STUDY OF ALMOST EVERYWHERE CONVERGENCE OF SERIES BY MEANS OF MARTINGALE METHODS

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ABSTRACT. Martingale methods are used to study the almost everywhere convergence of general function series. Applications are given to ergodic series, which improves recent results of Fan [9], and to dilated series, including Davenport series, which completes results of Gaposhkin [12] (see also [13]). Application is also given to the almost everywhere convergence with respect to Riesz products of lacunary series.

1. INTRODUCTION

Let us first state two problems which motivate the investigation in this paper. One comes from the classical analysis and the other from the ergodic theory.

Given a function $f \in L^1(\mathbb{R}/\mathbb{Z})$ (or equivalently a locally integrable 1-periodic function on the real line $\mathbb{R}$) such that $\int_0^1 f(x)dx = 0$, an increasing sequence of positive integers $(n_k)_{k\geq 0} \subset \mathbb{N}$ and a sequence of complex numbers $(a_k)_{k\geq 0} \subset \mathbb{C}$, one would like to investigate the convergence (almost everywhere convergence or L^1-convergence etc) of the following series, called *dilated series*,

$$\sum_{k=0}^{\infty} a_k f(n_k x). \tag{1}$$

If the series converges almost everywhere (a.e.) whence $(a_k) \in \ell^2$, we say that $\{f(n_k x)\}$ is a *convergence system*. The famous Carleson theorem states that both $\{\sin nx\}$ and $\{\cos nx\}$ are convergence systems. In general, one should find suitable conditions on f and on (n_k) for $\{f(n_k x)\}$ to be a convergence system. This is a long standing problem. One may consult the survey by Berkes and Weber [2] and, for recent progresses, Weber [22] and the references therein.

The other problem is the almost everywhere convergence of the so-called *ergodic series*

$$\sum_{k=0}^{\infty} a_k f(T^k x) \tag{2}$$

where T is a measure-preserving map on a probability space $(X, \mathcal{B}, \mu)$ and $f \in L^1(\mu)$ such that $\int f d\mu = 0$. To be more precise, we want to find sufficient conditions on the dynamical system $(X, \mathcal{B}, \mu, T)$ and/or on f, such that the series (2)

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converges for *any* sequence $(a_n)_{n\in\mathbb{N}} \in \ell^p$, where $1 \le p \le 2$ is fixed (our main interest is in the case $p = 2$). Sufficient conditions for the a.e. convergence with specific (regular) sequences $(a_n)_{n\in\mathbb{N}}$ may be found for instance in [4]. The fact that conditions have to be imposed to ensure the a.e. convergence of (2) may be illustrated by the following result of Dowker and Erdös [6]: if μ is an non-atomic ergodic measure, for every positive sequence $(a_n)_{n\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}} a_k = \infty$, there exists $f \in L^\infty(\mu)$ with $\int f d\mu = 0$ such that $\sum_{k\in\mathbb{N}} a_k f(T^k x)$ diverges a.e. Some sufficient conditions are recently found in [9] for $\{f \circ T^n\}$ be to be a convergence system.

In this paper, we will study series in a general setting. The above situations are special examples. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $(Z_n)_{n\ge 0} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$, $p \ge 1$, be a sequence of L^p-integrable random variables with $\mathbb{E}Z_n = 0$. We shall study the almost sure convergence of the random series

$$\sum_{n=0}^{\infty} Z_n(\omega). \tag{3}$$

Suppose that we are given an increasing filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$ such that $\mathcal{A}_0 = \{\emptyset, \Omega\}$ and $\mathcal{A}_\infty = \mathcal{A}$ where $\mathcal{A}_\infty := \bigvee_{n=0}^{\infty} \mathcal{A}_n$. We will analyze random variables using this filtration. For every $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathbb{E}^n := \mathbb{E}(\cdot|\mathcal{A}_n)$ and introduce the operator

$$\mathcal{D}_n = \mathbb{E}^{n+1} - \mathbb{E}^n.$$

If $Z \in L^p(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}Z = 0$ ($p \ge 1$), we have the decomposition

$$Z = \sum_{n=0}^{\infty} \mathcal{D}_n Z\,,$$

where the convergence takes place a.e. and in L^p. This is what we mean by the analysis based on the filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$ and it is a simple consequence of Doob's convergence theorem of martingales.

One of our main results is the following theorem, which is a special case of Theorem 5 corresponding to $p = 2$.

Theorem A. *Let $(Z_n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{A}, \mathbb{P})$ be such that $\mathbb{E}Z_n = 0$ for every $n \in \mathbb{N}$. Then the series $\sum_{n\in\mathbb{N}} Z_n$ converges $\mathbb{P}$-a.s. and in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ under the following set of conditions*

$$\sum_{k=0}^{\infty} \Big(\sum_{n=0}^{\infty} \|\mathcal{D}_{n+k} Z_n\|_2^2 \Big)^{1/2} < \infty \tag{4}$$

and

$$\sum_{k=1}^{\infty} \Big(\sum_{n=0}^{\infty} \|\mathcal{D}_{n} Z_{n+k}\|_2^2 \Big)^{1/2} < \infty\,. \tag{5}$$

One of ingredients in the proof of Theorem A is the Doob maximal inequality of martingales. Assume that $\{Z_n\} \subset L^2(\Omega, \mathcal{A}, P)$ are independent and $\mathbb{E}Z_n = 0$

for all n. It follows from Theorem A that $\sum \mathbb{E}|Z_n|^2 < \infty$ implies the almost sure convergence of $\sum Z_n$. This is a trivial application of Theorem A, because this known result of Kolmogorov is actually covered by the Doob convergence theorem of martingales.

