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# Inhomogeneous approximation with coprime integers and lattice orbits

Michel Laurent & Arnaldo Nogueira

ABSTRACT – Let  $(\xi, y)$  be a point in  $\mathbb{R}^2$  and  $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$  a function. We investigate the problem of the existence of infinitely many pairs  $p, q$  of coprime integers such that

$$|q\xi + p - y| \leq \psi(|q|).$$

We give both unconditional results which are valid for every real pair  $(\xi, y)$  with  $\xi$  irrational, and metrical results valid for almost all points  $(\xi, y)$ . We link the subject with density exponents of lattice orbits in  $\mathbb{R}^2$ .

## 1 Introduction and results

Minkowski has proved that for every real irrational number  $\xi$  and every real number  $y$  not belonging to  $\mathbb{Z}\xi + \mathbb{Z}$ , there exist infinitely many pairs of integers  $p, q$  such that

$$|q\xi + p - y| \leq \frac{1}{4|q|}.$$

See for instance Theorem II in Chapter 3 of Cassels' monograph [4]. The statement is optimal in the sense that the approximating function  $\ell \mapsto (4\ell)^{-1}$  cannot be decreased. Note that the restriction  $y \notin \mathbb{Z}\xi + \mathbb{Z}$  can be dropped at the cost of replacing the upper bound  $(4|q|)^{-1}$  by  $c|q|^{-1}$  for any constant  $c$  greater than  $1/\sqrt{5}$ . When  $y = 0$ , the primitive point  $(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})$  remains a solution to the above inequality, therefore we may moreover require that the pair of integers  $p, q$  be coprime. However, for a non-zero real number  $y$ , this extra requirement is far from being obvious to satisfy. In this direction, Chalk and Erdos [6] have obtained the following result:

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**Theorem** (Chalk-Erdos). *Let  $\xi$  be an irrational real number and let  $y$  be a real number. There exists an absolute constant  $c$  such that the inequality*

$$(1) \quad |q\xi + p - y| \leq \frac{c(\log q)^2}{q(\log \log q)^2}$$

*holds for infinitely many pairs of coprime integers  $(p, q)$  with  $q$  positive.*

We study more generally the diophantine inequation

$$|q\xi + p - y| \leq \psi(|q|)$$

for coprime integers  $p$  and  $q$ , where  $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$  is a given function. Two types of questions naturally arise. First, finding unconditional results which are valid for every real pair  $(\xi, y)$  with  $\xi$  irrational as (1), and secondly getting metrical results valid for almost all points  $(\xi, y)$ . Here is an example of the first kind.

**Theorem 1.** *Let  $\xi$  be an irrational real number and let  $y$  be a real number. There exist infinitely many integer quadruples  $(p_1, q_1, p_2, q_2)$  satisfying*

$$q_1 p_2 - p_1 q_2 = 1$$

*and*

$$(2) \quad |q_i \xi + p_i - y| \leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}} \leq \frac{c}{\sqrt{|q_i|}}, \quad (i = 1, 2),$$

*with  $c = 2\sqrt{3} \max(1, |\xi|)^{1/2} |y|^{1/2}$ .*

Theorem 1 will be deduced in Section 2 from our results [10] of effective density for  $SL(2, \mathbb{Z})$ -orbits in  $\mathbb{R}^2$ . The estimate (2) is best possible, up to the value of the constant  $c$ . However, the optimality of (1) remains unclear. We address the following

**Problem.** *Can we replace the approximating function  $\psi(\ell) = c(\log \ell)^2 / \ell(\log \log \ell)^2$  occurring in (1) by a smaller one, possibly  $\psi(\ell) = c\ell^{-1}$  ?*

We shall further discuss this problem in Section 4 for the function  $\psi(\ell) = 2\ell^{-1}$ , offering some hints and indicating the difficulties which then arise. It turns out that the approximating function  $\psi(\ell) = \ell^{-1}$  is permitted for almost all pairs  $(\xi, y)$  of real numbers relatively to Lebesgue measure. The last assertion follows from the following metrical statement:

**Theorem 2.** *Let  $\psi : \mathbb{N} \mapsto \mathbb{R}^+$  be a function. Assume that  $\psi$  is non-increasing, tends to 0 at infinity and that for every positive integer  $c$  there exists a positive real number  $c_1$  satisfying*

$$(3) \quad \psi(c\ell) \geq c_1\psi(\ell), \quad \forall \ell \geq 1.$$

*Furthermore assume that*

$$\sum_{\ell \geq 1} \psi(\ell) = +\infty.$$

*Then, for almost all pairs  $(\xi, y)$  of real numbers there exist infinitely many primitive points  $(p, q)$  such that*

$$(4) \quad q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q).$$

*If  $\sum_{\ell \geq 1} \psi(\ell)$  converges, the pairs  $(\xi, y)$  satisfying (4) for infinitely many primitive points  $(p, q)$  form a set of null Lebesgue measure.*

Note that we could have equivalently required in (4) that  $q$  be negative. Such a refinement could as well be achieved in the frame of Theorem 1, with a weaker approximating function of the form  $\psi(\ell) = \ell^{-\mu}$  for any given real number  $\mu < 1/3$ , by employing alternatively Theorem 5 in Section 9 of [10]. We leave the details of proof to the interested reader, arguing as in Section 2. For questions of density involving signs, see also [7].

The proof of Theorem 2 is given in Section 3. It combines standard tools from metrical number theory with the ergodic properties of the linear action of  $SL(2, \mathbb{Z})$  on  $\mathbb{R}^2$  [13]. We refer to Harman's book [8] for closely related results. See also the recent overview [1] and the monographs [14], [15].

