow 0$, in the variation of the action, S_U being considered as a functional of the field $\mathbf{X} \mapsto \delta\mathbf{X} = \epsilon\xi(\mathbf{X})$. This amounts to calculate the Gateaux derivative of S_U in the direction of the vector field ξ :

$$\frac{(\delta S_U)(\epsilon)}{\epsilon} := \left(\frac{d}{d\epsilon} S_U(\epsilon\xi) \right)_{\epsilon=0}. \quad (41)$$

The bounded open set U is assumed to be an *open domain* in the sense of Proposition 2 in Appendix A, in particular it has a smooth boundary ∂U . Because U is an open domain, the assumption “ $\xi(X) = 0$ if $X \notin U$ ” means exactly that the support of ξ is included in the closure $\bar{U} = U \cup \partial U$ (Corollary to Proposition 2).⁴ Since $\text{Supp}(\partial_\nu \xi^\mu) \subset \text{Supp}(\xi^\mu)$, this implies that all derivatives of ξ also vanish if $X \notin U$. It follows that the corresponding change in $g^{\mu\nu}$, as determined to the first order in ϵ {Eq. (94.2) in Ref. [4]}:

$$\delta g^{\mu\nu} = \epsilon (\xi^{\mu;\nu} + \xi^{\nu;\mu}) = \delta x^{\mu;\nu} + \delta x^{\nu;\mu}, \quad (42)$$

also vanishes if $X \notin U$. [Here, the $g^{\mu\nu}$ ’s are the components of the inverse of the metric’s component matrix $(g_{\mu\nu})$.] Thus, in particular, $\delta g^{\mu\nu} = 0$ on ∂U .⁵

One assumes moreover that the matter Lagrangian L depends on the matter fields and the metric and their first-order derivatives, but not on their higher-order derivatives. The same applies then to the “Lagrangian density” $\mathcal{L} := L\sqrt{-g}$, i.e., the latter is a smooth real function $\mathcal{L} = \mathcal{L}(\mathbf{q}^A, \mathbf{q}_\mu^A, \underline{g}^{\mu\nu}, \underline{g}_\rho^{\mu\nu})$. To

⁴ Considering “every compact subset K of V ” [10] is too general, because one needs to use the divergence theorem on K to eliminate a surface term, as is done below to get Eqs. (48) and (56), and as is done for the equation after Eq. (222) in Ref. [10]. One should assume that the compact set is a “closed domain”. A closed domain D is the closure of its interior $\overset{\circ}{D}$ (Proposition 2), hence it is equivalent to start from a bounded open domain U and to define $K = \bar{U}$, which is a compact domain, or to start from K and to define $U = \overset{\circ}{K}$. Then, of course, the smooth boundary ∂U has measure zero with respect to the invariant four-dimensional measure $\sqrt{-g} d^4 \mathbf{X}$. Hence, the action integral (34), as well as its variation (56) or (58), are unaltered if instead of U we consider its closure K as the integration domain, as considered in Ref. [10]. Thus, when the compact K considered in Ref. [10] is a closed domain, as it should be imposed, the definition of the action S_K and the condition imposed on the vector field X in Ref. [10] are equivalent to those considered here for S_U and ξ .

⁵ In Ref. [4], it is “set $\delta g^{\mu\nu} = 0$ at the integration limits” and, later on, it is stated that “the ξ^μ vanish at the limits of the integration”, without any explanation nor any link between the two statements. Note that assuming merely that $\xi^\mu = 0$ on the boundary ∂U does *not* imply that $\delta g^{\mu\nu} = 0$ on ∂U .

calculate the action (34), in the Lagrangian L the values (36) are assigned to \mathbf{q}^A and \mathbf{q}_μ^A . In addition, now one assigns the values $g^{\mu\nu}(\mathbf{X})$ and $g^{\mu\nu}_{,\rho}(\mathbf{X})$ to $\underline{g}^{\mu\nu}$ and $\underline{g}^{\mu\nu}_{,\rho}$ respectively. At this stage, it is usually admitted that the change in the action (34), δS_U , is given by

$$\delta S_U = \int_{\chi(U)} \left[\frac{\partial \mathcal{L}}{\partial \mathbf{q}^A} \delta \phi^A + \frac{\partial \mathcal{L}}{\partial (\mathbf{q}_\mu^A)} \delta (\phi_{,\mu}^A) + \frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial (\underline{g}^{\mu\nu}_{,\rho})} \delta (g^{\mu\nu}_{,\rho}) \right] d^4 \mathbf{X}, \quad (43)$$

without a justification nor a precise definition of δS_U , $\delta \phi^A$, etc. {Actually, the step (43) and some later intermediate steps are even skipped by Landau & Lifshitz [4].} However, recall that here not only the integrand but also the integration domain change in the action (34), so the first-order variation of the integral S_U involves *a priori* more than just the first-order approximation of the integrand. Let us be more precise. Denote provisionally the list of all fields (matter fields and metric) by ϕ^J ($J = 1, \dots, n+1$). Applying the definition (34) with the new chart (40), we write

$$S_\epsilon := S_U(\epsilon \boldsymbol{\xi}) = \int_{F_\epsilon(\Omega)} \tilde{\mathcal{L}}(\mathbf{X}, \epsilon) d^4 \mathbf{X}, \quad (44)$$

with $\Omega := \chi(U)$,

$$F_\epsilon := \chi_\epsilon \circ \chi^{-1}, \quad (45)$$

$$\tilde{\mathcal{L}}(\mathbf{X}, \epsilon) := \mathcal{L}(\phi_\epsilon^J(\mathbf{X}), \partial_\mu \phi_\epsilon^J(\mathbf{X})), \quad (46)$$

where $\phi_\epsilon^J : \chi_\epsilon(W) \rightarrow \mathbb{R}^{n_J}$ is the local expression of the field ϕ^J in the chart χ_ϵ . The integral (44) has a form that is well known, in particular, in continuum mechanics. The expression of its derivative is also well known:

$$\frac{d S_\epsilon}{d \epsilon} = \int_{F_\epsilon(\Omega)} \left[\frac{\partial \tilde{\mathcal{L}}}{\partial \epsilon} + \operatorname{div}(\tilde{\mathcal{L}} \boldsymbol{\xi}_\epsilon) \right] d^4 \mathbf{X}, \quad (47)$$

where $\boldsymbol{\xi}_\epsilon(\mathbf{X}) := \frac{\partial F_\epsilon}{\partial \epsilon}(\mathbf{X}_0)$, with $\mathbf{X}_0 := F_\epsilon^{-1}(\mathbf{X})$, is the “velocity field” at “time” ϵ . In particular, we get from (40) and these definitions that $\boldsymbol{\xi}_{\epsilon=0} = \boldsymbol{\xi}$. Therefore, applying the divergence theorem in (47) and since $\boldsymbol{\xi} = 0$ on ∂U (hence

$\xi = \mathbf{0}$ on $\partial\Omega$, because $\partial\Omega = \partial(\chi(U)) = \chi(\partial U)$, we get

$$\begin{aligned} \left(\frac{dS_\epsilon}{d\epsilon}\right)_{\epsilon=0} &= \int_{\Omega} \left(\frac{\partial\tilde{\mathcal{L}}}{\partial\epsilon}\right)_{\epsilon=0} d^4\mathbf{X} \\ &= \int_{\Omega} \left[\frac{\partial\mathcal{L}}{\partial\mathbf{q}^J} \left(\frac{\partial(\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right)_{\epsilon=0} + \frac{\partial\mathcal{L}}{\partial\mathbf{q}_\mu^J} \left(\frac{\partial(\partial_\mu\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right)_{\epsilon=0} \right] d^4\mathbf{X}. \end{aligned} \quad (48)$$

