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A 3D BOUNDARY OPTIMAL CONTROL FOR THE BIDOMAIN-BATH SYSTEM MODELING THE THORACIC SHOCK THERAPY FOR CARDIAC DEFIBRILLATION

MOSTAFA BENDAHDANE, NAGAI AH CHAMAKURI, ELOÏSE COMTE, AND BEDR'EDDINE AÏNSEBA

ABSTRACT. This work is dedicated to study the cardiac defibrillation problem by using an optimal thoracic electroshock treatment. The problem is formulated as an optimal control problem in a 3D domain surrounded by the bath and including the heart. The control corresponds to the thoracic electroshock and the model describing the electrical activity in the heart is the bidomain model. The bidomain model is coupled with the quasi-static Maxwell's equation to consider the effect of an external bathing medium. The existence and uniqueness of a weak solution for the direct problem is assessed as well as the existence of a weak solution for the adjoint problem. The numerical discretization is realized using a finite element method for the spatial discretization and linearly implicit Runge-Kutta methods for the temporal discretization of the partial differential equations. The numerical results are demonstrated for the termination of re-entry waves.

1. INTRODUCTION

One of the leading causes of death over the world are cardiovascular diseases. Knowledge and understanding of the electrical heart activity are important issues in order to establish new diagnostic techniques. In this work we would like to use control techniques by mean of high external stimulations over the body thorax in order to steer the heart electrical activity to a given level. This kind of treatment is used in cardiology to reset some arrhythmia disease (for more details see e.g. [32] and [33]). In literature, numerous investigations focused on electrophysiologically important issues such as the formation of reentrant arrhythmias such as spiral waves [17] and their degeneration into fibrillation [28], or the termination of turbulent electrical activity in the heart by applying strong electric fields (defibrillation) [4], the only known therapy to terminate otherwise lethal ventricular fibrillation.

In order to model the electrical heart activity, we distinguish the geometry of the bath and the electrical activation in the myocardium which is based on the bidomain model. This model was introduced by Tung [29] and this is widely considered to be among the most complete descriptions of bioelectric activity at the tissue and organ level ([30, 8]). The electrical heart activity and the volume conductor for the bath model is

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given by the following system:

$$\left\{ \begin{array}{ll}
 \beta c_m \partial_t u - \nabla \cdot (\mathbf{M}_i(x) \nabla u_i) + \beta I_{\text{ion}}(u, w) = I_i, & (t, x) \in \Omega_{T,H} := (0, T) \times \Omega_H, \\
 \beta c_m \partial_t u + \nabla \cdot (\mathbf{M}_e(x) \nabla u_e) + \beta I_{\text{ion}}(u, w) = I_e, & (t, x) \in \Omega_{T,H}, \\
 -\nabla \cdot (\mathbf{M}_s(x) \nabla u_s) = 0, & (t, x) \in \Omega_{T,B} := (0, T) \times \Omega_B, \\
 \partial_t w - H(u, w) = 0, & (t, x) \in \Omega_{T,H}, \\
 \\
 (\mathbf{M}_i(x) \nabla u_i) \cdot \eta = 0, & (t, x) \in \Sigma_{T,H} := (0, T) \times \Sigma_H, \\
 (\mathbf{M}_e(x) \nabla u_e) \cdot \eta = (\mathbf{M}_s(x) \nabla u_s) \cdot \eta, & (t, x) \in \Sigma_{T,H}, \\
 u_e = u_s, & (t, x) \in \Sigma_{T,H}, \\
 (\mathbf{M}_s(x) \nabla u_s) \cdot \eta_s = 0, & (t, x) \in \Sigma_{T,B} \setminus \Sigma_{T,H}, \\
 \\
 u(0, x) = u_0(x), & x \in \Omega_H, \\
 w(0, x) = w_0(x), & x \in \Omega_H.
 \end{array} \right. \quad (1.1)$$

The heart's spatial domain is represented by $\Omega_H \subset \mathbb{R}^3$ which is a bounded open subset, and by Σ_H we denote its piecewise smooth boundary. A distinction is made between the intracellular and extracellular tissues which are separated by the cardiac cellular membrane. The thorax is modeled by a volume conduction Ω_B with a piecewise smooth boundary $\Sigma_B := \Sigma_H \cup \Sigma$, where Σ is the thorax surface. The whole domain is denoted by the $\Omega = \Omega_H \cup \Omega_B$, we refer Figure 1 for the pictorial representation of such computational domain and other sub domains. For all $(x, t) \in \Omega_{T,H} := \Omega_H \times (0, T)$, $u_i = u_i(x, t)$, $u_e = u_e(x, t)$ stand for the intracellular and extracellular potentials respectively, and for all $(x, t) \in \Omega_{T,B} := \Omega_B \times (0, T)$, $u_s(x, t)$ stands for the bathing medium electric potential. The transmembrane potential is the difference $u = u(x, t) := u_i - u_e$. $\mathbf{M}_i(x)$ and $\mathbf{M}_j(x)$ are tensors which represent respectively the intracellular and extracellular conductivity of the tissue respectively. The diagonal matrix \mathbf{M}_s represents the conductivity tensor of the bathing medium.

The constant $c_m > 0$ is the surface capacitance of the membrane and β is the surface-to-volume ratio.

We denote by I_i and I_e the internal and the external current stimulus respectively. Moreover, $H(u, w)$ and $I_{\text{ion}}(u, w)$ are functions which correspond to the widely known FitzHugh-Nagumo model for the membrane and ionic currents (see for e.g. [14, 22]). For detailed exposition of the such several ionic models we refer to [15] and [27]. Recalling the definition of $H(u, w)$ and $I_{\text{ion}}(u, w)$, we know from [14] and [22] that the membrane kinetics can be simply reformulated by:

$$H(u, w) = au - bw, \quad (1.2)$$

$$I_{\text{ion}}(u, w) = -\lambda(w - u(1 - u)(u - \theta)), \quad (1.3)$$

where a, b, λ, θ are given parameters. Moreover, we impose the following zero mean condition for the extracellular potential in order to obtain uniqueness of the elliptic systems.

$$\int_{\Omega_H} u_e(t, x) \, dx = 0, \quad \text{for all } t \in (0, T). \quad (1.4)$$

Concerning the bidomain equations, we use also the principle of conservation of current between the intra- and extra-cellular domains in the same way of [7]. Because of the low-frequency response of human tissue, we suppose (as in [1]) that the potential on the bath is modeled by Laplace equation according to the theory of the Quasi-static Maxwell's equations but reconstructions using the bidomain equations are considerably better than those obtained by quasi-static heart model. For an isolated heart (with no coupling to a surrounding bath), the well-posedness results for a class of degenerate reaction-diffusion systems in electrocardiology has been proved in [7]. The existence of numerical solutions has been proved in [2] by using the finite volume method as well as its convergence to a weak solution. It is important to mention that the numerical approximation for solving a 2D bidomain-bath model (tridomain model) can also be done by the finite volume method, as in [1]. The heart activity can be measured using the electrocardiogram (ECG) which is a non-invasive method: electrodes, attached to the surface of the thorax, are recorded by

an external device. They detect the electrical activity of the heart over a period of time (see [20] for more details). With these measures, heart diseases can be diagnosed, in particular cardiac rhythm abnormality. The study of such problem has been the subject of a lot of interest and has received several contributions for many years. In electrocardiography, direct and inverse problems reduce to a quasi-static Poisson's equation with different boundary conditions (see for e.g. [30], [31], [19], [25]), where the reconstruction of the solution on the heart is formulated by a linear problem.

In cardiac electrophysiology, an optimal control problem constrained by the 2D tridomain equations has been investigated and the well-posedness of solution for this model and the related control problem has been proved in [1]. PDE constrained optimization techniques have been applied, for the first time, to the monodomain and bidomain model to predict optimized shock waveforms in 2D [11, 12] and more recently for the optimal control of bidomain-bath model using Mitchell-Shaefter model in 3D geometries [13, 10]. The present work, devoted to the rigorous study of mathematical analysis of such complex bidomain-bath model equations and numerical realization combined with Fitz-Hugh Nagumo ionic model in realistic 3D geometries. Our study of the problem of control (1.5) below, is new even if we utilize the method (for direct problem) in [7] for the well-posedness of the weak solution (recall that the model in [7] is for an isolated heart). Moreover, we prove here the existence of the weak solution to adjoint problem (see Section 4 below).

We consider a bidomain model coupled with quasi-static Maxwell's equations, adding a 3D control on the bathing boundary which is represented by a flow in a dynamic problem. This is formulated by the following model :

$$\left\{ \begin{array}{ll} \beta c_m \partial_t u - \nabla \cdot (\mathbf{M}_i(x) \nabla u_i) + \beta I_{\text{ion}}(u, w) = I_i, & (t, x) \in \Omega_{T,H} := (0, T) \times \Omega_H, \\ \beta c_m \partial_t u + \nabla \cdot (\mathbf{M}_e(x) \nabla u_e) + \beta I_{\text{ion}}(u, w) = I_e, & (t, x) \in \Omega_{T,H}, \\ \partial_t w - H(u, w) = 0 & (t, x) \in \Omega_{T,H} \\ -\nabla \cdot (\mathbf{M}_s(x) \nabla u_s) = 0, & (t, x) \in \Omega_{T,B} := (0, T) \times \Omega_B, \\ \\ (\mathbf{M}_i(x) \nabla u_i) \cdot \eta = 0, & (t, x) \in \Sigma_{T,H} := (0, T) \times \Sigma_H, \\ (\mathbf{M}_e(x) \nabla u_e) \cdot \eta = (\mathbf{M}_s(x) \nabla u_s) \cdot \eta, & (t, x) \in \Sigma_{T,H}, \\ u_e = u_s, & (t, x) \in \Sigma_{T,H}, \\ (\mathbf{M}_s(x) \nabla u_s) \cdot \eta_s = 0, & (t, x) \in \Sigma_{1,T} := (\Sigma \times (0, T)) \setminus \Sigma_{2,T}, \\ (\mathbf{M}_s(x) \nabla u_s) \cdot \eta_s = v, & (t, x) \in \Sigma_{2,T} := \Gamma \times (0, T), \\ \\ u(0, x) = u_0(x), & x \in \Omega_H, \\ w(0, x) = w_0(x), & x \in \Omega_H, \end{array} \right. \quad (1.5)$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is a part of the thorax surface, more precisely the site where the stimulus is applied. The boundary surfaces Γ_1 and Γ_2 act as anode and cathode respectively, see Figure 1 for pictorial representation. Herein, we denote by v the stimulus or the control that acts on Γ .

The structure of the paper is organized as follows : In Section 2, we will introduce the direct problem in electrocardiography and present the main results. The existence and uniqueness of the weak solution will be proved in Section 3. To prove this, we use the similar approach developed in [7] for an isolated heart: we introduce a regularized problem in order to avoid the degeneration of the solution. Thanks to this non degenerated problem and by a classical compactness method, we prove the existence for the degenerated problem. Furthermore, by an other Theorem, we prove also the uniqueness of this weak solution. Section 4 will be devoted to the optimal control. We introduce a functional useful for minimize, prove the existence of the control and obtain the existence of weak solution to adjoint problem. Moreover, the derivation of dual problem and the optimality conditions are also discussed. Finally, we demonstrate the numerical realization of the successful termination of the reentry waves in Section 6.

2. PRELIMINARIES AND WELL-POSEDNESS OF THE DIRECT PROBLEM

Before studying our problem, we make some assumptions. We assume $\mathbf{M}_j = \mathbf{M}_j(x) : \Omega_H \rightarrow \mathbb{R}$, $j = i, e$, and $\mathbf{M}_s = \mathbf{M}_s(x) : \Omega_B \rightarrow \mathbb{R}$, are C^1 functions and satisfy

$$\mathbf{M}_j(x_1)(\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) \geq C_M |\xi_1 - \xi_2|^2, \quad \mathbf{M}_s(x_2)(\xi_1 - \xi_2) \cdot (\xi_1 - \xi_2) \geq C_M |\xi_1 - \xi_2|^2, \quad (2.1)$$

for a.e. $x_1 \in \Omega_H$ and $x_2 \in \Omega_B$, $\forall \xi_1, \xi_2 \in \mathbb{R}^3$, and with C_M being a positive constant.

We note that the ionic current $I_{\text{ion}}(u, w)$ can be decomposed into $I_{\text{ion}}(u, w) = I_{1,\text{ion}}(u) + I_{2,\text{ion}}(w)$ where $I_{1,\text{ion}}(u) = \lambda u(1-u)(u-\theta)$ and $I_{2,\text{ion}}(w) = -\lambda w$.