We can also make an analysis using a decreasing filtration. Suppose that we are given a decreasing filtration $(\mathcal{B}_n)_{n\in\mathbb{N}}$ such that $\mathcal{B}_0 = \mathcal{A}$. Let $\mathcal{B}_\infty := \bigcap_{n=0}^\infty \mathcal{B}_n$. For every $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathbb{E}_n = \mathbb{E}(\cdot|\mathcal{B}_n)$ and introduce

$$\mathfrak{d}_n = \mathbb{E}_n - \mathbb{E}_{n+1}.$$

Let $p \geq 1$. For $Z \in L^p(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{E}(Z|\mathcal{B}_\infty) = 0$ which implies $\mathbb{E}Z = 0$, we have the decomposition

$$Z = \sum_{n=0}^\infty \mathfrak{d}_n Z\,,$$

where the series converges in L^p and $\mathbb{P}$-a.s.

Theorem B. Let $(Z_n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{A}, \mathbb{P})$ be such that $\mathbb{E}_\infty(Z_n) = 0$ for every $n \in \mathbb{N}$. Then the series $\sum_{n\in\mathbb{N}} g_n$ converges $\mathbb{P}$-a.s. and in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ under the following conditions

$$\sum_{k=0}^\infty \Big(\sum_{n=0}^\infty \|\mathfrak{d}_{n+k} Z_n\|_2^2 \Big)^{1/2} < \infty \tag{6}$$

and

$$\sum_{k=1}^\infty \Big(\sum_{n=0}^\infty \|\mathfrak{d}_n Z_{n+k}\|_2^2 \Big)^{1/2} < \infty\,. \tag{7}$$

As an application of Theorem A, we have the following theorem, in which the condition (8) is sharp (see Proposition 16).

Recall first that a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ is said to be *Hadamard lacunary* if $\inf_k n_{k+1}/n_k \geq q > 1$ for some q and that the *modulus of L^2-continuity* of $f \in L^2(\mathbb{R}/\mathbb{Z})$ is defined by $\omega_2(\delta, f) = \sup_{0\leq h\leq\delta} \|f(\cdot + h) - f(\cdot)\|_2$.

Theorem C. *Suppose that $f \in L^2(\mathbb{R}/\mathbb{Z})$ satisfies $\int f(x)dx = 0$ and*

$$\sum_{n\in\mathbb{N}} \frac{\omega_2(2^{-n}, f)}{\sqrt{n}} < \infty \tag{8}$$

and that $(n_k)_{k\in\mathbb{N}}$ is Hadamard lacunary. Then $\{f(n_k x)\}$ is a convergence system.

Let us look at a very interesting special system $\{f(n_k x)\}$ where f is a Davenport function. Let $\lambda > 0$. The function

$$f_\lambda(x) = \sum_{m=1}^\infty \frac{\sin 2\pi m x}{m^\lambda}$$

is well defined because the series converges everywhere. It is called Davenport function. When $\lambda > 1/2$, it is L^2-integrable and $\{f_\lambda(nx)\}_{n\geq 1}$ is a complete system in $L^2([0,1])$ ([23]). When $\lambda > 1$, $\{f_\lambda(nx)\}_{n\geq 1}$ is even a Riesz basis ([15, 18]). When $1/2 < \lambda \leq 1$, $\{f_\lambda(nx)\}_{n\geq 1}$ is not a Riesz basis, but Brémont proved that $\{f_\lambda(n_k x)\}_{n_k\geq 1}$ is a Riesz sequence (see Definition 1) for any Hadamard lacunary sequence $\{n_k\}$.

Theorem D *Let $\lambda > 1/2$. Suppose that $\{n_k\} \subset \mathbb{N}$ is lacunary in the sense of Hadamard. Then the so-called Davenport series $\sum_{k=1}^{\infty} a_k f_\lambda(n_k x)$ converges almost everywhere if ond only if $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.*

In this special case of Davenport series, the condition $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ is also proved necessary for the almost everywhere convergence. Both the sufficiency and necessity are new.

Now let us give an application of Theorem B. Consider a measure-theoretic dynamical system $(X, \mathcal{B}, \mu T)$. Let L be the associated transfer (or Perron-Frobenius) operator, which is defined by

$$\int f \cdot Lg d\mu = \int f \circ T \cdot g d\mu \qquad (\forall f \in L^\infty(\mu), \forall g \in L^1(\mu)). \tag{9}$$

As an application of Theorem B, we have the following theorem, in which the condition (H2) is sharp to some extent (see Proposition 18).

Theorem E *Assume that $(X, \mathcal{B}, T, \mu)$ is an ergodic measure-preserving dynamical system and $f \in L^2(\mu)$. Suppose*

(H1) $\lim_{m\to\infty} \mathbb{E}(f|T^{-m}\mathcal{B}) = 0$ *a.e., which implies* $\int f d\mu = 0$*;*

(H2) $\sum_{n=1}^{\infty} \frac{\|L^n f\|_2}{\sqrt{n}} < \infty$

Then $\{f(T^n x)\}$ is a convergence system.

This theorem improves a result in [9], where the norm $\|\cdot\|_\infty$ was used instead of the norm $\|\cdot\|_2$.