Theorem 2 is a metrical statement about pairs  $(\xi, y)$  of real numbers. A natural question is to understand what happens on each fiber when we fix either  $\xi$  or  $y$ . In this direction, here is a partial result which will be deduced from the explicit construction displayed in Section 4.

**Theorem 3.** *Let  $\xi$  be an irrational number and let  $(p_k/q_k)_{k \geq 0}$  be the sequence of its convergents. Assume that the series*

$$(5) \quad \sum_{k \geq 0} \frac{1}{\max(1, \log q_k)}$$

*diverges. Then for almost every real number  $y$  there exist infinitely many primitive points  $(p, q)$  satisfying*

$$|q\xi + p - y| \leq \frac{2}{|q|}.$$

*Moreover the series (5) diverges for almost every real number  $\xi$ .*

We now turn to the second part of the paper devoted to density exponents for lattice orbits in  $\mathbb{R}^2$ . As already mentioned, the approximating function  $\psi(\ell) = c\ell^{-1/2}$  occurring in Theorem 1 is directly connected to the density exponent  $1/2$  for  $SL(2, \mathbb{Z})$ -orbits. We intend to show that this exponent  $1/2$  is best possible in general.

We work in the more general setting of *lattices*  $\Gamma$  in  $SL(2, \mathbb{R})$ . Recall that a lattice  $\Gamma$  in  $SL(2, \mathbb{R})$  is a discrete subgroup for which the quotient  $\Gamma \backslash SL(2, \mathbb{R})$  has finite Haar measure. We view  $\mathbb{R}^2$  as a space of column vectors on which the group of matrices  $\Gamma$  acts by left multiplication. We equip  $\mathbb{R}^2$  with the supremum norm  $|\cdot|$ , and for any matrix  $\gamma \in \Gamma$ , we denote as well by  $|\gamma|$  the maximum of the absolute values of the entries of  $\gamma$ . Let us first give a

**Definition.** *Let  $\mathbf{x}$  and  $\mathbf{y}$  be two points in  $\mathbb{R}^2$ . We denote by  $\mu_\Gamma(\mathbf{x}, \mathbf{y})$  the supremum, possibly infinite, of the exponents  $\mu$  such that the inequality*

$$(6) \quad |\gamma\mathbf{x} - \mathbf{y}| \leq |\gamma|^{-\mu}$$

*has infinitely many solutions  $\gamma \in \Gamma$ .*

Note that for a fixed  $\mathbf{x} \in \mathbb{R}^2$ , the function  $\mathbf{y} \mapsto \mu_\Gamma(\mathbf{x}, \mathbf{y})$  is  $\Gamma$ -invariant. By the ergodicity of the action of  $\Gamma$  on  $\mathbb{R}^2$ , see [13], this function is therefore constant almost everywhere on  $\mathbb{R}^2$ . We denote by  $\mu_\Gamma(\mathbf{x})$  its generic value and we call  $\mu_\Gamma(\mathbf{x})$  the *generic density exponent* of the orbit  $\Gamma\mathbf{x}$ .

**Theorem 4.** *The upper bound  $\mu_\Gamma(\mathbf{x}) \leq 1/2$  holds true for any point  $\mathbf{x} \in \mathbb{R}^2$  such that the orbit  $\Gamma\mathbf{x}$  is dense in  $\mathbb{R}^2$ .*

In an equivalent way, Theorem 4 asserts that the upper bound  $\mu(\mathbf{x}, \mathbf{y}) \leq 1/2$  holds for almost all points  $\mathbf{y} \in \mathbb{R}^2$ . This bound was already known in the case of the unimodular group  $\Gamma = SL(2, \mathbb{Z})$  as a consequence of Theorem 3 in [10].

One may optimistically conjecture that  $\mu_\Gamma(\mathbf{x}) = 1/2$  for every point  $\mathbf{x}$  such that  $\Gamma\mathbf{x}$  is dense in  $\mathbb{R}^2$ , or at least for almost every point  $\mathbf{x} \in \mathbb{R}^2$ . In this direction, it follows from [10] that the lower bound

$$\mu_{SL(2, \mathbb{Z})}(\mathbf{x}) \geq \frac{1}{3}$$

holds for all points  $\mathbf{x}$  in  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$  with irrational slope. Weaker lower bounds can as well be deduced from [12] which are valid for any lattice  $\Gamma \subset SL(2, \mathbb{R})$ . Note that the function  $\mathbf{x} \mapsto \mu_\Gamma(\mathbf{x})$  is  $\Gamma$ -invariant since the quantity  $\mu_\Gamma(\mathbf{x})$  obviously depends only on the orbit  $\Gamma\mathbf{x}$ . Thus, the generic density exponent  $\mu_\Gamma(\mathbf{x})$  takes the same value for almost all points  $\mathbf{x} \in \mathbb{R}^2$ .

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## 2 Proof of Theorem 1

We first state a result obtained in [10]. In this section, we denote by  $\Gamma$  the lattice  $\text{SL}(2, \mathbb{Z})$ . For any point  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  in  $\mathbb{R}^2$  with irrational slope  $x_1/x_2$ , the orbit  $\Gamma\mathbf{x}$  is dense in  $\mathbb{R}^2$ . We have obtained in [10] effective results concerning the density of such an orbit. In particular, our estimates are essentially optimal when the target point  $\mathbf{y}$  has rational slope.

