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A FUNCTIONAL CLT FOR FIELDS OF COMMUTING TRANSFORMATIONS VIA MARTINGALE APPROXIMATION

CHRISTOPHE CUNY, JÉRÔME DEDECKER, AND DALIBOR VOLNÝ

Dedicated to the memory of Mikhail (Misha) Gordin

Abstract. We consider a field $f \circ T_1^{i_1} \circ \cdots \circ T_d^{i_d}$, where $T_1, \ldots, T_d$ are completely commuting transformations in the sense of Gordin. If one of these transformations is ergodic, we give sufficient conditions in the spirit of Hannan under which the partial sum process indexed by quadrants converges in distribution to a brownian sheet. The proof combines a martingale approximation approach with a recent CLT for martingale random fields due to Volný. We apply our results to completely commuting endomorphisms of the m-torus. In that case, the conditions can be expressed in terms of the L^2-modulus of continuity of f.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and T a (non-invertible) measure preserving map. Let U be the associated Koopman operator ($Uf = f \circ T$ for every $f \in L^1(\Omega, \mathcal{A}, \mu)$) and U^* be the associated Perron-Frobenius operator. In 1978, Gordin and Lifšic [10] (see also [8]) observed that if $f = (I - U^*)g$ for some $g \in L^2(\mu)$ (i.e. f is a coboundary for U^*), then one has a decomposition $f = (g - UU^*g) + (U - I)U^*g$ into the sum of a reverse martingale difference plus a coboundary (for U that time). This allows to prove the central limit theorem (CLT) and the weak invariance principle (WIP) for f from the corresponding results for stationary reverse martingale differences.

This fruitful approach presents a part of the martingale-approximation method, known also as Gordin's method and started with the seminal paper [8] from 1969. It has been further developped in many papers (to go beyond the coboundaries for U^*). Let us mention the following references where optimal or sharp results have been obtained concerning the CLT as well as other limit theorems: Hannan [13, 14], Heyde [15], Maxwell and Woodroofe [17], Peligrad and Utev [18], Gordin and Peligrad [11], Cuny [4].

Consider now a family of commuting measure preserving transformations $T_1, \ldots, T_d$. In 2009, Gordin [9] proved a decomposition analogous to the above one (see Section 4), when the transformations are completely commuting (see the next section for the definition) and $f = (I - U_1^*) \cdots (I - U_d^*)g$. However, probably by lack of a CLT for

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"multi-dimensional" reverse martingale differences, he did not derive any CLT from that decomposition.

Very recently, the third author [19] proved such a CLT. Actually, he worked in the setting of martingale differences but, as shown in Section 3, its proof applies equally in the reverse martingale case and yields also the weak invariance principle. In this paper, we provide a suitable reverse-martingale approximation under a condition in the spirit of Hannan, from which the CLT and the WIP follow. Note that the WIP under Hannan's condition has been recently obtained by Volný and Wang [20] in the case where the random field can be expressed as a function of an iid random field. In the one-dimensional setting this condition is known to be sharp (see for instance Dedecker [6]). The results of Volný and Wang can, using [20], be extended to random fields which are not Bernoulli.

We apply these results to prove a CLT and a WIP in the case where the transformations are commuting *dilating* endomorphisms of the m-dimensional torus. Note that, for such commuting endomorphisms (not necessarily dilating), the CLT has been obtained recently by mean of completely different technics under slightly stronger conditions, see Section 5 for a deeper discussion.

2. Setting of the paper

Let us describe our setting.

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space. Consider a family $\{T_1, \dots, T_d\}$ of measure preserving transformations on Ω.

Denote by $U_1, \dots, U_d$ the corresponding Koopman operators and by $U_1^*, \dots, U_d^*$ the associated adjoint operators, also known in that context as the Perron-Frobenius operators. Recall that those operators are characterized as follows

$$U_i f = f \circ T_i \,;$$
$$\int_\Omega U_i f \, g \, d\mu = \int_\Omega f \, U_i^* g \, d\mu \,,$$

for every positive measurable functions f, g and every $i \in \{1, \dots, d\}$.

Definition 1. *We say that the family $\{T_1, \dots, T_d\}$ (or the family $\{U_1, \dots, U_d\}$) is completely commuting if it is commuting and if moreover*

$$U_i U_j^* = U_j^* U_i \quad \textit{for any } i, j \in \{1, \dots, d\},\ i \neq j. \tag{1}$$

Notice that (as already observed by Gordin) this definition is slightly abusive since that property depends on μ. Since μ will be fixed in the sequel, we will not worry about that fact.

We consider now the natural filtrations associated with our transformations. For every $i \in \{1, \dots, d\}$ and every $n \in \mathbb{N}$, denote $\mathcal{F}_n^{(i)} := T_i^{-n}(\mathcal{A})$.

Then, it is well-known (and not hard to prove) that, for every $i \in \{1, \dots, d\}$ and every $n \in \mathbb{N}$, we have

$$\mathbb{E}(f|\mathcal{F}_n^{(i)}) = U_i^n (U_i^*)^n f \quad \text{for any } f \in L^1(\Omega, \mathcal{A}, \mu). \tag{2}$$

On the other hand, since the U_i's are clearly isometries (of any L^p, $1 \le p \le \infty$), we also have for every $i \in \{1, \dots, d\}$

$$U_i^* U_i f = f \quad \text{for any } f \in L^1(\Omega, \mathcal{A}, \mu). \tag{3}$$

The relevance of the property of complete commutation lies in the fact that for every $i, j \in \{1, \dots, d\}$ with $i \neq j$, and every $n \in \mathbb{N}$, the operator U_i and the operator of conditional expectation with respect to $\mathcal{F}_n^{(j)}$ are commuting.

3. An invariance principle for stationary d-fields of reverse martingale differences

In all of this section we suppose given a completely commuting family $\{T_1, \dots, T_d\}$ of transformations on the probability space $(\Omega, \mathcal{A}, \mu)$ and we make use of the previous notations.

We shall use the notation $\underline{n}$ to specify that $\underline{n}$ is a vector. Then, if $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ (with $\mathbb{N} = \{0, 1 \dots\}$, and later $\mathbb{N}^* = \{1, 2 \dots\}$), we shall use the notation

$$U^{\underline{n}} f = U_1^{n_1} \cdots U_d^{n_d} f \quad \text{for any } f \in L^1(\Omega, \mathcal{A}, \mu).$$

Definition 2. *We shall say that $(f_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ is a commuting stationary d-field of reverse martingale differences if there exists $f \in L^1(\Omega, \mathcal{A}, \mu)$ with $\mathbb{E}(f|\mathcal{F}_1^{(i)}) = 0$ for every $i \in \{1, \dots, d\}$ such that $f_{\underline{n}} = U^{\underline{n}} f$.*

Let $\{e_1, \dots, e_d\}$ be the canonical basis in $\mathbb{R}^d$. If $(f_{\underline{n}})_{\underline{n} \in \mathbb{N}^d}$ is a commuting stationary d-field of reverse martingale differences, then, for every $i \in \{1, \dots, d\}$, we have (with f as in the definition), $U_i^* f = 0$ and

$$\mathbb{E}(f_{\underline{n}}|\mathcal{F}_{n_i+m}^{(i)}) = U_i^{n_i+m} (U_i^*)^{n_i+m} U^{\underline{n}} f = U^{\underline{n}+me_i} (U_i^*)^k f = 0 \quad \text{for any } m \in \mathbb{N}^*, \underline{n} \in \mathbb{N}^d. \tag{4}$$

For every $\underline{k}, \underline{h} \in \mathbb{N}^d$, we shall write $\underline{k} \preceq \underline{h}$ if for every $i \in \{1, \dots, d\}$ we have $k_i \le h_i$. For every $\underline{n} \in \mathbb{N}^d$ and every $\underline{t} \in [0,1]^d$, we shall write $[\underline{nt}] := ([n_1 t_1], \dots, [n_d t_d])$, where $[\cdot]$ stands for the integer part.