We assume there exist constants $C_I, C'_I > 0$ such that :

$$\begin{cases} \frac{I_{1,\text{ion}}(u_1) - I_{2,\text{ion}}(u_2)}{u_1 - u_2} \geq -C_I, \\ 0 \leq \liminf_{u \rightarrow +\infty} \left| \frac{I_{1,\text{ion}}(u)}{u^3} \right| \leq \limsup_{u \rightarrow +\infty} \left| \frac{I_{1,\text{ion}}(u)}{u^3} \right| < C'_I, \end{cases} \quad (2.2)$$

for all $u_1, u_2, u \in \mathbb{R}$. Observe that the consequence of (2.2)

$$\begin{cases} C_1 |u|^3 < |I_{1,\text{ion}}(u)| < C_2 (|u|^3 + 1), \\ I_{1,\text{ion}}(u)u \geq -C_I |u|^2, \\ I_{1,\text{ion},u}(u) \geq -C_I, \end{cases} \quad (2.3)$$

for all $u \in \mathbb{R}$ and for some constants $C_1, C_2 > 0$, where $I_{1,\text{ion},u}$ is the derivative of $I_{1,\text{ion}}$ with respect to u .

Next we will use the following spaces. By $H^m(\Omega)$, we denote the usual Sobolev space of order m . Since the electrical potentials u_i and u_e are defined up to an additive constant, we use the quotient space $\tilde{H}^1(\Omega_H) = H^1(\Omega_H)/\{u \in H^1(\Omega), u \equiv \text{Const}\}$. Given $T > 0$ and $1 \leq p \leq \infty$, $L^p(0, T; \mathbb{R})$ denotes the space of L^p integrable functions from the interval $[0, T]$ into \mathbb{R} .

Now we recall the Aubin-Lions compactness result (see, e.g., [5, 18, 26]). Let X be a Banach space, and let X_0, X_1 be separable and reflexive Banach spaces. Suppose $X_0 \hookrightarrow X \hookrightarrow X_1$, with a compact embedding of X_0 into X . Let $(u_n)_{n \geq 1}$ be a sequence that is bounded in $L^\alpha(0, T; X_0)$ and for which $(\partial_t u_n)_{n \geq 1}$ is bounded in $L^\beta(0, T; X_1)$, with $1 < \alpha, \beta < \infty$. Then $(u_n)_{n \geq 1}$ is precompact in $L^\alpha(0, T; X)$.

Let us also recall the following well-known compactness result (see, e.g., [26]): Let $X \hookrightarrow Y \hookrightarrow Z$ be Banach spaces, with a compact embedding of X into Y . Let $(u_n)_{n \geq 1}$ be a sequence that is bounded in $L^\infty(0, T; X)$ and equicontinuous as Z -valued distributions. Then the sequence $(u_n)_{n \geq 1}$ is precompact in $C([0, T]; Y)$.

Now we define the weak solution to the bidomain-bath model problem (1.5):

Definition 2.1 (Weak solution). *A weak solution to system (1.5) is a five tuple function (u_i, u_e, u_s, u, w) such that $u \in L^2(0, T, H^1(\Omega_H)) \cap L^4(\Omega_{T,H})$, $\partial_t u \in L^2(0, T, (H^1(\Omega_H))^*) + L^{\frac{4}{3}}(\Omega_{T,H})$, $u_i, u_e \in L^2(0, T, \tilde{H}^1(\Omega_H))$, $u_s \in L^2(0, T, H^1(\Omega_B))$, $w \in C([0, T], L^2(\Omega_H))$, and satisfying the following weak formulation*

$$\begin{aligned} \iint_{\Omega_{T,H}} \beta c_m \partial_t u \phi_i + \iint_{\Omega_{T,H}} \mathbf{M}_i(x) \nabla u_i \nabla \phi_i + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u, w) \phi_i &= \iint_{\Omega_{T,H}} I_i \phi_i \\ \iint_{\Omega_{T,H}} \beta c_m \partial_t u \phi_e - \iint_{\Omega_H} \mathbf{M}_e(x) \nabla u_e \nabla \phi_e + \iint_{\Omega_B} \mathbf{M}_s(x) \nabla u_s \nabla \phi_s \\ &\quad - \iint_{\Sigma_{2,T}} v \phi_s + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u, w) \phi_e = \iint_{\Omega_{T,H}} I_e \phi_e \\ &\quad \iint_{\Omega_{T,H}} \partial_t w \phi_w - H(u, w) \phi_w = 0. \end{aligned}$$

for all $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H)) \cap L^4(\Omega_{T,H})$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$.

Our first main result concerning direct problem, is the following theorem (the second main result concerns the adjoint problem (4.3) below).

Theorem 2.1. *Assume conditions (1.4), (2.1) and (2.2) then if u_0 in $L^2(\Omega_H)$, $v \in L^2(\Sigma_{2,T})$ and $I_i, I_e \in L^2(\Omega_{T,H})$, then there exists a weak solution to the system (1.5). Moreover, the weak solution is unique.*

The purpose of the following section is to prove the existence and uniqueness of weak solution for our model. We prove first the existence of solutions for the nondegenerate systems in Subection 3.1. Main results are proved in Subsection 3.2 for the existence and Subsection 3.3 for the uniqueness.

3. WELL-POSEDNESS OF THE DIRECT PROBLEM (PROOF OF THEOREM 2.1)

3.1. Existence of the weak solution for the non degenerated problem. The purpose of this section is to prove the existence of weak solution for the bidomain-bath model using specific nondegenerate approximation systems. We show existence of solutions by applying the Faedo-Galerkin method, deriving a priori estimates, and then passing to the limit in the approximate solutions using monotonicity and compactness arguments.

The approximation model reads

$$\begin{cases} \beta c_m \partial_t u + \varepsilon \partial_t u_i - \nabla \cdot (\mathbf{M}_i(x) \nabla u_i) + \beta I_{\text{ion}}(u, w) = I_i & \text{in } \Omega_H \\ \beta c_m \partial_t u - \varepsilon \partial_t u_e + \nabla \cdot (\mathbf{M}_e(x) \nabla u_e) + \beta I_{\text{ion}}(u, w) = I_e & \text{in } \Omega_H \\ -\nabla \cdot (\mathbf{M}_s(x) \nabla u_s) = 0 & \text{in } \Omega_B \\ \partial_t w - H(u, w) = 0 & \text{in } \Omega_H \end{cases} \quad (3.1)$$

supplemented with initial and boundary conditions in (1.5). We consider the following spectral problem : find z in H^1 and a number λ such that for all ϕ in H^1

$$\begin{cases} (\nabla z, \nabla \phi)_{L^2(\Omega_H)} = \lambda(z, \phi), \\ (\nabla z) \cdot \eta = 0. \end{cases}$$

The problem possesses a sequence of eigenvalues $(\lambda_l)_{l=1}^{\infty}$ and the corresponding eigenfunctions $(e_l)_{l=1}^{\infty}$ orthogonal in H^1 and orthonormal in L^2 . We look for finite dimensional approximate solutions as the following sequences for $t > 0$:

$$\begin{cases} u_n(t, \cdot) = \sum_{l=1}^n c_{n,l}(t) e_l(\cdot), u_{n,j}(t, \cdot) = \sum_{l=1}^n c_{n,j,l}(t) e_l(\cdot), \text{ for } j = i, e, \\ u_{n,s}(t, \cdot) = \sum_{l=1}^n c_{n,s,l}(t) e_l(\cdot), w_n(t, \cdot) = \sum_{l=1}^n d_{n,l}(t) e_l(\cdot). \end{cases}$$

For all $k = 1, \dots, n$:

$$\begin{cases} (\beta c_m \partial_t u_n + \varepsilon \partial_t u_{n,i} - \nabla \cdot (\mathbf{M}_i \nabla u_{n,i}), e_k)_{L^2(\Omega_H)} = (I_i - \beta I_{\text{ion}}(u_n, w_n), e_k)_{L^2(\Omega_H)}, \\ (\beta c_m \partial_t u_n - \varepsilon \partial_t u_{n,e} + \nabla \cdot (\mathbf{M}_e \nabla u_{n,e}), e_k)_{L^2(\Omega_H)} = (I_e - \beta I_{\text{ion}}(u_n, w_n), e_k)_{L^2(\Omega_H)}, \\ -(\nabla \cdot (\mathbf{M}_s \nabla u_{n,s}), e_k)_{L^2(\Omega_B)} = 0, \\ (\partial_t w_n, e_k) = (H(u_n, w_n) e_k)_{L^2(\Omega_H)}, \end{cases}$$

with the initial conditions :

$$\begin{cases} u_n(0, \cdot) = u_{0,n}(\cdot) = \sum_{l=1}^n c_{n,l}(0) e_l(\cdot), \\ u_{n,j}(0, \cdot) = u_{0,n,i}(\cdot) = \sum_{l=1}^n c_{n,j,l}(0) e_l(\cdot), \\ w_n(0, \cdot) = w_{0,n}(\cdot) = \sum_{l=1}^n d_{n,l}(0) e_l(\cdot), \end{cases}$$

where

$$\begin{cases} c_{n,l}(0) = (u_0, e_l)_{L^2(\Omega_H)}, \\ c_{n,j,l}(0) = (u_{0,j}, e_l)_{L^2(\Omega_H)}, \\ d_{n,l}(0) = (w_0, e_l)_{L^2(\Omega_H)}, \end{cases}$$

for $j = i, e$. Next, we use a finite dimensional approximation of (v, I_i, I_e) :

$$I_{j,n}(t, \cdot) = \sum_{l=1}^n (I_j, e_l)_{L^2(\Omega_H)} e_l(\cdot) \text{ and } v_n(t, \cdot) = \sum_{l=1}^n (v, e_l)_{L^2(\Gamma)} e_l(\cdot),$$

for $j = i, e$. This implies (recall that $(e_l)_{l=1}^{\infty}$ is orthonormal in L^2)

$$\begin{aligned} \beta c_m c'_{n,k}(t) + \varepsilon c'_{n,i,k}(t) + \int_{\Omega_H} \mathbf{M}_i \nabla u_{n,i} \nabla e_k &= \int_{\Omega_H} (I_{i,n} - \beta I_{\text{ion}}(u_n, w_n)) e_k, \\ \beta c_m c'_{n,k}(t) - \varepsilon c'_{n,e,k}(t) - \int_{\Omega_H} \mathbf{M}_e \nabla u_{n,e} \nabla e_k + \int_{\Omega_B} \mathbf{M}_s \nabla u_{n,s} \nabla e_k - \int_{\Gamma} v_n e_k & \\ &= \int_{\Omega_H} (I_{e,n} - \beta I_{\text{ion}}(u_n, w_n)) e_k, \end{aligned} \quad (3.2)$$

$$d'_{n,k}(t) = \int_{\Omega_H} H(u_n, w_n) e_k.$$

Therefore, adding the two first equations of this system, we get

$$\begin{cases} (2\beta c_m + \varepsilon) c'_{n,k}(t) = F_k(t, \{c_{n,l}\}_{l=1}^n, \{c_{n,i,l}\}_{l=1}^n, \{c_{n,e,l}\}_{l=1}^n, \{c_{n,s,l}\}_{l=1}^n) \\ d'_{n,k}(t) = \int_{\Omega_H} H(u_n, w_n) e_k = F_k^w(t, \{c_{n,l}\}_{l=1}^n, \{d_{n,l}\}_{l=1}^n) \end{cases}$$

where

$$\begin{aligned} F_k(t, \{c_{n,l}\}_{l=1}^n, \{c_{n,i,l}\}_{l=1}^n, \{c_{n,e,l}\}_{l=1}^n, \{c_{n,s,l}\}_{l=1}^n) &= \int_{\Omega_H} (\mathbf{M}_e \nabla u_{n,e} - \mathbf{M}_i \nabla u_i) \cdot \nabla e_k \\ &- \int_{\Omega_B} \mathbf{M}_s \nabla u_{n,s} \cdot \nabla e_k + \int_{\Gamma} v_n e_k + \int_{\Omega_H} (I_{i,n} + I_{e,n} - 2\beta I_{\text{ion}}(u_n, w_n)) e_k. \end{aligned}$$

Using that $c'_{n,k} = \frac{F_k}{2\beta c_m + \varepsilon}$ and the system (3.2), we find:

$$\begin{cases} \varepsilon c'_{n,i}(t) = -\frac{\beta c_m F_k}{2\beta c_m + \varepsilon} - \int_{\Omega_H} \mathbf{M}_i \nabla u_{n,i} \nabla e_k + \int_{\Omega_H} (I_{i,n} - \beta I_{\text{ion}}(u_n, w_n)) e_k \\ \quad := F_k^i(t, \{c_{n,l}\}_{l=1}^n, \{c_{n,i,l}\}_{l=1}^n, \{d_{n,l}\}_{l=1}^n) \\ \varepsilon c'_{n,e}(t) = \frac{\beta c_m F_k}{2\beta c_m + \varepsilon} + \int_{\Omega_H} \mathbf{M}_e \nabla u_{n,e} \nabla e_k - \int_{\Omega_B} \mathbf{M}_s \nabla u_{n,s} \nabla e_k - \int_{\Sigma_2} v_n e_k - \int_{\Omega_H} (I_{e,n} - \beta I_{\text{ion}}(u_n, w_n)) e_k \\ \quad := F_k^e(t, \{c_{n,l}\}_{l=1}^n, \{c_{n,i,l}\}_{l=1}^n, \{c_{n,e,l}\}_{l=1}^n, \{c_{n,s,l}\}_{l=1}^n, \{d_{n,l}\}_{l=1}^n) \\ d'_n(t) = F_k^w(t, \{c_{n,l}\}_{l=1}^n, \{d_{n,l}\}_{l=1}^n). \end{cases}$$

Thanks to our assumptions the functions F_k , F_k^i , F_k^e and F_k^w are Caratheodory functions. Hence, according to the ODE theory, there exist functions $\{c_{n,l}\}_{l=1}^n$, $\{c_{n,i,l}\}_{l=1}^n$, $\{c_{n,e,l}\}_{l=1}^n$, $\{c_{n,s,l}\}_{l=1}^n$, $\{d_{n,l}\}_{l=1}^n$ absolutely continuous satisfying the equations. Thus, there exists a weak local solution for all $t \in (0, t_0)$

$$\begin{aligned} c_{n,l}(t) &= c_{n,l}(0) + \frac{1}{2\beta c_m + \varepsilon} \int_0^t F_k(\tau, \{c_{n,k}(\tau)\}_{l=1}^n, \{c_{n,i,k}(\tau)\}_{l=1}^n, \{d_{n,k}(\tau)\}_{l=1}^n) d\tau, \\ c_{n,j,l}(t) &= \frac{1}{\varepsilon} \int_0^t F_k^j(\tau, \{c_{n,k}(\tau)\}_{l=1}^n, \{c_{n,i,k}(\tau)\}_{l=1}^n, \{c_{n,e,k}(\tau)\}_{l=1}^n, \{c_{n,s,k}(\tau)\}_{l=1}^n, \{d_{n,k}(\tau)\}_{l=1}^n) d\tau \\ &\quad + c_{n,j,l}(0), \\ d_{n,l}(t) &= d_{n,l}(0) + \int_0^t F_k^w(\tau, \{c_{n,k}(\tau)\}_{l=1}^n, \{d_{n,k}(\tau)\}_{l=1}^n) d\tau, \end{aligned}$$

for $j = i, e$.