**Lemma 1.** *Let  $\mathbf{x}$  be a point in  $\mathbb{R}^2$  with irrational slope and  $\mathbf{y} = \begin{pmatrix} y \\ y \end{pmatrix}$  a point on the diagonal with  $y \neq 0$ . Then, there exist infinitely many matrices  $\gamma \in \Gamma$  such that*

$$(7) \quad |\gamma\mathbf{x} - \mathbf{y}| \leq \frac{c}{|\gamma|^{1/2}} \quad \text{with} \quad c = 2\sqrt{3}|\mathbf{x}|^{1/2}|y|^{1/2}.$$

**Proof.** The point  $\mathbf{y}$  has rational slope 1. Apply Theorem 1 (ii) of [10] with  $a = b = 1$ .  $\square$

Put  $\mathbf{x} = \begin{pmatrix} \xi \\ 1 \end{pmatrix}$ . The point  $\mathbf{x}$  has irrational slope  $\xi$  so that Lemma 1 may be applied. Write  $\gamma = \begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix}$  a matrix provided by Lemma 1. Then, the inequality (7) gives

$$\begin{aligned} \max(|q_1\xi + p_1 - y|, |q_2\xi + p_2 - y|) &\leq \frac{c}{\max(|p_1|, |p_2|, |q_1|, |q_2|)^{1/2}} \\ &\leq \frac{c}{\max(|q_1|, |q_2|)^{1/2}}. \end{aligned}$$

Therefore, both points  $(p_1, q_1)$  and  $(p_2, q_2)$  satisfy (2), and since the determinant  $q_1p_2 - q_2p_1 = 1$ , the two integer points  $(p_1, q_1)$  and  $(p_2, q_2)$  are primitive. As there exist infinitely many matrices  $\gamma$  verifying (7), we thus find infinitely many solutions to (2).

Assume now that the irrational number  $\xi$  has bounded partial quotients. Then, Theorem 4 in [10] gives us in the opposite direction a lower bound of the form

$$|\gamma\mathbf{x} - \mathbf{y}| \geq \frac{c'}{|\gamma|^{1/2}},$$

for some positive constant  $c'$  depending only upon  $(\xi, y)$ . Since  $|\gamma| \leq c'' \max(|q_1|, |q_2|)$  when (2) holds, the estimate (2) is optimal up to the value of  $c$ .

**Remark.** The single inequality  $|q_1\xi + p_1 - y| \leq \psi(|q_1|)$  geometrically means that the point  $\gamma\mathbf{x}$  falls inside a neighborhood of the vertical line  $x_1 = y$ . A better understanding of the shrinking target problem for the dense orbit  $\Gamma\mathbf{x}$ , not to a point  $\mathbf{y}$  as in [10] but to a line in  $\mathbb{R}^2$ , may possibly lead to a refinement of (1).

### 3 Proof of Theorem 2

It is convenient to view the pairs  $(\xi, y)$  occurring in Theorem 2 as column vectors  $\begin{pmatrix} \xi \\ y \end{pmatrix}$  in  $\mathbb{R}^2$ . We are concerned with the set  $\mathcal{E}(\psi)$  of vectors  $\begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2$  for which there exist infinitely many primitive integer points  $(p, q)$  such that

$$(8) \quad q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \psi(q).$$

For fixed  $p, q$ , denote by  $\mathcal{E}_{p,q}(\psi)$  the strip

$$\mathcal{E}_{p,q}(\psi) := \left\{ \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathbb{R}^2; \quad |q\xi + p - y| \leq \psi(q) \right\},$$

and for every positive integer  $q$ , let

$$\mathcal{E}_q(\psi) := \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p,q)=1}} \mathcal{E}_{p,q}(\psi)$$

be the union of all relevant strips involved in (8) for fixed  $q$ . Without loss of generality, we shall assume that  $\psi(q) \leq 1/2$ , so that the above union is disjoint. Then  $\mathcal{E}(\psi)$  is equal to the lim sup set

$$\mathcal{E}(\psi) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \mathcal{E}_q(\psi).$$

As usual when dealing with lim sup set in metrical theory, we first estimate Lebesgue measure of pairwise intersections of the subsets  $\mathcal{E}_q(\psi)$ ,  $q \geq 1$ . We establish next a new kind of zero-one law.

#### 3.1 Measuring intersections

In this section, we restrict our attention to points located in the unit square  $[0, 1]^2$ . We denote by  $\varphi$  the Euler totient function and by  $\lambda$  the Lebesgue measure on  $\mathbb{R}^2$ .

**Lemma 2.** *Let  $\psi : \mathbb{N} \rightarrow [0, 1/2]$  be a function.*

(i) *For every positive integer  $q$ , we have*

$$\lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = \frac{2\varphi(q)\psi(q)}{q}.$$

(ii) *Let  $q$  and  $s$  be distinct positive integers. Then, we have the upper bound*

$$\lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0, 1]^2) \leq 4\psi(q)\psi(s).$$

**Proof.** Denote by  $\chi_q$  the characteristic function of the interval  $[-\psi(q), \psi(q)]$ . Then the characteristic function  $\chi_{\mathcal{E}_q(\psi)}$  of the subset  $\mathcal{E}_q(\psi) \subset \mathbb{R}^2$  is equal to

$$\chi_{\mathcal{E}_q(\psi)}(\xi, y) = \sum_{\substack{p \in \mathbb{Z} \\ \gcd(p, q) = 1}} \chi_q(q\xi + p - y) = \sum_{\substack{p \in \mathbb{Z} \\ \gcd(p, q) = 1}} \chi_q(q\xi - p - y).$$