Let $(U^{\underline{n}} f)_{\underline{n} \in \mathbb{N}^d}$ be a random field on $(\Omega, \mathcal{A}, \mu)$. For every $\underline{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and every $\underline{t} = (t_1, \dots, t_d) \in [0,1]^d$ set

$$S_{\underline{n},\underline{t}}(f) := \sum_{0 \preceq \underline{k} \preceq [\underline{nt}]} \prod_{i=1}^d (k_i \wedge (n_i t_i - 1) - k_i + 1)\, U^{\underline{k}} f\,,$$

$$\text{and} \quad T_{\underline{n},\underline{t}}(f) := \frac{S_{\underline{n},\underline{t}}(f)}{(\prod_{i=1}^d n_i)^{1/2}}\,.$$

Theorem 1. *Let $(U^{\underline{n}}f)_{\underline{n}\in\mathbb{N}^d}$ be a commuting stationary d-field of reverse martingale differences with $f \in L^2(\Omega, \mathcal{A}, \mu)$. Assume that one of the U_i's is ergodic. Then, the process $(T_{\underline{n},\underline{t}}(f))_{\underline{t}\in[0,1]^d})_{\underline{n}\in\mathbb{N}^d}$ converges in law in $C([0,1]^d)$ to $(\|f\|_2 W_{\underline{t}})_{\underline{t}\in[0,1]^d}$, where $(W_{\underline{t}})_{\underline{t}\in[0,1]^d}$ is the standard d-dimensional brownian sheet.*

Remark. Recall that $(W_{\underline{t}})_{\underline{t}\in[0,1]^d}$ is the centered gaussian process characterized by $\mathbb{E}(W_{\underline{s}}W_{\underline{t}}) = \prod_{i=1}^d (s_i \wedge t_i)$. The ergodicity will be needed at the very end of the proof, in order to prove Lemma 4 below.

As usual we shall prove that result in two steps. The first one consists in proving tightness. The second one consists in proving convergence of the finite dimensional distributions.

3.1. **Proof of the tightness.** The tightness has been proved by Volný and Wang [20] in the case of martingale differences rather than *reverse* martingale differences. Their argument carry on to our setting provided that we have a maximal inequality of Cairoli's type for reverse martingale fields.

We shall just state the appropriate version of Cairoli's maximal inequality needed and refer to [20] for the proof of the tightness. We state it in the stationary case, but it holds in a more general setting.

Proposition 2 (Cairoli, [1])**.** *Let $(U^{\underline{n}}f)_{\underline{n}\in\mathbb{N}^d}$ be a commuting stationary d-field of reverse martingale differences with $f \in L^p(\Omega, \mathcal{A}, \mu)$, $p > 1$. Then, for every $\underline{n} \in \mathbb{N}^d$ we have*

$$\mathbb{E}\left(\max_{1\preceq\underline{k}\preceq\underline{n}}\left|\sum_{0\preceq\underline{i}\preceq\underline{k}} U^{\underline{i}}f\right|^p\right) \leq 2^{dp}\left(\frac{p}{p-1}\right)^d \mathbb{E}\left(\left|\sum_{0\preceq\underline{i}\preceq\underline{n}} U^{\underline{i}}f\right|^p\right) \tag{5}$$

Proof. We explain how to derive the result from Cairoli's original result when $d = 2$. Let $\underline{n} \in \mathbb{N}^2$. For every $0 \preceq \underline{k} \preceq \underline{n}$ write $g_{\underline{k}} := U^{\underline{n}-\underline{k}}f$. Then, $(\sum_{0\preceq\underline{i}\preceq\underline{k}} g_{\underline{i}})_{0\preceq\underline{k}\preceq\underline{n}}$ is a so-called orthomartingale. Moreover, we see that for every $0 \preceq \underline{k} \preceq \underline{n}$,

$$\begin{aligned}\sum_{0\preceq\underline{i}\preceq\underline{k}} U^{\underline{i}}f &= \sum_{0\preceq\underline{i}\preceq\underline{n}} U^{\underline{i}}f - \sum_{(k_1,0)\preceq\underline{i}\preceq\underline{n}} U^{\underline{i}}f - \sum_{(0,k_2)\preceq\underline{i}\preceq\underline{n}} U^{\underline{i}}f + \sum_{(k_1,k_2)\preceq\underline{i}\preceq\underline{n}} U^{\underline{i}}f \\ &= \sum_{0\preceq\underline{i}\preceq\underline{n}} g_{\underline{i}} - \sum_{0\preceq\underline{i}\preceq(n_1-k_1,0)} g_{\underline{i}} - \sum_{0\preceq\underline{i}\preceq(0,n_2k_2)} g_{\underline{i}} + \sum_{0\preceq\underline{i}\preceq(n_1-k_1,n_2-k_2)} g_{\underline{i}}\,.\end{aligned}$$

□

3.2. **Convergence of the finite dimensional distributions.** We have to prove that for every $(\underline{t}_k)_{0\leq k\leq L}$ with $\underline{t}_k \in [0,1]^d$,

$$\big(T_{\underline{n},\underline{t}_0}(f), \ldots, T_{\underline{n},\underline{t}_L}(f)\big) \Rightarrow \big(W_{\underline{t}_0}(f), \ldots W_{\underline{t}_L}(f)\big) \tag{6}$$

as $n_1, \ldots, n_d \to +\infty$.

Note first that, if

$$\widetilde{S}_{\underline{n},\underline{t}}(f) := \sum_{0\preceq \underline{k}\preceq [\underline{nt}]-\underline{1}} U^{\underline{k}} f \quad \text{and} \quad \widetilde{T}_{\underline{n},\underline{t}}(f) := \frac{\widetilde{S}_{\underline{n},\underline{t}}(f)}{(\prod_{i=1}^d n_i)^{1/2}}\,,$$

where $\underline{1} = (1,\dots,1)$, then $\|T_{\underline{n},\underline{t}}(f) - \widetilde{T}_{\underline{n},\underline{t}}(f)\|_2 \to 0$ as $n_1,\dots,n_d \to +\infty$. This follows easily from the fact that, using the reverse-martingale property,

$$\|S_{\underline{n},\underline{t}}(f) - \widetilde{S}_{\underline{n},\underline{t}}(f)\|_2^2 \le \|f\|_2^2 \left(\sum_{i=1}^d \prod_{j\in\{1,\dots,d\}, j\neq i} n_j \right).$$

Hence, is suffices to prove (6) with $\widetilde{T}_{\underline{n},\underline{t}_i}$ instead of $T_{\underline{n},\underline{t}_i}$. Let $(a_k)_{0\le k\le L}$ be $L+1$ real numbers. By the Cramer-Wold device, it suffices to prove that

$$\sum_{k=0}^L a_k \widetilde{T}_{\underline{n},\underline{t}_k}(f) \Rightarrow \sum_{k=0}^L a_k W_{\underline{t}_k}(f)\,.$$

As a second simple remark, note that the sum $\sum_{k=0}^L a_k \widetilde{T}_{\underline{n},\underline{t}_k}(f)$ can be written as a weighted sum over disjoint and adjacent rectangles. Hence, it suffices to prove the convergence in distribution for such rectangles.

We shall make the proof when $d=2$, the general case can be proved by induction. Let $t_0 = 0 < t_1 < \dots < t_K \le 1$ and $s_0 = 0 < s_1 < \dots < s_K \le 1$. Let also $(a_{k,\ell})_{1\le k,\ell\le K}$ be real numbers. From the remarks above, it suffices to prove that

$$\begin{aligned} V_{n_1,n_2} &= \frac{1}{\sqrt{n_1 n_2}} \sum_{k=1}^K \sum_{\ell=1}^K a_{k,\ell} \sum_{i=[n_1 t_{k-1}]}^{[nt_k]-1} \sum_{j=[n_2 s_{\ell-1}]}^{[ns_\ell]-1} U_1^i U_2^j(f) \\ &\Rightarrow \sum_{k=1}^K \sum_{\ell=1}^K a_{k,\ell} \left(W_{(t_k,s_\ell)}(f) + W_{(t_{k-1},s_{\ell-1})}(f) - W_{(t_k,s_{\ell-1})}(f) - W_{(t_{k-1},s_\ell)}(f) \right)\,. \end{aligned}$$

as $n_1, n_2 \to \infty$. Notice that the random variable on right hand is distributed according to $\mathcal{N}(0,\Gamma)$, with

$$\Gamma = \|f\|_2^2 \sum_{k=1}^K \sum_{\ell=1}^K a_{k,\ell}^2 (t_k - t_{k-1})(s_\ell - s_{\ell-1})\,.$$

Clearly, it suffices to prove the desired convergence in distribution when $n_1, n_2 \to +\infty$ along any sequence $(m_r, n_r)_{r\ge 1}$. Hence, let us fix a sequence $(m_r,n_r)_{r\ge1}$ such that $m_r, n_r \to +\infty$ as $r\to+\infty$. It remains to prove that

$$V_r = \frac{1}{\sqrt{m_r n_r}} \sum_{k=1}^K \sum_{\ell=1}^K a_{k,\ell} \sum_{i=[m_r t_{k-1}]}^{[m_r t_k]-1} \sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f) \Rightarrow \mathcal{N}(0,\Gamma) \tag{7}$$

Proof of (7)**.** Since one of the $U_i's$ is assumed to be ergodic, let us assume that U_2 is.

