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On the Complexity and Volume of Hyperbolic 3-Manifolds.

Thomas Delzant and Leonid Potyagailo

January 28, 2016

Abstract

We compare the volume of a hyperbolic 3-manifold M of finite volume and a complexity of its fundamental group.¹

1 Introduction.

Complexity of 3-manifolds and groups. One of the most striking corollaries of the recent solution of the geometrization conjecture for 3-manifolds is the fact that every aspherical 3-manifold is uniquely determined by its fundamental group. So it is natural to think that topological/geometrical invariants of such a manifold M are closely related to combinatorial invariants of its fundamental group $\pi_1 M$.

The study of a complexity of 3-manifolds goes back to the classical work of H. Kneser [K]. Recall that the Kneser complexity invariant $k(M)$ is defined to be the minimal number of simplices of a triangulation of the manifold M . The main result of Kneser is that this complexity serves as a bound of the number of embedded incompressible 2-spheres in M , and bounds the numbers of factors in a decomposition of M as a connected sum. A version of this complexity was used by W. Haken to prove the existence of hierarchies for a large class of compact 3-manifolds (called since then Haken manifolds). Another complexity $c(M)$ of the 3-manifold M was defined by S. Matveev as the minimal number of vertices of a special spine of M . It is shown in [Ma] that $k(M) = c(M)$ if M is a non-compact hyperbolic 3-manifold of finite volume.

The rank (minimal number of generators) is also a measure of complexity of a finitely generated group. According to the classical theorem of I. Grushko [Gr], the rank of a free product of

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groups is the sum of their ranks. This immediately implies that every finitely generated group is a free product of finitely many freely indecomposable factors, which is an algebraic analogue of the Kneser theorem.

For a finitely presented group G a measure of complexity of G was introduced in [De] as follows:

Definition 1.1. *The minimum of the number of relations of length 3 taken over all presentations of G containing only relations of length at most 3 is called (absolute) invariant T and is denoted by $T(G)$. \square*

T_A

Without loss of generality we can consider here only the presentations of G with relations of length at most 3. Indeed suppose the group G is defined by a presentation $\langle a_1, \dots, a_r; R_1, \dots, R_l \rangle$. Then triangulating the faces of the presentation complex having more than 3 edges we obtain that it contains $\sum_{i=1}^n (|R_i| - 2)$ triangles where $|R_i|$ is the length of R_i . So $T(G) = \min\{\sum (|R_i| - 2)\}$, where the minimum is taken over all presentations of G .

Note that $T(G) \leq t$ if and only if there exists a simply-connected simplicial 2-dimensional polyhedron P such that G acts freely and simplicially on P and the number of 2-dimensional simplices of the quotient $\Pi = P/G$ is at most t .

There is an obvious inequality between the Kneser complexity of a closed 3-manifold M and the invariant $T(\pi_1 M)$. Indeed, by contracting a maximal subtree of the 2-dimensional skeleton of a triangulation of M one obtains a triangular presentation of the group $\pi_1 M$. Since every 3-simplex has four 2-faces and each of them is counted twice we have

$$T(\pi_1 M) \leq 2k(M).$$

In order to compare the complexity of a manifold and that of its fundamental group, it is enough to find a function θ such that $\theta(\pi_1 M) \leq T(\pi_1 M)$. Note that the existence of such a function follows from G. Perelman's solution of the geometrization conjecture [Pe 1-3]. Indeed there could exist at most finitely many different 3-manifolds having the fundamental groups isomorphic to the same group G (for irreducible 3-manifolds with boundary this was shown much earlier in [Swa]). The question which still remains open is to describe the asymptotic behavior of the function θ .

Note that for certain finite cyclic groups the following inequality is shown in [PP, Theorem 3.9]:

$$\frac{\log_3 p}{\log_2 3} \leq T(\mathbb{Z}/2^k p) \leq \frac{7}{2}(\log_2 p + k),$$

where p is odd and $k \geq 0$. However, the above problem remains widely open for irreducible 3-manifolds with infinite fundamental group.

If M is a compact hyperbolic 3-manifold, D. Cooper showed [C] that for every simplicial map $f : S \rightarrow M$ of a 2-complex S realizing a fixed presentation of $\pi_1 M$ one has:

$$\text{Vol}M < \pi \cdot \text{Area}(f(S)), \tag{C}$$

where Vol and Area stand for the hyperbolic volume and for the hyperbolic area respectively. If in particular all relations of the presentation are of length at least 3 the Cooper inequality (C) implies:

$$\text{Vol}M < \pi \cdot (L - 2n),$$

where n is the number of the relations in the presentation and L is the sum of their word lengths. Since the group $\pi_1 M$ is without torsion, we can assume that all relations are always of length at least 3. So the inequality (C) also yields:

$$\text{Vol}M \leq \pi \cdot T(\pi_1 M). \tag{C'}$$

Note that the volume is not comparable with the absolute invariant T for the hyperbolic lattices acting on \mathbb{H}^3 . Indeed there exists infinite sequences of different compact hyperbolic 3-manifolds M_n obtained by Dehn filling on a fixed finite volume hyperbolic manifold M with cusps such that $\text{Vol}M_n < \text{Vol}M$ [Th]. The ranks of the groups $\pi_1 M_n$ are all bounded by $\text{rank}(\pi_1 M)$. Since $\pi_1 M_n$ are not isomorphic, we must have $T(\pi_1 M_n) \rightarrow \infty$. So the ratio $T(\pi_1 M)/\text{Vol}M$ is not bounded for hyperbolic 3-manifolds of finite volume. Therefore instead of the absolute invariant T we will use a relative version of it introduced in [De]:

Definition 1.2. *Let G be a finitely presented group, and \mathcal{E} be a family of subgroups. We say that $T(G, \mathcal{E}) \leq t$ if there exists a simply-connected 2-dimensional complex P such that G acts simplicially on P , the number of 2-faces of the quotient (an orbihedron) $\Pi = P/G$ is at most t , and the stabilizers of vertices of P are elements of \mathcal{E} .*

The main goal of the present paper is to obtain uniform constants comparing the volume of a hyperbolic 3-manifold M of finite volume and the relative invariant $T(\pi_1 M, E)$ where E is a family of virtually abelian (elementary) subgroups of $\pi_1 M$.

To finish our historical discussion we note that general "volume-type" invariants (i.e. positive numbers which are multiplicative on passing to subgroups of finite index) are important for studying group-theoretical properties like (finitely)-(co)-Hopfian properties [BGHM], [R]. We show below that the relative invariant $T(\cdot, \cdot)$ is an "asymptotically volume-type" invariant for the hyperbolic lattices of dimension 3, i.e. the ratio of the relative invariant of the lattice (with respect to the family of maximal elementary subgroups) to the volume of the manifold is uniformly bounded (see Corollary 1.5).

Main Results. Let M be a hyperbolic 3-manifold of finite volume. We consider the family E_μ of all elementary subgroups of $\pi_1 M$ having the translation length less than the Margulis constant $\mu = \mu(3)$. The family E_μ includes all parabolic subgroups of $\pi_1 M$ as well as cyclic loxodromic ones representing geodesics in M of length less than μ (see also the next Section).

The first result of the paper is the following:

Theorem A. *There exists a constant C such that for every hyperbolic 3-manifold M of finite volume the following inequality holds:*

$$T(\pi_1 M, E_\mu) \leq C \cdot \text{Vol}M \quad (*)$$

□

So for the relative invariant we have.

Corollary 1.3. *Let M_n be a sequence of different hyperbolic 3-manifolds obtained by Dehn surgery on a cusped hyperbolic 3-manifold of finite volume M . Then*

$$T(\pi_1 M_n, E_{\mu,n}) \leq C \cdot \text{Vol}M < +\infty,$$

where $E_{\mu,n}$ is the system of elementary subgroups of $\pi_1 M_n$ whose translation length is less than μ .