Observe that if  $\begin{pmatrix} \xi \\ y \end{pmatrix}$  belongs to  $[0, 1]^2$ , the indices  $p$  of non-vanishing terms occurring in the last sum are located in the interval  $-1 \leq p \leq q$ . Integrating first with respect to  $x$ , we find

$$\begin{aligned} \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) &= \int_0^1 \int_0^1 \chi_{\mathcal{E}_q(\psi)}(x, y) dx dy \\ &= \sum_{\substack{p \in \mathbb{Z} \\ -1 \leq p \leq q, \gcd(p, q) = 1}} \int_0^1 \int_0^1 \chi_q(qx - p - y) dx dy \\ &= \int_{1-\psi(q)}^1 \frac{-1 + y + \psi(q)}{q} dy + \sum_{\substack{1 \leq p \leq q-2 \\ \gcd(p, q) = 1}} \int_0^1 \frac{2\psi(q)}{q} dy \\ &\quad + \int_0^{1-\psi(q)} \frac{2\psi(q)}{q} dy + \int_{1-\psi(q)}^1 \frac{1 - y + \psi(q)}{q} dy \\ &= \frac{2\psi(q)\psi(q)}{q}. \end{aligned}$$

The first term appearing in the third equality of the above formula corresponds to the summation index  $p = -1$  and the two last ones to  $p = q - 1$ . We have thus proved (i).

For the second assertion, we majorize

$$\begin{aligned} \lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0, 1]^2) &= \int_0^1 \int_0^1 \chi_{\mathcal{E}_q(\psi)}(x, y) \chi_{\mathcal{E}_s(\psi)}(x, y) dx dy \\ &\leq \int_0^1 \int_0^1 \left( \sum_{p \in \mathbb{Z}} \chi_q(qx + p - y) \right) \left( \sum_{r \in \mathbb{Z}} \chi_s(sx + r - y) \right) dx dy \\ &= \int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dx dy, \end{aligned}$$

where  $\|\cdot\|$  stands as usual for the distance to the nearest integer. Now, (ii) follows from the probabilistic independence formula

$$\int_0^1 \int_0^1 \chi_q(\|qx - y\|) \chi_s(\|sx - y\|) dx dy = 4\psi(q)\psi(s),$$

obtained by Cassels on page 124 of [4] (see Proof (ii)). □



### 3.2 A zero-one law

We say that a subset of  $\mathbb{R}^2$  is a *null* set if it has Lebesgue measure 0. A set whose complementary is a null set is called a *full* set. The goal of this section is to prove the

**Proposition.** *Let  $\psi$  be an approximating function as in Theorem 2. Then the subset  $\mathcal{E}(\psi)$  is either a null set or a full set.*

For proving the proposition, it is convenient to introduce the larger subset

$$\mathcal{E}'(\psi) = \bigcup_{k \geq 1} \mathcal{E}(k\psi).$$

In other words,  $\mathcal{E}'(\psi)$  is the set of all points  $\begin{pmatrix} \xi \\ y \end{pmatrix}$  in  $\mathbb{R}^2$  for which there exist a positive real number  $\kappa$ , depending possibly on  $\begin{pmatrix} \xi \\ y \end{pmatrix}$ , and infinitely many primitive points  $(p, q)$  satisfying

$$(9) \quad q \geq 1 \quad \text{and} \quad |q\xi + p - y| \leq \kappa\psi(q).$$

Observe that  $\mathcal{E}(k\psi) \subseteq \mathcal{E}(k'\psi)$  if  $1 \leq k \leq k'$ . In particular,  $\mathcal{E}(\psi)$  is contained in  $\mathcal{E}'(\psi)$ .

**Lemma 3.** *Let  $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$  be a function tending to zero at infinity. Then the difference  $\mathcal{E}'(\psi) \setminus \mathcal{E}(\psi)$  is a set of null Lebesgue measure.*

**Proof.** We show that all sets  $\mathcal{E}(k\psi)$ ,  $k \geq 1$ , have the same Lebesgue measure. For every real number  $y$ , denote by  $\mathcal{E}(\psi, y) \subseteq \mathbb{R}$  the section of  $\mathcal{E}(\psi)$  on the horizontal line  $\mathbb{R} \times \{y\}$ , i.e.

$$\mathcal{E}(\psi, y) = \left\{ \xi \in \mathbb{R}; \begin{pmatrix} \xi \\ y \end{pmatrix} \in \mathcal{E}(\psi) \right\}.$$

Then, using (8), we can express

$$\mathcal{E}(\psi, y) = \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{\substack{p \in \mathbb{Z} \\ \gcd(p, q) = 1}} \left[ \frac{-p + y - \psi(q)}{q}, \frac{-p + y + \psi(q)}{q} \right]$$

as a limsup set of intervals. If we restrict to a bounded part of  $\mathcal{E}(\psi, y)$ , the above union over  $p$  reduces to a finite one. Observe that the centers  $\frac{-p+y}{q}$  of these intervals do not depend on  $\psi$ , and that their length is multiplied by the constant factor  $k$  when replacing  $\psi$  by  $k\psi$ . Appealing now to a result due to Cassels [5], we infer that all limsup sets  $\mathcal{E}(k\psi, y)$ ,  $k \geq 1$ , have the same Lebesgue measure. See also Corollary of Lemma 2.1 on page 30 of [8]. Notice that for fixed  $k$ , the length  $\frac{2k\psi(q)}{q}$  of the intervals