We will apply the following result of McLeish as stated in Hall and Heyde [12] (see Theorem 3.6 p. 77). This theorem is stated for an array of martingale differences, but a simple change of time gives the next proposition. We first mention what we mean by an array of reverse martingale differences.

Definition 3. *Let $((X_{r,k})_{0\le k\le p_r-1})_{r\ge 1}$ be an array of variables and $((\mathcal{G}_{r,k})_{0\le k\le p_r-1})_{r\ge 1}$ be an array of σ-algebras, such that for every $r\ge 1$, $(\mathcal{G}_{r,k})_{0\le k\le p_r-1}$ is decreasing. We say that $((X_{r,k},(\mathcal{G}_{r,k})_{0\le k\le p_r-1})_{r\ge 1}$ is an array of reverse martingale differences if $X_{r,k}$ is $\mathcal{G}_{r,k}$-measurable for every $r\ge 1$ and $1\le k\le p_r$ and if $\mathbb{E}(X_{r,k}|\mathcal{G}_{r,k+1})=0$ for every $r\ge 1$ and every $0\le k\le p_r-2$.*

Proposition 3 (McLeish)**.** *Let $((X_{r,k},(\mathcal{G}_{r,k})_{0\le k\le p_r-1})_{r\ge 1}$ be an array of reverse martingale differences in L^2. Assume that*

(i) $\sup_{0\le k\le p_r-1}|X_{r,k}|\xrightarrow{P}0$ *;*
(ii) $\sup_{0\le k\le p_r-1}\mathbb{E}(X_{r,k}^2)<\infty$ *;*
(iii) $\sum_{k=0}^{p_r-1}X_{r,k}^2\xrightarrow{P}V$ *for some* $V\ge 0$*;*

Then, $\sum_{0\le k\le p_r-1}X_{r,k}$ *converges in distribution to* $\mathcal{N}(0,V)$*.*

To apply this proposition, we write

$$\frac{1}{\sqrt{m_rn_r}}\sum_{i=0}^{m_r-1}\sum_{k=1}^{K}\mathbf{1}_{\{[m_rt_{k-1}]\le i\le[m_rt_k]-1\}}\left(\sum_{\ell=1}^{K}a_{k,\ell}\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_1^iU_2^j(f)\right):=\sum_{i=0}^{m_r-1}Z_{r,i}\,.$$

Note that $Z_{r,i}$ is a reverse martingale difference with respect to $\mathcal{G}_{r,i}=\mathcal{F}_i^{(1)}$. We shall now prove that $(Z_{r,i})_{0\le i\le m_r-1}$ satisfies *(i)*, *(ii)* and *(iii)* of Proposition 3 (with $V=\Gamma$ for *(iii)*).

Proof of *(i)* **and** *(ii)***.** We have

$$\sup_{0\le i\le m_r-1}|Z_{r,i}|\le\left(\sup_{1\le k\le K}\sum_{\ell=1}^{K}|a_{k,\ell}|\right)\sup_{1\le\ell\le K}\sup_{0\le i\le m_r-1}\frac{1}{\sqrt{m_rn_r}}\left|\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_1^iU_2^j(f)\right|.$$

Hence we just have to prove that for every $\varepsilon>0$, and every $1\le\ell\le K$,

$$\gamma_{\ell,\varepsilon,r}:=\mu\left(\sup_{0\le i\le m_r-1}\frac{1}{\sqrt{m_rn_r}}\left|\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_1^iU_2^j(f)\right|\ge\varepsilon\right)\underset{r\to+\infty}{\longrightarrow}0.$$

Using the stationarity and Markov inequality, we obtain that

$$\begin{aligned}\gamma_{j,\varepsilon,r}&\le m_r\mu\left(\frac{1}{\sqrt{m_rn_r}}\left|\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_2^j(f)\right|\ge\varepsilon\right)\\&\le\frac{1}{\varepsilon^2}\mathbb{E}\left(\left(\frac{1}{\sqrt{n_r}}\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_2^j(f)\right)^2\mathbf{1}_{\left|\sum_{j=[n_rs_{\ell-1}]}^{[n_rs_\ell]-1}U_2^j(f)\right|>\sqrt{m_rn_r}\varepsilon}\right).\end{aligned}$$

Notice that $(U_2^i f)_{0\le i\le m_r-1}$ is a stationary sequence of reverse martingale differences. Now, it is well-known (using stationarity, truncation and Burkholder inequality) that the family

$$(Y_{r,\ell})_{r\ge 1} := \left(\left(\frac{1}{\sqrt{n_r}}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_2^j(f)\right)^2\right)_{r\ge 1}$$

is uniformly integrable, and (i) easily follows.

In the same way, (ii) can be proved by using the stationarity and the fact that $(Y_{r,\ell})_{r\ge 1}$ is bounded in L^1.

Proof of (iii)**.** This is the difficult part. It suffices to prove that

$$\lim_{r\to\infty}\left\|\sum_{i=0}^{m_r-1} Z_{r,i}^2 - \Gamma\right\|_1 = 0\,.$$

Now

$$Z_{r,i}^2 = \sum_{k=1}^{K}\frac{1}{m_r}\mathbf{1}_{\{[m_r t_{k-1}]\le i\le [m_r t_k]-1\}}\left(\sum_{\ell=1}^{K} a_{k,\ell}\frac{1}{\sqrt{n_r}}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\right)^2 .$$

Hence, it suffices to prove that

$$\lim_{r\to\infty}\sum_{k=1}^{K}\left\|\frac{1}{m_r}\sum_{i=[m_r t_{k-1}]}^{[m_r t_k]-1}\left(\sum_{\ell=1}^{K} a_{k,\ell}\frac{1}{\sqrt{n_r}}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\right)^2 - \gamma_k\right\|_1 = 0$$

where $\gamma_k = \|f\|_2^2\sum_{\ell=1}^{K} a_{k,\ell}^2(t_k - t_{k-1})(s_\ell - s_{\ell-1})$. By stationarity, it suffices to prove that, for each $k\in\{1,\dots,K\}$,

$$\lim_{r\to\infty}\left\|\frac{1}{m_r}\sum_{i=0}^{[m_r t_k]-[m_r t_{k-1}]-1}\left(\sum_{\ell=1}^{K} a_{k,\ell}\frac{1}{\sqrt{n_r}}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\right)^2 - \gamma_k\right\|_1 = 0\,.$$

Setting $u_r = [m_r t_k] - [m_r t_{k-1}]$, this is equivalent to

$$\lim_{r\to\infty}\left\|\frac{1}{u_r}\sum_{i=0}^{u_r-1}\left(\sum_{\ell=1}^{K} a_{k,\ell}\frac{1}{\sqrt{n_r}}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\right)^2 - \|f\|_2^2\sum_{\ell=1}^{K} a_{k,\ell}^2(s_\ell - s_{\ell-1})\right\|_1 = 0\,. \tag{8}$$

In order to prove (8), let us admit the following lemma for a while.

Lemma 4. *Let $\varepsilon > 0$. If U_2 is ergodic, there exist integers $v\ge 1$ (large enough) and $p(v)$ (large enough), such that for every $n\ge p(v)$*

$$\left\|\frac{1}{v}\sum_{i=0}^{v-1}\left(\frac{1}{\sqrt{n}}\sum_{\ell=1}^{K} a_{k,\ell}\sum_{j=[n s_{\ell-1}]}^{[n s_\ell]-1} U_1^i U_2^j(f)\right)^2 - \Delta_k\right\|_1 < \varepsilon\,, \tag{9}$$

where $\Delta_k := \|f\|_2^2 \sum_{\ell=1}^K a_{k,\ell}^2 (s_\ell - s_{\ell-1})$.