The paper is organized as follows. In Section 2 we performs an analysis using increasing filtrations. Theorem A will be proved there, together with some more general results. Conditions in Theorem A will be converted into some more practical conditions and divergence will also discussed. Section 3 is parallel to Section 2, as Theorem B is parallel to Theorem A. There an analysis is made using decreasing filtration. But details are omitted and some details can also be found in [9]. Application of Theorem A to ergodic series is discussed in Section 4 and Theorem E will be proved there. Section 5 is devoted to dilated series (Theorem C and Theorem D) and Section 6 is devoted to lacunary series and their almost everywhere convergence with respect to Riesz product and to more general inhomogeneous equilibriums.

2. Analysis using increasing filtration

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $p \geq 1$. Let $(Z_n)_{n\geq 0} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$ be a sequence random variables such that $\mathbb{E}Z_n = 0$ for all n. We shall study the almost sure convergence of the random series $\sum_{n=0}^{\infty} Z_n(\omega)$. For $n \geq 0$, denote the partial sum

$$S_n = \sum_{k=0}^{n} Z_k.$$

Then define the maximal functions

$$S^*(\omega) = \sup_{n\in\mathbb{N}} |S_n(\omega)|, \quad S_N^*(\omega) = \max_{0\leq n\leq N} |S_n(\omega)| \ \ (\forall N \geq 0).$$

In this section, we give an analysis of the series by using an increasing filtration. In the next section, we do the same by using an decreasing filtration. Notice that there will be minor differences between two analysis. But applications will show that there is a question of how to choose a filtration. For example, for the dilated series, we will introduce an increasing filtration and for the ergodic series, there is a natural decreasing filtration.

2.1. **Decomposition relative to an increasing filtration.** Suppose that we are given an increasing filtration $(\mathcal{A}_n)_{n\in\mathbb{N}}$. Assume that $\mathcal{A}_0 = \{\emptyset, \Omega\}$ and $\mathcal{A}_\infty = \mathcal{A}$ where $\mathcal{A}_\infty := \bigvee_{n=0}^{\infty} \mathcal{A}_n$. For every $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathbb{E}^n := \mathbb{E}(\cdot|\mathcal{A}_n)$ and

$$\mathcal{D}_n = \mathbb{E}^{n+1} - \mathbb{E}^n.$$

The operators $\mathcal{D}_n$ have the following remarkable properties. The proofs, which are easy, are left to the reader.

Lemma 1. *Assume $h \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $f, g \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. The operators $\mathcal{D}_n$ have the following properties.*

(1) *For any $n \geq 0$, $\mathcal{D}_n$ is $\mathcal{A}_{n+1}$-measurable and $\mathbb{E}^n\mathcal{D}_n f = 0$.*
(2) *For any distinct integers n and m, $\mathcal{D}_n f$ and $\mathcal{D}_m g$ are orthogonal.*
(3) *For any $N_1 < N_2$ we have*

$$\sum_{n=N_1}^{N_2} \|\mathcal{D}_n f\|_2^2 = \|\mathbb{E}^{N_2+1} f - \mathbb{E}^{N_1} f\|_2^2.$$

The first assertion implies that for any sequence $(f_n) \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, $(\mathcal{D}_n f_n)$ is a sequence of martingale differences. The second assertions will be referred to as the orthogonality of the martingale difference.

For any integral random variable of zero mean, we may decompose it into martingale as follows. In the following lemma, we include an inequality of Bahr-Esseen-Rio and an inequality of Burkholder.

Lemma 2. *Let $Z \in L^p(\Omega, \mathcal{A}, \mathbb{P})$, $p \geq 1$ with $\mathbb{E}Z = 0$. We have the decomposition*

$$Z = \sum_{n=0}^{\infty} \mathcal{D}_n Z \,, \tag{10}$$

where the series converges in L^p and $\mathbb{P}$-a.s. Moreover we have

$$\|Z\|_p \leq \max(1, \sqrt{p-1}) \Big(\sum_{n=0}^{\infty} \|\mathcal{D}_n Z\|_p^{p'} \Big)^{1/p'} \,, \tag{11}$$

where $p' := \min(2, p)$ and, if $p > 1$,

$$\Big\| \Big(\sum_{n=0}^{\infty} |\mathcal{D}_n Z|^2 \Big)^{1/2} \Big\|_p \leq C_p \|Z\|_p \,. \tag{12}$$

Proof. By assumptions, $(\mathbb{E}^n Z)_{n\in\mathbb{N}}$ is a uniformly integrable martingale converging in L^p and $\mathbb{P}$-a.s. to $\mathbb{E}^\infty Z = Z$. Hence, (10) follows from the equality

$$\sum_{n=1}^{N} \mathcal{D}_n Z = \mathbb{E}^{N+1} Z - \mathbb{E}^0 Z$$

where $\mathbb{E}^0 Z = \mathbb{E}(Z|\mathcal{A}_0) = \mathbb{E}Z = 0$, for $\mathcal{A}_0 = \{\emptyset, \Omega\}$. Then, (11) follows from von Bahr-Esseen [1] when $1 \leq p \leq 2$ and from Rio [21] when $p \geq 2$, while (12) is the Burkholder inequality. □

Remark. The inequality (12) is nothing but the (reverse) Burkholder inequality. In particular, we have a converse inequality but we shall only need (12) in the sequel.

We call $\mathcal{D}_n Z$ the n-th order detail of Z with respect to $(\mathcal{A}_n)$.