In the next step, we will reserve the letter T for an arbitrary time in $[0, t_0)$ (and not the final time used elsewhere).

To prove the global existence of the Faedo Galerkin weak solution, we use the following Lemma :

Lemma 3.1. *If $u_{i,0}, u_{e,0} \in L^2(\Omega_H)$ and $v \in L^2(\Sigma_{2,T})$, then there exist constants c_1, c_2, c_3 not depending on n such that*

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega_H))} + \sum_{j=i,e} \|\sqrt{\varepsilon}u_{n,j}\|_{L^\infty(0,T;L^2(\Omega_H))} + \|w_n\|_{L^\infty(0,T;L^2(\Omega_H))} \leq c_1, \quad (3.3)$$

$$\sum_{j=i,e} \|\nabla u_{n,j}\|_{L^2(\Omega_{T,H})} + \|\nabla u_{n,s}\|_{L^2(\Omega_{T,B})} \leq c_2, \quad (3.4)$$

$$\|u_n\|_{L^4(\Omega_{T,H})} \leq c_3. \quad (3.5)$$

If $u_{i,0}, u_{e,0} \in \tilde{H}^1(\Omega_H)$, $u_0 \in L^4(\Omega_H)$, $v \in L^2(\Sigma_{2,T})$ and $I_i, I_e \in L^2(\Omega_{T,H})$, then there exists a constant $c_5 > 0$ such that :

$$\|\partial_t u_n\|_{L^2(\Omega_{T,H})} + \sum_{j=i,e} \|\sqrt{\varepsilon}\partial_t u_{n,j}\|_{L^2(\Omega_{T,H})} \leq c_5. \quad (3.6)$$

Proof. First, note that the Galerkin solutions satisfy the following weak formulation :

$$\begin{aligned} \beta c_m \int_{\Omega_H} \partial_t u_n \phi_{i,n} + \varepsilon \int_{\Omega_H} \partial_t u_{n,i} \phi_{i,n} + \int_{\Omega_H} \mathbf{M}_i \nabla u_{n,i} \cdot \nabla \phi_{i,n} &= \int_{\Omega_H} (I_{i,n} - \beta I_{\text{ion}}(u_n, w_n)) \phi_{i,n}, \\ \beta c_m \int_{\Omega_H} \partial_t u_n \phi_{e,n} - \varepsilon \int_{\Omega_H} \partial_t u_{n,e} \phi_{e,n} - \int_{\Omega_H} \mathbf{M}_e \nabla u_{n,e} \cdot \nabla \phi_{e,n} + \int_{\Omega_B} \mathbf{M}_s \nabla u_{n,s} \cdot \nabla \phi_{s,n} \\ - \int_{\Gamma} v_n \eta_s \phi_{s,n} &= \int_{\Omega_H} (I_{e,n} - \beta I_{\text{ion}}(u_n, w_n)) \phi_{e,n}, \\ \int_{\Omega_H} \partial_t w_n \phi_{w,n} &= \int_{\Omega_H} H(u_n, w_n) \phi_{w,n}, \end{aligned} \quad (3.7)$$

where

$$\phi_{j,n}(t, x) = \sum_{l=1}^n d_{j,n,l}(t) e_l(x) \text{ and } \phi_{w,n}(t, x) = \sum_{l=1}^n d_{n,l}(t) e_l(x),$$

for some given (absolutely continuous) coefficients $d_{j,n,l}(t), d_{n,l}(t)$ for $j = i, e, s$. Substituting $\phi_{i,n} = u_{n,i}, \phi_{e,n} = -u_{n,e}, \phi_{s,n} = u_{n,s}, \phi_{w,n} = w_n$ in (3.7) and adding the resulting equations, we get

$$\begin{aligned} \frac{\beta c_m}{2} \frac{d}{dt} \int_{\Omega_H} |u_n|^2 + \frac{\varepsilon}{2} \sum_{j=i,e} \frac{d}{dt} \int_{\Omega_H} |u_{n,j}|^2 + \frac{1}{2} \frac{d}{dt} \int_{\Omega_H} |w_n|^2 + \sum_{j=i,e} \int_{\Omega_H} \mathbf{M}_j(x) \nabla u_{n,j} \cdot \nabla u_{n,j} \\ + \int_{\Omega_B} \mathbf{M}_s(x) \nabla u_{n,s} \cdot \nabla u_{n,s} - \int_{\Sigma_2} v_n u_s + \int_{\Omega_H} \beta I_{\text{ion}}(u_n, w_n) u_n - \int_{\Omega_H} H(u_n, w_n) w_n \\ = \int_{\Omega_H} I_{i,n}(u_{n,i} - u_{n,e}) + \int_{\Omega_H} (I_{i,n} - I_{e,n}) u_{n,e}. \end{aligned} \quad (3.8)$$

Next, According to the trace theorem to the sixth integral and Poincaré inequality to the last integral, together with (1.4), (2.1), (2.3) and the Young inequality we deduce from (3.8)

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{\Omega_H} (\beta c_m |u_n|^2 + \varepsilon \sum_{j=i,e} |u_{n,j}|^2 + |w_n|^2) + \frac{C_M}{2} \sum_{j=i,e} \int_{\Omega_H} |\nabla u_{n,j}|^2 + \frac{C_M}{2} \int_{\Omega_B} |\nabla u_{n,s}|^2 \\
& \quad + \int_{\Omega_H} \beta I_{1,\text{ion}}(u_n) u_n + \beta C_I \int_{\Omega_H} |u_n|^2 \\
& \leq c_s \int_{\Gamma} |v_n|^2 + \int_{\Omega_H} \left(\frac{1}{2} + \beta C_I\right) |u_n|^2 + b \int_{\Omega_H} |w_n|^2 + \int_{\Omega_H} (a + \beta \lambda) u_n w_n + \int_{\Omega_H} \frac{1}{2} |I_{i,n}|^2 \\
& \quad + C_1 \int_{\Omega_H} |I_{i,n} - I_{e,n}|^2,
\end{aligned} \tag{3.9}$$

for some constant $C_1 > 0$. This implies

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega_H} (\beta c_m |u_n|^2 + \varepsilon \sum_{j=i,e} |u_{n,j}|^2 + |w_n|^2) \leq C_2 \int_{\Omega_H} |u_n|^2 + C_3 \int_{\Omega_H} |w_n|^2 + C_4, \tag{3.10}$$

for some constants $C_2, C_3, C_4 > 0$. Therefore an application of Gronwall inequality gives

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega_H))} + \|w_n\|_{L^\infty(0,T;L^2(\Omega_H))} + \sum_{j=i,e} \varepsilon \|u_{n,j}\|_{L^\infty L^2(0,T;L^2(\Omega))} \leq C_5, \tag{3.11}$$

for some constant $C_5 > 0$. Next we use (2.3) to deduce

$$\beta \iint_{\Omega_{T,H}} I_{\text{ion}}(u_n, w_n) u_n \leq \iint_{\Omega_{T,H}} (\beta I_{\text{ion}}(u_n, w_n) u_n + \beta C_I |u_n|^2) + \beta \iint_{\Omega_{T,H}} C_I |u_n|^2 \leq C_6,$$

for some constant C_6 . Therefore, there exists a constant $C_7 > 0$ such that

$$\|u_n\|_{L^4(\Omega_{T,H})} \leq C_7,$$

which proves (3.5). Moreover (3.11) and (3.9) imply that there exists a constant $C_8 > 0$ such that :

$$\sum_{j=i,e} \iint_{\Omega_{T,H}} |\nabla u_{n,j}|^2 + \iint_{\Omega_{T,B}} |\nabla u_{n,s}|^2 \leq C_8, \tag{3.12}$$

which proves (3.4).

To prove (3.6), we substitute $\phi_{i,n} = \partial_t u_{n,i}$, $\phi_{e,n} = -\partial_t u_{n,e}$, $\phi_{s,n} = u_{n,s}$, and $\phi_{w,n} = \partial_t w_n$ in (3.7), we integrate in time and we add the resulting equations to deduce

$$\begin{aligned}
& \int_0^T \int_{\Omega_H} \beta c_m |\partial_t u_n|^2 + \int_0^T \int_{\Omega_H} |\partial_t w_n|^2 + \int_0^T \int_{\Omega_H} \varepsilon \sum_{j=i,e} |\partial_t u_{n,j}|^2 + \int_0^T \int_{\Omega_B} \mathbf{M}_s(x) \nabla u_s \cdot \nabla u_s \\
& \quad + \int_0^T \int_{\Omega_H} \sum_{j=i,e} \mathbf{M}_j \nabla u_{n,j} \cdot \nabla (\partial_t u_{n,j}) - \int_0^T \int_{\Sigma_2} v_n u_s + \int_0^T \int_{\Omega_H} \beta I_{\text{ion}}(u_n, w_n) \partial_t u_n \\
& = \int_0^T \int_{\Omega_H} H(u_n, w_n) \partial_t w_n + \int_0^T \int_{\Omega_H} I_{i,n} \partial_t u_{n,i} - \int_0^T \int_{\Omega_H} I_{e,n} \partial_t u_{n,e} \\
& = \int_0^T \int_{\Omega_H} H(u_n, w_n) \partial_t w_n + \int_0^T \int_{\Omega_H} I_{i,n} \partial_t u_n + \int_0^T \int_{\Omega_H} (I_{i,n} - I_{e,n}) \partial_t u_{n,e}.
\end{aligned} \tag{3.13}$$

Now we set $\mathcal{M}_j(s) = \int_0^s \mathbf{M}_j(t) dt$ and $\mathcal{I}_1(s) = \int_0^s I_{1,\text{ion}}(t) dt$. Observe that

$$\begin{aligned}
\int_0^T \int_{\Omega_H} \mathbf{M}_j \nabla u_{n,j} \nabla (\partial_t u_{n,j}) &= \int_0^T \partial_t \left(\int_{\Omega_H} \mathcal{M}_j(\nabla u_{n,j}) \right) \\
&= \int_{\Omega_H} \mathcal{M}_j(\nabla u_{n,j}(T, x)) - \int_{\Omega_H} \mathcal{M}_j(\nabla u_{n,j}(0, x)),
\end{aligned}$$

and

$$\begin{aligned} \int_0^T \int_{\Omega_H} I_{1,\text{ion}}(u_n) \partial_t u_n &= \int_0^T \partial_t \left(\int_{\Omega_H} \mathcal{I}_1(u_n) dx \right) dt \\ &= \int_{\Omega_H} \mathcal{I}_1(u_n(T, x)) - \int_{\Omega_H} \mathcal{I}_1(u_n(0, x)). \end{aligned}$$

Using this and thanks to Young's inequality, (1.2), (1.4), we get

$$\begin{aligned} &\int_0^T \int_{\Omega_H} \beta c_m |\partial_t u_n|^2 + \int_0^T \int_{\Omega_H} |\partial_t w_n|^2 + \int_0^T \int_{\Omega_H} \varepsilon/2 \sum_{j=i,e} |\partial_t u_{n,j}|^2 + \int_{\Omega_H} \sum_{j=i,e} \mathcal{M}_j(\nabla u_{n,j}(T, x)) \\ &\quad + C_M \int_0^T \int_{\Omega_B} |\nabla u_{n,s}|^2 + \beta \int_{\Omega_H} \mathcal{I}_1(u_n(T, x)) + \int_0^T \int_{\Omega_H} \beta I_{2,\text{ion}}(w_n) \partial_t u_n \\ &\leq c_s \int_0^T \int_{\Gamma} |v_n|^2 + C \int_0^T \int_{\Omega_H} (|u_n|^2 + |w_n|^2) + \int_{\Omega_H} \sum_{j=i,e} \mathcal{M}_j(\nabla u_{n,j}(0, x)) + \beta \int_{\Omega_H} \mathcal{I}_1(u_n(0, x)) \\ &\quad + \int_0^T \int_{\Omega_H} \frac{\beta c_m}{2} |\partial_t u_n|^2 + \frac{1}{2} \int_0^T \int_{\Omega_H} |\partial_t w_n|^2 + C, \end{aligned} \tag{3.14}$$

for some constant $C > 0$. Finally we use

$$\begin{aligned} &\left(\mathcal{M}_j(\nabla u_{n,j}(T, x)) + \mathcal{I}_1(u_n(T, x)) + C_I |u_n(T, x)|^2 \right) \geq 0, \\ &\left(|\mathcal{M}_j(\nabla u_{n,j}(0, x))| + \mathcal{I}_1(u_n(0, x)) \right) \leq C \left(|\nabla u_{n,j}(0, x)|^2 + |u_n(0, x)|^4 \right), \end{aligned}$$

(for some constant $C > 0$) and (3.3) to get from (3.14)

$$\int_0^T \int_{\Omega_H} \beta c_m |\partial_t u_n|^2 + \int_0^T \int_{\Omega_H} |\partial_t w_n|^2 + \int_0^T \int_{\Omega_H} \varepsilon \sum_{j=i,e} |\partial_t u_{n,j}|^2 \leq C.$$

for some constant $C > 0$. This completes the proof of Lemma 3.1. \square

The next step is to show that the local solution constructed above can be extended to $[0, T)$ but this can be done exactly as in [7], so we omit the details.