□

Proof: The inequality (*) gives

$$T(\pi_1 M_n, E_{\mu,n}) \leq C \cdot \text{Vol}(M_n),$$

and by [Th] one has $\text{Vol}(M_n) < \text{Vol}M$. *QED.*

Our next result is the following :

Theorem B. *(Generalized Cooper inequality) Let E be a family of elementary subgroups of G . Then*

$$\text{Vol}M \leq \pi \cdot T(\pi_1 M, E) \quad (**)$$

□

Note that Theorem B gives a generalization of the Cooper inequality (C') for the relative invariant $T(\pi_1 M, E)$. We have the following:

Corollary 1.4. *Suppose $M_n \xrightarrow{f_n} M$ is a sequence of finite coverings over a finite volume hyperbolic 3-manifold M such that $\deg f_n \rightarrow +\infty$. Then $T(\pi_1 M_n, E_n) \rightarrow +\infty$, where E_n is the system of elementary subgroups of $\pi_1 M_n$. \square*

Proof: Since $\text{Vol}(M_n) \rightarrow \infty$, the statement follows immediately from (**) applied to the manifolds M_n and $E = E_n$. *QED.*

Theorems A and B together yield:

Corollary 1.5. *Let M be a finite volume hyperbolic 3-manifold. Let E_μ and E denote the families of elementary subgroups of $\pi_1 M$ with the translation length at most μ and any family of elementary subgroups respectively. Then for the constant C from Theorem A the following statements hold:*

i) $\frac{1}{C} \cdot T(\pi_1 M, E_\mu) \leq \text{Vol} M \leq \pi \cdot T(\pi_1 M, E_\mu).$

ii) $T(\pi_1 M, E_\mu) \leq C \cdot \pi \cdot T(\pi_1 M, E).$

iii) *If $M = M_{\mu\text{thick}}$, i.e. every loop in M of length less than μ is homotopically trivial, then*

$$T(\pi_1 M) \leq C \cdot \pi \cdot T(\pi_1 M, E).$$

\square

Let us now briefly describe the content of the paper. In Section 2 we provide some preliminary results needed in the future. The proof of Theorem B is given in Section 3. In Section 4 we prove Theorem A using some standard techniques and the results of Section 2. In the last Section 5 we discuss some open questions related to the present paper.

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2 Preliminary results.

In this Section we first recall few standard definitions. Then we prove a technical Lemma 2.3 which will be used further in order to apply the result of [De] (see Section 4, page 18).

Say that a group G *splits* as a graph of groups $X_* = (X, (C_e)_{e \in X^1}, (G_v)_{v \in X^0})$ if G is isomorphic to the fundamental group $\pi_1(X_*)$ in the sense of Serre [Se] where C_e and G_v are respectively edge and vertex groups of the graph X . The Bass-Serre tree of the splitting is the universal cover of the graph X . In a particular case when X has only one edge G splits as an amalgamated free product (resp. an HNN-extension) if X has two vertices (resp. one vertex).

Let G be a group acting on a tree T . A subset H of G is called *elliptic* (resp. *hyperbolic*) in T if H fixes a point in T (resp. does not fix a point in T).

Definition 2.1. *We say that G splits relatively to a family of subgroups (E_1, \dots, E_n) , or the pair $(G, (E_i)_{1 \leq i \leq n})$ splits, if G acts on a tree T such that every subgroup E_i is elliptic in this splitting ($i = 1, \dots, n$). The tree T is called $(G, (E_i)_{1 \leq i \leq n})$ -tree in this case. \square*

Note that the pair $(G, (E_i)_{1 \leq i \leq n})$ splits if and only if G splits as a graph of groups such that every subgroup E_i ($i = 1, \dots, n$) is conjugate into a vertex group of the graph.

Definition 2.2. *Suppose the pair $(G, (E_i)_{1 \leq i \leq n})$ splits as a graph of groups:*

$$G = \pi_1(X, C_e, G_v). \quad (1)$$

*The decomposition (1) such that all edge groups are non-trivial is called **reduced** if every vertex group G_v cannot be decomposed relatively to the subgroups $E_i < G_v$ as a graph of groups having one of the subgroups C_e as a vertex group.*

*The decomposition (1) is called **rigid** if whenever one has a $(G, (E_i)_{i \in \{1, \dots, n\}})$ -tree T^* such that the subgroup C_e contains a non-trivial edge stabilizer then C_e acts elliptically on T^* . \square*

Recall that for a constant μ any n -dimensional hyperbolic manifold M can be decomposed into a thick and thin parts : $M = M_{\mu\text{thick}} \sqcup M_{\mu\text{thin}}$ such that the injectivity radius at each point of $M_{\mu\text{thin}}$ is less than $\mu/2$, and $M_{\mu\text{thick}} = M \setminus M_{\mu\text{thin}}$. By the Margulis Lemma, for each dimension n there exists the Margulis constant $\mu(n)$ such that if $\mu < \mu(n)$ then the components of $M_{\mu\text{thin}}$ are either parabolic cusps or regular neighborhoods (tubes) of the closed geodesics of M of length less than μ .

We will always assume that the constant μ is less than $\mu(n)$ and denote by $E_\mu = E_\mu(\pi_1 M)$ the system of subgroups of $\pi_1 M_{\mu\text{thin}}$. We will need the following:

Lemma 2.3. *Let H be a group admitting the following splitting as a graph of groups:*

$$H = \pi_1(X, C_e, G_v), \quad (2)$$

where each vertex group G_v is a lattice in $\text{Isom}(\mathbb{H}^n)$ ($n > 2$) and C_e is an elementary subgroup of G_v . Then (2) is a reduced and rigid splitting of the pair (H, \mathcal{E}) where $\mathcal{E} \subset \bigcup_v E(G_v)$, and $E(G_v)$ is the family of elementary subgroups of G_v .

Remark 2.4. *The above Lemma will be used in Section 4 in a very particular geometric situation when the group H is the fundamental group of the double of the thick part $M_{\mu\text{thick}}$ of M along its boundary and $\mathcal{E} = \{\text{id}\}$. \square*

Proof: We first claim that it is enough to prove that every vertex group G_v of the graph X cannot split non-trivially over an elementary subgroup. Indeed, if it is the case then obviously (2) is reduced. If it is not rigid, then the couple (H, \mathcal{E}) acts on a simplicial tree T^* such that one of the groups C_e contains an edge stabiliser C_e^* of T^* and therefore acts hyperbolically on T^* . It follows that the vertex group G_v containing C_e also acts hyperbolically on T^* and so is decomposable over elementary subgroups.

Let us now fix a vertex v and set $G = G_v$. The Lemma now follows from the following statement:

Sublemma 2.5. *[Be] Let G be the fundamental group of a Riemannian manifold M of finite volume of dimension $n > 2$ with pinched sectional curvature within $[a, b]$ for $a \leq b < 0$. Then G does not split over a virtually nilpotent group.*

Proof: We provide here a direct proof of this Sublemma in the case of the constant curvature. Suppose, on the contrary, that

$$G = A *_C B \quad \text{or} \quad G = A *_C \tilde{C}, \tag{3}$$

where C is an elementary subgroup. Let \tilde{C} be the maximal elementary subgroup containing C . The group \tilde{C} is virtually abelian and contains a maximal abelian subgroup \tilde{C}_0 of finite index. We have the following

Claim 2.6. *The group \tilde{C}_0 is separable in G .*

*Proof:*² Recall that the subgroup \tilde{C}_0 is said *separable* if $\forall g \in G \setminus \tilde{C}_0$ there exists a subgroup of finite index $G_0 < G$ such that $\tilde{C}_0 < G_0$ and $g \notin G_0$. Since \tilde{C}_0 is a maximal abelian subgroup of G , and $g \notin \tilde{C}_0$, it follows that there exists $h \in \tilde{C}_0$ such that $\gamma = gh_0g^{-1}h_0^{-1} \neq 1$. The group G is residually finite, so there exists an epimorphism $\tau : G \rightarrow K$ to a finite group K such that $\tau(\gamma) \neq 1$. Since $\tau(\tilde{C}_0)$ is abelian, $\tau(\gamma) \notin \tau(\tilde{C}_0)$ and the subgroup $G_0 = \tau^{-1}(\tau(\tilde{C}_0))$ satisfies our Claim. *QED.*

²The argument is due to M. Kapovich and one of the authors is thankful for sharing it with him (about 20 years ago).