Hence, defining simply the variations of the fields to the first order in ϵ :

$$\delta\phi^J(\mathbf{X}) := \epsilon \left(\frac{\partial(\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right)_{\epsilon=0}, \quad \delta(\phi_{,\mu}^J)(\mathbf{X}) := \epsilon \left(\frac{\partial(\partial_\mu\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right)_{\epsilon=0}, \quad (49)$$

and remembering that the ϕ^J 's are the matter fields ϕ^A for $J = A = 1, \dots, n$, while ϕ^{n+1} is the metric field $g^{\mu\nu}$, we see that Eq. (48), together with the definition (41), proves the ‘‘obvious’’ Eq. (43). We see then from Eq. (47) that Eq. (43) is in general *false* if the boundary condition $\xi = \mathbf{0}$ on $\partial\Omega$ is not valid.

If we write

$$\left(\frac{\partial(\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right)_{\epsilon=0} = \lim_{\epsilon \rightarrow 0} \frac{\phi_\epsilon^J(\mathbf{X}) - \phi_0^J(\mathbf{X})}{\epsilon}, \quad (50)$$

and if we remember that ϕ_ϵ^J is the local expression of the field ϕ^J in the chart (40), that follows the flow of the vector field ξ at small values of ϵ , we recognize (at least in the case that ϕ^J is a tensor field) the definition of the Lie derivative [11] — or rather, of its opposite. That is,

$$\delta\phi^J(\mathbf{X}) = -\epsilon L_\xi\phi^J(\mathbf{X}). \quad (51)$$

We note also that

$$\frac{\partial(\partial_\mu\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon} = \partial_\mu \left(\frac{\partial(\phi_\epsilon^J(\mathbf{X}))}{\partial\epsilon}\right), \quad (52)$$

hence

$$\delta(\phi_{,\mu}^J)(\mathbf{X}) = \partial_\mu(\delta\phi^J(\mathbf{X})). \quad (53)$$

In practical terms, $\delta\phi^J(\mathbf{X})$ can be computed in two steps [4]: first, one computes the difference $\phi_\epsilon^J(\mathbf{X}') - \phi^J(\mathbf{X})$ between the local expressions in the initial and the modified chart at events that *correspond* together through the transition map, i.e. $\mathbf{X}' = F_\epsilon(\mathbf{X})$. Second, one uses a first-order Taylor expansion

to find the difference $\phi_\epsilon^J(\mathbf{X}) - \phi^J(\mathbf{X})$, i.e., at one and the same event. For instance, this gives indeed the expression (42) for the metric [4]. As another example, consider a vector field V , with components V^μ in the initial chart. We find without difficulty

$$\delta V^\mu = \delta x^\mu_{;\nu} V^\nu - V^\mu_{;\nu} \delta x^\nu = \delta x^\mu_{;\nu} V^\nu - V^\mu_{;\nu} \delta x^\nu, \quad (54)$$

where the second equality occurs due to the symmetry of the Christoffel symbols (i.e., due to the fact that the Levi-Civita connection has no torsion). This does coincide with the Lie derivative definition (51).

In a last step, let us assume that the matter fields obey the Euler-Lagrange equations (37). Just like for the derivation of the latter equations from the stationarity of the action (34): because $\delta(\phi^A_{;\mu}) = (\delta\phi^A)_{;\mu}$ in view of (53), one may transform the second term in (43) and use the divergence theorem to make a surface term appear in it; and that surface term vanishes because $\delta\phi^A = 0$ on $\partial\mathcal{U}$ [as can be checked for a vector on (54) and for the contravariant metric tensor on (42)]. It then follows from (37) that the first two terms in (43) cancel one another, thus ⁶

$$\delta S_{\mathcal{U}} = \int_{\chi(\mathcal{U})} \left[\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial (g^{\mu\nu}_{;\rho})} \delta (g^{\mu\nu}_{;\rho}) \right] d^4 \mathbf{X}, \quad \mathcal{L} := L\sqrt{-g}. \quad (55)$$

In the same way, because $\delta(g^{\mu\nu}_{;\rho}) = (\delta g^{\mu\nu})_{;\rho}$, one may transform the second term in (55) and use the divergence theorem to make a surface term appear in it, and that surface term vanishes because $\delta g^{\mu\nu} = 0$ on $\partial\mathcal{U}$. One thus gets [4]:

$$\delta S_{\mathcal{U}} = \int_{\chi(\mathcal{U})} \left[\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left(\frac{\partial \mathcal{L}}{\partial (g^{\mu\nu}_{;\rho})} \right) \right] \delta g^{\mu\nu} d^4 \mathbf{X}. \quad (56)$$

We have thereby proved the following:

Theorem 1. *Assume that the bounded subset \mathcal{U} of the spacetime is an open domain in the sense of Proposition 2. In the domain $\mathcal{W} \supset \bar{\mathcal{U}}$ of some chart χ ,*

⁶ All sixteen $g^{\mu\nu}$'s ($0 \leq \mu \leq 3$, $0 \leq \nu \leq 3$) are considered as independent variables in \mathcal{L} for the calculation of $\partial \mathcal{L} / \partial \underline{g}^{\mu\nu}$, even though $g^{\mu\nu} = g^{\nu\mu}$: see Note ‡ on p. 269 in Ref. [4]. Thus, all sixty-four $g^{\mu\nu}_{;\rho}$'s ($0 \leq \mu \leq 3$, $0 \leq \nu \leq 3$, $0 \leq \rho \leq 3$) are also considered as independent variables for the calculation of $\partial \mathcal{L} / \partial \underline{g}^{\mu\nu}_{;\rho}$.

define a one-parameter family of new charts by (40), associated with a smooth vector field ξ that is defined over W and vanishes for $X \notin U$. Assume that the smooth matter Lagrangian has the form $L = L(\mathbf{q}^A, \mathbf{q}_\mu^A, \underline{g}^{\mu\nu}, \underline{g}_\rho^{\mu\nu})$. Then:

- (i) The first-order variation δS_U of the action, defined in Eq. (41), is given by Eq. (43). In this equation, the variations $\delta\phi^J$ and $\delta(\phi_{,\mu}^J)$ ($J = 1, \dots, n+1$, with $\phi^{n+1} = (g^{\mu\nu})$) are defined by Eq. (49) or equivalently by Eqs. (51) and (53).
- (ii) If the matter fields ϕ^A ($A = 1, \dots, n$) obey the Euler-Lagrange equations (37), then δS_U is given by Eq. (56) right above.