Let

$$F_{r,i} = \sum_{\ell=1}^K a_{k,\ell} \frac{1}{\sqrt{n_r}} \sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\,.$$

Let $r \geq 1$, such that $u_r \geq v$ and write $u_r = vq_r + t_r$, with $q_r \geq 1$ and $t_r \in \{0, \dots, v-1\}$. We have that

$$\frac{1}{u_r}\sum_{i=0}^{u_r-1} F_{r,i}^2 - \Delta_k = \frac{v}{u_r}\sum_{k=0}^{q_r-1}\left(\frac{1}{v}\sum_{i=kv}^{(k+1)v-1} F_{r,i}^2 - \Delta_k\right) + \frac{1}{u_r}\sum_{i=vq_r}^{u_r-1} F_{r,i}^2 - \frac{t_r\Delta_k}{u_r} \tag{10}$$

By stationarity, and Lemma 4, we see that, for $n_r \geq p(v)$

$$\begin{aligned} &\left\| \frac{v}{u_r}\sum_{k=0}^{q_r-1}\left(\frac{1}{v}\sum_{i=kv}^{(k+1)v-1} F_{r,i}^2 - \Delta_k\right)\right\|_1 \\ &\qquad \leq \left\| \frac{1}{v}\sum_{i=0}^{v-1}\left(\frac{1}{\sqrt{n_r}}\sum_{\ell=1}^K a_{k,\ell}\sum_{j=[n_r s_{\ell-1}]}^{[n_r s_\ell]-1} U_1^i U_2^j(f)\right)^2 - \Delta_k\right\|_1 < \varepsilon\,. \end{aligned} \tag{11}$$

On another hand, for a fixed v, one has

$$\lim_{r\to\infty}\left\| \frac{1}{u_r}\sum_{i=vq_r}^{u_r-1} F_{r,i}^2 - \frac{t_r\Delta_k}{u_r}\right\|_1 = 0\,. \tag{12}$$

From (10), (11) and (12), we see that (8) holds. This completes the proof of (*iii*). □

Proof of Lemma 4 The proof relies on the following convergence in law.

Lemma 5. *Let* $v \geq 1$. *The sequence of random vectors*

$$\left(\frac{1}{\sqrt{n}}\sum_{\ell=1}^K a_{k,\ell}\sum_{j=[ns_{\ell-1}]}^{[ns_\ell]-1} U_1^i U_2^j(f)\right)_{1\leq i\leq v}$$

converges in distribution to $(N_i)_{1\leq i\leq v}$, *where the* N_i*'s are iid with common distribution* $\mathcal{N}(0, \|f\|_2^2 \sum_{\ell=1}^K a_{k,\ell}^2 (s_\ell - s_{\ell-1}))$.

Lemma 4 follows easily from Lemma 5, a truncation argument and the law of large numbers in L^1 for $(N_i^2)_{1\leq i\leq v}$ (with $v \to +\infty$).

Lemma 5 can be proved by applying Proposition 3, but it is shorter to notice that it is a consequence of the WIP for stationary and ergodic $\mathbb{R}^v$-valued reverse martingale differences (note that it is the only place where we use the ergodicity of U_2). Indeed,

letting $V(f) = (U_1^1(f), \dots, U_1^v(f))'$, it follows from the WIP that

$$\left(\frac{1}{\sqrt{n}} \sum_{j=0}^{[ns_1]-1} V(U_2^j(f)), \dots, \frac{1}{\sqrt{n}} \sum_{j=[ns_{K-1}]}^{[ns_K]-1} V(U_2^j(f)) \right)$$

converges in distribution to $(G_1, \dots, G_K)$, where the G_ℓ's are independent Gaussian random vectors with respective covariance matrix $(s_\ell - s_{\ell-1})\mathbb{E}(V(f)V(f)')$. Now since $\mathbb{E}(U_1^i(f)U_1^j(f)) = 0$ if $i \neq j$, we see that $\mathbb{E}(V(f)V(f)') = \|f\|_2^2 \mathrm{Id}_v$, where Id_v is the identity $v \times v$ matrix. Lemma 5 follows straightforwardly.

4. Reverse martingale approximation

We shall consider again a completely commuting family of (non invertible) measure preserving transformations $T_1, \dots, T_d$.

In all that section we assume the following property

$$\|(U_i^*)^n f\|_2 \underset{n \to +\infty}{\longrightarrow} 0 \quad \text{for any } i \in \{1, \dots, d\}, \text{ and any } f \in L^2(\Omega, \mathcal{A}, \mu) \text{ with } \mathbb{E}(f) = 0\,.$$

This property is equivalent to the fact that each T_i is exact (see Definition 4.14 in Walters [21]).

We shall now prove a (reverse) martingale approximation result under a condition in the spirit of Hannan. For $f \in L^2(\mu)$ set

$$\begin{aligned} \|f\|_{\mathcal{X}_2} :&= \sum_{n_1, \dots, n_d \in \mathbb{N}} \left\| \prod_{i=1}^d (U_i^{n_i}(U_i^*)^{n_i} - U_i^{n_i+1}(U_i^*)^{n_i+1}) f \right\|_2 \\ &= \sum_{n_1, \dots, n_d \in \mathbb{N}} \left\| \prod_{i=1}^d ((U_i^*)^{n_i} - U_i(U_i^*)^{n_i+1}) f \right\|_2 , \end{aligned} \tag{13}$$

and

$$\mathcal{X}_2 := \{ f \in L^2(\mu) \,:\, \|f\|_{\mathcal{X}_2} < \infty \}\,.$$

It is not hard to prove as in the one-dimensional case (see for instance the proof of Proposition 12 in [5]) that a sufficient condition for f to be in $\mathcal{X}_2$ is that

$$\sum_{n_1, \dots, n_d \in \mathbb{N}} \frac{\|(U_1^*)^{n_1} \cdots (U_d^*)^{n_d} f\|_2}{(n_1 \cdots n_d)^{1/2}} < \infty\,. \tag{14}$$

We first prove a maximal inequality. Its statement, as well as its proof, are analogous to Lemma 5.2 of [20].

Lemma 6. *Let $f \in \mathcal{X}_2$. Then,*

$$\mathbb{E}\left(\max_{1 \preceq \underline{k} \preceq \underline{n}} \left| \sum_{0 \preceq \underline{i} \preceq \underline{k}} U^{\underline{i}} f \right|^2 \right) \leq 2^{3d} (n_1 \cdots n_d) \|f\|_{\mathcal{X}_2}^2\,. \tag{15}$$

Proof. Let $f \in \mathcal{X}_2$. Using that for every $i \in \{1, \dots, d\}$, $\|(U_i^*)^n f\|_2 \longrightarrow 0$ as $n \to \infty$, we obtain the following orthogonal decomposition

$$f = \sum_{m_1,\dots,m_d \in \mathbb{N}} \prod_{i=1}^{d} \left(U_i^{m_i} (U_i^*)^{m_i} - U_i^{m_i+1} (U_i^*)^{m_i+1} \right) f := \sum_{\underline{m} \in \mathbb{N}^d} f_{\underline{m}} \,. \tag{16}$$

Then, clearly

$$\max_{1 \preceq \underline{k} \preceq \underline{n}} \left| \sum_{0 \preceq \underline{i} \preceq \underline{k}} U^{\underline{i}} f \right| \leq \sum_{\underline{m} \in \mathbb{N}^d} \max_{1 \preceq \underline{k} \preceq \underline{n}} \left| \sum_{0 \preceq \underline{i} \preceq \underline{k}} U^{\underline{i}} f_{\underline{m}} \right|$$

Now, for every $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, $(U^{\underline{i}} f_{\underline{m}})_{\underline{i} \in \mathbb{N}^d}$ is a commuting stationary d-field of reverse martingale differences associated with $f_{\underline{m}} \in L^2(\Omega, \mathcal{A}_{\underline{m}}, \mu)$, where $\mathcal{A}_{\underline{m}} = T_1^{-m_1} \circ \cdots \circ T_d^{-m_d}(\mathcal{A})$. Hence, the result follows from Proposition 2. □

We shall also need the following lemma.

Lemma 7. *Let T_1 be a measure preserving transformation and U_1 the associated Koopman operator. Let $f, g \in L^2(\Omega, \mathcal{A}, \mu)$ be such that for $h \in \{f, g\}$,*

$$\sum_{n \geq 0} \left\| \left((U_1^*)^n - U_1 (U_1^*)^{n+1} \right) h \right\|_2 < \infty \,. \tag{17}$$

Then, $\sum_{k \in \mathbb{Z}} |\mathbb{E}(U_1^{k^+} f \, U_1^{k^-} g)| < \infty$, where $k^+ = \max\{0, k\}$ and $k^- = \max\{0, -k\}$. Moreover, writing $\tilde{f} := \sum_{k \geq 0} ((U_1^*)^k - U_1 (U_1^*)^{k+1}) f$ and $\tilde{g} := \sum_{k \geq 0} ((U_1^*)^k - U_1 (U_1^*)^{k+1}) g$,

$$\mathbb{E}\left(\tilde{f} \tilde{g} \right) = \sum_{k \in \mathbb{Z}} \mathbb{E}\left(U_1^{k^+} f \, U_1^{k^-} g \right) \,. \tag{18}$$

Proof. The absolute convergence of the series will follow from the proof. Hence, we only prove (18). For every $f \in L^2(\Omega, \mathcal{A}, \mu)$ such that $\mathbb{E}(f) = 0$, using that $\|(U_1^*)^n f\|_2 \underset{n \to +\infty}{\longrightarrow} 0$, we have

$$f = \sum_{n \in \mathbb{N}} \left(U_1^n (U_1^*)^n - U_1^{n+1} (U_1^*)^{n+1} \right) f \,, \tag{19}$$

where the summands are orthogonal.