$\left[ \frac{-p+y-k\psi(q)}{q}, \frac{-p+y+k\psi(q)}{q} \right]$  tend to 0 as  $q$  tends to infinity, as required by Lemma 2.1. By Fubini, the fibered sets

$$\mathcal{E}(k\psi) = \coprod_{y \in \mathbb{R}} \left( \mathcal{E}(k\psi, y) \times \{y\} \right), \quad k \geq 1,$$

have as well the same Lebesgue measure in  $\mathbb{R}^2$ .  $\square$

**Lemma 4.** *Let  $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$  be a non-increasing function satisfying (3). Then  $\mathcal{E}'(\psi)$  is either a null or a full set.*

**Proof.** It is based on the following observation. Let  $\begin{pmatrix} \xi \\ y \end{pmatrix}$  belong to  $\mathcal{E}'(\psi)$  and let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a matrix in  $SL(2, \mathbb{Z})$  such that  $c\xi + d > 0$ . Then the point  $\begin{pmatrix} \xi' \\ y' \end{pmatrix}$  with coordinates

$$\xi' = \frac{a\xi + b}{c\xi + d} \quad \text{and} \quad y' = \frac{y}{c\xi + d}$$

belongs to  $\mathcal{E}'(\psi)$ . Indeed, substituting

$$(10) \quad q = aq' + cp', \quad p = bq' + dp'$$

in (9) and dividing by  $c\xi + d$ , we obtain the inequalities

$$(11) \quad q' \geq 1 \quad \text{and} \quad |q'\xi' + p' - y'| \leq \frac{\kappa}{c\xi + d} \psi(q) \leq \kappa' \psi(q'),$$

for some  $\kappa' > 0$  independent of  $q'$ . The positivity of  $q'$  is proved as follows. Note that (9) implies the estimate

$$p = -q\xi + \mathcal{O}_{\xi, y}(1).$$

Then, inverting the linear substitution (10), we find

$$q' = dq - cp = q(c\xi + d) + \mathcal{O}_{\gamma, \xi, y}(1).$$

Since we have assumed that  $c\xi + d > 0$ , the term  $q(c\xi + d)$  is arbitrarily large when  $q$  is large enough. The condition (3) now shows that  $\psi(q) \asymp \psi(q')$ . Thus (11) is satisfied for infinitely many primitive points  $(p', q')$ , since the linear substitution (10) is unimodular. We have shown that  $\begin{pmatrix} \xi' \\ y' \end{pmatrix}$  belongs to  $\mathcal{E}'(\psi)$ .

We now prove that the intersection  $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$  is either a full or a null subset of the half plane  $\mathbb{R} \times \mathbb{R}^+$ . To that purpose, we consider the map

$$\Phi : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+, \quad \text{defined by} \quad \Phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x/y \\ 1/y \end{pmatrix}.$$

Clearly  $\Phi$  is a continuous involution of  $\mathbb{R} \times \mathbb{R}^+$ . The image

$$\Omega := \Phi\left(\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)\right)$$

is formed by all points of the type

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\xi}{y} \\ \frac{1}{y} \end{pmatrix},$$

where  $\begin{pmatrix} \xi \\ y \end{pmatrix}$  ranges over  $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ . Now, the above condition  $c\xi + d > 0$  is obviously equivalent to  $cu + dv > 0$  since  $y$  is positive. Then, the point

$$\Phi \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} \frac{au+bv}{cu+dv} \\ \frac{1}{cu+dv} \end{pmatrix} = \begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix}$$

belongs to  $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$ , by the preceding observation. Applying the involution  $\Phi$ , we find that

$$\Phi \left( \begin{pmatrix} \frac{a\xi+b}{c\xi+d} \\ \frac{y}{c\xi+d} \end{pmatrix} \right) = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

belongs to  $\Omega$ . In other words, setting  $\Gamma = SL(2, \mathbb{Z})$ , we have established the inclusion

$$(\Gamma\Omega) \cap (\mathbb{R} \times \mathbb{R}^+) \subseteq \Omega.$$

Since the reversed inclusion is obvious, the equality  $\Omega = (\Gamma\Omega) \cap (\mathbb{R} \times \mathbb{R}^+)$  holds in fact. Assuming that  $\Omega$  is not a null set, the ergodicity of the linear action of  $\Gamma$  on  $\mathbb{R}^2$  [13] shows that  $\Gamma\Omega$  is a full set in  $\mathbb{R}^2$ . Hence  $\Omega$  is a full set in the half plane  $\mathbb{R} \times \mathbb{R}^+$ . Transforming now  $\Omega$  by  $\Phi$ , we find that

$$\Phi(\Omega) = \mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+),$$

is as well a full set in  $\mathbb{R} \times \mathbb{R}^+$ , thus proving the claim.

We finally use another transformation to carry the zero-one law from the positive half plane  $\mathbb{R} \times \mathbb{R}^+$  to the negative one  $\mathbb{R} \times \mathbb{R}^-$ . Writing (9) in the equivalent form

$$q \geq 1 \quad \text{and} \quad |q(-\xi) + (-p) - (-y)| \leq \kappa\psi(q),$$

shows that  $\mathcal{E}'(\psi)$  is invariant under the symmetry  $\begin{pmatrix} \xi \\ y \end{pmatrix} \mapsto \begin{pmatrix} -\xi \\ -y \end{pmatrix}$  which maps  $\mathbb{R} \times \mathbb{R}^+$  onto  $\mathbb{R} \times \mathbb{R}^-$ . Therefore  $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^-)$  is a null or a full set in  $\mathbb{R} \times \mathbb{R}^-$  when  $\mathcal{E}'(\psi) \cap (\mathbb{R} \times \mathbb{R}^+)$  is accordingly a null or a full set in  $\mathbb{R} \times \mathbb{R}^+$ .  $\square$

Now, the combination of Lemma 3 and Lemma 4 obviously yields our proposition.