Denote $C_0 = C \cap \tilde{C}_0$ (the maximal abelian subgroup of C). We have $\tilde{C} = \bigcup_{i=1}^m c_i C_0 \cup C_0$. So by the Claim we can find a subgroup of finite index G_0 of G containing C_0 such that $c_i \notin G_0$ ($i = 1, \dots, m$). Then $G_0 \cap \tilde{C} = C_0$ is an abelian group and by the Subgroup Theorem [SW] we have that G_0 splits as :

$$G_0 = A_0 *_{C'_0} B_0 \text{ or } G_0 = A_0 *_{C'_0}, \quad (3')$$

where $C'_0 < C_0$ is also abelian. Suppose first that $G_0 = A_0 *_{C'_0} B_0$. Since G_0 is not elementary group, one of the vertex subgroups of this splitting, say A_0 is not elementary too. Then for any fixed $c \in C'_0$ the map $\varphi : G_0 \rightarrow (cA_0c^{-1}) *_{C'_0} B_0$, such that $\varphi|_{A_0} = cA_0c^{-1}$ and $\varphi|_{B_0} = \text{id}$ is an exterior automorphism (as c commutes with every element of C'_0) of infinite order. So the group of the exterior automorphisms $\text{Out}(G_0)$ is infinite. This contradicts to the Mostow rigidity as G_0 is still a lattice. In the case of HNN-extension $G_0 = A_0 *_{C'_0} = \langle A_0, t \mid tC'_0t^{-1} = \psi(C'_0) \rangle$ suppose first that t does not belong to the centralizer $Z(C'_0)$ of C'_0 in G_0 . Then we put $\varphi|_{A_0} = cA_0c^{-1}$ for some $c \in C'_0$ such that $[c, t] \neq 1$ and $\varphi(t) = t$. Since $t \notin Z(C'_0)$ we obtain again that φ is an infinite order exterior automorphism which is impossible. If, finally, $t \in Z(C'_0)$ then put $\varphi|_{A_0} = \text{id}$ and $\varphi(t) = t^2$ and it is easy to see that $G'_0 = \varphi(G_0)$ is a subgroup of index 2 of G_0 isomorphic to G_0 . Then $\text{Vol}(\mathbb{H}^n/\varphi(G_0)) < +\infty$ and again by Mostow rigidity we must have $\text{Vol}(\mathbb{H}^n/G_0) = \text{Vol}(\mathbb{H}^n/\varphi(G_0))$, and so $\varphi : G_0 \rightarrow G_0$ should be surjective. A contradiction. The Sublemma 2.5 and Lemma 2.3 follow. *QED.*

3 Proof of the generalized Cooper inequality.

The aim of this Section is to prove Theorem B stated in the Introduction:

Theorem B. (*Generalized Cooper inequality*) *Let E be a family of elementary subgroups of G . Then*

$$\text{Vol}M \leq \pi \cdot T(\pi_1 M, E) \quad (**)$$

□

Proof: For the sake of clarity we first briefly describe the strategy of the proof. We start (Lemma 3.1) by constructing a simplicial map $F : \Pi' \rightarrow M$ where Π' is a sub-orbihedron of an orbihedron Π whose fundamental group is G . The main step in the proof is Proposition 3.3 below (which we call *simplicial blow-up procedure*) where we prove that $F|_{\Pi'}$ can be approximated by a simplicial map $\psi_\varepsilon : \Sigma_\varepsilon \rightarrow M$ of a polyhedron Σ_ε such that $\pi_1(\Sigma_\varepsilon) \cong G$ and $|\text{Area}(F(\Pi')) - \text{Area}(\psi_\varepsilon(\Sigma_\varepsilon))| < \varepsilon$ ($\forall \varepsilon > 0$). Then applying the Cooper inequality (C) to $\psi_\varepsilon(\Sigma_\varepsilon)$ we will obtain the inequality (**).

Let M be a hyperbolic manifold \mathbb{H}^3/G where $G < \text{Isom}(\mathbb{H}^3)$ is a lattice. Let us also fix a simply-connected 2-dimensional simplicial polyhedron P admitting a simplicial action of G such that the vertex stabilizers are elements of the system E . Let us also assume that the quotient $\Pi = P/G$ is a finite orbihedron.

Lemma 3.1. *There exists a G -equivariant simplicial continuous map $f : P \rightarrow \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that the images of the 2-simplices of P are geodesic triangles or ideal triangles of \mathbb{H}^3 .*

Proof: Let us first construct a G -equivariant continuous map $f : P \rightarrow \overline{\mathbb{H}^3} = \mathbb{H}^3 \cup \partial\mathbb{H}^3$ such that the image of the fixed points for the action G on P belong to $\partial\mathbb{H}^3$. To do it we apply the construction from [DePo, Lemma 1.6] where instead of a tree as the goal space we will use the hyperbolic space \mathbb{H}^3 . Let us first construct a map $\rho : E \rightarrow \mathbb{H}^3$ as follows. Since the group G is torsion-free we can assume that all non-trivial groups in E are infinite. Then for every elementary group $E_0 \in E$ we put $\rho(E_0) = x \in \partial\mathbb{H}^3$ to be one of the fixed points for the action of E_0 on $\partial\mathbb{H}^3$ (by fixing a point $O \in \partial\mathbb{H}^3$ for the image of the trivial group $\rho(id)$). The map ρ has the following obvious properties :

- a) $\forall E_1, E_2 \in E$ if $E_1 \cap E_2 \neq \emptyset$ then $\rho(E_1) = \rho(E_2)$;
- b) if \tilde{E}_0 is a maximal elementary subgroup then $\rho(E_0) = \rho(\tilde{E}_0)$ and $\rho(g\tilde{E}_0g^{-1}) = g\rho(\tilde{E}_0)$ ($g \in G$).

We now choose the set of G -non-equivalent vertices $\{p_1, \dots, p_l\} \subset P$ representing all vertices of $\Pi = P/G$. We first construct a map f on zero-skeleton $P^{(0)}$ of the complex P by putting $f(p_i) = \rho(E_i)$ and then extend it equivariantly $f(gp_i) = gf(p_i)$ ($g \in G$).

Suppose now $y = (q_1, q_2)$ ($q_1, q_2 \in P^{(0)}$) is an edge of P . To define f on y we distinguish two cases: 1) $H = \text{Stab}(y) \neq 1$ and 2) $H = 1$.

In the first case we have necessarily that $E_{q_1} \cap E_{q_2} = H_0$ is an infinite elementary group where E_{q_i} is the stabilizer of q_i . Then there exist $g_i \in G$ such that $q_i = g_i(p_{k_i})$ ($i = 1, 2$). So $E_{q_i} = g_i E_{p_{k_i}} g_i^{-1}$ and $g_1 E_{p_{k_1}} g_1^{-1} \cap g_2 E_{p_{k_2}} g_2^{-1} = H_0$. It follows that $E_{p_{k_1}} \cap g_1^{-1} g_2 E_{p_{k_1}} g_2^{-1} g_1$ is an infinite group and, therefore $f(p_{k_1}) = g_1^{-1} g_2 (f(p_{k_2}))$ implying that

$$f(q_1) = f(g_1 p_{k_1}) = f(g_2 p_{k_2}) = f(q_2).$$

In the case 2) the stabilizer of the infinite geodesic $l =]f(q_1), f(q_2)[\subset \mathcal{P}$ is trivial so we extend $f : y \rightarrow l$ by a piecewise-linear homeomorphism. Having defined the map f as above on the maximal set of non-equivalent edges of $P^{(1)}$ under G , we extend it equivariantly to the 1-skeleton $P^{(1)}$ by putting $f(gy) = gf(y)$ ($g \in G$). Finally we extend f piecewise linearly to the 2-skeleton $P^{(2)}$.

We obtain a G -equivariant continuous map $f : P \rightarrow \overline{\mathbb{H}^3}$ such that the all 2-faces of the simplicial complex $f(P) \cap \mathbb{H}^3$ are ideal geodesic triangles. The Lemma is proved. *QED.*

Remarks 3.2. 1. Note that the above Lemma is true in any dimension. We restricted our consideration to dimension 3 since the further argument will only concern this case.