For any Z_n, we have the following decomposition

$$Z_n = X_n + Y_n \quad \text{with} \quad X_n = Z_n - \mathbb{E}^n Z_n, \quad Y_n = \mathbb{E}^n Z_n.$$

Notice that $\mathbb{E}^n X_n = 0$ i.e. X_n is conditionally centered, and Y_n is $\mathcal{A}_n$-measurable with $\mathbb{E}(Y_n) = 0$. Our random series is thus decomposed into two random series

$$\sum_{n=0}^{\infty} Z_n = \sum_{n=0}^{\infty} X_n + \sum_{n=0}^{\infty} Y_n.$$

The convergence of $\sum Z_n$ is reduced to those of $\sum X_n$ and $\sum Y_n$. In the sequel, we separately study these two series.

2.2. **Conditionally centered series $\sum X_n$.** The maximal functions associated to the series $\sum X_n$ will be denoted by $S^*_{N,X}$ and S^*_X. Recall that we write $p' = \min(2,p)$ for $p \geq 1$.

Proposition 3. *Let $(X_n)_{n\in\mathbb{N}} \subset L^p(\Omega,\mathcal{A},\mathbb{P})$ be such that $\mathbb{E}^n(X_n) = 0$ for every $n \in \mathbb{N}$ $(p > 1)$. Then, for every $N \in \mathbb{N}$, we have the following maximal inequality*

$$\big\|S^*_{N,X}\big\|_p \leq \frac{p}{p-1}\max(1,\sqrt{p-1})\sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{N}\|\mathcal{D}_{\ell+k}X_\ell\|_p^{p'}\Big)^{1/p'}. \tag{13}$$

Consequently, the series $\sum_{n\in\mathbb{N}} X_n$ converges $\mathbb{P}$-a.s. and in $L^p(\Omega,\mathcal{A},\mathbb{P})$ under the following condition:

$$\sum_{k=0}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_{n+k}X_n\|_p^{p'}\Big)^{1/p'} < \infty\,. \tag{14}$$

Moreover, when (14) *holds, we have $S^*_X \in L^p(\Omega,\mathcal{A},\mathbb{P})$.*

Proof. By the decomposition in Lemma 2 applied to each X_ℓ and using the fact $\mathbb{E}^\ell(X_\ell) = 0$ which implies that, for every $k \leq \ell$, $\mathbb{E}^k X_\ell = \mathbb{E}^k(\mathbb{E}^\ell X_\ell) = 0$, hence that $\mathcal{D}_k X_\ell = 0$ for $k < \ell$, we have

$$X_\ell = \sum_{k\geq\ell}\mathcal{D}_k X_\ell = \sum_{k=0}^{\infty}\mathcal{D}_{k+\ell}X_\ell\,.$$

Let $N \in \mathbb{N}$ and $0 \leq n \leq N$. We obtain that

$$|S_n| = \left|\sum_{\ell=0}^{n}X_\ell\right| = \left|\sum_{\ell=0}^{n}\sum_{k=0}^{\infty}\mathcal{D}_{\ell+k}X_\ell\right| \leq \sum_{k=0}^{\infty}\max_{0\leq m\leq N}\left|\sum_{\ell=0}^{m}\mathcal{D}_{\ell+k}X_\ell\right|.$$

Now, for every $k \in \mathbb{N}$ fixed, the sequence $(\sum_{\ell=0}^{m}\mathcal{D}_{\ell+k}X_\ell)_{m\in\mathbb{N}}$ is a martingale. Hence, by Doob's maximal inequality and the von Bahr-Esseen-Rio inequality (11) in Lemma 2, we have

$$\|S^*_N\|_p \leq \frac{p}{p-1}\max(1,\sqrt{p-1})\sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{N}\|\mathcal{D}_{\ell+k}X_\ell\|_p^{p'}\Big)^{1/p'}.$$

Thus we have proved the maximal inequality. Consequently, for any $N' \leq N''$, setting $K_p := \frac{p}{p-1}\max(1,\sqrt{p-1})$, we have

$$\|\max_{N'\leq p,q\leq N''}|S_p - S_q|\|_p \leq 2\|\max_{N'\leq n\leq N''}|S_n - S_{N'}|\|_p \leq 2K_p\sum_{k=0}^{\infty}\Big(\sum_{\ell=N'+1}^{N''}\|\mathcal{D}_{\ell+k}X_\ell\|_p^{p'}\Big)^{1/p'}.$$

Letting $N'' \to +\infty$, we infer that

$$\|\sup_{p,q\geq N'}|S_p - S_q|\|_p \leq 2K_p\sum_{k=0}^{\infty}\Big(\sum_{\ell\geq N'+1}\|\mathcal{D}_{\ell+k}X_\ell\|_p^{p'}\Big)^{1/p'} < \infty\,, \tag{15}$$

by (14). By the Lebesgue dominated convergence theorem on $\mathbb{N}$, applied to the counting measure, we deduce that

$$\| \sup_{p,q\geq N'} |S_p - S_q| \|_p \underset{N'\to+\infty}{\longrightarrow} 0\,.$$

By the Fatou lemma we infer that almost surely

$$\sup_{p,q\geq N'} |S_p - S_q| \underset{N'\to+\infty}{\longrightarrow} 0\,,$$

which finishes the proof. □

Remark. If $p \geq 2$, then $p' = 2$ and the condition (14) with $p \geq 2$ is stronger than the condition (14) with $p = 2$. Hence, the relevance of the proposition when $p \geq 2$ lies in the integrability of the maximal function S_X^*. While if one is only concerned with the a.s. convergence it is better to apply the proposition with $p = 2$. However, as we shall see, the control of S^* in L^p with $p > 2$ will allows us to study the divergence of the series $\sum X_n$.