Note that the consequence of Lemma 3.1 and the Aubin-Lions compactness lemma (see for e.g. [5] and [18]) is the following convergence (at the cost of extracting subsequences, which we do not bother to relabel), we can assume there exist limit functions $(u_\varepsilon, u_{i,\varepsilon}, u_{e,\varepsilon}, u_{s,\varepsilon}, w_\varepsilon)$ such that, for ε fixed :

$$\begin{cases} u_n \rightarrow u \text{ almost everywhere in } \Omega_{T,H} \text{ and strongly in } L^2(\Omega_{T,H}), \\ u_n \rightharpoonup u_\varepsilon \text{ in } L^2(0, T; H^1(\Omega_H)) \cap L^4(\Omega_{T,H}), \\ u_{n,i} \rightharpoonup u_{i,\varepsilon} \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ u_{n,e} \rightharpoonup u_{e,\varepsilon} \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ u_{n,s} \rightharpoonup u_{s,\varepsilon} \text{ weakly in } L^2(0, T; H^1(\Omega_B)), \\ w_n \rightarrow w_\varepsilon \text{ strongly in } L^2(\Omega_{T,H}). \end{cases} \tag{3.15}$$

Finally, using (3.15) sending n to ∞ in the following weak formulation delivers the existence of a weak global solution for ε fixed:

$$\begin{aligned}
& \iint_{\Omega_{T,H}} \beta c_m \partial_t u_n \phi_i + \varepsilon \iint_{\Omega_{T,H}} \partial_t u_{n,i} \phi_i + \iint_{\Omega_{T,H}} \mathbf{M}_i(x) \nabla u_{n,i} \nabla \phi_i \\
& \quad + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u_n, w_n) \phi_i = \iint_{\Omega_{T,H}} I_{i,n} \phi_i \\
& \iint_{\Omega_{T,H}} \beta c_m \partial_t u_n \phi_e - \varepsilon \iint_{\Omega_{T,H}} \partial_t u_{n,e} \phi_e - \iint_{\Omega_H} \mathbf{M}_e(x) \nabla u_{n,e} \nabla \phi_e \\
& \quad + \iint_{\Omega_B} \mathbf{M}_s(x) \nabla u_{n,s} \nabla \phi_s - \iint_{\Sigma_{2,T}} v_n \phi_s + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u_n, w_n) \phi_e = \iint_{\Omega_{T,H}} I_e \phi_e \\
& \iint_{\Omega_{T,H}} \partial_t w_n \phi_w - H(u_n, w_n) \phi_w = 0.
\end{aligned} \tag{3.16}$$

for all $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H)) \cap L^4(\Omega_{T,H})$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$.

3.2. Existence of the weak solution for the degenerate problem. We will now prove the existence of the weak solution to our degenerate system (the bidomain-bath model). Note that by the (weak) lower semi-continuity properties of norms, the estimates in Lemma 3.1 hold with $(u_n, u_{n,i}, u_{n,e}, u_{n,s}, w_n)$ replaced by $(u_\varepsilon, u_{i,\varepsilon}, u_{e,\varepsilon}, u_{s,\varepsilon}, w_\varepsilon)$, respectively. Moreover, the constants c_1, c_2, c_3, c_4 are independent of ε . Therefore we have the following lemma:

Lemma 3.2. *If $u_{i,\varepsilon,0}, u_{e,\varepsilon,0} \in L^2(\Omega_H)$, then there exist constants $cst_1, cst_2, cst_3, cst_4$ not depending on ε such that*

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_H))} + \sum_{j=i,e} \|\sqrt{\varepsilon} u_{j,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_H))} + \|w_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_H))} \leq cst_1, \tag{3.17}$$

$$\sum_{j=i,e} \|\nabla u_{j,\varepsilon}\|_{L^2(\Omega_{T,H})} + \|\nabla u_{s,\varepsilon}\|_{L^2(\Omega_{T,B})} \leq cst_2, \tag{3.18}$$

$$\|u_\varepsilon\|_{L^4(\Omega_{T,H})} \leq cst_3. \tag{3.19}$$

If $u_{i,\varepsilon,0}, u_{e,\varepsilon,0} \in \tilde{H}^1(\Omega_H)$ and $u_{\varepsilon,0} \in L^4(\Omega_H)$, then there exists a constant $cst_5 > 0$ such that :

$$\|\partial_t u_\varepsilon\|_{L^2(\Omega_{T,H})} + \sum_{j=i,e} \|\sqrt{\varepsilon} \partial_t u_{j,\varepsilon}\|_{L^2(\Omega_{T,H})} \leq cst_4. \tag{3.20}$$

In view of Lemma 3.1 and and the Aubin-Lions compactness lemma [5, 18], we can assume there exist limit functions (u, u_i, u_e, u_s, w) such that (at the cost of extracting subsequences, which we do not bother to relabel) the following convergences hold as $\varepsilon \rightarrow 0$

$$\begin{cases} u_\varepsilon \rightarrow u \text{ almost everywhere in } \Omega_{H,T} \text{ and strongly in } L^2(\Omega_{T,H}), \\ u_\varepsilon \rightharpoonup u \text{ in } L^2(0, T; H^1(\Omega_H)) \cap L^4(\Omega_{T,H}), \\ u_{i,\varepsilon} \rightharpoonup u_i \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ u_{e,\varepsilon} \rightharpoonup u_e \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ u_{s,\varepsilon} \rightharpoonup u_s \text{ weakly in } L^2(0, T; H^1(\Omega_B)), \\ w_\varepsilon \rightarrow w \text{ strongly in } L^2(\Omega_{T,H}). \end{cases} \tag{3.21}$$

Moreover if $u_{i,\varepsilon,0}, u_{e,\varepsilon,0} \in \tilde{H}^1(\Omega_H)$ and $u_{\varepsilon,0} \in L^4(\Omega_H)$, we find

$$\begin{cases} \partial_t u_\varepsilon \rightharpoonup \partial_t u \text{ in } L^2(\Omega_{TH}), \\ \varepsilon \partial_t u_{j,\varepsilon} \rightharpoonup 0 \text{ in } L^2(\Omega_{TH}) \text{ for } j = i, e. \end{cases} \quad (3.22)$$

Thus, using (3.21)- (3.22) and sending ε to 0 in the weak formulation (3.16), where $(u_n, u_{n,i}, u_{n,e}, u_{n,s}, w_n)$ replaced by $(u_\varepsilon, u_{i,\varepsilon}, u_{e,\varepsilon}, u_{s,\varepsilon}, w_\varepsilon)$, we get the existence of a weak solution $(u_\rho, u_{i,\rho}, u_{e,\rho}, u_{s,\rho}, w_\rho)$ satisfying

$$\begin{aligned} & \iint_{\Omega_{T,H}} \beta c_m \partial_t u_\rho \phi_i + \iint_{\Omega_{T,H}} \mathbf{M}_i(x) \nabla u_{i,\rho} \nabla \phi_i + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u_\rho, w_\rho) \phi_i = \iint_{\Omega_{T,H}} I_i \phi_i \\ & \iint_{\Omega_{T,H}} \beta c_m \partial_t u_\rho \phi_e - \iint_{\Omega_{T,H}} \mathbf{M}_e(x) \nabla u_{e,\rho} \nabla \phi_e + \iint_{\Omega_{T,B}} \mathbf{M}_s(x) \nabla u_{s,\rho} \nabla \phi_s \\ & \quad - \iint_{\Sigma_{2,T}} v \phi_s + \iint_{\Omega_{T,H}} \beta I_{\text{ion}}(u_\rho, w_\rho) \phi_e = \iint_{\Omega_{T,H}} I_e \phi_e \\ & \iint_{\Omega_{T,H}} (\partial_t w_\rho \phi_w - H(u_\rho, w_\rho) \phi_w) = 0. \end{aligned} \quad (3.23)$$

for all $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H)) \cap L^4(\Omega_{T,H})$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$.

In the case $u_0 \in L^2(\Omega_H)$, we approximate the initial data u_0 by a sequence $(u_{0,\rho})_{\rho>0}$ of functions satisfying

$$u_{0,\rho} \in C_0^\infty(\Omega_H), \quad \|u_{0,\rho}\|_{L^2(\Omega_H)} \leq \|u_0\|_{L^2(\Omega_H)}, \quad \text{and } u_{0,\rho} \rightarrow u_0 \text{ as } \rho \rightarrow 0.$$

Note that by the (weak) lower semicontinuity properties of norms, the estimates (3.3), (3.4) and (3.5) in Lemma 3.1 hold with $(u_n, u_{n,i}, u_{n,e}, u_{n,s}, w_n)$ replaced by $(u_\rho, u_{i,\rho}, u_{e,\rho}, u_{s,\rho}, w_\rho)$, respectively (the constants c_1, c_2, c_3 are independent of ρ). Moreover the convergence (3.21) holds where $(u_n, u_{n,i}, u_{n,e}, u_{n,s}, w_n)$ replaced by $(u_\rho, u_{i,\rho}, u_{e,\rho}, u_{s,\rho}, w_\rho)$. Finally using the compactness argument in [26] (observe that $u_\rho \in L^2(0, T; H^1(\Omega_H)) \cap L^4(\Omega_{H,T})$ and $\partial_t u_\rho \in L^2(0, T; (H^1(\Omega_H))^*) + L^{\frac{4}{3}}(\Omega_{H,T})$ independently of ρ) and sending ρ to 0 in the weak formulation (3.23), we get the existence of the weak solution (u, u_i, u_e, u_s, w) for the degenerated problem (1.5) in the sense of Definition 2.1.

3.3. Uniqueness of the weak solution. Now, the purpose is to prove uniqueness of the weak solution to our degenerate problem (1.5).

Theorem 3.3. *Assume conditions (2.1) and (2.2). Let $(u_{i,1}, u_{e,1}, u_{s,1}, u_1, w_1)$ and $(u_{i,2}, u_{e,2}, u_{s,2}, u_2, w_2)$ be two weak solutions to system (1.5), with data $u_{j,0}, v_j, I_{i,j}$ and $I_{e,j}$ for $j = 1, 2$. Then for any $t \in (0, T]$,*

$$\begin{aligned} & \int_{\Omega_H} (u_1 - u_2)^2(t) + \int_{\Omega_H} (w_1 - w_2)^2(t) \leq \exp\left(\frac{2C_I + 1 + a}{\beta c_m} t\right) \int_{\Omega_H} |u_{1,0}(x) - u_{2,0}(x)|^2 dx \\ & \quad + \exp((2b + a)t) \int_{\Omega_H} |w_{1,0}(x) - w_{2,0}(x)|^2 dx \\ & \quad + \int_0^t \exp\left(\frac{2C_I + 1 + a}{\beta c_m} (t - s)\right) \left(\int_{\Omega_H} |I_{i,1} - I_{i,2}|^2 dx + c_s \int_{\Sigma_2} (v_1 - v_2)^2 dy \right) ds \\ & \quad + C \int_0^t \exp((2b + a)(t - s)) \int_{\Omega_H} |I_{e,1} - I_{e,2}|^2 dx ds. \end{aligned}$$

In particular, there exists at most one weak solution to the (1.5) model.