2. If the system E contains only parabolic subgroups one can claim that the action of G on $f(P) \cap \mathbb{H}^3$ is in addition proper. Indeed, using the convex hull $\mathcal{P} \subset \mathbb{H}^3$ of the maximal family of non-equivalent parabolic points constructed in [EP] the above argument gives the map $f : P \rightarrow \overline{\mathcal{P}} \subset \overline{\mathbb{H}^3}$. By [EP, Proposition 3.5] the set of faces of \mathcal{P} is locally finite in \mathbb{H}^3 . Since the boundary of each face of the 2-orbihedron $f(P)$ constructed above belongs to $\partial\mathcal{P}$, we obtain that the set of 2-faces of $f(P) \subset \mathbb{H}^3$ is locally finite in this case. \square

If now W is the set of the fixed points for the action of G on P , we put $P' = P \setminus W$ and $Q' = f(P') = f(P) \cap \mathbb{H}^3$. Let also $\nu : P \rightarrow \Pi$ and $\pi : \mathbb{H}^3 \rightarrow M = \mathbb{H}^3/G$ denote the natural projections. Then by Lemma 3.1 the map f projects to a simplicial map $F : (\Pi' = P'/G) \rightarrow Q'/G \subset M$ such that the following diagram is commutative:

$$\begin{array}{ccc} P' & \xrightarrow{f|_{P'}} & Q' \subset \mathbb{H}^3 \\ \nu \downarrow & & \pi \downarrow \\ \Pi' & \xrightarrow{F} & Q'/G \subset M \end{array}$$

Note that, if Π is a simplicial polyhedron, it is proved in [C] that the hyperbolic area of $F(\Pi)$ bounds the volume of the manifold M . This argument does not work if Π is an orbihedron but not a polyhedron. Indeed the complex Q' above is not necessarily simply connected. So the group G is not isomorphic to $\pi_1(Q'/G)$ but is a non-trivial quotient of it. Our goal now is to construct a new simplicial polyhedron Σ with the fundamental group G whose image into M has area arbitrarily close to that of $F(\Pi')$.

Proposition 3.3. (simplicial blow-up procedure). For every $\varepsilon > 0$ there exists a 2-dimensional complex Σ_ε and a simplicial map $\varphi_\varepsilon : \Sigma_\varepsilon \rightarrow M$ such that

1) The induced map $\varphi_\varepsilon : \pi_1 \Sigma_\varepsilon \rightarrow M$ is an isomorphism.

and

2) For the hyperbolic area one has:

$$|\text{Area}(\varphi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(F(\Pi'))| < \varepsilon.$$

Proof of the Proposition: Let Π be a finite orbihedron with elementary vertex groups and such that $\pi_1^{\text{orb}}(\Pi) \cong G$. Let us fix a vertex σ of Π and let $\tilde{\sigma} \in \nu^{-1}(\sigma)$ be its lift in P . We denote by G_σ the group of the vertex σ in G . By Lemma 3.1 the point $f(\tilde{\sigma}) \in \partial\mathbb{H}^3$ is fixed by the elementary group G_σ . We will distinguish between the two cases when the group G_σ is loxodromic cyclic or parabolic subgroup of rank 2.

Case 1. The group G_σ is loxodromic.

Let $V \subset \Pi$ be a regular neighborhood of the vertex σ . Then the punctured neighborhood $V \setminus \sigma$ is homotopically equivalent to the one-skeleton $L^{(1)}$ of the link L of σ .

We will call *realization* of L a graph $\Lambda \subset V \setminus \sigma$ such that the canonical map $L \rightarrow \Lambda$ is a homeomorphism. Let us fix a maximal tree T in Λ , and let y_i be the edges from $\Lambda \setminus T$ which generate the group $\pi_1(L)$ ($i = 1, \dots, k$).

By its very definition, the G -equivariant map $f : P \rightarrow \mathbb{H}^3$ sends the edges of P to geodesics of \mathbb{H}^3 . So let $G_\sigma = \langle g \rangle$ and let $\gamma \subset M$ be the corresponding closed geodesic in M . We denote by $A_g \subset \mathbb{H}^3$ the axis of the element g and by g^+, g^- its fixed points on $\partial\mathbb{H}^3$. Let us assume that $f(\tilde{\sigma}) = g^+$. For $X \subset M$ we denote by $\text{diam}(X)$ the diameter of X in the hyperbolic metric of M .

Recall that the map $f : P \rightarrow \mathbb{H}^3 \cup \partial\mathbb{H}^3$ constructed in Lemma 3.1 induces the map $F : \Pi' \rightarrow M$. We start with the following:

Step 1. *For every $\eta > 0$ there exists a realization Λ of L in Π such that for the maximal tree T of Λ one has*

$$\text{diam}(F(T)) < \eta,$$

Furthermore, for every edge $y_i \in \Lambda \setminus T$ its image $F(y_i)$ is contained in a η -neighborhood $N_\eta(\gamma) \subset M$ of the geodesic γ ($i=1, \dots, k$).

Proof: We fix a sufficiently small neighborhood V of a vertex σ in Π (the "smallness" will be specified later on). Let $\tilde{\sigma} \in \nu^{-1}(\sigma)$ be its lift to P and let $\tilde{\Lambda}$ and \tilde{T} be the lifts of Λ and T to a neighborhood $\tilde{V} \subset \nu^{-1}(V)$ of $\tilde{\sigma}$. We are going first to show that, up to decreasing V , the image $f(\tilde{T})$ belongs to a sufficiently small horosphere in \mathbb{H}^3 centered at the point g^+ .

Let α be an edge of Π having σ as a vertex and $\tilde{\alpha}$ be its lift starting at a point $\tilde{\sigma}$. Then $a = f(\tilde{\alpha}) \subset \mathbb{H}^3$ is the geodesic ray ending at the point g^+ , let $a(t)$ be its parametrization. For a given t_0 we fix a horosphere S_{t_0} based at g^+ and passing through the point $a(t_0)$. Suppose there is a simplex in P having two edges $\tilde{\alpha} = [\tilde{\sigma}, s]$, $\tilde{\alpha}_1 = [\tilde{\sigma}, s_1]$ at the vertex $\tilde{\sigma}$ and an edge $[s, s_1]$ in Λ . The horosphere S_{t_0} is the level set of the Busemann function β_{g^+} based at the point g^+ . So for the geodesic rays $a = f(\tilde{\alpha})$ and $a_1 = f(\tilde{\alpha}_1)$ issuing from the point g^+ we have that the points $f(s) = a(t_0)$ and $f(s_1) = a_1(t_0)$ belong to the horosphere S_{t_0} . Proceeding in this way for all simplices whose edges share the vertex σ , we obtain that $f(\tilde{T}^{(0)}) \subset S_{t_0} \subset \mathbb{H}^3$. Since Λ is finite, so is the tree \tilde{T} . By choosing t_0 sufficiently large ($t_0 > \Delta$) we may assume that $d(\alpha_i(t_0), \alpha_j(t_0)) < \eta$ and $d(\alpha_i(t_0), A_g) < \eta$ ($i, j = 1, \dots, k$). We now connect all the vertices of

$f(\tilde{T})$ by geodesic segments $b_i \subset \mathbb{H}^3$. By convexity, and up to increasing the parameter t_0 , we also have $d(b_i, A_g) < \eta$.

By Lemma 3.1 the map f sends the lifts $\tilde{y}_i \in \tilde{T}$ of the edges $y_i \in \Lambda \setminus T$ simplicially to b_i ($i = 1, \dots, k$); and f maps G_σ -equivariantly the preimage $\tilde{\Lambda} = \nu^{-1}(\Lambda)$ to \mathbb{H}^3 . Hence the map f projects to the map $F : \Lambda \rightarrow M$ satisfying the claim of Step 1. \square

Step 2. *Definition of the polyhedron $\check{\Pi}$*

Using the initial orbihedron Π we will construct a new polyhedron $\check{\Pi}$ having the following properties :

- a) $\Pi^{(0)} = \check{\Pi}^{(0)}$ and $\Pi = \check{\Pi}$ outside of V ;
- b) $\pi_1(L^*) = G_\sigma$, where L^* is the link of σ in $\check{\Pi}$;
- c) $\pi_1(\check{\Pi}) \cong G$.

The graph Λ realizes the link of the vertex σ so there exists an epimorphism $\pi_1(\Lambda) \rightarrow \langle g \rangle$. Every edge $y_i \in \Lambda \setminus T$ which is a generator of the group $\pi_1 \Lambda$ is mapped onto $g^{n_{y_i}}$ in G_σ ($i = 1, \dots, k$). We now subdivide each edge y_i by edges y_{ij} ($i = 1, \dots, k, j = 1, \dots, n_{y_i}$), and denote by Λ' the obtained graph. Let S be a circle considered as a graph with one edge e and one vertex u . Then there exists a simplicial map from Λ' to S mapping simplicially each edge y_{ij} onto S .