2.3. **Adapted series $\sum Y_n$.** The maximal functions associated to the series $\sum Y_n$ will be denoted by $S_{N,Y}^*$ and S_Y^*.

Proposition 4. *Let $(Y_n)_{n\geq 1} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$ be such that Y_n is $\mathcal{A}_n$-measurable and $\mathbb{E}Y_n = 0$ for every $n \geq 1$ ($p > 1$). Then, for every $N \geq 1$,*

$$\|S_{N,Y}^*\|_p \leq K_p \sum_{k=1}^{N} \Big(\sum_{\ell=k}^{N} \|\mathcal{D}_{\ell-k} Y_\ell\|_p^{p'} \Big)^{1/p'} \tag{16}$$

Consequently, the series $\sum_{n\in\mathbb{N}} Y_n$ converges $\mathbb{P}$-a.s. and in $L^p(\Omega, \mathcal{A}_0, \mathbb{P})$ under the following condition

$$\sum_{k=1}^{\infty} \Big(\sum_{n=0}^{\infty} \|\mathcal{D}_n Y_{n+k}\|_p^{p'} \Big)^{1/p'} < \infty. \tag{17}$$

Moreover, when (17) holds, we have $S_Y^* \in L^p(\Omega, \mathcal{A}, \mathbb{P})$.

Proof. The proof is similar to that of Proposition 3. By assumption, we infer that for every $\ell \geq 1$,

$$Y_\ell = \sum_{k=0}^{\ell-1} \mathcal{D}_k Y_\ell = \sum_{k=1}^{\ell} \mathcal{D}_{\ell-k} Y_\ell\,.$$

Hence,

$$|\sum_{\ell=1}^{n} Y_\ell| = |\sum_{\ell=1}^{n} \sum_{k=1}^{\ell} \mathcal{D}_{\ell-k} Y_\ell| \leq \sum_{k=1}^{N} \max_{k\leq m\leq N} |\sum_{\ell=k}^{m} \mathcal{D}_{\ell-k} Y_\ell|\,.$$

Since, for every fixed $k \in \mathbb{N}$, $(\sum_{\ell=k}^{m} \mathcal{D}_{\ell-k} Y_\ell)_{m\geq 1}$ is a martingale, then (16) follows from the Doob maximal inequality and the Bahr-Esseen-Rio inequality (11).

The proof of the $\mathbb{P}$-a.s. and L^p-convergence may be done as for Proposition 3. □

2.4. **Convergence and integrability of general series $\sum Z_n$.** Combining Proposition 3 and Proposition 4 and using the decomposition $Z_n = X_n + Y_n$, we derive the following theorem.

Theorem 5. *Let $p > 1$. Let $(Z_n)_{n\in\mathbb{N}} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$ be such that $\mathbb{E}Z_n = 0$ for every $n \in \mathbb{N}$. Then, for every $N \in \mathbb{N}$,*

$$\|S^*_{N,Z}\|_p \le K_p\Big(\sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{N}\|\mathcal{D}_{\ell+k}Z_\ell\|_p^{p'}\Big)^{1/p'} + \sum_{k=1}^{N}\Big(\sum_{\ell=k}^{N}\|\mathcal{D}_{\ell-k}Z_\ell\|_p^{p'}\Big)^{1/p'}\Big)\,. \tag{18}$$

Consequently, the series $\sum_{n\in\mathbb{N}} Z_n$ converges $\mathbb{P}$-a.s. and in $L^p(\Omega, \mathcal{A}, \mathbb{P})$ under the following set of conditions

$$\sum_{k=0}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_{n+k}Z_n\|_p^{p'}\Big)^{1/p'} < \infty \tag{19}$$

and

$$\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_{n}Z_{n+k}\|_p^{p'}\Big)^{1/p'} < \infty\,. \tag{20}$$

Moreover, if both conditions (19) *and* (20) *are satisfied, then $S^*_Z \in L^p(\Omega, \mathcal{A}, \mathbb{P})$.*

Remark. If (Z_n) is adapted to $(\mathcal{A}_n)$, then the condition (19) is trivially satisfied.

Proof. Recall that $X_n = Z_n - \mathbb{E}^n Z_n$ and $Y_n = \mathbb{E}^n Z_n$. The theorem follows immediately from the decomposition $Z_n = X_n + Y_n$, Proposition 3 and Proposition 4 and the following simple facts

$$S^*_N \le S^*_{N,X} + S^*_{N,Y},$$

$$\mathcal{D}_{\ell+k}X_\ell = \mathbb{E}^{\ell+k+1}(Z_\ell - \mathbb{E}^\ell Z_\ell) - \mathbb{E}^{\ell+k}(Z_\ell - \mathbb{E}^\ell Z_\ell) = \mathcal{D}_{\ell+k}Z_\ell,$$

$$\mathcal{D}_{\ell-k}Y_\ell = \mathbb{E}^{\ell-k+1}(\mathbb{E}^\ell Z_\ell) - \mathbb{E}^{\ell-k}(\mathbb{E}^\ell Z_\ell) = \mathcal{D}_{\ell-k}Z_\ell.$$

□

We call (19) the condition on higher order details and (20) the condition on lower order details.