### 3.3 Concluding the proof of Theorem 2

Assume first that  $\sum \psi(\ell)$  converges. We have to show that the set

$$\mathcal{E}(\psi) = \limsup_{q \rightarrow +\infty} \mathcal{E}_q(\psi)$$

has null Lebesgue measure. Lemma 2 shows that the partial sums

$$\sum_{q=1}^Q \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = 2 \sum_{q=1}^Q \frac{\varphi(q)\psi(q)}{q} \leq 2 \sum_{q=1}^Q \psi(q)$$

converge (\*). Then, Borel-Cantelli Lemma ensures that the lim sup set  $\mathcal{E}(\psi) \cap [0, 1]^2$  is a null set. Thus  $\mathcal{E}(\psi)$  cannot be a full set. Now, the above proposition tells us that  $\mathcal{E}(\psi)$  is a null set.

We now consider the case of a divergent series  $\sum \psi(\ell)$ . Observe that the estimate

$$(12) \quad \frac{1}{2} \sum_{q=1}^Q \psi(q) \leq \sum_{q=1}^Q \frac{\varphi(q)\psi(q)}{q} \leq \sum_{q=1}^Q \psi(q)$$

holds true for any large integer  $Q$ , since the sequence  $\psi(\ell)_{\ell \geq 1}$  is non-increasing. The right inequality is obvious, while the left one easily follows from Abel summation process. See for instance Chapter 2 of [8], where full details are provided. By Lemma 2 and (12), the sums

$$\sum_{q=1}^Q \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2) = 2 \sum_{q=1}^Q \frac{\varphi(q)\psi(q)}{q} \geq \sum_{q=1}^Q \psi(q)$$

are then unbounded. Then, using a classical converse to Borel-Cantelli Lemma, we have the lower bound

$$(13) \quad \begin{aligned} \lambda(\mathcal{E}(\psi) \cap [0, 1]^2) &= \lambda\left(\limsup_{q \rightarrow +\infty} (\mathcal{E}_q(\psi) \cap [0, 1]^2)\right) \\ &\geq \limsup_{Q \rightarrow +\infty} \frac{\left(\sum_{q=1}^Q \lambda(\mathcal{E}_q(\psi) \cap [0, 1]^2)\right)^2}{\sum_{q=1}^Q \sum_{s=1}^Q \lambda(\mathcal{E}_q(\psi) \cap \mathcal{E}_s(\psi) \cap [0, 1]^2)}. \end{aligned}$$

See for instance Lemma 2.3 in [8]. Lemma 2 and (12) now show that the numerator on the right hand side of (13) equals

$$4 \left( \sum_{q=1}^Q \frac{\varphi(q)\psi(q)}{q} \right)^2 \geq \left( \sum_{q=1}^Q \psi(q) \right)^2,$$

---

(\*) Here again we assume without loss of generality that  $\psi(q) \leq 1/2$  for every  $q \geq 1$ , so that Lemma 2 may be applied.

when  $Q$  is large, while the denominator is bounded from above by

$$4 \sum_{\substack{q=1, s=1 \\ q \neq s}}^Q \psi(q)\psi(s) + 2 \sum_{q=1}^Q \psi(q) \leq 4 \left( \sum_{q=1}^Q \psi(q) \right)^2 + 2 \sum_{q=1}^Q \psi(q).$$

Thus (13) yields the lower bound

$$\lambda(\mathcal{E}(\psi) \cap [0, 1]^2) \geq \frac{1}{4}.$$

Hence  $\mathcal{E}(\psi)$  is not a null set; it is thus a full set according to our proposition.

## 4 An approach to our problem

In this section, we apply a transference principle between homogeneous and inhomogeneous approximation, as displayed in Chapter V of [4] and in [3], for constructing explicit integer solutions of the inequality

$$(14) \quad |q\xi + p - y| \leq \frac{2}{|q|}.$$

Let  $(p_k/q_k)_{k \geq 0}$  be the sequence of convergents to the irrational number  $\xi$ . The theory of continued fractions, see for instance the monograph [9], tells us that

$$(15) \quad |q_k \xi - p_k| \leq \frac{1}{q_{k+1}} \quad \text{and} \quad p_k q_{k+1} - p_{k+1} q_k = (-1)^{k+1},$$

for any  $k \geq 0$ . Setting  $\nu_k = (-1)^{k+1} q_k y$ , we thus have the relations

$$(16) \quad \nu_k q_{k+1} + \nu_{k+1} q_k = 0 \quad \text{and} \quad \nu_k (q_{k+1} \xi - p_{k+1}) + \nu_{k+1} (q_k \xi - p_k) = y.$$

Now, let  $n_k$  be anyone of the two integers  $\lfloor \nu_k \rfloor$  and  $\lceil \nu_k \rceil$  (†). Then,

$$(17) \quad |\nu_k - n_k| < 1,$$

and  $n_k$  is either equal to  $(-1)^{k+1} \lfloor y q_k \rfloor$  or to  $(-1)^{k+1} \lceil y q_k \rceil$ . Setting

$$(18) \quad p = -n_k p_{k+1} - n_{k+1} p_k \quad \text{and} \quad q = n_k q_{k+1} + n_{k+1} q_k,$$