To construct polyhedron $\check{\Pi}$, we replace the neighborhood V by the cone of the above map. Namely, we first delete the vertex σ from Π as well as all edges connecting σ with L . Then we connect the vertices of the edge y_{ij} with the vertex $u \in S$ by edges which we call *vertical* ($i = 1, \dots, k, j = 1, \dots, n_{y_i}$). So $\check{\Pi}$ is the union of $\Pi \setminus V$ and the rectangles R_{ij} , which are bounded by y_{ij} , two vertical edges and the loop S . The set of rectangles $\{R_{ij} \mid i = 1, \dots, k, j = 1, \dots, n_{y_i}\}$ realizes the epimorphism $\pi_1(L) \rightarrow G_\sigma$. By Van-Kampen theorem we have $\pi_1(\check{\Pi}) \cong G$, and the conditions a)-c) follow. \square

Step 3. *There exists a constant c (depending only on the topology of Π) such that for all $\eta > 0$, there exists a map $\check{F} : \check{\Pi} \rightarrow M$ such that*

1) \check{F} induces an isomorphism on the fundamental groups,

2) $\check{F}|_{\check{\Pi} \setminus V} = F$,

3) $\sum_{ij} \text{Area}(\check{F}(R_{ij})) < c \cdot \eta$. (2)

\square

Proof: We choose a neighborhood V of the singular point σ and put $\check{F} = F|_{\Pi \setminus V}$. Using Step 2 we transform the orbihedron Π to $\check{\Pi}$ in the neighborhood V and let P^\sim be the universal covering of $\check{\Pi}$. Note that, by construction, P^\sim is obtained by adding the G -orbit of the rectangles R_{ij} to the preimage $\check{\Lambda}' = \nu^{-1}(\Lambda')$ of the graph Λ' ($i = 1, \dots, k, j = 1, \dots, n_{y_i}$).

We will now extend the map f defined on $P \setminus V$ to the polyhedron $P^\sim \setminus P$ as follows. We first subdivide every segment b_i in n_{y_i} geodesic subsegments $b_{ij} \subset b_i$ corresponding to the edges y_{ij} . Then we project orthogonally each b_{ij} to A_g and let $\tilde{\gamma} \subset A_g$ denote its image. Let $\tau_{ij} \subset \mathbb{H}^3$ be the rectangle formed by $b_{ij}, \tilde{\gamma}$ and these two orthogonal segments from b_{ij} to A_g whose lengths are by Step 1 less than η . We extend the map f simplicially to a map \check{F} sending the rectangle $\nu^{-1}(R_{ij})$ to the rectangle τ_{ij} ($i = 1, \dots, k, j = 1, \dots, n_{y_i}$). Note that by construction the lift \check{S} of the circle S is mapped on $\tilde{\gamma}$. The map \check{F} descends to a map $\check{F} : \Pi_* \setminus \Pi \rightarrow N_\eta(\gamma)$. It induces the epimorphism $\pi_1 \check{\Pi} \rightarrow G$.

Let us now make the area estimates for the added rectangles τ_{ij} . Each rectangle $\tau = \tau_{ij}$ has four vertices A, B, C, D in \mathbb{H}^3 where $B = gA, D = g(C)$ and the segment $[A, B] \subset A_g$ is the orthogonal projection of $[C, D]$ on A_g . The rectangle τ is bounded by these two segments and two perpendicular segments $l_1 = [A, C]$ and $l_2 = [B, D]$ to the geodesic A_g ($l_2 = g(l_1)$). We have $\tau \subset ABC'D$ where $\angle BDC' = \frac{\pi}{2}$ and $\beta = \angle BC'D < \frac{\pi}{2}$. Then by [Be, Theorem 7.17.1] one has $\cos(\beta) \leq \sinh(d(B, D)) \cdot \sinh l(\gamma)$, where $l(\gamma)$ is the translation length of γ . Therefore $\text{Area}(\tau) < \frac{\pi}{2} - \beta$, and $\sin(\text{Area}(\tau)) \leq \sinh \eta \cdot \sinh l(\gamma)$. Summing up over all segments b_{ij} we arrive to the formula (2). This proves Case 1. \square

Case 2. The group G_σ is parabolic.

The proof is similar and even simpler in this case. Let again T be the maximal tree of the graph Λ realizing the link L of the vertex σ . We start by embedding a lift $\check{T}^{(0)}$ of the zero-skeleton of T^0 into a horosphere $S_{t_0} \subset \mathbb{H}^3$ based at the parabolic fixed point $p \in \partial \mathbb{H}^3$ of the group $G_\sigma = \langle g_1, g_2 \rangle \cong \mathbb{Z} + \mathbb{Z}$. Then, using Lemma 3.1, we construct an embedding $f : \check{\Lambda}^{(0)} \rightarrow S_{t_0}$ of the zero-skeleton of the graph $\check{\Lambda} = \nu^{-1}(\Lambda)$ into the same horosphere S_{t_0} invariant under G_σ (which was not so in the previous case). Since the number of vertices of \check{T} is finite, for any $\eta > 0$ we can choose a horosphere S_{t_0} ($t_0 > \Delta$) such that $\text{diam} \check{T} < \eta$. Fixing a point $O \in S_{t_0}$, we can also assume that $d(O, \check{T}^{(0)}) < \eta$.

Now, let us modify the orbihedron Π in the neighborhood V of σ . First we delete the vertex σ from Π and all edges connecting σ with the graph Λ . We then add to the obtained orbihedron a torus \mathcal{T} with two intersecting loops C_1 and C_2 representing the generators of $\pi_1(T, u)$ where $u \in C_1 \cap C_2$. To realize the epimorphism $\pi_1 \Lambda \rightarrow G_\sigma$ in M we proceed as before. For any edge $y \in \Lambda \setminus T$ corresponding to the element $g = ng_1 + mg_2$ in G_σ we add a rectangle R bounded by y , two edges connecting the end points of y with u and a loop $C \subset \mathcal{T}$ representing the element g in $\pi_1(T, u)$. Let $\check{\Pi}$ denote the obtained orbihedron.

Coming back to \mathbb{H}^3 , let us assume for simplicity that $p = \infty$ and the horosphere S_{t_0} is a

euclidean plane. By Lemma 3.1 the map f sends the edges $\tilde{y}_i \in \tilde{\Lambda} \setminus \tilde{T}$ to the geodesic edges b_i connecting the vertices of $f(\tilde{T})$.

We now construct the rectangles τ_i by projecting the end points of the edges b_i to the corresponding vertices of the Euclidean lattice given by the orbit $G_\sigma O$. Let us briefly describe this procedure in case of one rectangle τ . Suppose that the edge $y \in \Lambda \setminus T$ represents the element $g = ng_1 + mg_2 \in G_\sigma$. Let A and gA be vertices of $f(\tilde{T})$ belonging to S_{t_0} connected by a geodesic segment b corresponding to y . Let $\tau \subset \mathbb{H}^3$ be the geodesic bounded by the edges $b, l = [O, A], gl, gb$. We extend the map $\tilde{F} : \tilde{R} \rightarrow \tau$ where \tilde{R} is a lift of the corresponding rectangle R added to Π . The map \tilde{F} descends now to a simplicial map $\check{F} : \check{\Pi} \rightarrow M$ sending the torus \mathcal{T} into a cusp neighborhood of the manifold M . Since the rectangle τ belongs to η -neighborhood of the horosphere S_{t_0} , its area, being close to the Euclidean one, is bounded by $c \cdot \eta^2$ for some constant $c > 0$. Summing up over all edges y_i we obtain that the area of added rectangles does not exceed $k \cdot c \cdot \eta^2$. This proves Case 2. \square

To finish the proof of Proposition 3.3, we note that the initial orbihedron Π is finite, so it has a finite number of vertices v_1, \dots, v_l whose vertex groups are either loxodromic or parabolic. So for a fixed $\varepsilon > 0$, we apply the above simplicial "blow-up" procedure in a neighborhood of each vertex v_i ($i = 1, \dots, l$). Finally, we obtain a 2-complex Σ_ε ; and the simplicial map $\phi_\varepsilon : \Sigma_\varepsilon \rightarrow M$ which induces an isomorphism on the fundamental groups and such that $|\text{Area}(\phi_\varepsilon(\Sigma_\varepsilon)) - \text{Area}(f(\Pi'))| < \psi(\eta)$, where ψ is a continuous function such that $\lim_{\eta \rightarrow 0} \psi(\eta) = 0$. So for η sufficiently small we have $\psi(\eta) < \varepsilon$ which proves the Proposition. *QED.*