Proof. Note that the following equations hold for all test functions $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H)) \cap L^4(\Omega_{T,H})$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$:

$$\iint_{\Omega_{t,H}} \beta c_m \partial_s (u_1 - u_2) \phi_i + \iint_{\Omega_{t,H}} \mathbf{M}_i(\nabla u_{i,1} - \nabla u_{i,2}) \cdot \nabla \phi_i + \iint_{\Omega_{t,H}} \beta (I_{\text{ion},1} - I_{\text{ion},2}) \phi_i$$

$$= \iint_{\Omega_{t,H}} (I_{i,1} - I_{i,2}) \phi_i$$

$$\begin{aligned} & \iint_{\Omega_{t,H}} \beta c_m \partial_s (u_1 - u_2) \phi_e - \iint_{\Omega_{t,H}} \mathbf{M}_e (\nabla u_{e,1} - \nabla u_{e,2}) \cdot \nabla \phi_e + \iint_{\Omega_{t,B}} \mathbf{M}_s (\nabla u_{s,1} - \nabla u_{s,2}) \cdot \nabla \phi_s \\ & - \iint_{\Sigma_{2,t}} \mathbf{M}_s (\nabla u_{s,1} - \nabla u_{s,2}) \cdot \eta_s \phi_s + \iint_{\Omega_{t,H}} \beta (I_{\text{ion},1} - I_{\text{ion},2}) \phi_e = \iint_{\Omega_{t,H}} (I_{e,1} - I_{e,2}) \phi_e, \end{aligned}$$

$$\iint_{\Omega_{t,H}} \partial_s (w_1 - w_2) \phi_w = \iint_{\Omega_{t,H}} (H(u_1, w_1) - H(u_2, w_2)) \phi_w.$$

Herein, $\Omega_{t,H} = (0, t) \times \Omega_H$, $\Omega_{t,B} = (0, t) \times \Omega_B$ and $\Sigma_{2,t} = (0, t) \times \Gamma$ for $t \in (0, T)$. Note that from Lemma 2.3 in [3], there exists a family of linear operators $(\Theta_\varepsilon)_\varepsilon$ from $L^2(0, T; H^1(\Omega_H))$ into $D(\mathbb{R} \times \mathbb{R}^3)$ such that for all $u \in L^2(0, T; H^1(\Omega_H)) \cap L^4(\Omega_{T,H})$

$\Theta_\varepsilon(u)$ converges strongly to u in $L^2(0, T; H^1(\Omega_H)) \cap L^4(\Omega_{T,H})$.

Substituting $\phi_i = \Theta_\varepsilon(u_{i,1} - u_{i,2})$, $\phi_e = -\Theta_\varepsilon(u_{e,1} - u_{e,2})$, $\phi_s = u_{s,1} - u_{s,2}$, $\phi_w = w_1 - w_2$ in the equations, using the linearity of $\Theta_\varepsilon(\cdot)$ adding the resulting equations, send $\varepsilon \rightarrow 0$ and integrating in time, we find :

$$\begin{aligned} & \int_{\Omega_H} \frac{\beta c_m}{2} |u_1 - u_2|^2(t) + \int_{\Omega_H} \frac{1}{2} |w_1 - w_2|^2(t) + \sum_{j=i,e} \iint_{\Omega_{t,H}} \mathbf{M}_j (\nabla u_{j,1} - \nabla u_{j,2}) \cdot (\nabla u_{j,1} - \nabla u_{j,2}) \\ & + \iint_{\Omega_{t,B}} \mathbf{M}_s (\nabla u_{s,1} - \nabla u_{s,2}) \cdot (\nabla u_{s,1} - \nabla u_{s,2}) \\ & + \iint_{\Omega_{t,H}} \beta (I_{\text{ion},1}(u_1, w_1) - I_{\text{ion},2}(u_2, w_2))(u_1 - u_2) \\ & = \iint_{\Sigma_{2,t}} (v_1 - v_2)(u_{s,1} - u_{s,2}) + \iint_{\Omega_{t,H}} (H(u_1, w_1) - H(u_2, w_2))(w_1 - w_2) \\ & + \int_{\Omega_H} \frac{\beta c_m}{2} |u_{1,0} - u_{2,0}|^2 + \int_{\Omega_H} \frac{1}{2} |w_{1,0} - w_{2,0}|^2 \\ & + \iint_{\Omega_{t,H}} (I_{i,1} - I_{i,2})(u_{i,1} - u_{i,2}) - \iint_{\Omega_{t,H}} (I_{e,1} - I_{e,2})(u_{e,1} - u_{e,2}). \end{aligned} \tag{3.24}$$

Now we use (2.1), (2.2), (1.4), Young inequality, trace theorem to deduce from (3.24)

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_H} (\beta c_m |u_1 - u_2|^2 + |w_1 - w_2|^2)(t) + \sum_{j=i,e} C_M \iint_{\Omega_{t,H}} |\nabla u_{j,1} - \nabla u_{j,2}|^2 \\
& \quad + C_M \iint_{\Omega_{t,B}} |\nabla u_{s,1} - \nabla u_{s,2}|^2 \\
& \leq \iint_{\Sigma_{2,t}} (v_1 - v_2)(u_{s,1} - u_{s,2}) + \iint_{\Omega_{t,H}} (H(u_1, w_1) - H(u_2, w_2))(w_1 - w_2) \\
& \quad + \int_{\Omega_H} \frac{\beta c_m}{2} |u_{1,0} - u_{2,0}|^2 + \int_{\Omega_H} \frac{1}{2} |w_{1,0} - w_{2,0}|^2 + \iint_{\Omega_{t,H}} C_I |u_1 - u_2|^2 \\
& \quad + \iint_{\Omega_{t,H}} \beta \lambda (w_1 - w_2)(u_1 - u_2) + \iint_{\Omega_{t,H}} |I_{i,1} - I_{i,2}|^2 + \frac{1}{2} \iint_{\Omega_{t,H}} |u_1 - u_2|^2 \\
& \quad + C \iint_{\Omega_{t,H}} |I_{e,2} - I_{e,1}|^2 + \frac{C_M}{2} \iint_{\Omega_{t,H}} |\nabla u_{e,1} - \nabla u_{e,2}|^2 \\
& \leq c_s \iint_{\Sigma_{2,t}} |v_1 - v_2|^2 + \frac{C_M}{2} \iint_{\Omega_{t,B}} |\nabla u_{s,1} - \nabla u_{s,2}|^2 + \iint_{\Omega_{t,H}} b |w_1 - w_2|^2 + \frac{a}{2} \iint_{\Omega_{t,H}} |u_1 - u_2|^2 \\
& \quad + \frac{a}{2} \iint_{\Omega_{t,H}} |w_1 - w_2|^2 + \int_{\Omega_H} \frac{\beta c_m}{2} |u_{1,0} - u_{2,0}|^2 + \int_{\Omega_H} \frac{1}{2} |w_{1,0} - w_{2,0}|^2 \\
& \quad + \iint_{\Omega_{t,H}} C_I |u_1 - u_2|^2 + \iint_{\Omega_{t,H}} |I_{i,1} - I_{i,2}|^2 + \frac{1}{2} \iint_{\Omega_{t,H}} |u_1 - u_2|^2 \\
& \quad + C \iint_{\Omega_{t,H}} |I_{e,2} - I_{e,1}|^2 + \frac{C_M}{2} \iint_{\Omega_{t,H}} |\nabla u_{e,1} - \nabla u_{e,2}|^2.
\end{aligned} \tag{3.25}$$

This implies

$$\begin{aligned}
& \int_{\Omega_H} \frac{\beta c_m}{2} |u_1 - u_2|^2(t) + \int_{\Omega_H} \frac{1}{2} |w_1 - w_2|^2(t) \\
& \leq c_s \iint_{\Sigma_{2,t}} |v_1 - v_2|^2 + \int_{\Omega_H} \frac{\beta c_m}{2} |u_{1,0} - u_{2,0}|^2 + \int_{\Omega_H} \frac{1}{2} |w_{1,0} - w_{2,0}|^2 \\
& \quad + \iint_{\Omega_{t,H}} (C_I + \frac{1}{2} + \frac{a}{2}) |u_1 - u_2|^2 + \iint_{\Omega_{t,H}} (b + \frac{a}{2}) |w_1 - w_2|^2 + \iint_{\Omega_{t,H}} |I_{i,1} - I_{i,2}|^2 \\
& \quad + C \iint_{\Omega_H} |I_{e,2} - I_{e,1}|^2.
\end{aligned}$$

Finally an application of Gronwall inequality yields

$$\begin{aligned}
& \int_{\Omega_H} |u_1 - u_2|^2(t) dx + \int_{\Omega_H} |w_1 - w_2|^2(t) dx \\
& \leq \exp\left(\frac{2C_I + 1 + a}{\beta c_m} t\right) \int_{\Omega_H} |u_{1,0}(x) - u_{2,0}(x)|^2 dx \\
& \quad + \int_0^t \exp\left(\frac{2C_I + 1 + a}{\beta c_m} (t-s)\right) \left[\int_{\Omega_H} |I_{i,1} - I_{i,2}|^2 dx + \int_{\Sigma_2} c_s |v_1 - v_2|^2 dy \right] ds \\
& \quad + \exp((2b+a)t) \int_{\Omega_H} |w_{1,0}(x) - w_{2,0}(x)|^2 dx + C \int_0^t \exp((2b+a)(t-s)) \int_{\Omega_H} |I_{e,1} - I_{e,2}|^2 dx ds.
\end{aligned}$$

This proves our uniqueness theorem. \square

4. OPTIMAL CONTROL OF THE HEART ACTIVITY BY EXTERNAL STIMULATIONS

In this section we will prove the existence of the solution for the following optimal control problem.

$$\tilde{J}(u, v) := \iint_{\Omega_{T,H}} |u - u_d|^2 dxdt + \frac{\alpha}{2} \iint_{\Sigma_{2,T}} |v|^2 dsdt, \quad (4.1)$$

where \tilde{J} is the cost functional and α denotes the regularization parameter. Recall that v is the control and u_d is the desired state. More precisely, the cardiac defibrillation will aim at driving the transmembrane voltage $u(x, t)$ to a desired state u_d at the intracellular space by properly applying the control v at the boundary of the bath domain. The main aim is finding an optimal external current v which is as small as possible compared to the ad-hoc strategies while still leading to a defibrillation. In computations, the control acts at boundary of the bathing ($\Gamma \subset \Sigma_B$), see Figure 1.

We define the solution operator $S : L^2(\Sigma_2) \rightarrow L^2(0, T, \tilde{H}^1(\Omega_H)) \times L^2(0, T, \tilde{H}^1(\Omega_H)) \times L^2(0, T, H^1(\Omega_B)) \times L^2(0, T, H^1(\Omega_H)) \cap L^4(\Omega_{T,H}) \times C([0, T], L^2(\Omega_H))$ by $S(v) = (u_i, u_e, u_s, u, w)$ where the (u_i, u_e, u_s, u, w) is the solution to the bidomain-bath equations. Introducing the reduced cost functional as follows

$$J(v) := \tilde{J}(u, v). \quad (4.2)$$

The reduced cost functional is utilized in the following subsections.

4.1. Existence of the control. In this subsection, we show that our optimal control problem has a solution that we characterize using relaxation techniques to get the optimality system. We use the following theorem :

Theorem 4.1. *Given $u_0 \in L^2(\Omega_H)$, $v \in L^2(\Sigma_{2,T})$ and $u_d \in L^2(\Omega_{T,H})$, there exists a solution v^* of the optimal control problem (4.2).*

Proof. The goal is to prove that there exists v^* such that $J(v^*) = \inf_v J(v)$. Observe that the functional J is bounded (recall that $u \in L^2(\Omega_{T,H})$)

$$0 \leq \inf J(v) < \infty.$$

Thus the control v is bounded in $L^2(\Sigma_{2,T})$. Since the functional J is bounded, it means that there exists an infimum m such that

$$\inf J(v) = m.$$

This implies that there exists a sequence $(v_n)_n$ such that $J(v_n) \rightarrow m$ as $m \in [0, +\infty[$, and

$$m \leq J(v_n) \leq m + \frac{1}{n}.$$

Note that

$$J(v_n) - J(v) = \iint_{\Omega_{T,H}} |u_n - u_d|^2 + \frac{\alpha}{2} \iint_{\Sigma_{2,T}} |v_n|^2 - \iint_{\Omega_{T,H}} |u - u_d|^2 - \frac{\alpha}{2} \iint_{\Sigma_{2,T}} |v|^2.$$

Since the sequence $(v_n)_n$ is bounded we can extract a subsequence still denoted $(v_n)_n$ such that

$$v_n \rightharpoonup v^* \quad \text{weakly in } L^2(\Sigma_{2,T}).$$

Next we use the convergences (3.21) where ε replaced by n and the strong convergence of $(u_n)_n$ in $L^2(\Omega_{T,H})$, we get (because the functional J is lower-semicontinuous on the L^2 weak sense)

$$J(v^*) \leq \liminf_{n \rightarrow \infty} J(v_n)$$

Finally we obtain

$$\min_v J(v) \leq J(v^*) \leq \liminf_{n \rightarrow \infty} J(v_n) = \min_v J(v).$$