Equation (56) leads one to define an object \mathbf{T} (usually called ‘‘Hilbert energy-momentum tensor’’, e.g. [8], though not in Ref. [4]) by its components [4]:

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} := \frac{\partial\mathcal{L}}{\partial\underline{g}^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left(\frac{\partial\mathcal{L}}{\partial(\underline{g}_\rho^{\mu\nu})} \right), \quad \mathcal{L} := L\sqrt{-g}. \quad (57)$$

The symmetry of this object: $T_{\nu\mu} = T_{\mu\nu}$, follows from the symmetry of the metric *and* the invariance of L under general coordinate changes.⁷ As shown by Theorem 1: For the object \mathbf{T} whose components are defined by Eq. (57), we have ‘‘on shell’’ for any regular bounded open set U and for any coordinate change $\delta x^\mu = \epsilon\xi^\mu$ such that $\xi^\mu(X)$ vanishes for $X \notin U$:

$$\delta S_U = \frac{1}{2} \int_{\chi(U)} T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} \, d^4 \mathbf{X}. \quad (58)$$

If the Lagrangian function L is *invariant under general coordinate changes*, then the action S_U in Eq. (34) is invariant too, hence the change δS_U given by Eq. (56) or (58) is zero for any possible coordinate change. Assume, moreover, that the object \mathbf{T} given by (57) turns out to be indeed a tensor. (This tensorial character does not seem to be proved in the literature, but see Subsect. 4.5 below.) Then, using the expression (42) of $\delta g^{\mu\nu}$ in terms of the vector field ξ , and since by assumption the latter vanishes on ∂U , one gets from (58) [4]:

$$T_\mu{}^\nu{}_{;\nu} = 0. \quad (59)$$

⁷ Reminding Note (6), the symmetry of the metric does not by itself imply the symmetry of the components (57): check e.g. $L = (g^{12})^2$, which of course is not invariant.

4.4 Reminder on the meaning of Eq. (59)

In contrast with (19) [$T_{\mu}{}^{\nu}{}_{,\nu} = 0$, with partial derivatives], Eq. (59) [with covariant derivatives] “does not generally express any conservation law whatever” [4]. Fock [1] used similar words: he noted that the four scalar equations contained in (59) “do not by themselves lead to conservation laws”. To explain it quickly, the presence of covariant derivatives gives to Eq. (59) the form of (19) *plus source terms*, which are the terms linear in the T-tensor itself (that involve the connection coefficients). Nevertheless, Eq. (59) can be rewritten in the form of (19) after introducing some “pseudo-tensor of the gravitational field” \mathbf{t} . But the definition of \mathbf{t} is not unique. And \mathbf{t} behaves as a tensor only for linear coordinate transformations. As a result, it is generally agreed that Eq. (59) can lead only (under special assumptions, e.g. an asymptotically flat spacetime) to *global* conservation laws, see e.g. [4, 12]. However, in order to be able to investigate the energy balance in any spatial domain, one would need to know uniquely the relevant energy density and its flux. And one would need that they obey a true and local conservation equation. (This is indeed the case in most fields of physics, e.g. in mechanics, thermodynamics, electrodynamics, chemistry, ..., as well as in Newtonian gravitation — as shown in Sect. 2 — and also in several alternative relativistic theories of gravitation in a flat spacetime, e.g. [13, 14, 15], including a preferred-frame scalar theory with a mechanism for gravity [16].)

What is thus lacking in theories based on Eq. (59), which include general relativity and its numerous variants or extensions, is not merely an exact local concept of the *gravitational* energy. As we recalled, in special relativity the local conservation equation (19) for the energy-momentum tensor can be rewritten as two local conservation equations of the type (18): a scalar one for the energy and a vector one for the three-momentum. We believe that the local concept of energy is indissolubly bound with the existence of a true local conservation equation of the type (18), as it exists both in non-relativistic physics and in special relativity. Since, as we recalled, the rewriting of Eq. (59) as an exact local conservation equation having the form (19) is neither tensorial nor unique, we consider that Eq. (59) does not provide an exact local concept for any form of *material energy*, either: assuming the definition of the Hilbert tensor \mathbf{T} is unique (that is proved in Subsect. 4.5), one could define the material energy density as, say, $w := T^{00}$. But another one could prefer to choose $w' = T_{00}$, and a third one, $w'' := T_0^0$. In the absence of an exact local conservation equation of the type (19), nobody can tell which choice is correct.

Another point is of course that, for a given bounded spatial domain Ω , the time evolution of, say, the integral of $w := T^{00}$ [the l.h.s. of Eq. (24)] is not equal to a flux through the boundary $\partial\Omega$, but is also affected by source terms depending on the gravitational field, for which there is no univoque definition of the energy-momentum tensor.

However, according to Padmanabhan [17], there exists a suitable definition for what we will call an *energy current* four-vector \mathcal{G} of the gravitational field, such that the total energy current $\mathcal{P} := \mathcal{G} + \mathbf{P}$ is conserved:

$$\mathcal{P}^{\mu}_{;\mu} = \mathcal{G}^{\mu}_{;\mu} + P^{\mu}_{;\mu} = 0, \quad (60)$$

where the four-vector field \mathbf{P} , with components

$$P^{\mu} := +T^{\mu}_{\nu} v^{\nu}, \quad (61)$$

is the matter energy current associated with the matter distribution, for the observer(s) having the four-velocity field \mathbf{v} . {The plus sign is with the $(+---)$ signature that we use. This four-vector \mathbf{P} should not be confused with the three-vector \mathbf{P} in Eq. (23)₁. The gravitational current \mathcal{G} of Ref. [17] also depends on \mathbf{v} .} Since, in coordinates adapted to the reference fluid defined by \mathbf{v} , we have [5, 18]

$$v^0 = \frac{1}{\sqrt{g_{00}}}, \quad v^j = 0, \quad (62)$$

we note that, in any such adapted coordinates:

$$P^{\mu} = T^{\mu}_0 / \sqrt{g_{00}}. \quad (63)$$

In particular, in the Minkowski spacetime, and taking for \mathbf{v} the four-velocity field of some inertial reference frame, the gravitational current \mathcal{G} vanishes, hence the conservation (60) of the (matter) current is exactly the same equation as the local energy conservation (20) with the definitions (21). Whence our use of the expression “matter energy current” to designate the four-vector field (61). Thus, according to the result [17], the situation regarding the energy and momentum conservation would be nearly the same in GR as it is in the ether theory [16], namely: (i) there is a (scalar) local conservation equation of the type (18) for the total (material plus gravitational) energy; (ii) the energy density and its flux involved in that conservation equation depend on the reference frame; ⁸ (iii) there is no equivalent local conservation equation

⁸ Actually, in the theory [16], the energy density and its flux have been defined only in the preferred reference frame assumed by the theory.

for the spatial momentum [by this we mean a (frame-dependent) space vector equation, thus three scalar equations, as Eq. (22)]. Anyway, as discussed in Ref. [16], the conservation of the total momentum, when it takes place, precludes a conservation of the momentum of *matter*.

4.5 Uniqueness and tensoriality of the Hilbert tensor

In addition to the difficulty described in the foregoing subsection, which did not seem solvable in the framework of the said theories (but may be solved by the work [17]), there is a point that needs clarification. In a curved spacetime, the Hilbert tensor field \mathbf{T} is taken as the *source of the gravitational field* — in general relativity and in many other relativistic theories of gravity. Clearly, that source has to be *locally* defined: it is not the global value (the space integral) of \mathbf{T} that matters to determine the gravitational field, but indeed the distribution of its local value. However, could not the Hilbert tensor be subject to “relocalizations”, due to the fact that the Lagrangian determining the equations of motion is not unique?