Proof of Theorem B. Let G be the fundamental group of a hyperbolic 3-manifold M of finite volume. Let $\Pi = P/G$ be a finite orbihedron realizing the invariant $T(G, E)$, i.e. $\pi_1^{\text{orb}}(\Pi) \cong G$, all vertex groups of Π are elementary and $|\Pi^{(2)}| = T(G, E)$. Hence $\text{Area}(F(\Pi')) = \pi \cdot T(G, E)$. Then by Proposition 3.3 for any $\varepsilon > 0$ there exists a 2-polyhedron Σ_ε and a map $\psi_\varepsilon : \Sigma_\varepsilon \rightarrow M$ which induces an isomorphism on the fundamental groups and such that

$$\text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi \cdot T(G, E) + \varepsilon$$

By the Cooper inequality (C) (see the Introduction) we have $\text{Vol}M < \text{Area}(\psi_\varepsilon(\Sigma_\varepsilon)) < \pi \cdot T(G, E) + \varepsilon$ ($\forall \varepsilon > 0$). It follows that $\text{Vol}M \leq \pi \cdot T(G, E)$. Theorem B is proved. *QED.*

4 Proof of Theorem A.

The goal of this Section is the proof of

Theorem A. *There exists a constant C such that for every hyperbolic 3-manifold M of finite volume the following inequality holds:*

$$T(G, E_\mu) \leq C \cdot \text{Vol}M \tag{*}$$

□

We start with the following Lemma dealing with n -dimensional hyperbolic manifolds :

Lemma 4.1. *Let M be a n -dimensional hyperbolic manifold of finite volume. Then there exists a 2-dimensional triangular complex W containing a 2-subcomplex L such that $\pi_1(W) \cong \pi_1 M_{\mu\text{thick}}$ and $\pi_1(L) \cong \pi_1(\partial M_{\mu\text{thick}})$. Furthermore we have*

$$|W^{(2)}| \leq \sigma \cdot \text{Vol}(M),$$

where $|W^{(2)}|$ is the number of 2-simplices of W and $\sigma = \sigma(\mu)$ is a constant depending only on μ .

Proof: The Lemma is a quite standard fact, proved for $n = 3$ in [Th] and more generally in [G], [BGLM], [Ge]. We provide a short proof of it for the sake of completeness. Consider a maximal set $\mathcal{A}_0 = \{a_i\}$ of points satisfying $a_i \in \partial M_{\mu\text{thick}}$ such that $d(a_i, a_j) > \mu/4$, where $d(\cdot, \cdot)$ is the hyperbolic distance of M restricted to $M_{\mu\text{thick}}$. We now complete the set \mathcal{A}_0 to the set \mathcal{A} which is a maximal set of points satisfying $\{a_i \in M_{\mu\text{thick}} \mid d(a_i, a_j) > \mu/4\}$. So $\mathcal{A} \cap \partial M_{\mu\text{thick}} = \mathcal{A}_0$.

By the triangle inequality we obtain

$$B(a_i, \mu/8) \cap B(a_j, \mu/8) = \emptyset \text{ if } i \neq j,$$

where $B(a_i, \mu/2)$ is an embedded ball in M (isometric to a ball in \mathbb{H}^n) centered at a_i of radius $\mu/2$. By the maximality of \mathcal{A}_0 and \mathcal{A} we have $M_{\mu\text{thick}} \subset \mathcal{U} = \bigcup_{a_i \in \mathcal{A}} B(a_i, \mu/4)$ and $\partial M_{\mu\text{thick}} \subset \mathcal{U}_0 =$

$$\bigcup_{a_i \in \mathcal{A}_0} B(a_i, \mu/4).$$

Recall that the nerve $N\mathcal{U}$ of the covering \mathcal{U} is constructed as follows. The vertex set $N\mathcal{U}^{(0)}$ is \mathcal{A} . The vertices $a_{i_1}, \dots, a_{i_{k+1}}$ span a k -simplex if for the corresponding balls we have

$\bigcap_{j=1}^{k+1} B(a_{i_j}, \mu/4) \neq \emptyset$. Since the covering \mathcal{U} is given by balls embedded in M , the intersection of

every finite subset of them is simply connected. So the nerve $N\mathcal{U}$ is homotopy equivalent to \mathcal{U} [Hat, Corollary 4G.3].

Note that $M_{\mu\text{thick}} \hookrightarrow \mathcal{U} \hookrightarrow M_{\frac{\mu}{2}\text{thick}}$. Indeed if $x \in \partial B(a_i, \mu/4)$ then by the triangle inequality we have $B(x, \mu/4) \subset B(a_i, \mu/2)$, and so both balls are embedded in M . Then $x \in M_{\frac{\mu}{2}\text{thick}}$. By the Margulis lemma the embedding $M_{\mu\text{thick}} \hookrightarrow M_{\frac{\mu}{2}\text{thick}}$ is a homotopy equivalence. It implies that \mathcal{U} , and so $N\mathcal{U}$ is homotopy equivalent to $M_{\mu\text{thick}}$. Let W denote the 2-skeleton of $N\mathcal{U}$. Then it is a standard topology fact that W carries the fundamental group of $N\mathcal{U}$ [Hat]. Therefore, $\pi_1 W \cong \pi_1 M_{\mu\text{thick}}$.

Let now L be the 2-skeleton of $N\mathcal{U}_0$. Since $N\mathcal{U}_0$ is a subcomplex of $N\mathcal{U}$ we have that L is a subcomplex of W . Similarly we first show the set $V = \partial M_{\mu\text{thick}} \cap (\bigcap_{i \in I} B(a_i, \mu/4))$ is simply

connected for every finite subset $\{a_i \in \mathcal{A}_0 \mid i \in I\}$. Suppose by contradiction that $\alpha \subset V$ is a non-trivial loop. The embedding $\pi_1(\partial M_{\mu\text{thick}}) \hookrightarrow \pi_1 M_{\mu\text{thick}}$ is injective so the loop α is not trivial in $\pi_1 M_{\mu\text{thick}}$ either. Since $M_{\mu\text{thick}} \hookrightarrow M_{\frac{\mu}{2}\text{thick}}$ is a homotopy equivalence α is not trivial in $M_{\frac{\mu}{2}\text{thick}}$ and so in $\mathcal{U} \subset M_{\frac{\mu}{2}\text{thick}}$. The latter one is impossible as the intersection of every finite subset of balls of \mathcal{U} is simply connected. By [Hat, Corollary 4G.3] the nerve $N\mathcal{U}_0$ is homotopy equivalent to \mathcal{U}_0 . If now $y \in \mathcal{U}_0$ then $d(y, \partial M_{\mu\text{thick}}) \leq \mu/4$ so there exists $y_0 \notin M_{\mu\text{thick}}$ such that $d(y, y_0) \leq \mu/2$. Then since $B(y_0, \mu/2) \subset B(y, \mu)$ we must have that $y \notin M_{2\mu\text{thick}}$. Therefore \mathcal{U}_0 belongs to a regular neighborhood of $\partial M_{\mu\text{thick}}$ of radius $\mu/4$ embedded into $M_{\frac{\mu}{2}\text{thick}} \setminus M_{2\mu\text{thick}}$. Since $M_{\frac{\mu}{2}\text{thick}}$ and $M_{2\mu\text{thick}}$ are homotopy equivalent it follows that \mathcal{U}_0 is homotopy equivalent to $\partial M_{\mu\text{thick}}$. Then $\pi_1 L \cong \pi_1(N\mathcal{U}_0) \cong \pi_1\mathcal{U}_0$ and so $\pi_1 L \cong \pi_1(\partial M_{\mu\text{thick}})$.

It remains to count the number of 2-faces of W . We have for the cardinality $|\mathcal{A}|$ of the set \mathcal{A} :

$$|\mathcal{A}| \leq \frac{\text{Vol}(M_{\mu\text{thick}})}{\text{Vol}(B(\mu/8))} \leq \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))} ,$$

where $B(\mu)$ denotes a ball of radius μ in the hyperbolic space \mathbb{H}^n . The number of faces of W containing a point of \mathcal{A} as a vertex is at most $m = \frac{\text{Vol}(B(\mu/2))}{\text{Vol}(B(\mu/8))}$. Then

$$|W^{(2)}| \leq C_m^2 \frac{\text{Vol}(M)}{\text{Vol}(B(\mu/8))} = \sigma \cdot \text{Vol}(M) ,$$

where $\sigma = \sigma(\mu) = \frac{C_m^2}{\text{Vol}(B(\mu/8))}$. This completes the proof of the Lemma. \square

Suppose now that M is a compact 3-manifold whose interior admits a hyperbolic structure of finite volume. Let $\mu = \mu(3)$ denote the 3-dimensional Margulis constant.