Let

$$A_{k,\ell}(f, g) = \mathbb{E}\left(\left(\left((U_1^*)^k - U_1 (U_1^*)^{k+1} \right) f \right) \left(\left((U_1^*)^\ell - U_1 (U_1^*)^{\ell+1} \right) g \right) \right)$$

Using (17) to permute $\mathbb{E}$ and $\sum$, we have (with absolute convergence)

$$\mathbb{E}\left(\tilde{f} \tilde{g} \right) = \sum_{k, \ell \in \mathbb{N}} A_{k,\ell}(f, g) = \sum_{k, \ell \,:\, k > \ell} A_{k,\ell}(f, g) + \sum_{k, \ell \,:\, k < \ell} A_{k,\ell}(f, g) + \sum_{k, \ell \,:\, k = \ell} A_{k,\ell}(f, g) \,.$$

The first two sums on right hand are symmetric one from the other, hence we shall deal only with the second one. Since $\mathbb{E}\left(\left((U_1^*)^k - U_1 (U_1^*)^{k+1} \right) f \right) \left(U_1 (U_1^*)^{\ell+1} g \right) = 0$ for

$\ell \geq k$,

$$\sum_{k\in\mathbb{N}}\sum_{\ell>k} A_{k,\ell}(f,g) = \sum_{k\in\mathbb{N}}\sum_{m\in\mathbb{N}^*} \mathbb{E}\left(\left(\left(U_1^k(U_1^*)^k - U_1^{k+1}(U_1^*)^{k+1}\right) f\right)(U_1^*)^m g\right)$$
$$= \sum_{m\in\mathbb{N}^*} \mathbb{E}\left(U_1^m f\, g\right) ,$$

where we have used (19). In the same way

$$\sum_{k\in\mathbb{N}} A_{k,k}(f,g) = \sum_{k\in\mathbb{N}} \mathbb{E}\left(\left(\left(U_1^k(U_1^*)^k - U_1^{k+1}(U_1^*)^{k+1}\right) f\right) g\right) = \mathbb{E}(fg)\,.$$

Theorem 8. *Let $f \in \mathcal{X}_2$. Then, there exists a commuting stationary d-field of reverse martingale differences $(U^{\underline{n}}(d))_{\underline{n}\in\mathbb{N}^d}$ with $d \in L^2(\mu)$ such that*

$$\mathbb{E}\left(\max_{0\preceq\underline{k}\preceq\underline{n}}\left|\sum_{0\preceq\underline{i}\preceq\underline{k}} U^{\underline{i}}f - U^{\underline{i}}d\right|^2\right) = o(n_1\cdots n_d), \quad \text{as } n_1,\dots,n_d \to +\infty\,, \tag{20}$$

and $\|d\|_2^2 = \sum_{\underline{n}\in\mathbb{N}^d}\mathbb{E}(U^{\underline{n}^+}f\, U^{\underline{n}^-}f)$, where we use the notations $\underline{n}^+ = (n_1^+,\dots,n_d^+)$ and $\underline{n}^- = (n_1^-,\dots,n_d^-)$.

Proof. By (19) we see that $\|\cdot\|_{\mathcal{X}_2}$ is definite on $\mathcal{X}_2$ hence that it is a norm. Moreover, $(\mathcal{X}_2, \|\cdot\|_{\mathcal{X}_2})$ is a Banach space.

Let $i \in \{1,\dots,d\}$. We easily see that $\|(U_i^*)^n f\|_{\mathcal{X}_2} \longrightarrow 0$ as $n \to +\infty$. Hence, U_i^* is mean ergodic on $\mathcal{X}_2$ (with no fixed points), that is

$$\mathcal{X}_2 = \overline{(I-U_i^*)\mathcal{X}_2}^{\mathcal{X}_2}\,.$$

Then, it follows that

$$\mathcal{X}_2 = \overline{\prod_{1\leq i\leq d}(I-U_i^*)\mathcal{X}_2}^{\mathcal{X}_2}\,. \tag{21}$$

Define a linear operator $\mathcal{D}$ on $\mathcal{X}_2$ by setting

$$\mathcal{D}f := \sum_{\underline{n}\in\mathbb{N}^d}\prod_{i=1}^d\left((U_i^*)^{n_i} - U_i(U_i^*)^{n_i+1}\right) f\,.$$

Let us observe that if $f = \prod_{1\leq i\leq d}(I-U_i^*)g$ with $g \in \mathcal{X}_2$, then $\mathcal{D}f = \prod_{1\leq i\leq d}(I-U_iU_i^*)g$.

Obviously,

$$\|\mathcal{D}(f)\|_2 \leq \|f\|_{\mathcal{X}_2}\,. \tag{22}$$

Let us prove (20) with $d = \mathcal{D}(f)$. Let us admit for a while that (20) holds whenever f belongs to $\prod_{1\leq i\leq d}(I-U_i^*)\mathcal{X}_2$. Let us show then that (20) holds for every $f \in \mathcal{X}_2$.

Let $f \in \mathcal{X}_2$. Let $\varepsilon > 0$. By (21), there exists $g \in \mathcal{X}_2$ such that

$$\left\| f - \prod_{1 \le i \le d} (I - U_i^*) g \right\|_{\mathcal{X}_2} < \varepsilon \,.$$

For every $\underline{n} \in \mathbb{N}^d$, we have, setting $\tilde{g} := \prod_{1 \le i \le d} (I - U_i^*) g$,

$$\left| \sum_{0 \preceq \underline{k} \preceq \underline{n}} \big(U^{\underline{k}} f - U^{\underline{k}} \mathcal{D} f \big) \right| \le \left| \sum_{0 \preceq \underline{k} \preceq \underline{n}} \big(U^{\underline{k}} f - U^{\underline{k}} \tilde{g} \big) \right| + \left| \sum_{0 \preceq \underline{k} \preceq \underline{n}} \big(U^{\underline{k}} h - U^{\underline{k}} \mathcal{D} \tilde{g} \big) \right| + \left| \sum_{0 \preceq \underline{k} \preceq \underline{n}} \big(U^{\underline{k}} \mathcal{D} (f - \tilde{g}) \big) \right| .$$

Using (15) to deal with the first term above and (22) and (5) to deal with the third term, and since we admit for the moment that (20) holds for $\tilde{g}$, we infer that

$$\limsup_{n_1, \dots, n_d \to +\infty} \frac{1}{n_1 \cdots n_d} \mathbb{E} \left(\max_{1 \preceq \underline{k} \preceq \underline{n}} \left| \sum_{0 \preceq \underline{i} \preceq \underline{k}} U^{\underline{i}} f - U^{\underline{i}} d \right|^2 \right) \le C \varepsilon \,,$$

and (20) follows by letting $\varepsilon \to 0$.

It remains to deal with the case where $f = \prod_{1 \le i \le d} (I - U_i^*) g$, for some $g \in \mathcal{X}_2$.

To do so we use the following simple identity (see also Gordin [9], Proposition 1):

$$\text{for } i \in \{1, \dots, d\}, \quad I - U_i^* = I - U_i U_i^* + (U_i - I) U_i^* \,.$$

Let $f = \prod_{1 \le i \le d} (I - U_i^*) g$ with $g \in \mathcal{X}_2$. We have

$$\begin{aligned} f &= \mathcal{D} f + \sum_{\mathcal{E} \subset \{1, \dots, d\}, \mathcal{E} \neq \emptyset} \prod_{i \in \mathcal{E}^c} (I - U_i U_i^*) \prod_{j \in \mathcal{E}} (U_j - I) U_j^* \, g \\ &:= \mathcal{D} f + h \,. \end{aligned} \tag{23}$$

The proof relies on the fact that the remainder in (23) (i.e. h) behaves like a coboundary in some "directions" and like a sum of reverse martingale differences in the other "directions". For the sake of simplicity, we only prove the results for $d = 2$, but the general case can be handled similarly.