Let us add to the Lagrangian L a total divergence: $L \leftrightarrow L' = L + D$ with

$$D = \operatorname{div} \mathbf{V} = \frac{1}{\sqrt{-g}} \partial_\rho (V^\rho \sqrt{-g}), \quad (64)$$

where $V^\mu = V^\mu(\mathbf{q}^A, \mathbf{q}_\mu^A, \underline{g}^{\mu\nu}, \underline{g}_\rho^{\mu\nu})$ is a spacetime vector field. [Of course, the partial derivatives in Eq. (64) apply once the relevant fields have been substituted for the arguments of V^μ , see before Eq. (43).] Then the Euler-Lagrange equations (37) stay *unchanged*, see e.g. Ref. [7]. Note that, of course, the modified Lagrangian L' is also an invariant scalar if L is. But, *a priori*, should not the T-tensor generally change? This would indirectly contradict a statement of Forger & Römer [10], according to which the energy-momentum tensor field “ \mathbf{T} is the rank 2 tensor field on space-time M depending on the fields of the theory which satisfies

$$\delta_g \int_K d^n \mathbf{X} \sqrt{|\det g|} L = -\frac{1}{2} \int_K d^n \mathbf{X} \sqrt{|\det g|} T^{\mu\nu} \delta g_{\mu\nu} \quad (65)$$

for every compact subset K of M and for every variation $\delta g_{\mu\nu}$ of the metric tensor with support contained in K ”. [Here, n is the dimension of the spacetime, thus $n = 4$ as far as we know. For us, K is the compact closure of the bounded *open domain* U , $K := \bar{U} = U \cup \partial U$; see Note 4.] Indeed, from this statement

one easily concludes that a total divergence does not change $T^{\mu\nu}$ {see [10], and see a detailed proof here around Eq. (79)}. However:

(a) It is not precised what is meant exactly by a “variation $\delta g_{\mu\nu}$ of the metric tensor” and what is meant exactly by δ_g (applied to the action integral) in the statement reproduced above, included in Theorem 4.2 of Ref. [10]. In the arguments (pp. 360–361) which lead the authors to state that theorem, the same situation is considered as in Subsect. 4.3 hereabove. I.e., the variation of the metric occurs due to an infinitesimal coordinate change (or equivalently due to an infinitesimal diffeomorphism), generated by a *vector field* ξ (noted X in Ref. [10]), with support contained in K . (The variation of the metric is thus appropriately noted $\delta_X g_{\mu\nu}$ in these arguments, and the variation of the action is noted $\delta_X S_K$.) It is in this precise situation that one can at the same time derive the basic equation (56) and state that actually $\delta S_U = 0$ — which is needed to derive the “covariant conservation” (59).

(b) In that situation [i.e., the variation of the metric resulting thus from a coordinate change], Eq. (65) is equivalent to our Eq. (58). However, we do not see how it could be the case that the validity of Eq. (58) “for every [regular] compact subset K of M and for every variation $\delta g_{\mu\nu}$ of the metric tensor with support contained in K ” would *characterize* (determine) some field object having components $T_{\mu\nu}$. Indeed, since the action is invariant under coordinate changes, it follows that the l.h.s. of (58) is automatically *zero* (for every [regular] compact subset K etc.). Hence, for example, $T_{\mu\nu} \equiv 0$ *is a solution of (58) (for every [regular] compact subset K etc.), as well as is (57)*.

(c) In that same situation, the following argument in Ref. [10] does not work. According to this argument, “ \mathbf{T} does not change when L is modified by the addition of a total divergence (...), simply because the addition of such a term does not affect the l.h.s. of [Eq. (65)].” The last statement is true, but since in that situation the l.h.s. of Eq. (65) is always zero, it can not prove that \mathbf{T} does not change.

(d) Therefore, it seems that, instead of the foregoing situation, the “variation $\delta g_{\mu\nu}$ of the metric tensor” alluded to in Theorem 4.2 of Ref. [10] be a variation of the metric itself (thus even in a fixed coordinate system):

$$g_{(\epsilon h)}^{\mu\nu}(\mathbf{X}) = g^{\mu\nu}(\mathbf{X}) + \epsilon h^{\mu\nu}(\mathbf{X}), \quad (66)$$

where the field \mathbf{h} , with components $h^{\mu\nu}$, is a given field of symmetric (2 0) tensors defined on U . (The matter fields are thus left unchanged.) However, if that is indeed the case, then Eq. (65) has a different meaning than Eq. (58): for example, the variation $\delta g^{\mu\nu}$ of the metric now depends on the ten independent parameters $h^{\mu\nu}$ ($0 \leq \mu \leq \nu \leq 3$) instead of merely the four parameters ξ^μ ($\mu = 0, \dots, 3$) as is the case in the arguments which lead to the statement of Theorem 4.2 in Ref. [10], as well as in Subsect. 4.3 hereabove. Thus, the validity of Eq. (65) in that different situation has to be proved separately. This proof takes Points (i) and (ii) of the following theorem, whose conclusive part is its Point (iii).

Theorem 2. *Let $L = L(\mathbf{q}^A, \mathbf{q}_\mu^A, \underline{g}^{\mu\nu}, \underline{g}_\rho^{\mu\nu})$ be a Lagrangian that is defined and smooth whenever the determinant of the matrix $(\underline{g}^{\mu\nu})$ is negative, and that is invariant under general coordinate changes. Let \bar{U} be a bounded open domain of the spacetime manifold V and let $K = \bar{U} = U \cup \partial U$ be its compact closure. Let E be the vector space of the symmetric (2 0) tensor fields \mathbf{g} which are defined and continuous on K and which are \mathcal{C}^1 on U . Let E^* be the subset of E made of the tensor fields $\mathbf{g} \in E$ such that, for any chart χ defined in a neighborhood of K (assuming there does exist such charts), we have $g^{-1} := \det G < 0$ over the domain $\chi(K)$, where G is the component matrix $G = (g^{\mu\nu})$. The matter fields ϕ^A ($A = 1, \dots, n$) being given functions which are defined and continuous on K and which are \mathcal{C}^1 on U , define an invariant functional S on E^* by*

$$S(\mathbf{g}) := \int_{\Omega} \mathcal{L}(\phi^A(\mathbf{X}), \phi_{,\mu}^A(\mathbf{X}), g^{\mu\nu}(\mathbf{X}), g_{,\rho}^{\mu\nu}(\mathbf{X})) d^4 \mathbf{X}, \quad \Omega := \chi(U), \quad \mathcal{L} := \sqrt{-g}L. \quad (67)$$

(i) *Given any two tensor fields $\mathbf{g} \in E^*$ and $\mathbf{h} \in E$, there is a number $a = a(\mathbf{g}, \mathbf{h}) > 0$ such that, for $\epsilon \in]-a, +a[$, the tensor field $\mathbf{g} + \epsilon\mathbf{h}$ is in E^* . We have:*

$$\Delta_{\mathbf{g}\mathbf{h}} S := \left(\frac{d}{d\epsilon} S(\mathbf{g} + \epsilon\mathbf{h}) \right)_{\epsilon=0} = \int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} h^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial \left(\underline{g}_{\rho}^{\mu\nu} \right)} h_{,\rho}^{\mu\nu} \right]_{\mathbf{g}} d^4 \mathbf{X}, \quad (68)$$

where the subscript \mathbf{g} means that, at any $\mathbf{X} \in \Omega$, one considers the derivatives of the function \mathcal{L} for the values $\underline{g}^{\mu\nu} = g^{\mu\nu}(\mathbf{X})$, $\underline{g}_{\rho}^{\mu\nu} = g_{,\rho}^{\mu\nu}(\mathbf{X})$ of its arguments.