We are going to apply a result of [De] which uses a bit different definition of the relative invariant. For the reader convenience we recall it:

Definition 4.2. [De] *Let \mathcal{E} be a finite system of subgroups of a finitely generated group G . We say that $T_0(G, \mathcal{E}) \leq t$ if G acts simplicially on a simply connected polyhedron P such that $|P^{(2)}/G| \leq t$; every element of \mathcal{E} fixes a vertex in P ; and every vertex stabilizer of P is conjugate to an element of \mathcal{E} .*

Remark 4.3. *The invariant $T_0(G, \mathcal{E})$ is a partial case of the invariant $T(G, E)$ introduced in 1.2. The latter one does not require that every element of a system E fixes a vertex of P and there is no any restriction on the finiteness of the conjugacy classes of the vertex stabilizers for the action $G \curvearrowright P$. We use the notations $T_0(G, \mathcal{E})$ and $T(G, E)$ for these two invariants. Notice that nothing changes for the absolute invariant $T(G)$.*

Let l_1, \dots, l_k be the set of closed geodesics in M of length less than μ . Then by [Ko] the manifold $M' = M \setminus \bigcup_i^k l_i$ is a complete hyperbolic manifold of finite volume and $\pi_1 M_{\mu\text{thick}} \cong \pi_1 M'$.

Denote by \mathcal{E}_μ the set of the fundamental groups of the components of $M_{\mu\text{thin}}$. Denote also by \mathcal{E}'_μ the system $\pi_1(\partial M_{\mu\text{thick}})$ of the fundamental groups of the boundary components of the thick part $M_{\mu\text{thick}}$. We have the following :

Lemma 4.4.

$$T_0(\pi_1 M, \mathcal{E}_\mu) \leq T_0(\pi_1 M', \mathcal{E}'_\mu) \leq T_0(\pi_1 M, \mathcal{E}_\mu) + 2k. \quad (4)$$

Proof: 1) Consider first the left inequality. Let $G = \pi_1 M$ and $G' = \pi_1(M')$. Let us fix a two-dimensional (G', \mathcal{E}'_μ) -orbihedron Π' containing $T_0(G', \mathcal{E}'_\mu)$ triangular 2-faces. Its orbihedral universal covering is a simply connected 2-polyhedron P' which admits a simplicial action by G' whose set of conjugacy classes of the vertex stabilizers is \mathcal{E}'_μ [H]. Let $N(l_i)$ be a regular neighborhood of the geodesic l_i in M ($i = 1, \dots, k$). Denote by $H_i = \langle \alpha_i, \beta_i \rangle$ the fundamental group of the torus $T_i = \partial N(l_i)$ where α_i is freely homotopic to l_i in $N(l_i)$. By assumption H_i fixes a point $x_i \in P'$. The group G is the quotient of G' by adding the relations $\beta_i = 1$ ($i = 1, \dots, k$). Denote by $p : G' \rightarrow G$ the epimorphism whose kernel is the subgroup H of G' normally generated by β_i ($i = 1, \dots, k$). Let P be the underlying space of the orbihedron P'/H , and π denote the natural projection $P' \rightarrow P'/H$. Since the stabilizers of the points $\pi(x_i)$ ($i = 1, \dots, k$) are trivial in G , the group G acts on the polyhedron P and the action is simplicial.

To prove the simple connectedness of P is enough to show that every loop $v = \pi(\beta_i)$ is trivial in P ($i = 1, \dots, k$). Let $v' \subset P'$ be a lift of v whose endpoints are y' and $\beta_i(y')$ ($y' \in P'$). Denote by s' a path in P' connecting the points x_i and y' . Hence $u' = s'v'\beta_i(s')$ is a loop in P' as $\beta_i(x_i) = x_i$ (where \bar{t} denotes the inverse path for a path t). Since P' is simply connected the loop u' is trivial in P' , so $u = \pi(u')$ is trivial in P . We have $u = sv\bar{s}$ where $s = \pi(s')$. Therefore v is a trivial loop in P too. So P is simply connected.

Let us now show that every vertex stabilizer for the action $G \curvearrowright P$ is conjugate into \mathcal{E}_μ . The system \mathcal{E}_μ is the image of \mathcal{E}'_μ under the equivariant epimorphism p . Thus every element in \mathcal{E}_μ fixes a point in P . Furthermore if $g(y) = y$ for $g \in G$ and $y \in P$ then there exists $y' \in P'$ and $g' \in G'$ such that $\pi(y') = y$, $\pi(g'(y')) = g(y)$ and $p(g') = g$. Therefore $g'(y') = h(y')$ for some $h \in H$. It follows that $h^{-1}g'$ is conjugate into some element of \mathcal{E}'_μ , and hence g is conjugate in G into some element of \mathcal{E}_μ .

Since the 2-faces of P are images by π of 2-faces of P' we have : $|\Pi^{(2)} = P/G| \leq |\Pi'^{(2)} = P'/G'|$. Thus $T_0(\pi_1 M, \mathcal{E}_\mu) \leq T_0(\pi_1 M', \mathcal{E}'_\mu)$.

2) Let Π be the 2-orbihedron which realizes $T_0(\pi_1 M, \mathcal{E}_\mu)$, and let P be its universal cover. To obtain a $(\pi_1 M', \mathcal{E}'_\mu)$ -orbihedron we modify P as follows. Let $H_i = \langle h_i \rangle$ be the loxodromic subgroup corresponding to the geodesic $l_i \subset M$ of length less than μ ($i = 1, \dots, k$). Let $x_i \in P$

be a vertex fixed by the subgroup H_i . Notice that the group G' is generated by G and elements β_i such that $[h_i, \beta_i] = 1$ ($i = 1, \dots, k$). So we add to Π a new loop β_i (by identifying it with the corresponding element in G) and glue a disk whose boundary is the loop corresponding to $[h_i, \beta_i]$. By triangulating each such a disk we add $2k$ new triangles to $\Pi^{(2)}$. Thus the universal cover P' is obtained by adding to P a vertex y_i and its orbit $\{Gy_i\}$, so that the points $\beta_i h_i g y_i$ are identified with $h_i \beta_i g y_i$. We further add the rectangle gD_i ($g \in G$) whose vertices are $h_i g y_i, \beta_i h_i g y_i, \beta_i g y_i, g y_i$ and subdivide it by one of the diagonal edges, say $(h_i g y_i, \beta_i g y_i)$ ($i = 1, \dots, k$). The construction gives a new 2-complex P' on which the pair (G', \mathcal{E}'_μ) acts simplicially. We claim that P' is simply connected. Indeed if α is a loop on it, since P is simply connected, α is homotopic to a product of loops belonging to the disks gD_i so α is a trivial loop. Since the 2-orbihedron $\Pi' = P'/G'$ contains $|\Pi^{(2)}| + 2k$ faces, we obtain $T_0(\pi_1 M', \mathcal{E}'_\mu) \leq T_0(\pi_1 M, \mathcal{E}_\mu) + 2k$ which was promised. *QED.*

Remark 4.5. *It is worth pointing out that in the context of volumes of hyperbolic 3-manifolds the following inequality (similar to (5)) is known:*

$$\text{Vol}M < \text{Vol}(M') < k \cdot (C_1(R) \cdot \text{Vol}(M) + C_2(R)), \quad (\dagger)$$

where R is the maximum of radii of the embedded tubes around the short geodesics l_i ($i = 1, \dots, k$) and $C_i(R)$ are functions of R ($i = 1, 2$). The left part of (\dagger) is classical and due to W. Thurston [Th], the right one is recently proved by I. Agol, P. A. Storm, and W. Thurston [AST] \square

Proof of Theorem A: By Lemma 4.1 there exists a 2-dimensional complex W containing a 2-dimensional subcomplex L such that $\pi_1 W \cong \pi_1 M_{\mu\text{thick}}$, $\pi_1 L \cong \pi_1(\partial M_{\mu\text{thick}})$ and $|W^{(2)}| < \sigma \cdot \text{Vol}M$ for some uniform constant σ . Consider now the double $N = DM_{\mu\text{thick}}$ of the manifold $M_{\mu\text{thick}}$ along the boundary $\partial M_{\mu\text{thick}}$, and the double V of W along L . Then by Van-Kampen theorem $\pi_1 V \cong \pi_1 N$. So $T(\pi_1 N) \leq |V^{(2)}| < 2\sigma \cdot \text{Vol}M$. The group $\pi_1 N$ splits as the graph of groups whose two vertex groups are $\pi_1 M_{\mu\text{thick}}$. The edge groups of the graph of groups are given by the system \mathcal{E}_μ . As $\pi_1 M_{\mu\text{thick}} \cong \pi_1 M'$ and M' is a complete hyperbolic 3-manifold of finite volume it follows from Lemma 2.3 that the above splitting is reduced and rigid for the pair $(\pi_1 N, \{\text{id}\})$. So by [De, Theorem II.1)] we have:

$$T(\pi_1 N) \geq 2T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}'_\mu). \quad (5)$$