2.5. **Practical criteria.** In order to apply Theorem 5, it is sometimes more convenient to use the sufficient conditions in the next lemma. The proof of the lemma will use the Burkholder inequality.

The condition (21) suggests that we need some order of approximation of Z_k by $\mathbb{E}(Z_k|\mathcal{A}_n)$ as n tends to infinity and the condition (22) suggests that the conditional expectation $\mathbb{E}(\cdot|\mathcal{A}_n)$) would contract in some order on the space $L^p_0(\Omega, \mathcal{A}, P)$ consisting of p-integrable variables with zero mean; sometimes it is really the case (see Lemma 21, see also Theorem 2 in [8] and Lemma 2 in [10]).

Lemma 6. *Let $(Z_n)_{n\in\mathbb{N}} \subset L^p(\Omega,\mathcal{A},\mathbb{P})$, $p>1$, with $\mathbb{E}Z_n = 0$ for all n. The condition* (19) *on the higher order details is satisfied if*

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=0}^{\infty}\|Z_k - \mathbb{E}^{2^\ell+k-1}Z_k\|_p^{p'}\Big)^{1/p'} < \infty\,. \tag{21}$$

The condition (20) *on the lower order details is satisfied if*

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=2^\ell}^{\infty}\|\mathbb{E}^{k+1-2^\ell}Z_k\|_p^{p'}\Big)^{1/p'} < \infty\,. \tag{22}$$

Proof. We make the proof when $p\geq 2$, then $p'=2$. The proof when $1<p<2$ may be done similarly. Look first at the condition (19) on the higher order of details. Cutting the sum over k into dyadic blocks and applying Cauchy-Schwarz inequality to each block we get

$$\sum_{k=0}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_{n+k}Z_n\|_p^2\Big)^{1/2} \leq \sum_{\ell=0}^{\infty}2^{\ell/2}\Big(\sum_{k=2^\ell-1}^{2^{\ell+1}-2}\sum_{n=0}^{\infty}\|\mathcal{D}_{n+k}Z_n\|_p^2\Big)^{1/2}.$$

Now, using successively the Hölder inequality (notice that $p/2\geq 1$), the trivial inequality $\|\cdot\|_{\ell^p}\leq\|\cdot\|_{\ell^2}$ and the reverse Burkholder inequality (12), we get that

$$\begin{aligned}
\sum_{k=2^\ell-1}^{2^{\ell+1}-2}\|\mathcal{D}_{n+k}Z_n\|_p^2 &\leq 2^{\ell(1-2/p)}\Big(\sum_{k=2^\ell-1}^{2^{\ell+1}-2}\|\mathcal{D}_{n+k}Z_n\|_p^p\Big)^{2/p}\\
&= 2^{\ell(1-2/p)}\Big(\mathbb{E}\big(\sum_{k=2^\ell-1}^{2^{\ell+1}-2}(\mathcal{D}_{n+k}Z_n)^p\big)\Big)^{2/p}\\
&\leq 2^{\ell(1-2/p)}\Big\|\big(\sum_{k\geq 2^\ell-1}(\mathcal{D}_{n+k}Z_n)^2\big)^{1/2}\Big\|_p^2\\
&\leq C_p 2^{\ell(1-2/p)}\|Z_n-\mathbb{E}^{n+2^\ell-1}Z_n\|_p^2\,.
\end{aligned}$$

Thus the first assertion follows. Similarly, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_nZ_{n+k}\|_p^2\Big)^{1/2} &= \sum_{k=1}^{\infty}\Big(\sum_{n=k}^{\infty}\mathcal{D}_{n-k}Z_n\|_p^2\Big)^{1/2}\\
&\leq \sum_{\ell=0}^{\infty}2^{\ell/2}\Big(\sum_{k=2^\ell}^{2^{\ell+1}-1}\sum_{n=k}^{\infty}\mathcal{D}_{n-k}Z_n\|_p^2\Big)^{1/2}.
\end{aligned}$$

Now, we first change the order of summation to get

$$\sum_{k=2^\ell}^{2^{\ell+1}-1}\sum_{n=k}^{\infty}\|\mathcal{D}_{n-k}Z_n\|_p^2 = \sum_{n=2^\ell}^{\infty}\sum_{k=2^\ell}^{\min(n,2^{\ell+1}-1)}\|\mathcal{D}_{n-k}Z_n\|_p^2\,.$$

Then, using the same arguments as above, we have

$$\begin{aligned}\sum_{k=2^\ell}^{\min(n,2^{\ell+1}-1)} \|\mathcal{D}_{n-k}Z_n\|_p^2 &\leq C_p 2^{\ell(1-2/p)}\|\mathbb{E}^{n+1-2^\ell}Z_n - \mathbb{E}^0 Z_n\|_p^2 \\ &= C_p 2^{\ell(1-2/p)}\|\mathbb{E}^{n+1-2^\ell}Z_n\|_p^2.\end{aligned}$$

Finally we get

$$\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\mathcal{D}_n Z_{n+k}\|_p^2\Big)^{1/2} \leq C_p\sum_{\ell=0}^{\infty}2^{\ell(1-1/p)}\Big(\sum_{k=2^\ell}^{\infty}\|\mathbb{E}^{k+1-2^\ell}Z_k\|_p^2\Big)^{1/2}.$$

□

In view of the above results and of Lemma 5 of [9], one expects to have better integrability properties for S_Z^* when $(Z_n) \subset L^\infty(\Omega, \mathcal{A}, \mathbb{P})$.