(ii) *If $\mathbf{g} \in E^*$, $\mathbf{h} \in E$, and $\mathbf{h}(X) = 0$ when $X \in \partial U$, then we have, in any chart*

χ whose domain contains K :

$$\Delta_{\mathbf{g}\mathbf{h}} S = \frac{1}{2} \int_{\Omega} T_{\mu\nu} h^{\mu\nu} \sqrt{-g} \, d^4 \mathbf{X}, \quad (69)$$

where the $T_{\mu\nu}$'s are defined in Eq. (57), the derivatives being taken as in the subscript notation \mathbf{g} above.

(iii) Equation (69) determines uniquely the continuous functions $\mathbf{X} \mapsto T_{\mu\nu}(\mathbf{X})$, $\chi(K) \rightarrow \mathbb{R}$. It follows that the ‘‘Hilbert tensor’’ with components (57) is not modified by the addition of a four-divergence. Moreover, this is indeed a (0 2) tensor field.

Proof. As a preliminary, recall that, if the matrix $G = (g^{\mu\nu})$ of some (2 0) tensor field \mathbf{g} in one chart χ verifies $g^{-1} := \det G < 0$ over $\chi(K)$, then the corresponding matrix G' in any other chart χ' in the atlas of V , whose domain also contains K , verifies $g'^{-1} := \det G' < 0$ over $\chi'(K)$: indeed the Jacobian matrix $J = (\frac{\partial x'^{\mu}}{\partial x^{\nu}})$ is invertible for these two compatible charts, and we have $g'^{-1} = g^{-1} (\det J)^2$. Also remind that the invariance of the functional (67) under the change of the chart follows from the definition of a Lagrangian that is invariant under general coordinate changes, Eq. (35), and from the invariance of the four-volume measure $dV_4 := \sqrt{-g} \, d^4 \mathbf{X}$.

(i) Let $\mathbf{g} \in E^*$ and $\mathbf{h} \in E$, thus in particular these are two continuous functions defined over the compact set K . The real function $M \mapsto \phi(M) := \det M$ is defined and \mathcal{C}^1 over the vector space $M(4, \mathbb{R})$ of the real 4×4 matrices. Choose a chart χ defined in a neighborhood of K . For $\mathbf{X} \in \chi(K)$, we note $G(\mathbf{X})$ the matrix $(g^{\mu\nu}(\mathbf{X}))$. Let us note also $H(\mathbf{X}) = (h^{\mu\nu}(\mathbf{X}))$. Since $\mathbf{g} \in E^*$, we have $\phi(G(\mathbf{X})) < 0$ for $\mathbf{X} \in \chi(K)$. Because $\phi \circ G$ is a continuous function over the compact $\chi(K)$, it is bounded and reaches its bounds. Hence, for some number $d < 0$, we have $\phi(G(\mathbf{X})) \leq d$ for $\mathbf{X} \in \chi(K)$. The set of matrices $C = \{M = G(\mathbf{X}) + \epsilon H(\mathbf{X}); \mathbf{X} \in \chi(K) \text{ and } |\epsilon| \leq 1\}$ is compact, hence we have $\text{Sup}\{\|\phi'(M)\|; M \in C\} = A < \infty$. Also, $\text{Sup}\{\|H(\mathbf{X})\|; \mathbf{X} \in \chi(K)\} = B < \infty$. Therefore, we have for any $\mathbf{X} \in \chi(K)$ and any $\epsilon \in]-1, 1[$:

$$|\phi(G(\mathbf{X}) + \epsilon H(\mathbf{X})) - \phi(G(\mathbf{X}))| \leq A |\epsilon| \|H(\mathbf{X})\| \leq |\epsilon| AB. \quad (70)$$

Hence, there is a number $a > 0$ such that, for $\epsilon \in]-a, +a[$, we have $\phi(G(\mathbf{X}) + \epsilon H(\mathbf{X})) < d/2 < 0$ for any $\mathbf{X} \in \chi(K)$. Thus, for $\epsilon \in]-a, +a[$, we have $\mathbf{g} + \epsilon \mathbf{h} \in E^*$, as announced. Since that statement does not depend on the chart,

the number $a > 0$ does not depend on the chart, either, thus $a = a(\mathbf{g}, \mathbf{h}) > 0$.

Denoting henceforth $\mathbf{g}_{(\epsilon\mathbf{h})} := \mathbf{g} + \epsilon\mathbf{h}$ for brevity, we define a \mathcal{C}^1 function f from $\Omega \times]-a, +a[$ into \mathbb{R} by setting

$$f(\mathbf{X}, \epsilon) = \mathcal{L}(\phi^A(\mathbf{X}), \phi_{,\mu}^A(\mathbf{X}), g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X}), \partial_\rho g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X})). \quad (71)$$

Using the definition (66), we have:

$$\frac{\partial (g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X}))}{\partial \epsilon} = h^{\mu\nu}(\mathbf{X}), \quad \frac{\partial (\partial_\rho g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X}))}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} (g_{,\rho}^{\mu\nu}(\mathbf{X}) + \epsilon h_{,\rho}^{\mu\nu}(\mathbf{X})) = h_{,\rho}^{\mu\nu}(\mathbf{X}). \quad (72)$$

Therefore, we get

$$\begin{aligned} \frac{\partial f}{\partial \epsilon}(\mathbf{X}, \epsilon) &= \left(\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} \right)_{g_{(\epsilon\mathbf{h})}} \frac{\partial (g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X}))}{\partial \epsilon} + \left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_{\rho}^{\mu\nu})} \right)_{g_{(\epsilon\mathbf{h})}} \frac{\partial (\partial_\rho g_{(\epsilon\mathbf{h})}^{\mu\nu}(\mathbf{X}))}{\partial \epsilon} \\ &= \left(\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} \right)_{g_{(\epsilon\mathbf{h})}} h^{\mu\nu}(\mathbf{X}) + \left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_{\rho}^{\mu\nu})} \right)_{g_{(\epsilon\mathbf{h})}} h_{,\rho}^{\mu\nu}(\mathbf{X}). \end{aligned} \quad (73)$$

From the definitions (67) and (71), we have $S(\mathbf{g}_{(\epsilon\mathbf{h})}) = \int_{\Omega} f(\mathbf{X}, \epsilon) \, d^4 \mathbf{X}$, hence

$$\frac{d}{d\epsilon} S(\mathbf{g}_{(\epsilon\mathbf{h})}) = \int_{\Omega} \frac{\partial f}{\partial \epsilon}(\mathbf{X}, \epsilon) \, d^4 \mathbf{X}. \quad (74)$$

Since at $\epsilon = 0$ we have $\mathbf{g}_{(\epsilon\mathbf{h})} = \mathbf{g}$, Eq. (68) follows from Eqs. (73) and (74).

(ii) The second term in the integrand on the r.h.s. of (68) can be written as

$$\begin{aligned} \left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_{\rho}^{\mu\nu})} \right)_{\mathbf{g}} h_{,\rho}^{\mu\nu}(\mathbf{X}) &= \frac{\partial}{\partial x^\rho} \left(\left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_{\rho}^{\mu\nu})} \right)_{\mathbf{g}} h^{\mu\nu}(\mathbf{X}) \right) \\ &\quad - h^{\mu\nu}(\mathbf{X}) \frac{\partial}{\partial x^\rho} \left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_{\rho}^{\mu\nu})} \right)_{\mathbf{g}}. \end{aligned} \quad (75)$$

The first term on the r.h.s. of (75) is a divergence in \mathbb{R}^4 and its integral on Ω vanishes if $h^{\mu\nu}(\mathbf{X}) = 0$ ($\mu, \nu = 0, \dots, 3$) on $\partial\Omega$, thus if $\mathbf{h}(X) = 0$ on ∂U .