---

(†) As usual  $\lfloor x \rfloor$  and  $\lceil x \rceil$  stand respectively for the floor and the ceiling of the real number  $x$ . Then  $\lceil x \rceil = \lfloor x \rfloor + 1$ , unless  $x$  is an integer in which case  $\lfloor x \rfloor = \lceil x \rceil = x$ .

we deduce from (16) the expressions

$$(19) \quad \begin{aligned} q\xi + p - y &= n_k(q_{k+1}\xi - p_{k+1}) + n_{k+1}(q_k\xi - p_k) - y \\ &= (n_k - \nu_k)(q_{k+1}\xi - p_{k+1}) + (n_{k+1} - \nu_{k+1})(q_k\xi - p_k) \end{aligned}$$

and

$$(20) \quad q = (n_k - \nu_k)q_{k+1} + (n_{k+1} - \nu_{k+1})q_k.$$

Recall that  $q_k\xi - p_k$  and  $q_{k+1}\xi - p_{k+1}$  have opposite signs. Assuming that  $n_k - \nu_k$  and  $n_{k+1} - \nu_{k+1}$  have the same sign, we infer from the formulas (19), (20) and from (15), (17) that

$$(21) \quad |q\xi + p - y| < \frac{1}{q_{k+1}} \quad \text{and} \quad |q| < 2q_{k+1}.$$

Otherwise, we have

$$(22) \quad |q\xi + p - y| < \frac{2}{q_{k+1}} \quad \text{and} \quad |q| < q_{k+1}.$$

The inequalities (21) and (22) obviously imply (14).

Since the linear substitution (18) is unimodular, the integers  $p$  and  $q$  are coprime if and only if  $n_k$  and  $n_{k+1}$  are coprime. Recall that the two choices  $n_k = \lfloor \nu_k \rfloor$  and  $n_k = \lceil \nu_k \rceil$  are admissible, both for  $n_k$  and  $n_{k+1}$ . It thus remains to find indices  $k$  for which at least one of the coprimality conditions

$$(23) \quad \begin{aligned} &\gcd(\lfloor yq_k \rfloor, \lfloor yq_{k+1} \rfloor) = 1 \quad \text{or} \quad \gcd(\lceil yq_k \rceil, \lceil yq_{k+1} \rceil) = 1 \\ \text{or} \quad &\gcd(\lfloor yq_k \rfloor, \lceil yq_{k+1} \rceil) = 1 \quad \text{or} \quad \gcd(\lceil yq_k \rceil, \lfloor yq_{k+1} \rfloor) = 1, \end{aligned}$$

is verified. Note that (23) obviously fails for all  $k \geq 0$  when  $y$  is an integer not equal to 1 or to  $-1$ . Otherwise, the contingent existence of infinitely many indices  $k$  satisfying (23) is a non-trivial problem that we leave hanging.

Let us mention that the proof of (1) in [6] follows the same idea, finding a primitive integer point inside the square centered at the point  $(\nu_k, \nu_{k+1}) \in \mathbb{R}^2$  with side  $C \log |\nu_k| / \log \log |\nu_k|$  for some suitable large absolute constant  $C$ .

## 4.1 Proof of Theorem 3

We quote the following metrical result due to Harman (Theorem 8.3 in [8]). Assume that the series (5) diverges. Then for almost all positive real numbers  $y$ , there exist infinitely many indices  $k$  such that the integer part  $\lfloor yq_k \rfloor$  is a prime number. These

indices  $k$  fulfill (23) since, assuming for simplicity that  $y$  is irrational, either  $\lfloor yq_{k+1} \rfloor$  or  $\lceil yq_{k+1} \rceil = \lfloor yq_{k+1} \rfloor + 1$  is not divisible by  $\lfloor yq_k \rfloor$  and is thus relatively prime with  $\lfloor yq_k \rfloor$ . Hence (14) has infinitely many coprime solutions  $(p, q)$  for almost every positive real number  $y$ . Writing now (14) in the equivalent form

$$|(-q)\xi + (-p) - (-y)| \leq \frac{2}{|q|}$$

shows that,  $\xi$  being given, the set of all real numbers  $y$  for which (14) has infinitely many coprime solutions is invariant by the symmetry  $y \mapsto -y$ . The first assertion is thus established. To complete the proof, note that

$$\lim_{k \rightarrow +\infty} \frac{\log q_k}{k} = \frac{\pi^2}{12 \log 2}$$

for almost every  $\xi$  by Khintchine-Levy Theorem (see equation (4.18) in [2]). Thus the series (5) diverges for almost every  $\xi$ .

## 5 Generic density exponents

We prove in this section Theorem 4, as a consequence of Borel-Cantelli Lemma combined with the following counting result.

**Lemma 5.** *Let  $\mathbf{x}$  be a point in  $\mathbb{R}^2$  whose orbit  $\Gamma\mathbf{x}$  is dense in  $\mathbb{R}^2$ . For every symmetric compact set  $\Omega$  in  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$  there exists  $c > 0$  such that*

$$\text{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq T\} \leq cT$$

for any real number  $T \geq 1$ .