This concludes the proof of theorem 4.1. □

4.2. Optimal conditions and dual problem. Now, we define the Lagrangian as :

$$\begin{aligned}
L(u_i, u_e, u_s, w, v, p_i, p_e, p_s, p_w, p_v, p_1, p_2) = & \\
& \iint_{\Omega_{T,H}} |u - u_d|^2 dx dt + \frac{\alpha}{2} \iint_{\Sigma_{2,T}} |v|^2 dx dt \\
& + \iint_{\Omega_{T,H}} (\beta c_m \partial_t u + \beta I_{\text{ion}}(u, w))(p_i - p_e) dx dt - \iint_{\Omega_{T,H}} (I_i p_i - I_e p_e) dx dt \\
& - \iint_{\Omega_{T,H}} \nabla \cdot (\mathbf{M}_i(x) \nabla p_i) u_i dx dt - \iint_{\Sigma_{T,H}} \mathbf{M}_i(y) \nabla u_i \cdot \eta p_i dy dt + \iint_{\Sigma_{T,H}} \mathbf{M}_i(y) u_i (\nabla p_i) \cdot \eta dy dt \\
& - \iint_{\Omega_{T,H}} \nabla \cdot (\mathbf{M}_e(x) \nabla p_e) u_e dx dt - \iint_{\Sigma_{T,H}} \mathbf{M}_e(y) \nabla u_e \cdot \eta p_e dy dt + \iint_{\Sigma_{T,H}} \mathbf{M}_e(y) u_e (\nabla p_e) \cdot \eta dy dt \\
& - \iint_{\Omega_{T,B}} \nabla \cdot (\mathbf{M}_s(x) \nabla p_s) u_s dx dt - \int_0^T \int_{\Sigma_B} \mathbf{M}_s(y) \nabla u_s \cdot \eta p_s dy dt + \int_0^T \int_{\Sigma_B} \mathbf{M}_s(y) u_s (\nabla p_s) \cdot \eta dy dt \\
& + \iint_{\Omega_{T,H}} (\partial_t w - H(u, w)) p_w dx dt + \iint_{\Sigma_{2,T}} (\mathbf{M}_s(y) \nabla u_s - v) p_v dy dt \\
& + \iint_{\Sigma_{T,H}} (\mathbf{M}_e(y) \nabla u_e - \mathbf{M}_s(y) \nabla u_s) p_1 dy dt + \iint_{\Sigma_{T,H}} (u_e - u_s) p_2 dy dt.
\end{aligned}$$

The first order optimality system is given by the Karush-Kuhn-Tucker (KKT) conditions which result from equating the partial derivatives of L with respect to u_i, u_e, u_s and w equal to zero. First, observe that

$$\begin{aligned}
\left(\frac{\partial L}{\partial u_i}, \delta u_i \right) = & \iint_{\Omega_{T,H}} \left\{ -\beta c_m (\partial_t p_i - \partial_t p_e) - \nabla \cdot (\mathbf{M}_i(x) \nabla p_i) - H_u(u, w) p_w \right. \\
& \left. + 2(u - u_d) + \beta I_{\text{ion}u}(p_i - p_e) \right\} \delta u_i dx dt,
\end{aligned}$$

with boundary condition

$$\begin{aligned}
(\mathbf{M}_i \nabla p_i) \cdot \eta &= 0 \quad \text{on } \Sigma_{T,H}, \\
p_i(T, \cdot) &= p_e(T, \cdot) \quad \text{in } \Omega_H.
\end{aligned}$$

Moreover, we get

$$\begin{aligned}
\left(\frac{\partial L}{\partial u_e}, \delta u_e \right) = & \iint_{\Omega_{T,H}} \left\{ -\beta c_m (\partial_t p_i - \partial_t p_e) + \nabla \cdot (\mathbf{M}_e(x) \nabla p_e) + a p_w \right. \\
& \left. - 2(u - u_d) + \beta I_{\text{ion}u}(p_i - p_e) \right\} \delta u_e dx dt,
\end{aligned}$$

completed with the following boundary conditions

$$\begin{aligned}
p_1 &= p_e, \text{ on } \Sigma_{T,H} \\
p_2 &= (\mathbf{M}_e \nabla p_e) \cdot \eta, \text{ on } \Sigma_{T,H}.
\end{aligned}$$

Next, we have

$$\left(\frac{\partial L}{\partial w}, \delta w \right) = \iint_{\Omega_{T,H}} (-\partial_t p_w - H_w(u, w) p_w + \beta I_{\text{ion}w}(p_i - p_e)) \delta w dx dt.$$

Furthermore, we find

$$\begin{aligned}
\left(\frac{\partial L}{\partial u_s}, \delta u_s \right) = & - \iint_{\Omega_{T,B}} \nabla \cdot (\mathbf{M}_s(x) \nabla p_s) \delta u_s dx dt + \alpha \iint_{\Sigma_{2,T}} v \delta v \\
& - \iint_{\Sigma_{2,T}} p_v \delta v dy dt.
\end{aligned}$$

Herein, we impose the boundary condition

$$\alpha v = p_v \text{ on } \Sigma_{2,T}.$$

Herein $I_{\text{ion}u}$, $I_{\text{ion}w}$, H_u and H_w are the derivatives of I_{ion} and H with respect to u , w , respectively. Collecting the previous results, we get the following adjoint equations :

$$\begin{cases} -\beta c_m \partial_t p - \nabla \cdot (\mathbf{M}_i(x) \nabla p_i) - H_u(u, w) p_w + \beta I_{\text{ion}u} p + 2(u - u_d) = 0, & (t, x) \in \Omega_{T,H}, \\ -\beta c_m \partial_t p + \nabla \cdot (\mathbf{M}_e(s) \nabla p_e) + H_u(u, w) p_w + \beta I_{\text{ion}u} p - 2(u - u_d) = 0, & (t, x) \in \Omega_{T,H}, \\ -\nabla \cdot (\mathbf{M}_s(x) \nabla p_s) = 0 & (t, x) \in \Omega_{T,B}, \\ -\partial_t p_w - H_w(u, w) p_w + \beta I_{\text{ion}w} p = 0 & (t, x) \in \Omega_{T,H}, \end{cases} \quad (4.3)$$

where $p := p_i - p_e$, completed with the following conditions (boundary and final time) :

$$\begin{cases} p(\cdot, T) = 0 & \text{in } \Omega_H, \\ (\mathbf{M}_i(y) \nabla p_i) \cdot \eta = 0 & (t, y) \in \Sigma_{T,H}, \\ (\mathbf{M}_e(s) \nabla p_e) \cdot \eta = (\mathbf{M}_s(y) \nabla p_s) \cdot \eta & (t, y) \in \Sigma_{T,H}, \\ p_e = p_s & \text{on } \Sigma_{T,H}, \\ (\mathbf{M}_s(y) \nabla p_s) \cdot \eta = 0 & (t, y) \in \Sigma_{1,T}, \\ (\mathbf{M}_s(y) \nabla p_s) \cdot \eta = 0 & (t, x) \in \Sigma_{2,T}, \\ p_s = p_v = \alpha v & \text{on } \Sigma_{2,T}. \end{cases} \quad (4.4)$$

Note that in (4.4), we enforce the continuity conditions for the adjoint variables and their derivatives on the heart-bath surface. To find the optimal conditions, we calculate the gradient of the functional $J(v)$:

$$\left(\frac{\partial L}{\partial v}, \delta v \right) = \iint_{\Sigma_{2,T}} (\alpha v - p_v) \delta v dy dt \text{ and } \nabla J(v) = \frac{\partial L}{\partial v}.$$

The optimality condition is then

$$\alpha v = p_v \quad \text{on } \Sigma_{2,T}.$$

Finally, we introduce the condition of compatibility : we suppose that p_e has a zero-mean :

$$\int_{\Omega_H} p_e(t, x) = 0, \quad \text{for all } t \in (0, T). \quad (4.5)$$

4.3. Existence of the solution of adjoint problem. In this subsection, we sketch the proof of the existence of the solution of adjoint system. Now we define our weak solution to adjoint problem (4.3):

Definition 4.1 (Weak solution). *A weak solution to system (1.5) is a five tuple function $(p_i, p_e, p_s, p := p_i - p_e, p_w)$ such that $p \in L^2(0, T, H^1(\Omega_H))$, $p, p_i, p_e \in L^2(0, T, \tilde{H}^1(\Omega_H))$, $\partial_t p \in L^2(0, T, (H^1(\Omega_H))')$, $p_s \in L^2(0, T, H^1(\Omega_B))$, $p_w \in C([0, T], L^2(\Omega_H))$, and satisfying the following weak formulation*

$$\begin{aligned} & \iint_{\Omega_{T,H}} -\beta c_m \partial_t p \phi_i + \iint_{\Omega_{T,H}} \mathbf{M}_i(x) \nabla p_i \nabla \phi_i + \iint_{\Omega_{T,H}} \beta I_{\text{ion}u}(u, w) p \phi_i \\ & - \iint_{\Omega_{T,H}} H_u(u, w) p_w \phi_i + \iint_{\Omega_{T,H}} 2(u - u_d) \phi_i = 0, \\ & \iint_{\Omega_{T,H}} -\beta c_m \partial_t p_e - \iint_{\Omega_H} \mathbf{M}_e(x) \nabla p_e \nabla \phi_e + \iint_{\Omega_B} \mathbf{M}_s(x) \nabla p_s \nabla \phi_s \\ & + \iint_{\Omega_{T,H}} \beta I_{\text{ion}u}(u, w) p \phi_e + \iint_{\Omega_{T,H}} H_u(u, w) p_w \phi_e - \iint_{\Omega_{T,H}} 2(u - u_d) \phi_e = 0 \\ & \iint_{\Omega_{T,H}} -\partial_t p_w \phi_w - H_w(u, w) p_w \phi_w + \beta I_{\text{ion}w}(u, w) p \phi_w = 0. \end{aligned}$$

for all $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H))$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$.

Our second main result is the following existence theorem to the adjoint problem.

Theorem 4.2. *Assume conditions (2.1) and (2.2), then there exists a weak solution to the system (4.3). Moreover, the weak solution is unique.*

The proof of Theorem 4.2 (the existence of weak solution) is based on Faedo-Galerkin method. Let us indicate its main steps. To prove existence of the solution, we regularize the system (4.3) in the spirit of the system (3.1), in the left-hand side of the first equation of this system we add the term $\varepsilon \partial_t p_i$ (the term $-\varepsilon \partial_t p_e$ is added into the analogous equation written for p_e).

We look for finite dimensional approximate solution to the problem (4.3) (we complete the system (4.3) with the boundary and initial conditions (4.4): as sequences defined for $t \geq 0$ and $x \in \bar{\Omega}$ by

$$\left\{ \begin{array}{l} p_n(t, x) = \sum_{l=1}^n c_{n,l}(t) e_l(x), p_{n,j}(t, x) = \sum_{l=1}^n c_{n,j,l}(t) e_l(x), \text{ for } j = i, e, \\ p_{n,s}(t, x) = \sum_{l=1}^n c_{n,s,l}(t) e_l(x), p_{w_n}(t, x) = \sum_{l=1}^n d_{n,l}(t) e_l(x). \end{array} \right.$$

For all $k = 1, \dots, n$:

$$\left\{ \begin{array}{l} (-\beta c_m \partial_t p_n - \varepsilon \partial_t p_{n,i} - \nabla \cdot (\mathbf{M}_i \nabla p_{n,i}), e_k)_{L^2(\Omega_H)} = (H_u(u, w) p_{w_n} - \beta I_{\text{ion}u} p_n + 2(u - u_d), e_k)_{L^2(\Omega_H)}, \\ (-\beta c_m \partial_t p_n + \varepsilon \partial_t p_{n,e} + \nabla \cdot (\mathbf{M}_e \nabla p_{n,e}), e_k)_{L^2(\Omega_H)} = (-H_u(u, w) p_{w_n} - \beta I_{\text{ion}u} p_n + 2(u - u_d), e_k)_{L^2(\Omega_H)}, \\ -(\nabla \cdot (\mathbf{M}_s \nabla p_{n,s}), e_k)_{L^2(\Omega_B)} = 0, \\ (-\partial_t p_{w_n}, e_k) = (H_w(u, w) p_{w_n} + \beta I_{\text{ion}w} p_n, e_k)_{L^2(\Omega_H)}, \end{array} \right.$$

with the final time conditions :

$$\left\{ \begin{array}{l} p_n(T, x) = 0, \\ p_{n,j}(T, x) = p_{T,n,j}(x) = \sum_{l=1}^n c_{n,j,l}(T) e_l(x), \\ p_{w_n}(T, x) = 0. \end{array} \right.$$

for $j = i, e$. Proceeding exactly as in Subsection 3.1, we prove easily the existence interval $(0, T]$ for the Faedo-Galerkin solutions p_n , $p_{n,j}$ and p_{w_n} for $j = i, e, s$. In the next step, we will prove the global existence of the Faedo Galerkin weak solution. We multiply $p_{i,n}$, $-p_{e,n}$, $p_{s,n}$ and p_{w_n} by the first, second and the third equations in (3.1), respectively. Then integrating over (t, T) and using Young and Gronwall inequalities, we get for $t \in (0, T]$:

If $p_{i,T}, p_{e,T} \in L^2(\Omega_H)$, then there exist constants $c_1, c_2 > 0$ not depending on n such that

$$\|p_n\|_{L^\infty(t,T;L^2(\Omega_H))} + \sum_{j=i,e} \|\sqrt{\varepsilon} p_{n,j}\|_{L^\infty(t,T;L^2(\Omega_H))} + \|p_{w_n}\|_{L^\infty(t,T;L^2(\Omega_H))} \leq c_1, \quad (4.6)$$

$$\sum_{j=i,e} \|\nabla p_{n,j}\|_{L^2(t,T;L^2(\Omega_H))} + \|\nabla p_{n,s}\|_{L^2(t,T;L^2(\Omega_B))} \leq c_2. \quad (4.7)$$

Moreover, if $p_{i,T}, p_{e,T} \in \tilde{H}^1(\Omega_H)$, then there exists a constant $c_3 > 0$ such that :

$$\|\partial_t p_n\|_{L^2(t,T;L^2(\Omega_H))} + \sum_{j=i,e} \|\sqrt{\varepsilon} \partial_t p_{n,j}\|_{L^2(t,T;L^2(\Omega_H))} \leq c_3. \quad (4.8)$$

The next step is to show that the local solution constructed above can be extended to $(0, T]$ but this can be done exactly as in [7], so we omit the details. Note that the consequence of Lemma 3.1 is the following convergence (at the cost of extracting subsequences, which we do not bother to relabel), we can assume there exist limit functions $(p_\varepsilon, p_{i,\varepsilon}, p_{e,\varepsilon}, p_{s,\varepsilon}, p_{w_\varepsilon})$ such that, for ε fixed :

$$\left\{ \begin{array}{l} p_n \rightharpoonup p_\varepsilon \text{ in } L^2(0, T; H^1(\Omega_H)), \\ p_{n,i} \rightharpoonup p_{i,\varepsilon} \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ p_{n,e} \rightharpoonup p_{e,\varepsilon} \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ p_{n,s} \rightharpoonup p_{s,\varepsilon} \text{ weakly in } L^2(0, T; H^1(\Omega_B)), \\ p_\varepsilon \rightarrow p \text{ almost everywhere in } \Omega_{H,T} \text{ and strongly in } L^2, \\ p_{w_n} \rightarrow p_{w_\varepsilon} \text{ strongly in } L^2(\Omega_{T,H}). \end{array} \right. \quad (4.9)$$

Next, using (4.9) sending n to ∞ in the following weak formulation delivers the existence of a weak global solution for ε fixed:

$$\begin{aligned}
& \iint_{\Omega_{T,H}} -\beta c_m \partial_t p_n \phi_i - \varepsilon \iint_{\Omega_{T,H}} \partial_t p_{n,i} \phi_i + \iint_{\Omega_{T,H}} \mathbf{M}_i(x) \nabla p_{n,i} \cdot \nabla \phi_i \\
&= \iint_{\Omega_{T,H}} (H_u(u, w) p_{w_n} - \beta I_{\text{ion}u} p_n + -2(u - u_d)) \phi_i \\
& \iint_{\Omega_{T,H}} -\beta c_m \partial_t p_n \phi_e + \varepsilon \iint_{\Omega_{T,H}} \partial_t p_{n,e} \phi_e - \iint_{\Omega_H} \mathbf{M}_e(x) \nabla p_{n,e} \cdot \nabla \phi_e \\
&+ \iint_{\Omega_B} \mathbf{M}_s(x) \nabla p_{n,s} \cdot \nabla \phi_s = \iint_{\Omega_{T,H}} (-H_u(u, w) p_{w_n} - \beta I_{\text{ion}u} p_n + 2(u - u_d)) \phi_e \\
& \iint_{\Omega_{T,H}} -\partial_t w_n \phi_w = \iint_{\Omega_{T,H}} (-H_u(u, w) p_{w_n} - \beta I_{\text{ion}u} p_n + 2(u - u_d)) \phi_w.
\end{aligned} \tag{4.10}$$

for all $\phi_i, \phi_e \in L^2(0, T, \tilde{H}^1(\Omega_H))$, $\phi_s \in L^2(0, T, H^1(\Omega_B))$ and $\phi_w \in C([0, T], L^2(\Omega_H))$. The next goal is to send ε to 0. Note that, by the (weak) lower semicontinuity properties of norms, the estimates in (4.6) and (4.7) hold with $(p_n, p_{n,i}, p_{n,e}, p_{n,s}, p_{w_n})$ replaced by $(p_\varepsilon, p_{i,\varepsilon}, p_{e,\varepsilon}, p_{s,\varepsilon}, p_{w_\varepsilon})$, respectively. Therefore we have the following estimates:

If $p_{i,\varepsilon,T}, u_{e,\varepsilon,T} \in L^2(\Omega_H)$, then there exist constants $cst_1, cst_2, cst_3, cst_4 > 0$ not depending on ε such that

$$\|p_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_H))} + \sum_{j=i,e,s} \|\sqrt{\varepsilon} p_{j,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_H))} \leq cst_1 \tag{4.11}$$

$$\sum_{j=i,e} \|\nabla p_{j,\varepsilon}\|_{L^2(\Omega_{T,H})} + \|\nabla p_{s,\varepsilon}\|_{L^2(\Omega_{T,B})} \leq cst_2 \tag{4.12}$$

Moreover, if $u_{i,\varepsilon,0}, u_{e,\varepsilon,0} \in \tilde{H}^1(\Omega_H)$ and $u_{e,0} \in L^4(\Omega_H)$, then there exists a constant $cst_5 > 0$ such that :

$$\|\partial_t u_\varepsilon\|_{L^2(\Omega_H)} + \sum_{j=i,e,s} \|\sqrt{\varepsilon} \partial_t u_{\varepsilon,j}\|_{L^2(\Omega_H)} \leq cst_5. \tag{4.13}$$

In view of these estimations, we can assume there exist limit functions (p, p_i, p_e, p_s, p_w) such that (at the cost of extracting subsequences, which we do not bother to relabel) the following convergences hold as $\varepsilon \rightarrow 0$

$$\begin{cases} p_\varepsilon \rightharpoonup p \text{ in } L^2(0, T; H^1(\Omega_H)), \\ p_{i,\varepsilon} \rightharpoonup p_i \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ p_{e,\varepsilon} \rightharpoonup p_e \text{ weakly in } L^2(0, T; \tilde{H}^1(\Omega_H)), \\ p_{s,\varepsilon} \rightharpoonup p_s \text{ weakly in } L^2(0, T; H^1(\Omega_B)), \\ p_\varepsilon \rightarrow p \text{ almost everywhere in } \Omega_{H,T} \text{ and strongly in } L^2, \\ p_{w_\varepsilon} \rightarrow p_w \text{ strongly in } L^2(\Omega_{T,H}). \end{cases} \tag{4.14}$$

Moreover if $p_{i,\varepsilon,T}, p_{e,\varepsilon,T} \in \tilde{H}^1(\Omega_H)$, we find

$$\begin{cases} \partial_t p_\varepsilon \rightharpoonup \partial_t p \text{ in } L^2(\Omega_{TH}), \\ \varepsilon \partial_t p_{j,\varepsilon} \rightharpoonup 0 \text{ in } L^2(\Omega_{TH}) \text{ for } j = i, e. \end{cases} \tag{4.15}$$

Thus, using (4.14)- (4.15) and sending ε to 0 in the weak formulation (4.10), where $(p_n, p_{n,i}, p_{n,e}, p_{n,s}, p_{w_n})$ replaced by $(p_\varepsilon, p_{i,\varepsilon}, p_{e,\varepsilon}, p_{s,\varepsilon}, p_{w_\varepsilon})$, we get the existence of a weak solution (p, p_i, p_e, p_s, p_w) .

Note that, to ensure the well-posedness of adjoint problem, it is necessary to have some growth conditions for the various derivatives introduced in the derivation of the adjoint system (4.3). For this we utilized the condition (2.3) for the ionic function $I_{\text{ion}}(u, w)$ (i.e. $I_{1,\text{ion},u}(u) \geq -C_I$). Since $H(u, w)$ is a linear function that it's derivative with respect to u or w is constant ($H_u(u, w) = a$ and $H_w(u, w) = -b$). The uniqueness of the adjoint solution can be obtained by using these growth conditions (for the derivatives of the functions $I_{\text{ion}}(u, w)$ and $H(u, w)$) and the techniques in Theorem 3.3. Comparing to the proof of the uniqueness to

the direct problem, here we can use the test function only $L^2(H^1)$ (recall that the test functions for direct problem are in $L^2(H^1) \cap L^4$, refer Definition 2.1).

5. NUMERICAL APPROACH

In this section we demonstrate the numerical approach to solve the optimization problem (4.2) subject to the bidomain-bath equations. Here, we provide the brief overview of the numerical discretization of PDEs which arise solving the complete optimality system. For computer implementation, we use the following elliptic-parabolic form of the bidomain equations subject to the boundary and transmission conditions stated in (1.5):

$$0 = \nabla \cdot (\mathbf{M}_s(x) \nabla u_s) \quad \text{in } \Omega_{T,B} \quad (5.1)$$

$$0 = \nabla \cdot (\mathbf{M}_i(x) + \mathbf{M}_e(x)) \nabla u_e + \nabla \cdot (\mathbf{M}_i(x) \nabla u) \quad \text{in } \Omega_{T,H} \quad (5.2)$$

$$\partial_t u = \nabla \cdot (\mathbf{M}_i(x) \nabla u) + \nabla \cdot (\mathbf{M}_i \nabla u_e) - I_{\text{ion}}(u, w) + I_{tr}(x, t) \quad \text{in } \Omega_{T,H} \quad (5.3)$$

$$\partial_t w = H(u, w) \quad \text{in } \Omega_{T,H}, \quad (5.4)$$

where the term I_{tr} is the transmembrane current density stimulus as delivered by the intracellular electrode and $G(u, w)$ determines the evolution of the gating variables. To model $G(u, w)$, we use the modified FHN model [21] for our optimal control simulations. For the spatial discretization of partial differential equations in the primal and dual equations, we employed the piecewise bilinear finite element method. The higher order Rosenbrock time stepping methods are used for the temporal discretization, specifically we used the ROWDA [16] method which has 3 internal stages to solve the algebraic system at each time step, see for more details in [13, Section 4]. Here we briefly mention the algorithmic procedure to solve the primal equations, for the complete implementation details we request the readers to refer [13].

- 1: Initialize the state solution $\mathbf{u}_0, \mathbf{w}_0$ which was created using the S1-S2 protocol during the pre-shock stage. Set final termination time $T = 4$ msec.
- 2: Project the transmembrane solution from the tissue domain (\mathbf{u} on Ω_H) to the integrated domain Ω by using inter-processor communication in parallel environment, as explained in [13]. Here zero entries are padded in the global solution vector which corresponds to the nodal points at the bath domain.
- 3: Use the membrane potential solution \mathbf{u}^i of the integrated domain at time t^i , then solve the discretized elliptic system for \mathbf{u}_{se}^{i+1} at time t^{i+1} by using the stabilized saddle point approach, see for more details [12] to incorporate the zero mean condition into the solution procedure. After the full discretization of the PDEs we obtain a system of linear algebraic equations and to solve the linear system we employed a Conjugate Gradient (CG) method with AMG preconditioner [9], which is developed using a greedy heuristic algorithm for the aggregation based on a strength of connection criterion.
- 4: Communicate the extracellular potential solution from the complete domain (\mathbf{u}_{se} on Ω) to the cardiac tissue domain (\mathbf{u}_e on Ω_H) at time t^{i+1} .
- 5: Finally, by utilizing the computed solutions \mathbf{u}_e^{i+1} solve the discretized parabolic equation (5.3) for \mathbf{u}^{i+1} at time t^{i+1} by applying the ROWDA. We used a BiCGSTAB method with Jacobi preconditioning to solve the linear algebraic system which arrived after the time discretization at each internal stage of the ROWDA method.
- 6: Update the simulation time $t^{i+1} = t^i + \delta t$ and go to step 2 for the evaluation of next time step solution.

To solve the complete optimality system we used the Newton-CG optimization, see for complete details in [13]. The line search algorithm based on an Armijo type condition. In our computations, the termination of the optimization algorithm is based on the following condition:

$$\|\nabla J(v^k)\|_{L^2} \leq 10^{-3} \cdot |J(v^k)| \quad \text{or} \quad |J(v^k) - J(v^{k-1})| \leq 10^{-4} \quad (5.5)$$

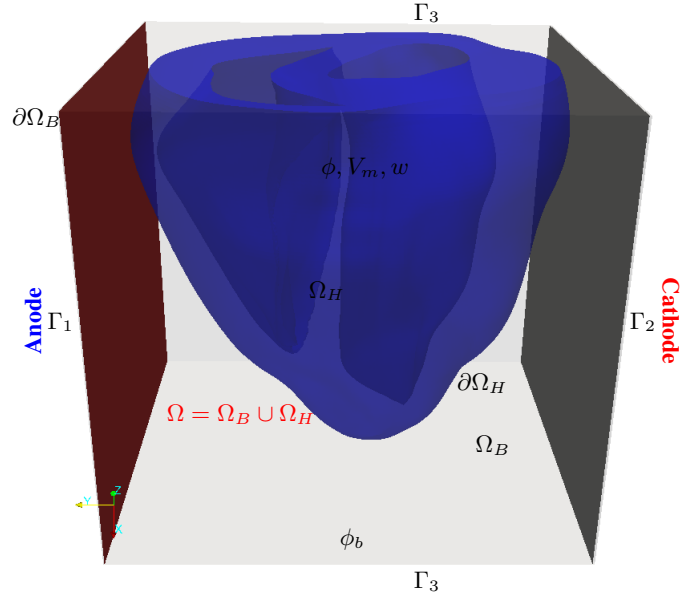


FIGURE 1. Schematic picture of the computational domain.