[Remind: $\partial\Omega = \partial(\chi(U)) = \chi(\partial U)$.] Hence, in that case, (68) rewrites as

$$\Delta_{\mathbf{g}} \mathbf{h} S = \int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \underline{g}^{\mu\nu}} - \frac{\partial}{\partial x^\rho} \left(\frac{\partial \mathcal{L}}{\partial (\underline{g}_\rho^{\mu\nu})} \right) \right]_{\mathbf{g}} h^{\mu\nu} d^4 \mathbf{X}. \quad (76)$$

In view of Eq. (57), this is Eq. (69).

(iii) Consider a given tensor field $\mathbf{g} \in \mathbf{E}^*$ and, in a given chart χ whose domain contains K , let $\mathbf{X} \mapsto T_{\mu\nu}(\mathbf{X})$ and $\mathbf{X} \mapsto \tilde{T}_{\mu\nu}(\mathbf{X})$ be two sets of functions ($\mu, \nu = 0, \dots, 3$) defined and continuous over $\chi(K)$, each set being symmetric, such that both verify Eq. (69) for any tensor field $\mathbf{h} \in \mathbf{E}$ that vanishes on ∂U . [We do *not* assume that either $T_{\mu\nu}$ or $\tilde{T}_{\mu\nu}$ is given by Eq. (57).] We claim that $T_{\mu\nu} = \tilde{T}_{\mu\nu}$ over $\chi(K)$. Denoting $\delta T_{\mu\nu} := \tilde{T}_{\mu\nu} - T_{\mu\nu}$, we thus have for any such tensor field \mathbf{h} :

$$\int_{\Omega} \delta T_{\mu\nu} h^{\mu\nu} \sqrt{-g} d^4 \mathbf{X} = 0. \quad (77)$$

Consider a given pair (μ_0, ν_0) of indices. Let φ be any real function which is defined and continuous over $\chi(K)$, which is \mathcal{C}^1 over Ω , and that has compact support $K' \subset \Omega$. Hence, $\varphi(\mathbf{X}) \neq 0$ implies $\mathbf{X} \in \Omega$. If $\mathbf{X} \in \partial\Omega$, we have $\mathbf{X} \notin \Omega$ since Ω is open, hence $\varphi(\mathbf{X}) = 0$. Therefore, by setting $h^{\mu\nu} = \frac{1}{2}(\delta_{\mu_0}^\mu \delta_{\nu_0}^\nu + \delta_{\nu_0}^\mu \delta_{\mu_0}^\nu) \varphi$ in the chart χ , we define a tensor field $\mathbf{h} \in \mathbf{E}$ such that $\mathbf{h}(X) = 0$ for $X \in \partial U$. We can thus apply (77) to get

$$\int_{\Omega} (\delta T_{\mu_0\nu_0} + \delta T_{\nu_0\mu_0}) \sqrt{-g} \varphi d^4 \mathbf{X} = \int_{\Omega} (2\delta T_{\mu_0\nu_0} \sqrt{-g}) \varphi d^4 \mathbf{X} = 0. \quad (78)$$

(The second equality follows from the symmetry of $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$.) Since this is true for any such function φ , we deduce that $\delta T_{\mu_0\nu_0} \sqrt{-g}$ is zero almost everywhere in Ω . But since this is a continuous function, it is zero everywhere in the open set $\Omega \subset \mathbb{R}^4$, and therefore it is zero also in its closure $\overline{\chi(U)} = \chi(\overline{U}) = \chi(K)$. Then, since $\sqrt{-g} \neq 0$ over $\chi(K)$, we have $\delta T_{\mu_0\nu_0} = 0$ over $\chi(K)$. This proves our precise statement about uniqueness at the beginning of this paragraph.

Now suppose the Lagrangian is a four-divergence: $L = ((\sqrt{-g} V^\mu)_{,\mu}) / \sqrt{-g}$ with $V^\mu = V^\mu(\mathbf{q}^A, \mathbf{q}_\mu^A, \underline{g}^{\mu\nu}, \underline{g}_\rho^{\mu\nu})$. Then the integral (67) rewrites as:

$$S(\mathbf{g} + \epsilon \mathbf{h}) = \int_{\partial\Omega} U^\mu(\phi^A(\mathbf{X}), \phi_{,\mu}^A(\mathbf{X}), g_{(\epsilon \mathbf{h})}^{\mu\nu}(\mathbf{X}), (g_{(\epsilon \mathbf{h})}^{\mu\nu})_{,\rho}(\mathbf{X})) dS_\mu(\mathbf{X}) \quad (79)$$

(setting $U^\mu := \sqrt{-g} V^\mu$, and with $\mathbf{g}_{(\epsilon\mathbf{h})} := \mathbf{g} + \epsilon\mathbf{h}$ and $dS_\mu := \varepsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma$) — when this integral makes sense, which is true if $\mathbf{g} \in \mathbf{E}^*$, $\mathbf{h} \in \mathbf{E}$, and $|\epsilon| < a = a(\mathbf{g}, \mathbf{h})$. If, moreover, $\mathbf{h}(X) = 0$ on ∂U , we have $\mathbf{g}_{(\epsilon\mathbf{h})}(X) = \mathbf{g}(X)$ for any $X \in \partial U$, so that the integral (79) does not depend on $\epsilon \in]-a, a[$. Therefore, the l.h.s. of Eq. (69) is zero. Since we have shown that this equation determines uniquely the functions $\mathbf{X} \mapsto T_{\mu\nu}(\mathbf{X}), \chi(K) \rightarrow \mathbb{R}$, these functions are zero.

Let us finally prove the actual tensoriality of the “Hilbert tensor”, whose components are defined by Eq. (57). Considering now any two charts χ and χ' whose domain contains K , Eq. (69) is true for any tensor field $\mathbf{h} \in \mathbf{E}$ that vanishes on ∂U , using either χ or χ' on the r.h.s. (with primes for χ'). Since the l.h.s. of (69) is invariant as is the action, so is the r.h.s.; i.e., we have for any such \mathbf{h} :

$$\int_{\Omega} T_{\mu\nu} h^{\mu\nu} \sqrt{-g} d^4 \mathbf{X} = \int_{\Omega'} T'_{\mu\nu} h'^{\mu\nu} \sqrt{-g'} d^4 \mathbf{X}'. \quad (80)$$

Composing with the reverse coordinate maps χ^{-1} and χ'^{-1} , we may regard $T_{\mu\nu}, h^{\mu\nu}, T'_{\mu\nu}, h'^{\mu\nu}$ as functions defined over $K = \chi^{-1}(\chi(K)) = \chi'^{-1}(\chi'(K))$, and we have for any such \mathbf{h} :

$$\int_U T_{\mu\nu} h^{\mu\nu} dV_4 = \int_U T'_{\mu\nu} h'^{\mu\nu} dV_4. \quad (81)$$