Then by Lemma 4.4 $T_0(\pi_1 M_{\mu\text{thick}}, \mathcal{E}'_\mu) \geq T_0(\pi_1 M, \mathcal{E}_\mu)$, and therefore

$$T_0(\pi_1 M, \mathcal{E}_\mu) < \sigma \cdot \text{Vol}M.$$

Recall that the initial system E_μ includes all elementary subgroups of $\pi_1 M$ whose translation length is less than μ . The set \mathcal{E}_μ is the set of all conjugacy classes of the subgroups of E_μ . So $T(\pi_1 M, E_\mu) \leq T_0(\pi_1 M, \mathcal{E}_\mu)$. We finally obtain

$$T(\pi_1 M, E_\mu) < C \cdot \text{Vol}M,$$

where $C = \sigma$. Theorem A is proved. □

5 Concluding remarks and questions.

The finiteness theorem of Wang affirms that there are only finitely many hyperbolic manifolds of dimension greater than 3 having the volume bounded by a fixed constant [W]. So it is natural to compare the volume of a hyperbolic manifold $M = \mathbb{H}^n/\Gamma$ with the absolute invariant $T(\Gamma)$. In the case $n > 3$ the inequality

$$\text{const} \cdot T(\Gamma) \leq \text{Vol}M$$

follows from [Ge, Thm 1.7] (see also Section 2 above, where instead of $T(\pi_1 M, E)$ one needs to consider $T(\pi_1 M)$ and use the fact that $\pi_1 M_{\mu\text{thick}} \cong \pi_1 M$). However, the result [C] is not known in higher dimensions. Thus we have the following :

Question 5.1. *Is there a constant C_n such that for every lattice Γ in $\text{Isom}(\mathbb{H}^n)$ one has*

$$\text{Vol}(\Gamma) \leq C_n \cdot T(\Gamma) ?$$

Remark 5.2. *(M. Gromov, T. Januszkiewicz) The answer is positive if M is a compact hyperbolic manifold of dimensions 4 and 6. Indeed if first $\dim M = 4$ then by the Gauss-Bonnet formula one has $\text{Vol}M = \frac{\Omega_4}{2} \cdot \chi(M)$, where Ω_4 is the volume of the standard unit 4-sphere. Hence $\text{Vol}M = \frac{\Omega_4}{2} \cdot (2 - 2b_1 + b_2)$ where $b_i = \text{rank}(H_i(M, \mathbb{Z}))$ is the i -th Betti number of M ($i = 1, 2$). Since $b_2 < T(\pi_1 M)$, one has $\text{Vol}M < \frac{\Omega_4}{2} \cdot (2 + b_2) < \Omega_4 \cdot T(\pi_1 M)$ (as $T(\pi_1 M) > 1$).*

The same argument also works in dimension 6 where one obtains $\text{Vol}M = \frac{\Omega_6}{2} \cdot (2 - 2b_1 + 2b_2 - b_3) < \frac{\Omega_6}{2} (2 + 2b_2) \leq \Omega_6 \cdot T(\pi_1 M)$. □

Recently it was shown by D. Gabai, R. Meyerhoff, and P. Milley that the Matveev-Weeks 3-manifold M_0 is the unique closed 3-manifold of the smallest volume [GMM]. Furthermore, C. Cao and R. Meyerhoff found cusped 3-manifolds $m003$ and $m004$ of the smallest volume [CM], [GMM]. In this context we have the following :

Question 5.3. *Is the minimum of the invariant $T(\pi_1 M, E_\mu)$ on the set of compact hyperbolic 3-manifolds attained on the manifold M_0 ? Is the minimal relative invariant $T(\pi_1 M, E_\mu)$ on the set of cusped finite volume 3-manifolds attained on the manifolds $m003$ and $m004$?*

References

- [AST] I. Agol, P. A. Storm, and W. Thurston, *Lower bounds on volumes of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 20 (2007), no. 4, 1053-1077.
- [Be] I.-Belegradek, *On Mostow rigidity for variable negative curvature*, Topology 41 (2002), no. 2, 341-361.
- [BGHM] M.R. Bridson, D. Groves, J.A. Hillman, G.J. Martin, *Cofinitely Hopfian groups, open mappings and knot complements*, Groups, Geometry and Dynamics, 4 (2010), 693-707.
- [BGLM] M. Burger, T. Gelander, A. Lubotzky, S. Mozes, *Counting hyperbolic manifolds*, GAFA, 12 (2002), no. 6, 1161-1173.
- [C] D. Cooper, *The volume of a closed hyperbolic 3-manifold is bounded by π times the length of any presentation of its fundamental group*, Proc. Amer. Math. Soc. 127 (1999), no. 3, 941-942.
- [CM] C. Cao, R. Meyerhoff, *The Orientable Cusped Hyperbolic 3-Manifolds of Minimum Volume*, Inventiones Math. 146 (2001) 451-478.
- [De] T. Delzant, *Décomposition d'un groupe en produit libre ou somme amalgamée*, J. Reine Angew. Math. 470 (1996), 153-180.
- [EP] D. B. A. Epstein, R. C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*, J. Differential Geom. 27 (1988), no. 1, 67-80.
- [GMM] D. Gabai, R. Meyerhoff, P. Milley, *Minimum volume cusped hyperbolic three-manifolds*, J. Amer. Math. Soc. 22 (2009), no. 4, 1157-1215.
- [Ge] T. Gelander, *Homotopy type and volume of locally symmetric manifolds*, Duke Math. J. 124 (2004), no. 3, 459-515.
- [G] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Birkhäuser Boston, 1999.
- [Gr] I. Grushko, *On the generators of a free product of groups*, Matem. Sbornik, N.S. 8(1940), 169-182.
- [H] A. Haefliger. *Complex of groups and orbihedra*. in *Group theory from a geometric point of view*, E. Ghys, A. Haefliger A. Verjovski Ed, World Scientific, 1991.
- [Hat] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

- [He] J. Hempel, *3-Manifolds*, Annals of Mathematics Studies, Vol. 86, Princeton University Press, Princeton, NJ, 1976.
- [J] W. Jaco, *Lectures on three-manifold topology*, CBMS Conference. 43, American Mathematical Society, Providence, 1980.
- [K] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, J. B. Deutsch. Math. Verein, vol. 38 (1929), p. 248-260.
- [Ko] S. Kojima, *Isometry transformations of hyperbolic 3-manifolds*, Topology and its Applications, 29 (1988), 297-307.
- [Ma] S. Matveev, *Algorithmic topology and classification of 3-manifolds*, Springer ACM-monographs, 2003, v. 9.
- [Pe1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, Preprint, 2002.
- [Pe2] G. Perelman, *Ricci flow with surgery on three-manifolds*, Preprint, 2003.
- [Pe3] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, Preprint, 2003.
- [PP] E. Pervova, C. Petronio, *Complexity and T -invariant of Abelian and Milnor groups, and complexity of 3-manifolds*, Math. Nachr. 281 (2008), no. 8, 1182-1195.
- [R] A. Reznikov, *Volumes of discrete groups and topological complexity of homology spheres*, Math. Annalen, 306(1996), 547-554,
- [Se] J.P. Serre, *Arbres, Amalgames, SL_2* , Astérisque no. 46, Soc.Math. France, 1977.
- [SW] P. Scott, T. Wall, *Topological methods in group theory. Homological group theory*, Proc. Sympos., Durham, 1977, pp. 137–203, London Math. Soc. Lecture Note Ser., 36, Cambridge Univ. Press, Cambridge-New York, 1979.
- [Swa] G. Swarup, *Two finiteness properties in 3-manifolds*, Bull. London Math. Soc. 12 (1980), no. 4, 296-302.
- [Th] W. Thurston, *Geometry and topology on 3-manifolds*, Preprint, Princeton University, 1978.
- [W] H.C. Wang, *Topics on totally discontinuous groups*, In "Symmetric Spaces" (W. Boothby, G. Weiss, eds.), M. Dekker (1972), 460-487.

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