**Proof.** Ledrappier [11] has shown that the limit formula

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \sum_{\gamma \in \Gamma, |\gamma| \leq T} f(\gamma\mathbf{x}) = \frac{4}{|\mathbf{x}| \text{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int \frac{f(\mathbf{y})}{|\mathbf{y}|} d\mathbf{y}$$

holds for any even continuous function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  having compact support on  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ , with a suitable normalisation of Haar measure on  $SL(2, \mathbb{R})$ . Approximating uniformly from above and from below the characteristic function of  $\Omega$  by even continuous functions, we deduce that

$$\lim_{T \rightarrow +\infty} \frac{\text{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq T\}}{T} = \frac{4}{|\mathbf{x}| \text{vol}(\Gamma \setminus SL(2, \mathbb{R}))} \int_{\Omega} \frac{d\mathbf{y}}{|\mathbf{y}|}.$$

Lemma 5 immediately follows. □

For any point  $\mathbf{y} \in \mathbb{R}^2$  and any positive real number  $r$ , we denote by

$$B(\mathbf{y}, r) = \{\mathbf{z} \in \mathbb{R}^2; |\mathbf{z} - \mathbf{y}| \leq r\}$$

the closed disc centered at  $\mathbf{y}$  with radius  $r$ .

**Lemma 6.** *Let  $\mathbf{x}$  be a point in  $\mathbb{R}^2$  whose orbit  $\Gamma\mathbf{x}$  is dense,  $\Omega$  a symmetric compact set in  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$  and  $\mu$  a real number  $> 1/2$ . For every integer  $n \geq 1$ , put*

$$\mathcal{B}_n = \bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| = n, \gamma\mathbf{x} \in \Omega}} B(\gamma\mathbf{x}, n^{-\mu}).$$

Then the set

$$\mathcal{B} := \limsup_{n \rightarrow +\infty} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{n \geq N} \mathcal{B}_n = \bigcap_{N \geq 1} \bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| \geq N, \gamma\mathbf{x} \in \Omega}} B(\gamma\mathbf{x}, |\gamma|^{-\mu})$$

has null Lebesgue measure.

**Proof.** We apply Borel-Cantelli Lemma and we prove that the series  $\sum_{n \geq 1} \lambda(\mathcal{B}_n)$  converges if  $\mu > 1/2$ .

For every positive integer  $n$ , set

$$M_n = \text{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| = n\}.$$

Lemma 5 gives us the upper bound

$$(24) \quad M_1 + \cdots + M_n = \text{Card}\{\gamma \in \Gamma; \gamma\mathbf{x} \in \Omega, |\gamma| \leq n\} \leq cn,$$

for some  $c > 0$  independent of  $n \geq 1$ . Since a ball of radius  $r$  has Lebesgue measure  $4r^2$ , we trivially bound from above

$$\lambda(\mathcal{B}_n) \leq \sum_{\substack{\gamma \in \Gamma \\ |\gamma| = n, \gamma\mathbf{x} \in \Omega}} 4n^{-2\mu} = 4M_n n^{-2\mu}.$$

Summing by parts, we deduce from (24) that

$$\begin{aligned} \sum_{n=1}^N \frac{M_n}{n^{2\mu}} &= \sum_{n=1}^{N-1} (M_1 + \cdots + M_n) \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{M_1 + \cdots + M_N}{N^{2\mu}} \\ &\leq c \sum_{n=1}^{N-1} n \left( \frac{1}{n^{2\mu}} - \frac{1}{(n+1)^{2\mu}} \right) + \frac{cN}{N^{2\mu}} = c \sum_{n=1}^N \frac{1}{n^{2\mu}}. \end{aligned}$$



The partial sums

$$\sum_{n=1}^N \lambda(\mathcal{B}_n) \leq 4 \sum_{n=1}^N \frac{M_n}{n^{2\mu}} \leq 4c \sum_{n=1}^N \frac{1}{n^{2\mu}}$$

thus converge if  $\mu > 1/2$ . □

## 5.1 Proof of Theorem 4

We argue by contradiction and suppose on the contrary that  $\mu_\Gamma(\mathbf{x}) > 1/2$ . Fix a real number  $\mu$  with  $1/2 < \mu < \mu_\Gamma(\mathbf{x})$ . Then for almost all points  $\mathbf{y} \in \mathbb{R}^2$ , we have  $\mu(\mathbf{x}, \mathbf{y}) > \mu$ . This means that there exist infinitely many  $\gamma \in \Gamma$  satisfying (6), or equivalently that  $\mathbf{y}$  belongs to infinitely many balls of the form  $B(\gamma\mathbf{x}, |\gamma|^{-\mu})$ . We now restrict our attention to points  $\mathbf{y}$  with  $\mu(\mathbf{x}, \mathbf{y}) > \mu$  lying in an annulus

$$\Omega' = \{\mathbf{z} \in \mathbb{R}^2; a' \leq |\mathbf{z}| \leq b'\},$$

where  $b' > a' > 0$  are arbitrarily fixed. Since  $\mathbf{y}$  belongs to the intersection  $\Omega' \cap B(\gamma\mathbf{x}, |\gamma|^{-\mu})$ , we deduce from the triangle inequality the estimate

$$a' - |\gamma|^{-\mu} \leq |\gamma\mathbf{x}| \leq b' + |\gamma|^{-\mu}.$$

Fixing  $a < a'$  and  $b > b'$ , the center  $\gamma\mathbf{x}$  then lies in the larger annulus

$$\Omega = \{\mathbf{z} \in \mathbb{R}^2; a \leq |\mathbf{z}| \leq b\},$$

provided that  $|\gamma|$  is large enough. It follows that  $\mathbf{y}$  falls inside the union of balls

$$\bigcup_{\substack{\gamma \in \Gamma \\ |\gamma| \geq N, \gamma\mathbf{x} \in \Omega}} B(\gamma\mathbf{x}, |\gamma|^{-\mu})$$

considered in Lemma 6 for every integer  $N$  large enough, and thus  $\mathbf{y}$  belongs to  $\mathcal{B}$ . However, Lemma 6 asserts that  $\mathcal{B}$  is a null set which is a contradiction.

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