For any given tensor field $\mathbf{h}_0 \in \mathbf{E}$, set $f := T_{\mu\nu} h_0^{\mu\nu}$ and $f' := T'_{\mu\nu} h_0'^{\mu\nu}$, which are thus two continuous functions on K . Consider any open domain W with $W \subset U$. Take any function ϕ which is defined and continuous over K , which is \mathcal{C}^1 over U , and that has compact support $K' \subset W$. Define $\mathbf{h} := \phi \mathbf{h}_0$. This is a tensor field that belongs to \mathbf{E} and vanishes on ∂W . Therefore, the open domain U being arbitrary in the already proved point (ii) and hence in (81), we can apply (81) with W instead of U . We thus get:

$$\int_W f \phi dV_4 = \int_W f' \phi dV_4. \quad (82)$$

Because this is true for any such function ϕ , it follows that we have $f = f'$ almost everywhere in W . Since these are continuous functions, we have $f = f'$ in W . And since this is true for any open domain W with $W \subset U$, we have $f = f'$ in U , and hence also in $K = \bar{U}$. That is, $T_{\mu\nu} h_0^{\mu\nu}$ is invariant under coordinate changes, for whatever tensor field $\mathbf{h}_0 \in \mathbf{E}$. Considering a given

point $X \in K$, we define a linear form Φ on the vector space \mathcal{T}_0^2 of the (2 0) tensors at X , by setting:

$$\forall \underline{h} \in \mathcal{T}_0^2, \Phi(\underline{h}) := T_{\mu\nu}(X)\underline{h}^{\mu\nu}, \quad (83)$$

which is thus independent of the chart. But the dual space of \mathcal{T}_0^2 is known (and easily checked) to be the vector space \mathcal{T}_2^0 of the (0 2) tensors \underline{S} at X . Hence, there is a unique tensor $\underline{S} \in \mathcal{T}_2^0$ for which, in any chart, we have

$$\forall \underline{h} \in \mathcal{T}_0^2, \Phi(\underline{h}) = S_{\mu\nu}\underline{h}^{\mu\nu}. \quad (84)$$

From (83) and (84), it follows that, in any chart, the numbers $T_{\mu\nu}(X)$ are the components of the unique tensor $\underline{S} \in \mathcal{T}_2^0$, Q.E.D. \square

5 A uniqueness result for the energy balance

5.1 Is the energy balance equation unique?

We begin with a discussion of this question for a system of isentropically deformable media in Newtonian gravity (NG). The energy balance (11) established in Section 2 for the *matter field equations* of NG has the form

$$\partial_\mu V^\mu = \text{field source} := -\rho \frac{\partial U}{\partial t}, \quad (85)$$

with the four-components column vector (V^μ) being here the “matter current” made with the matter energy density and flux:

$$(V^\mu) = (w_m, \Phi_m). \quad (86)$$

As we saw, Eq. (85) [i.e. Eq. (11)] is verified as soon as the following three equations are verified among the matter field equations: Newton’s second law (1), the isentropy equation (5), and the continuity equation (9). For instance, we did not use the “constitutive equation” that relates the stress tensor to some deformation tensor, or (for a barotropic fluid) that relates the pressure with the density. We note that, in view of Eqs. (12) and (13), the matter current (V^μ) is polynomial in the local values of the fields ϕ^A ($A = 1, \dots, n$) = $(\rho, \mathbf{v}, \Pi, \boldsymbol{\sigma}, U)$ that appear in those equations. (Thus, *assigning in this section — contrary to Section 4 — a different number A to different components of a given vector*

or tensor field: $n = 12$ here. The gravitational potential U plays the same role as does the metric tensor in a Lagrangian for the matter fields in a curved spacetime, as was the case in the foregoing section.) Now we ask if we can find a different expression for the matter current, say V'^μ , for which the l.h.s. of (85) would be *always* the same as with the current (86), so that the same balance equation (85) would be valid with V'^μ , when it is with V^μ . Thus: Can we change the matter current V^μ for another one $V'^\mu = V^\mu + W^\mu$, also polynomial with respect to the local values of the fields at any spacetime point X , $q^A = \phi^A(X)$, so that the l.h.s. of (85) would be unchanged *for whatever values of the fields*? I.e., can we find a column four-vector W^μ which would be polynomial in the q^A 's, and such that we would have $\partial_\mu W^\mu \equiv 0$?

5.2 A uniqueness result

Thus, let W^μ be an order- N polynomial in the field values $q^A = \phi^A(X)$, its coefficients being allowed to depend on the spacetime position X :

$$W^\mu(X, q^A) = C_0^\mu(X) + C_{1A}^\mu(X) q^A + \dots + C_{N A_1 \dots A_N}^\mu(X) q^{A_1} \dots q^{A_N} \quad (A_1 \leq \dots \leq A_N). \quad (87)$$

Assume that its 4-divergence vanishes identically, $\partial_\mu W^\mu \equiv 0$:

$$0 \equiv C_{0,\mu}^\mu + C_{1A,\mu}^\mu q^A + C_{1A}^\mu q_\mu^A + \dots + C_{N A_1 \dots A_N, \mu}^\mu q^{A_1} \dots q^{A_N} + C_{N A_1 \dots A_N}^\mu (q_\mu^{A_1} q^{A_2} \dots q^{A_N} + \dots + q^{A_1} \dots q^{A_{N-1}} q_\mu^{A_N}) \quad (q_\mu^A := \phi_{,\mu}^A(X)). \quad (88)$$

I.e., at any spacetime point $X_0(x^\rho)$, Eq. (88) is valid for whatever *possible* values q^A and q_μ^A of the fields and their derivatives at X_0 . But, for *whatever* values q^A and q_μ^A of these variables, there exists smooth functions $X \mapsto \phi^A(X)$ ($A = 1, \dots, n$), defined in some neighborhood U of X_0 , such that we have

$$\phi^A(X_0) = q^A \quad (A = 1, \dots, n), \quad \text{and} \quad \frac{\partial \phi^A}{\partial x^\mu}(X_0) = q_\mu^A \quad (A = 1, \dots, n; \mu = 0, \dots, 3). \quad (89)$$

Thus our assumption means that on the r.h.s. of (88) the polynomial function in the real variables q^A and q_μ^A ($A = 1, \dots, n; \mu = 0, \dots, 3$) is identically zero. Hence its coefficients are all zero. In particular:

$$C_{1A}^\mu(X_0) = 0, \dots, C_{N A_1 \dots A_N}^\mu(X_0) = 0. \quad (90)$$

Thus all coefficients in (87) are zero — except perhaps C_0^μ , with $C_{0,\mu}^\mu = 0$.

We thus got that we *cannot* alter the analytical expression of w_m and Φ_m on the l.h.s. of the matter energy balance (11). [Apart from arbitrarily adding a zero-divergence vector field C_0^μ that is *independent* of the matter fields — this is indeed obviously possible, but we can get rid of this by asking that the matter current (V^μ) be polynomial in the fields *and have no zero-order term*, as is indeed the case in all concrete examples.] The gravitational energy balance (15) has just the same form:

$$\partial_\mu V^\mu = \text{matter source} := \rho \frac{\partial U}{\partial t}, \quad (91)$$

where $V^\mu = (w_g, \Phi_g)$ is polynomial in the gravitational field ϕ^A ($A = 1, \dots, 4$) = $(\partial_\mu U)$. It is valid when the gravitational field equation is. Therefore, similarly as we found for the matter field energy balance, we cannot alter the analytical expression (15) of the gravitational energy balance.