Theorem 7. *Let $(Z_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega,\mathcal{A},\mathbb{P})$ with $\mathbb{E}Z_n = 0$ for all n. Assume that*

$$\Delta_1 := \sum_{\ell=0}^{\infty}\Big(\sum_{k=0}^{\infty}\|Z_k - \mathbb{E}^{\ell+k}Z_k\|_\infty^2\Big)^{1/2} < \infty\,. \tag{23}$$

and

$$\Delta_2 := \sum_{\ell=0}^{\infty}\Big(\sum_{k=\ell}^{\infty}\|\mathbb{E}^{k+1-\ell}Z_k\|_\infty^2\Big)^{1/2} < \infty\,. \tag{24}$$

Then $\sum_{n\in\mathbb{N}} Z_n$ converges $\mathbb{P}$-a.e. and in any L^p with $p \geq 1$. Moreover, we have

$$\mathbb{E}(\mathrm{e}^{\beta(S_Z^*)^2}) < \infty,$$

for every $\beta < \frac{1}{4e(\Delta_1+\Delta_2)^2}$.

Proof. We follow a standard strategy to prove the exponential integrability S_Z^* by estimating the p-th moments of S_Z^*. To estimate the p-th moments we shall not use Lemma 6 which yields badly behaving constants C_p, as $p \to \infty$, due to the use of Burkholder's inequality. Hence, we shall use Theorem 5 instead.

First remark that $\|S_Z^*\|_p$ $(p \geq 2)$ is bounded by K_p times the sum of the two terms in (19) and (20). Notice that

$$\|\mathcal{D}_{\ell+k}Z_\ell\|_p \leq \|\mathbb{E}^{\ell+k+1}Z_\ell - Z_\ell\|_p + \|\mathbb{E}^{\ell+k}Z_\ell - Z_\ell\|_p.$$

Then, using Minkowski's inequality in ℓ^2 we obtain that for every $N \in \mathbb{N}$,

$$\sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{\infty}\|\mathcal{D}_{\ell+k}Z_\ell\|_p^2\Big)^{1/2} \leq 2\sum_{k=0}^{\infty}\Big(\sum_{\ell=0}^{\infty}\|Z_\ell - \mathbb{E}^{\ell+k}Z_\ell\|_\infty^2\Big)^{1/2} = 2\Delta_1\,.$$

On the other hand, using

$$\|\mathcal{D}_n Z_{n+k}\|_p \leq 2\|\mathbb{E}^{n+1}Z_{n+k}\|_p,$$

we have

$$\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathcal{D}_n Z_{n+k}\|_p^2\Big)^{1/2}\le 2\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathbb{E}^{n+1}Z_{n+k}\|_\infty^2\Big)^{1/2}=2\Delta_2\,.$$

Thus we have proved $\|S_Z^*\|_p\le 2K_p\Delta$ with $\Delta=\Delta_1+\Delta_2$ for $p\ge 2$. Let $\beta>0$. We have

$$\mathbb{E}(\mathrm{e}^{\beta(S_Z^*)^2})=\sum_{p=0}^{\infty}\frac{\beta^p\|S_Z^*\|_{2p}^{2p}}{p!}\le 1+\sum_{p=1}^{\infty}\frac{\beta^p(2K_{2p}\Delta)^{2p}}{p!}\,.$$

Since $K_{2p}^{2p}\le\big(\frac{2p}{2p-1}\big)^{2p}p^p\sim \mathrm{e}p^p$ and, by Stirling's formula, $p!\sim\big(\frac{p}{\mathrm{e}}\big)^p\sqrt{2\pi p}$, we infer that $\mathbb{E}(\mathrm{e}^{\beta(S_Z^*)^2})<\infty$ as soon as $\beta<(4\mathrm{e}\Delta^2)^{-1}$. □

2.6. **Series of the form $\sum_{n=0}^{\infty}a_nW_n$.** In our applications, we shall be concerned with the situation where

$$Z_n=a_nW_n$$

with $(a_n)_{n\in\mathbb{N}}$ a deterministic sequence and $(W_n)_{n\in\mathbb{N}}$ a stationary sequence or at least a sequence behaving somehow closely to a stationary one. When $p\ge 2$ (resp. when $1<p<2$), we are going to control the L^p-moment of $\sum a_nW_n$ by the ℓ^2-moment (resp. the ℓ^p-moment) of (a_n). Before stating the result, let us state Cauchy's condensation principle whose proof is easy.

Lemma 8. *Let $(u_n)_{n\in\mathbb{N}}$ be positive numbers such that $u_{n+m}\le Ku_n$ for every $n,m\in\mathbb{N}$ and for some $K>0$. Then the series $\sum_{\ell\in\mathbb{N}}u_\ell$ converges if and only if the series $\sum_{\ell\in\mathbb{N}}2^\ell u_{2^\ell}$ converges.*

Theorem 9. *Let $1<p\le\infty$. Let $(W_n)_{n\in\mathbb{N}}\subset L^p(\Omega,\mathcal{A},\mathbb{P})$ be such that*

$$\widetilde{\Delta}_p:=\sum_{\ell=0}^{\infty}\frac{1}{(\ell+1)^{1/p}}\Big(\sup_{k\in\mathbb{N}}\|W_k-\mathbb{E}^{k+\ell-1}W_k\|_p+\sup_{m\in\mathbb{N}}\|\mathbb{E}^{m+1}W_{m+\ell}\|_p\Big)<\infty\,. \tag{25}$$