If this condition was not satisfied within a prescribed number of 12 iterations, the algorithm was terminated. An Armijo type condition is imposed in the line search algorithm. The complete optimization code was developed based on the public domain FEM software package DUNE [6].

6. NUMERICAL RESULTS

In this section we demonstrate the numerical results based on the realistic 3D geometries to support our numerical analysis. In our computations, full three spatial dimensional rabbit ventricle geometry is considered which is generated based on the histological images [23]. The size of the whole domain, $\Omega = \Omega_H \cup \Omega_B$, is $2.91 \times 3.12 \times 2.8 \text{ cm}^3$. The computational domain of the whole geometry comprises of 5,082,272 tetrahedral elements and 862,515 nodal points. The embedded cardiac tissue domain consists of 3,073,529 tetrahedrons and 547,680 nodal points. The computational domain and various relevant subdomains are depicted in Figure 1.

The conductivity values were chosen to arrive at physiologically relevant conduction velocities of 0.62 m/s and 0.38 m/s along and transverse to the principal fiber axes, respectively, and to keep anisotropy ratios within the range of values reported in experimental studies [24]. The presented numerical results were computed on a Linux cluster consisting of ten nodes where each node consists of 2 quad-core AMD Opteron processors 8356 clocked at 2.3 GHz and equipped with 1TB RAM. All presented results are based on the parallel Newton-CG algorithm using 64 cores.

Here we briefly explain the complete time horizons which are used in our simulations. To induce the reentry, we followed the S1-S2 protocol, see [10] for further details. At the initial time $t = 0$ msec the first stimulus (S1) is created at the bottom boundary of the cardiac tissue which introduces a planar wavefront. The second stimulus is activated at $t = 185$ msec at the point $(-0.185, 0.61, -0.55)$ with a sphere volume of 0.008 cm^3 which introduces a reentrant wave front wave in the simulation. The solution at $t = 520$ msec was then chosen as the initial state for the optimal control experiment. We run the optimization algorithm for the simulation time of 4 msec and then we run the post shock simulation for checking the successful defibrillation. For brevity, the three temporal horizons: reentry induction, optimization duration, and post shock simulation as shown in Figure 2.

The initial solution of the extracellular potential on the integrated domain, the transmembrane voltage and the gating solution is depicted in Figure 3 at simulation time 520 msec. In our simulations, the direct

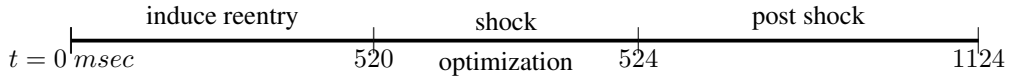
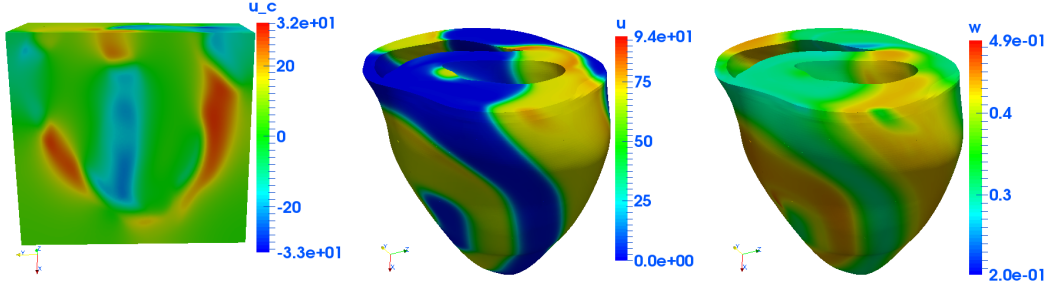
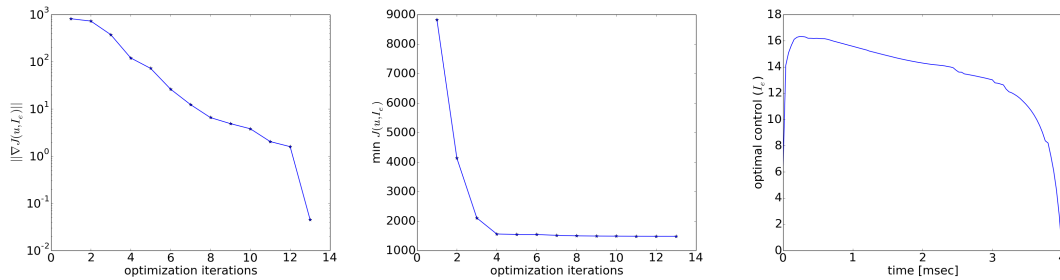


FIGURE 2. Different time horizons considered in the computations.

simulation was carried out until time $t = 3000$ msec to ensure that the induced reentry is maintained for a prolonged period of time.

FIGURE 3. The initial solution of state variables u_c , u and w at time 520 msec.

The norm of the gradient and the minimization values of the cost functional and the optimal control is depicted in Figure 4. The L^2 -norm of the gradient reduces slowly at the beginning than the later. At the end of the optimization iterations we observe that optimization algorithm takes full step length and converges superlinearly. The corresponding value of the minimization of the cost functional is shown in middle panel of the Figure 4. Here the optimization algorithm is terminated after 13 iterations due to less progress in the minimization values of the cost functional. The optimal control value ($I_e(t)$) is illustrated in right panel of the Figure 4. As mentioned earlier that the desired trajectory was created using the constant current of 18 mA/cm³ for the shock period time duration. We can observe that the optimal control approach computes the optimal state solution by constructing the optimal control trajectory during the time horizon. The initial guess for the optimization algorithm was chosen 6 mA/cm³. At the beginning of the shock period, the optimal control requires more current then the later period of time and it needs less current at the end of shock period. The optimal control approach computes the total current ($\int_0^T I_e(t)dt$) is 54.28 mA/cm³ against the ad-hoc strategy total current which is 72 mA/cm³.

FIGURE 4. The L^2 -norm of the gradient of the cost functional at the left, the minimization value of the cost functional $J(u, I_e)$ at the middle and the optimal control $I_e(t)$ at the right.

The uncontrolled solution of the transmembrane voltage is shown at various time instances in Figure 5. Here we can observe that the two re-entrant waves appear in the computational domain, one is at the front

face and the other is at the back face of the ventricle geometry, see first Figure 5. The reentrant excitation waves remain in the computational domain at later times and this can be attributed to a ventricle fibrillation in the real life situations.

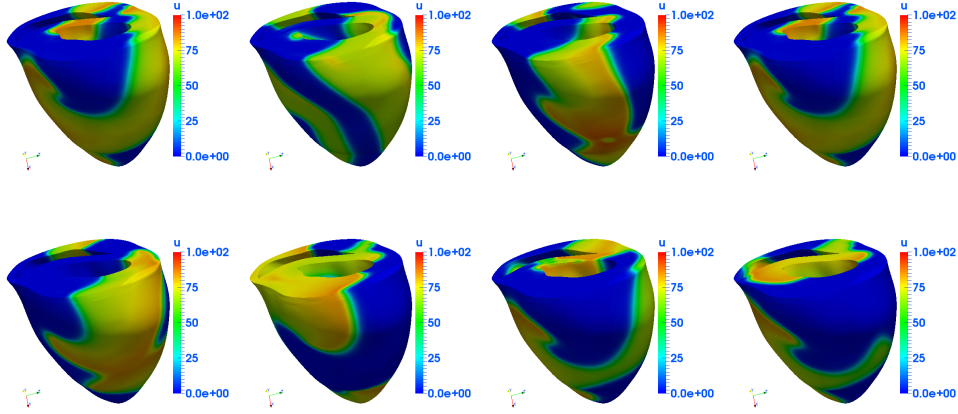


FIGURE 5. The uncontrolled solution of transmembrane voltage u at times $t= 529.04, 540, 578, 627, 683, 729, 789$ and 884 msec respectively.

The optimal state solution of the transmembrane voltage is depicted in Figure 6 at different time instances. We can observe that there were lot of virtual electrode polarizations present once the external stimulus applied to the bidomain equations, see left panel of Figure 6. At the end of the shock period, at the appearing of the virtual electrode polarization regions the membrane potential strength was increased and effectively reduced the excitable gap at computational domain.

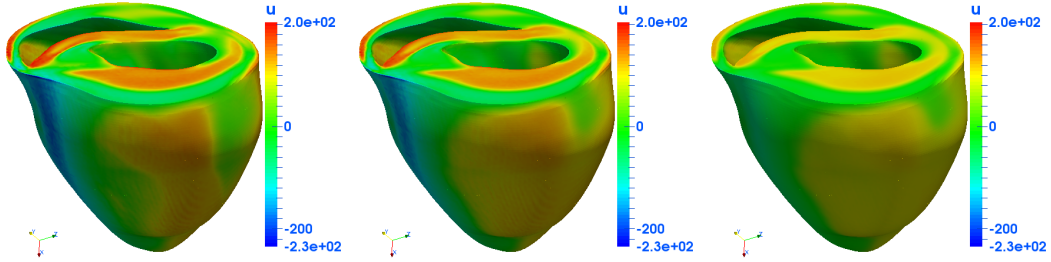


FIGURE 6. The optimal state solution of transmembrane voltage at times $520.8, 522.4$ and 524 (Here the color bar scale is fixed according the transmembrane voltage solution range at time 520.8).

The optimal state solution of the transmembrane voltage is depicted in Figure 6. Here we can observe that at first time instance, $t = 529.04$ during the post shock simulation, the existed virtual electrodes polarization wave fronts slowly disappear and then it forms a reentry wave different to the initial wave front, see time instance $t = 540$ and 576 msec. We can see that there were two more excitation waves fronts appeared before disappearing completely. All the reentrant waves were successfully terminated at the time instant 878 msec. The complete optimization algorithm took approximately 2 days and 8 hours on 64 cores.

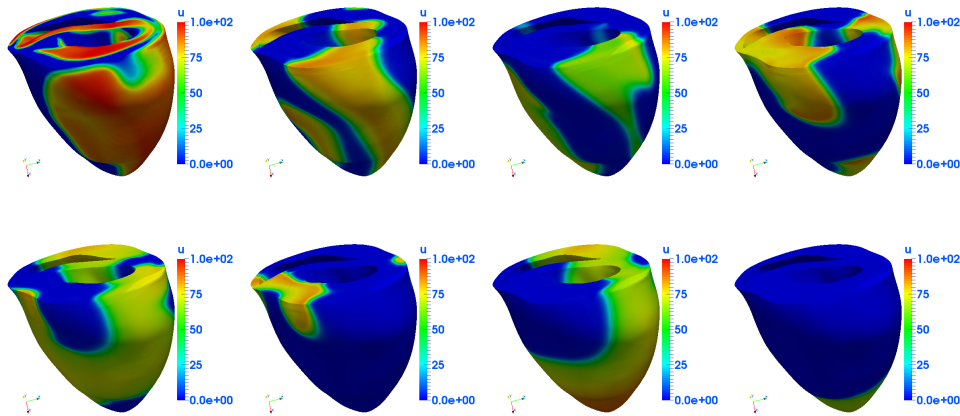


FIGURE 7. The controlled solution of transmembrane voltage u during the post shock simulation at times $t= 529.04, 540, 576, 627, 683, 729, 816$ and 864 msec respectively (Here the color bar scale values are rescaled according to the uncontrolled solution for better comparison).

7. CONCLUSION

In this paper, an optimal control problem constrained by the bidomain-bath equations in electrocardiology is investigated. The well-posedness analysis of the direct and adjoint problems are performed and the convergence of the Faedo-Galerkin scheme is addressed in detail. The theory was developed for the phenomenological ionic models in our study, specifically we incorporated Fitz-Hugh Nagumo model. The derivation of the complete optimality system was demonstrated. Moreover, the existence of the optimal control solution and first order optimality conditions were discussed as well. This provides the motivation to study the well-posedness of the optimal control of complex physiological models in future. We develop a rigorous study of mathematical analysis of such complex bidomain-bath model equations and numerical realization combined with Fitz-Hugh Nagumo ionic model in realistic 3D geometries. Our study of the problem of control (1.5), is new even if we utilize the method for direct problem in [7] (for an isolated heart) for the well-posedness of the weak solution. Moreover, we prove the existence of the weak solution to adjoint problem.

The numerical results were provided for the successful cardiac defibrillation using the Roger-McCulloch model which is a modified model of Fitz-Hugh Nagumo ionic model. Our numerical study suggests that the optimal control approach requires less total current as compared to the ad-hoc strategy total current which is essential in the clinical practice to avoid the adverse effects on the patients.

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