5.3 Generalization

These results are clearly general. Consider e.g. the Maxwell electromagnetic field instead of the Newtonian gravitational field. The energy balance of the e.m. field is:

$$\frac{\partial w_{\text{em}}}{\partial t} + \text{div } \Phi_{\text{em}} = -\mathbf{j} \cdot \mathbf{E}, \quad (92)$$

with $w_{\text{em}} := \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}$ the *volume energy density of the electromagnetic field*, and $\Phi_{\text{em}} := \frac{\mathbf{E} \wedge \mathbf{B}}{4\pi}$ the *electromagnetic energy flux*. The same uniqueness result says that we cannot find an alternative expression for w_{em} and Φ_{em} on the l.h.s., which would be valid for whatever values of the fields \mathbf{E} and \mathbf{B} and their first derivatives.

6 Conclusion

The classical concept of energy emerges from an analysis of the power done, first in the case of a mass point and then for the case of a volume element in a continuous medium. We have argued that, in the case of a continuous medium or a system of fields, the meaning of the energy conservation is primarily local: it says that, in *any* bounded domain, the energy loss or gain is due only to a well-identified flux that goes through the boundary of that domain. Thus it expresses in a general way the Lavoisier principle: “Nothing is lost, nothing is created, everything transforms.” While it is of course interesting also and

even often important to have global energy conservation laws, this interest is limited by two facts: i) an exact global conservation law can be hoped, strictly speaking, only for the Universe as a whole, because there are energy exchanges at all scales — but physics can not be reduced to cosmology. ii) A global energy conservation law says merely that one number is a constant: the total energy; in the most favorable case with global conservation of the energy-momentum and angular momentum, ten numbers are constant. In the relevant case of a system of fields, however, there is an infinite number of degrees of freedom, so this is only a small part of the information needed.

We have tried to precisely state and prove the main results regarding the derivation of the Hilbert tensor from the invariance of the action in generally-covariant theories. We hope to have proved these results in a convincing way, keeping the mathematical sophistication to the minimum needed. The Hilbert tensor's theory is beautiful and is essential to general relativity. It is important also in relativistic quantum mechanics. One should note, however, that historically the main examples of the energy-momentum tensor have been derived from the corresponding local energy conservation equations [19]. Whence the interest in examining the uniqueness of this kind of equations.

A Appendix: Regular domains

Definition 1 [20]. *Let M be a differentiable manifold, with dimension d . One will call closed domain of M , any closed subset D of M such that, for any $x \in D$,*

*either (i) there is an open subset W of M such that $x \in W \subset D$,
or (ii) there is a chart (W, φ) with $\varphi(x) = \mathbf{0}$ and $\varphi(y) = (y^1, \dots, y^d)$ for $y \in W$, such that*

$$W \cap D = \{y \in W; y^1 \leq 0\}, \quad \text{i.e.} \quad \varphi(W \cap D) = \{\mathbf{y} \in \varphi(W); y^1 \leq 0\}. \quad (93)$$

Proposition 1 [20]. *Let D be a closed domain of a d -dimensional differentiable manifold M . In case (i), the point x is in $\overset{\circ}{D}$, the interior of D (i.e. the largest open set U of M , such that $U \subset D$). In case (ii), the point x is in ∂D , the boundary of D , which is a $(d - 1)$ -dimensional submanifold of the differentiable manifold M .*

Recall that the (topological) boundary of any subset A of M is defined to be $\partial A := \overline{A} \cap \overline{\mathbb{C}A}$, where the overbar means the adherence (or closure) in M and $\mathbb{C}A$ means the complementary set of A in M . It is easy to prove (cf. [21]) that we have always

$$\overset{\circ}{A} \cap \partial A = \emptyset, \quad (94)$$

$$\overset{\circ}{A} \cup \partial A = \overline{A}. \quad (95)$$

If M is an oriented manifold, then the Stokes theorem (and thus also the divergence theorem) applies to any differential $(d - 1)$ -form (respectively to any continuously differentiable vector field), in any closed domain D of M with its boundary ∂D [20].

Proposition 2. *Let D be a closed domain (see Definition 1) of a d -dimensional differentiable manifold M and let $U := \overset{\circ}{D}$ be its interior. We have*

$$\overline{U} = D, \quad (96)$$

i.e., a closed domain is the closure of its interior. We thus call $U := \overset{\circ}{D}$ an open domain of M . Moreover, we have

$$\overline{\mathbb{C}D} = \mathbb{C}U. \quad (97)$$

Proof. Since D is a closed set such that $U \subset D$, we have $\overline{U} \subset D$. Due to (95), in order to prove that $D \subset \overline{U}$, we just have to prove that $\partial D \subset \overline{U}$. If $x \in \partial D$, we may apply to it Case (ii) of Definition 1. Let A be any open neighborhood of x . We will show that it intersects both U and $\mathbb{C}D$; to show this, we may assume that $A \subset W$, with W the domain of the chart φ . Thus $\varphi(A)$ is an open neighborhood of $\varphi(x) = \mathbf{0}$ in \mathbb{R}^d , hence it contains a ball $|y^j| < r$ ($j = 1, \dots, d$).
a) Take first $\mathbf{y}_0 = (y^j)$ such that $-r < y^1 < 0$ and $y^j = 0$ for $j = 2, \dots, d$: then $\mathbf{y}_0 \in \varphi(A) \subset \varphi(W)$. But we get from (93) that any point $\mathbf{y} \in \varphi(W)$ such that $y^1 < 0$ is in the interior of $\varphi(D \cap W)$. Thus $\mathbf{y}_0 \in \varphi(A)$ is in the interior of $\varphi(D \cap W)$, or equivalently $y_0 = \varphi^{-1}(\mathbf{y}_0) \in A$ is in the interior of $D \cap W$, hence in $U = \overset{\circ}{D}$. So $\partial D \subset \overline{U}$, hence (96) is proved.
b) On the other hand, take now $\mathbf{y}_1 = (y^j)$ such that $0 < y^1 < r$ and $y^j = 0$ for $j = 2, \dots, d$: also $\mathbf{y}_1 \in \varphi(A) \subset \varphi(W)$, but we get from (93) that $y_1 = \varphi^{-1}(\mathbf{y}_1) \in A$ is in $\mathbb{C}D$. Thus we have also $\partial D \subset \overline{\mathbb{C}D}$, whence $\overline{\mathbb{C}D} \cup \partial D = \overline{\mathbb{C}D}$. But, from (94) and (95), we have $D \cap \mathbb{C}\partial D = U$, or $(\mathbb{C}D) \cup \partial D = \mathbb{C}U$. Therefore,

$$\mathbb{C}U = \overline{\mathbb{C}U} = \overline{\mathbb{C}D} \cup \partial D = \overline{\mathbb{C}D}, \quad (98)$$

which proves (97). \square

Corollary. Let U be an open domain of M and let f be a continuous real function defined in a neighborhood of $D = \bar{U}$. In order that $\text{Supp } f \subset D$, it is necessary and sufficient that $f(x) = 0$ if $x \notin U$.

Proof. The support of f , $\text{Supp } f$, is defined to be the smallest closed set containing the set of the points x such that $f(x) \neq 0$, or equivalently $\complement \text{Supp } f$ is the largest open set Ω such that $f|_{\Omega} = 0$. Therefore, $\complement D$ being an open set,

$$(\text{Supp } f \subset D) \Leftrightarrow (f|_{\complement D} = 0). \quad (99)$$

Since f is continuous, $f|_{\complement D} = 0$ is equivalent to $f|_{\overline{\complement D}} = 0$ — that is, from (97), to $f|_{\complement U} = 0$. \square

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