We have

$$f - \mathcal{D} f = (I - U_1 U_1^*)(U_2 - I) U_2^* g + (I - U_2 U_2^*)(U_1 - I) U_1^* g + (U_1 - I)(U_2 - I) U_1^* U_2^* g \,.$$

For every $0 \le k_1 \le n_1$ and every $0 \le k_2 \le n_2$, we have

$$\sum_{0 \le i_1 \le k_1} U_1^{i_1} \sum_{0 \le i_2 \le k_2} U_2^{i_2} (I - U_2 U_2^*)(U_1 - I) U_1^* g = (U_1^{k_1} - I) \sum_{0 \le i_2 \le k_2} U_1^{i_1} (I - U_2 U_2^*) U_1^* g \,.$$

Define $Z_{k_1,n_2} := \max_{1\le k_2\le n_2} |\sum_{0\le i_2\le k_2} U_2^{i_2}(I-U_2U_2^*)U_1^{k_1}U_1^*g|$. For every $A>0$, we have

$$\max_{1\le k_1\le n_1}\max_{1\le k_2\le n_2}\left|\sum_{0\le i_1\le k_1} U_1^{i_1}\sum_{0\le i_2\le k_2} U_2^{i_2}(I-U_2U_2^*)(U_1-I)U_1^*g\right|^2$$
$$\le 2|Z_{0,n_2}|^2 + 2n_2A^2 + 2\sum_{0\le k_1\le n_1}\left|Z_{k_1,k_2}\mathbf{1}_{\{|Z_{k_1,n_2}|\ge A\sqrt{n_2}\}}\right|^2,$$

and, by stationarity

$$\mathbb{E}\left(\max_{1\le k_2\le n_2}\max_{1\le k_1\le n_1}\left|\sum_{0\le i_1\le k_1} U_1^{i_1}\sum_{0\le i_2\le k_2} U_2^{i_2}(I-U_2U_2^*)(U_1-I)U_1^*g\right|^2\right)$$
$$\le 2\mathbb{E}\left(|Z_{0,n_2}|^2\right) + 2n_2A^2 + 2n_1\mathbb{E}\left(\left|Z_{0,n_2}\mathbf{1}_{\{|Z_{0,n_2}|\ge A\sqrt{n_2}\}}\right|^2\right).$$

Since, $(Z_{0,n_2}^2/n_2)_{n_2\ge1}$ is uniformly integrable, it follows that

$$\frac{1}{n_1n_2}\mathbb{E}\left(\max_{1\le k_2\le n_2}\max_{1\le k_1\le n_1}\left|\sum_{0\le i_1\le k_1} U_1^{i_1}\sum_{0\le i_2\le k_2} U_2^{i_2}(I-U_2U_2^*)(U_1-I)U_1^*g\right|^2\right)\underset{n_1,n_2\to+\infty}{\longrightarrow}0\,.$$

We may deal similarly with the sum associated with the term $(I-U_1U_1^*)(U_2-I)U_2^*g$. To deal with the sum associated with the term $(U_1-I)-U_2-I)U_1^*)U_2^*g$ is somehow easier.

To finish the proof of the theorem, it remains to identify $\|\mathcal{D}f\|_2^2$. But, this follows by applying inductively Lemma 7, noticing that

$$\sum_{\underline{n}\in\mathbb{N}^d}\prod_{i=1}^d\left((U_i^*)^{n_i}-U_i(U_i^*)^{n_i+1}\right)=\prod_{i=1}^d\sum_{k_i\in\mathbb{N}}\left((U_i^*)^{k_i}-U_i(U_i^*)^{k_i+1}\right)$$

and using the fact that $(U_1,\dots,U_d)$ is completely commuting. □

5. Expanding endomorphisms of the m-dimensional torus

Let A be a $m\times m$ ($m\ge1$) matrix with integer entries. We say that A is expanding if all its eigenvalues have modulus strictly greater than 1.

A induces a transformation θ_A of the m-dimensional torus $[0,1)^m$, preserving the Lebesgue-Haar measure λ. We denote by U_A the corresponding Koopman operator, and by U_A^* the Perron-Frobenius operator.

Let us give a simple condition under which θ_A and θ_B are completely commuting.

Lemma 9. *Let A and B be two expanding $m\times m$ ($m\ge1$) matrices with integer entries. Assume that A and B commutes and that they have coprime determinants. Then, θ_A and θ_B are completely commuting.*

Proof. We have to prove that $U_A^* U_B = U_B U_A^*$. Let Γ be representatives of $\mathbb{Z}^m/A\mathbb{Z}^m$ with distinct images in $\mathbb{Z}^m/A\mathbb{Z}^m$. By (34) (using that A^{-1} and B commute), it suffices to prove that B induces a bijection (modulo $\mathbb{Z}^m$) of the set $(A^{-1}\gamma)_{\gamma\in\Gamma}$.

Let $\gamma, \gamma' \in \Gamma$ be such that there exists $\beta \in \mathbb{Z}^m$ such that $BA^{-1}\gamma = BA^{-1}\gamma' + \beta$. Set $\delta := \gamma - \gamma' \in \mathbb{Z}^m$. Writing $A^{-1} = (\det A)^{-1}\tilde{A}$, where $\tilde{A}$ is the adjugate matrix of A (with integer entries) and similarly, $B^{-1} = (\det B)^{-1}\tilde{B}$, we see that

$$(\det B)\tilde{A}\delta = (\det A)\tilde{B}\beta\,.$$

Since $\det B \wedge \det A = 1$, by Gauss lemma, we infer that $\det A$ divides all entries $\tilde{A}\delta$, hence that $A^{-1}\delta \in \mathbb{Z}^m$ and $\delta \in A\mathbb{Z}^m$. By definition of Γ, we see that $\gamma = \gamma'$ and the lemma is proved. □

The fact that A and B have coprime determinants is by no mean necessary for θ_A and θ_B to be completely commuting as one may see from the following basic example: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

The next proposition is an easy consequence of a result by Fan [7] (see Proposition 13 of the Appendix). Recall that the modulus of continuity in L^2 is given by

$$\Omega_{2,f}(\delta) := \sup_{0\le|x|\le\delta} \|f(\cdot + x) - f\|_2\ ,$$

where $|x|$ stands for the euclidean norm of $x \in [0,1)^m$.

Proposition 10. *Let $A_1, \dots, A_d$ be commuting expanding $m \times m$ matrices with integral entries. Let $\lambda_{min} > 1$ be the infimum of the modulus of their eigenvalues. There exists $C > 0$ such that, for every $f \in L^2(\lambda)$ and every $n_1, \dots, n_d \in \mathbb{N}$,*

$$\|(U_{A_1}^*)^{n_1} \cdots (U_{A_d}^*)^{n_d} f\|_2 \le \Omega_{2,f}\left(C\lambda_{min}^{-(n_1+\cdots+n_d)}\right)\ . \tag{24}$$

Proof. Let us first notice that $(U_{A_1}^*)^{n_1} \cdots (U_{A_d}^*)^{n_d}$ is the Perron-Frobenius operator associated with $\theta_{{A_d}^{n_d}\cdots{A_1}^{n_1}}$. Now, since the matrices are commuting, it follows from standard linear algebra results, that the set of the eigenvalues of ${A_d}^{n_d}\cdots{A_1}^{n_1}$ is included in $\{{\lambda_d}^{n_d}\cdots{\lambda_1}^{n_1}\ :\ \lambda_i \text{ is an eigenvalue of } A_i\}$. In particular, ${A_d}^{n_d}\cdots{A_1}^{n_1}$ is an expanding matrix. Then, the result follows from Proposition 13, noticing that

$$\Delta\left(A_1^{-n_1}\cdots A_d^{-n_d}([0,1]^m)\right) \le C\lambda_{min}^{-(n_1+\cdots+n_d)}\,,$$

where $\Delta\left(A_1^{-n_1}\cdots A_d^{-n_d}([0,1]^m)\right)$ is the diameter of $A_1^{-n_1}\cdots A_d^{-n_d}([0,1]^m)$. □

We shall use the following notation: $U^{\underline{i}} = U_{A_1}^{i_1}\cdots U_{A_d}^{i_d}$ for every $(i_1,\dots,i_d) \in \mathbb{N}^d$.

Theorem 11. *Let $d \ge 1$. Let $f \in L^2([0,1)^m, \lambda)$ centered such that*

$$\int_0^1 \frac{|\log(t)|^{(d-2)/2}}{t}\,\Omega_{2,f}(t)\,dt < \infty\,. \tag{25}$$

Let $A_1, \dots, A_d$ be expanding $m \times m$ matrices with integral entries. Assume that they are commuting and that their determinants are pairwise coprime. Then, there exists

a stationary d-field of reverse martingale differences $(U^{\underline{n}}(d))_{\overline{n}\in\mathbb{N}^d}$ *with* $d \in L^2(\lambda)$ *such that*

$$\mathbb{E}\left(\max_{1\preceq \underline{k}\preceq \underline{n}}\left|\sum_{0\preceq \underline{i}\preceq \underline{k}} U^{\underline{i}}f - U^{\underline{i}}d\right|^2\right) = o(n_1\cdots n_d) \quad \text{as } n_1,\dots,n_d\to+\infty\,. \tag{26}$$

In particular, we have an invariance principle.

Remark. We see that if $\Omega_{2,f}(t) = o(|\log(t)|^{-d/2-\varepsilon})$ $(t\to 0)$ (25) holds. A CLT has been obtained by Cohen and Conze [2] under the condition $\Omega_{2,f}(t) = o(|\log(t)|^{-d-\varepsilon})$ $(t\to 0)$. However, the results of [2] apply to general arrays and to commuting families of general endomorphisms (including for instance automorphisms) inducing a totally ergodic $\mathbb{N}^d$-action (see their paper for more informations).

Proof. It follows from (31) and a density argument that $\|(U^*_{A_i})^n f\|_2 \to 0$ as $n\to+\infty$ for every centered $f\in L^2(\lambda)$ and every $i\in\{1,\dots,d\}$. Hence, by Theorem 8, we just have to check that (14) holds. Using (31), we see that (14) will hold provided that

$$\sum_{n_1,\dots,n_d\in\mathbb{N}^*} \frac{\Omega_{2,f}(C\lambda_{min}^{-(n_1+\cdots+n_d)})}{(n_1\cdots n_d)^{1/2}} < \infty\,. \tag{27}$$

Now, making the change of index $n_1+\cdots+n_d\to n_d$,

$$\sum_{n_1,\dots,n_d\in\mathbb{N}^*} \frac{\Omega_{2,f}(C\lambda_{min}^{-(n_1+\cdots+n_d)})}{(n_1\cdots n_d)^{1/2}} = \sum_{n_1,\dots,n_{d-1}\in\mathbb{N}^*}\ \sum_{n_d\geq n-1+\cdots+n_{d-1}+1} \frac{\Omega_{2,f}(C\lambda_{min}^{-n_d})}{(n_1\cdots(n_d-n_1-\cdots-n_{d-1}))^{1/2}}\,.$$

Notice that for every $m\geq 1$, we have

$$\sum_{1\leq n_1\leq m-1}\frac{1}{\sqrt{n_1(m-n_1)}} \leq \sqrt{2/m}\sum_{1\leq n_1\leq m/2}\frac{1}{\sqrt{n_1}} + \sqrt{2/m}\sum_{m/2\leq n_1\leq m-1}\frac{1}{\sqrt{m-n_1}} \leq D\,,$$

for a constant $D>0$ independent of m.

Hence (27) holds if and only if

$$\sum_{n_d\geq 1}\ \sum_{n_2,\dots,n_{d-1}\in\mathbb{N}^*\,:\,n_2+\cdots+n_{d-1}\leq n_d-1} \frac{\Omega_{2,f}(C\lambda_{min}^{n_d})}{\sqrt{n_2\cdots n_{d-2}}}$$
$$\leq \sum_{n_d\geq 1}\left(\sum_{1\leq j\leq n_d}\frac{1}{\sqrt{j}}\right)^{d-2}\Omega_{2,f}(C\lambda_{min}^{n_d}) \leq C\sum_{n_d\geq 1} n_d^{(d-2)/2}\Omega_{2,f}(C\lambda_{min}^{n_d}) < \infty\,,$$

which is equivalent to (25) by comparing series and integrals. □

We shall now give a sufficient condition in terms of Fourier coefficients.

Lemma 12. *Let $f \in L^2([0,1)^m)$ with Fourier coefficients $(c_{\underline{n}}(f))_{\underline{n}\in\mathbb{Z}^m}$. For every $k \geq 1$, define*

$$A_k(f) := \max_{1\leq i\leq m} \sum_{\underline{n}\in\mathbb{Z}^m\,:\,|n_i|\geq k} |c_n(f)|^2 \,.$$

If

$$\sum_{k\geq 1} \frac{(\log k)^{(d-2)/2}}{k} (A_k(f))^{1/2} < \infty \tag{28}$$

then (25) *holds.*

Remark. Note that (28) holds as soon as $A_k(f) = O((\log k)^{-d-\varepsilon})$, for some $\varepsilon > 0$. Conversely, if (28) is satisfied, since $(A_k(f))_{k\geq 1}$ is non increasing, we see that, $A_k(f) = o((\log k)^{-d})$. Note also that Levin [16] proved the CLT (he also announced the weak invariance principle in [16]) under a condition that is easily seen to be equivalent to

$$\sum_{k\geq 1} \frac{(\log k)^{d-1}}{k} (A_k(f))^{1/2} < \infty \,.$$

He worked in the same setting as Cohen and Conze [2].

Proof. Let us first notice that (25) is equivalent to

$$\sum_{k\geq 1} \frac{(\log k)^{(d-2)/2}}{k} \Omega_{2,f}(1/k) < \infty \,.$$

Let $k \geq 1$ and $\underline{x} \in \mathbb{R}^m$ with $|\underline{x}| \leq 1/k$. We have

$$\|f(\cdot + \underline{x}) - f\|_2^2 = \sum_{\underline{n}\in\mathbb{Z}^m} |c_{\underline{n}}(f)|^2 |1 - \mathrm{e}^{2i\pi\langle \underline{n},\underline{x}\rangle}|^2 \,.$$

In particular, majorizing $|1 - \mathrm{e}^{2i\pi\langle \underline{n},\underline{x}\rangle}|$ either by 2 or by $2\pi|\underline{n}|\,|\underline{x}|$, we see that

$$\begin{aligned}(\Omega_{2,f}(1/k))^2 &\leq \sum_{\underline{n}\in\{-k,\dots,k\}^m} |c_{\underline{n}}(f)|^2 \frac{4\pi^2|\underline{n}|^2}{k^2} + C_m \sum_{i=1}^m \sum_{\underline{n}\in\mathbb{Z}^m\,:|n_i|\geq k} |c_{\underline{n}}(f)|^2 \\ &\leq \frac{\tilde{C}_m}{k^2} \sum_{i=1}^m \sum_{\underline{n}\in\mathbb{Z}^m\,:|n_i|\leq k} n_i^2 |c_{\underline{n}}(f)|^2 + C_m \sum_{i=1}^m \sum_{\underline{n}\in\mathbb{Z}^m\,:|n_i|\geq k} |c_{\underline{n}}(f)|^2 \,,\end{aligned}$$

for two positive constants C_m and $\bar{C}_m$. The second sum on right hand can be handled directly by using (28), and it remains to prove that

$$\sum_{k\geq 1} \frac{(\log k)^{(d-2)/2}}{k^2} (B_k(f))^{1/2} < \infty \,, \tag{29}$$

where

$$B_k(f) := \max_{1\leq i\leq m} \sum_{\underline{n}\in\mathbb{Z}^m\,:\,|n_i|\leq k} n_i^2 |c_n(f)|^2 \,.$$

Let us prove that (28) implies (29). Let

$$A_k^{(i)} := \sum_{\underline{n}\in\mathbb{Z}^m\,:\,|n_i|\geq k} |c_n(f)|^2 \,.$$

We have

$$\sum_{\underline{n}\in\mathbb{Z}^m\,:\,|n_i|\leq k} n_i^2|c_n(f)|^2 = \sum_{j=0}^{k} j^2(A_j^{(i)} - A_{j+1}^{(i)}) = \sum_{j=1}^{k} A_j^{(i)}(j^2-(j-1)^2) - k^2 A_{k+1}^{(i)} \,.$$

Hence, we infer that (29) hold as soon as

$$B := \sum_{k\geq 1} \frac{(\log k)^{(d-2)/2}}{k^2} \left(\sum_{j=1}^{k} jA_j(f)\right)^{1/2} < \infty \,, \tag{30}$$

To prove that (28) implies (30), we first notice that (28) is equivalent to

$$\sum_{n\geq 0} n^{(d-2)/2}\sqrt{A_{2^n}(f)} < \infty$$

(to see this it suffices to use the monoticity of A_k). Next

$$B = \sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}} \frac{(\log k)^{(d-2)/2}}{k^2} \left(\sum_{j=1}^{k} jA_j(f)\right)^{1/2} \leq C\sum_{n=0}^{\infty} \frac{n^{(d-2)/2}}{2^n} \left(\sum_{j=1}^{2^{n+1}} jA_j(f)\right)^{1/2} .$$

In the same way, using the monoticity of A_k,

$$\left(\sum_{j=1}^{2^{n+1}} jA_j(f)\right)^{1/2} = \left(\sum_{k=0}^{n}\sum_{j=2^k}^{2^{k+1}} jA_j\right)^{1/2} \leq \left(2\sum_{k=0}^{n} 2^{2k}A_{2^k}\right)^{1/2} \leq \sqrt{2}\sum_{k=0}^{n} 2^k\sqrt{A_{2^k}} \,.$$

It follows that

$$B \leq C\sum_{n=0}^{\infty} \frac{n^{(d-2)/2}}{2^n}\sum_{k=0}^{n} 2^k\sqrt{A_{2^k}} \leq D\sum_{k\geq 0} k^{(d-2)/2}\sqrt{A_{2^k}(f)} < \infty$$

which completes the proof. □

6. Appendix

Proposition 10 of Section 5 is a consequence of the following proposition, due to Fan [7]. We shall give the proof for the sake of completeness and because the reference [7] is hard to obtain.

Proposition 13 (Fan, [7])**.** *There exists $C > 0$ such that for every expanding $m \times m$ matrices A with integral entries, for every $f \in L^2(\lambda)$ with $\lambda(f) = 0$,*

$$\|U_A^* f\|_2 \leq \Omega_{2,f}\left(\Delta\left(A^{-1}([0,1]^m)\right)\right) \,, \tag{31}$$

where $\Delta\left(A^{-1}([0,1]^m)\right)$ is the diameter of $A^{-1}([0,1]^m)$.

Proof. Let us recall some facts about tiling and Perron-Frobenius operators associated with expanding matrices.

Let A be an expanding $m \times m$ matrix. Let $\Gamma \subset \mathbb{Z}^m$ be representatives of $\mathbb{Z}^m/A\mathbb{Z}^m$. Then, see e.g. [3], there exists a unique compact set $K \subset \mathbb{R}^m$, such that

$$K = \cup_{\gamma\in\Gamma}(A^{-1}K + A^{-1}\gamma) \tag{32}$$

and an integer $q \geq 1$ such that

$$\sum_{\underline{n}\in\mathbb{Z}^m} \mathbf{1}_{K+\underline{n}} = q \qquad \lambda\text{-almost everywhere.}$$

Moreover, for every $\gamma, \gamma' \in \Gamma$ with $\gamma \neq \gamma'$, $\lambda\big((A^{-1}K + A^{-1}\gamma) \cap (A^{-1}K + A^{-1}\gamma')\big) = 0$. Using that, clearly,

$$\mathbf{1}_K = \sum_{\underline{n}\in\mathbb{Z}^d} \mathbf{1}_{((K+\underline{n})\cap[0,1]^m)-\underline{n}}\,,$$

we infer that for every locally integrable f,

$$q\int_{[0,1]^m} f d\lambda = \int_K f d\lambda\,. \tag{33}$$

It follows then that the Perron-Frobenius operator U_A^* is given by

$$U_A^* f(x) = \frac{1}{\det A}\sum_{\gamma\in\Gamma} f\left(A^{-1}x + A^{-1}\gamma\right)\,. \tag{34}$$

Let $f \in L^2(\lambda)$ with $\lambda(f) = 0$. Using that U_A^* preserves λ and making the change of variable $y \to y + x$, we infer that

$$U_A^* f(x) = \frac{1}{q\det A}\sum_{\gamma\in\Gamma}\int_K \big(f(A^{-1}x + A^{-1}\gamma) - f(A^{-1}(y+x) + A^{-1}\gamma)\big)\ \lambda(dy)\,.$$

Hence, using Jensen's inequality (recall that $\lambda(K) = q$),

$$(U_A^* f)^2 \leq \frac{1}{q\det A}\sum_{\gamma\in\Gamma}\int_K \big(f(A^{-1}x + A^{-1}\gamma) - f(A^{-1}(y+x) + A^{-1}\gamma)\big)^2\ \lambda(dy)\,.$$

Making the change of variable $x \to A^{-1}x + A^{-1}\gamma$, we infer that

$$\begin{aligned}
\|U_A^* f\|_2^2 &\leq \sum_{\gamma\in\Gamma}\left(\int_K \lambda(dy)\int_K \big(f(A^{-1}x + A^{-1}\gamma) - f(A^{-1}(y+x) + A^{-1}\gamma)\big)^2\ \lambda(dx)\right)^{1/2}\\
&\leq \frac{1}{q^2}\sum_{\gamma\in\Gamma}\int_K \lambda(dy)\int_{A^{-1}K+A^{-1}\gamma}\big(f(x) - f(x + A^{-1}y)\big)^2\ \lambda(dx)\\
&= \int_{[0,1]^m}\big\|f - f(\cdot + A^{-1}y)\big\|_2^2\ \lambda(dy)\,,
\end{aligned}$$

where we have used (32) and (33). The announced result clearly follows. □

References

[1] R. Cairoli, *Une inégalité pour martingales à indices multiples et ses applications* (French) Séminaire de Probabilités, IV (Univ. Strasbourg, 1968/69) (1970) 1-27 Lecture Notes in Mathematics, Vol. 124. Springer, Berlin.

[2] G. Cohen and J.-P. Conze, *CLT for random walks of commuting endomorphisms on compact abelian groups*, accepted for publicaion in J. Theoret. Probab, DOI: 10.1007/s10959-015-0631-y.

[3] J.-P. Conze, L. Hervé, L. and A. Raugi, *Pavages auto-affines, opérateurs de transfert et critères de réseau dans* $\mathbb{R}^d$, (French) Bol. Soc. Brasil. Mat. (N.S.) 28 (1997), no. 1, 1-42.

[4] C. Cuny, *A compact LIL for martingales in 2-smooth Banach spaces with applications*, Bernoulli 21 (2015), no. 1, 374-400.

[5] C. Cuny and M. Peligrad, *Central limit theorem started at a point for stationary processes and additive functionals of reversible Markov chains*, J. Theoret. Probab. 25 (2012), no. 1, 171-188.

[6] J. Dedecker, *On the optimality of McLeish's conditions for the central limit theorem*, C. R. Math. Acad. Sci. Paris 353 (2015), no. 6, 557-561.

[7] A. Fan, *Decay of correlation for expanding toral endomorphisms.*

[8] M. I. Gordin, *The central limit theorem for stationary processes*, (Russian) Dokl. Akad. Nauk SSSR 188 1969 739-741.

[9] M. I. Gordin, *Martingale-co-boundary representation for a class of stationary random fields*, (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 364 (2009), Veroyatnost i Statistika. 14.2, 88–108, 236; translation in J. Math. Sci. (N. Y.) 163 (2009), no. 4, 363-374.

[10] M. I. Gordin and B. A. Lifšic, *Central limit theorem for stationary Markov processes*, (Russian) Dokl. Akad. Nauk SSSR 239 (1978), no. 4, 766-767.

[11] M. I. Gordin and M. Peligrad, *On the functional central limit theorem via martingale approximation*, Bernoulli 17 (2011) 424–440.

[12] P. Hall and C. C. Heyde, Martingale limit theory and its application. Probability and Mathematical Statistics, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980.

[13] E. J. Hannan, *Central limit theorems for time series regression*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 26 (1973) 157-170.

[14] E. J. Hannan, *The central limit theorem for time series regression*, Stochastic Process. Appl. 9 (1979) 281-289.

[15] C. C. Heyde, *On the central limit theorem for stationary processes*, Z. Wahrscheinlichkietstheorie und Verw. Gebiete 30 (1974), 315-320.

[16] M. B. Levin, *Central limit theorem for Zd+-actions by toral endomorphisms*, Electron. J. Probab. 18 (2013), no. 35, 42 pp.

[17] M. Maxwell and M. Woodroofe, *Central limit theorems for additive functionals of Markov chains*, Ann. Probab. 28 (2000) 713-724.

[18] M. Peligrad and S. Utev, *A new maximal inequality and invariance principle for stationary sequences*, Ann. Probab. 33 (2005) 798-815.

[19] D. Volný, *A central limit theorem for fields of martingale differences*, arXiv:1504.02439, accepted for publication in C. R. Math. Acad. Sci. Paris

[20] D. Volný and Y. Wang, *An invariance principle for stationary random fields under Hannan's condition*, Stochastic Process. Appl. 124 (2014), no. 12, 4012-4029.

[21] P. Walters, An introduction to ergodic theory. Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

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