Then the series $\sum_{n\in\mathbb{N}}a_nW_n$ converges $\mathbb{P}$-a.e. for every $(a_n)_{n\in\mathbb{N}}\in\ell^{p'}$. Moreover, when $1<p<\infty$, there exists $C_p>0$ (independent from $(a_n)_{n\in\mathbb{N}}$ and $(W_n)_{n\in\mathbb{N}}$) such that

$$\|\sup_{N\in\mathbb{N}}|\sum_{n=0}^{N}a_nW_n|\,\|_p\le C_p\widetilde{\Delta}_p\|(a_n)_{n\in\mathbb{N}}\|_{\ell^{p'}}\,; \tag{26}$$

and when $p=\infty$, we have

$$\mathbb{E}\exp\big(\beta\sup_{N\in\mathbb{N}}|\sum_{n=0}^{N}a_nW_n|^2\big)<\infty$$

for every $\beta<(4\mathrm{e}\widetilde{\Delta}_\infty\|(a_n)_{n\in\mathbb{N}}\|_{\ell^2})^{-1}$.

Proof. Let $1 < p < \infty$. By Theorem 5 and Lemma 6, we only have to check that

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)} \Big(\sup_{k\in\mathbb{N}} \|W_k - \mathbb{E}^{k+2^\ell-1} W_k\|_p + \sup_{m\in\mathbb{N}} \|\mathbb{E}^{m+1} W_{m+2^\ell}\|_p \Big) < \infty \,. \tag{27}$$

This is actually equivalent to the condition (25), by the Cauchy condensation principle. Let us check the condition in the Cauchy condensation principle. Notice that

$$\sup_{m\in\mathbb{N}} \|\mathbb{E}^{m+1} W_{m+\ell+1}\|_p \le \sup_{m\in\mathbb{N}} \|\mathbb{E}^{m+2} W_{m+\ell+1}\|_p \le \sup_{m\in\mathbb{N}} \|\mathbb{E}^{m+1} W_{m+\ell}\|_p \,,$$

and that, by the Burkholder inequality (12), for every $k, \ell \in \mathbb{N}$ and every $m \ge 1$

$$\begin{aligned} C_p \|W_k - \mathbb{E}^{k+\ell-1} W_k\|_p &\ge \|\big(W_k - \mathbb{E}^{k+\ell+m-1} W_k)^2 + (\mathbb{E}^{k+\ell+m-1} W_k - \mathbb{E}^{k+\ell-1} W_k)^2\big)^{1/2}\|_p \\ &\ge \|W_k - \mathbb{E}^{k+\ell+m-1} W_k\|_p \end{aligned}$$

The case $p = \infty$ can be proved similarly basing on Theorem 7. □

To conclude this section we shall prove that, when $p \ge 2$, there are situations where the condition $(a_n)_{n\in\mathbb{N}} \in \ell^2$ in the above theorem is also necessary for the $\mathbb{P}$-a.e. convergence of $\sum_{n\in\mathbb{N}} a_n W_n$.

Definition 1. *We say that a sequence $(W_n)_{n\in\mathbb{N}} \subset L^2(\Omega, \mathcal{A}, \mathbb{P})$ is a Riesz system if there exists $C > 0$ such that for every $(b_n)_{n\in\mathbb{N}} \in \ell^2$,*

$$C^{-1} \|(b_n)_{n\in\mathbb{N}}\|_{\ell^2} \le \|\sum_{n\in\mathbb{N}} b_n W_n\|_2 \le C \|(b_n)_{n\in\mathbb{N}}\|_{\ell^2} \,.$$

If moreover $(W_n)_{n\in\mathbb{N}}$ is complete in $L^2(\Omega, \mathcal{A}, \mathbb{P})$, we say that is a Riesz basis.

Theorem 10. *Let $2 < p \le \infty$. Suppose that $(W_n)_{n\in\mathbb{N}} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$ such that*

$$\sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)^{1/p}} \Big(\sup_{k\in\mathbb{N}} \|W_k - \mathbb{E}^{k+\ell-1} W_k\|_p + \sup_{m\in\mathbb{N}} \|\mathbb{E}^{m+1} W_{m+\ell}\|_p \Big) < \infty \tag{28}$$

and that $(W_n)_{n\in\mathbb{N}}$ is is a Riesz sequence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Then the series $\sum_{n\in\mathbb{N}} a_n W_n$ does not converge $\mathbb{P}$-a.e. for every $(a_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} |a_n|^2 = \infty$.

Remark. We do not know whether the series is $\mathbb{P}$-a.e. divergent under the above conditions.

Proof. We proceed as in the proof of Theorem 2.6 of [9]. Recall first the following Paley-Zygmund inequality (see Kahane [16] for the case $q = 2$, the proof being the same for general $q > 1$): Let $Z \in L^q(\Omega, \mathcal{A}, \mathbb{P})$ be non negative ($q > 1$). For any $0 < \lambda < 1$, we have

$$\mathbb{P}(Z \ge \lambda \mathbb{E}Z) \ge \Big((1-\lambda) \frac{\mathbb{E}Z}{\|Z\|_q} \Big)^{q/(q-1)} \,.$$

We apply the inequality with $q = p/2$ and $Z = Z_N = \sup_{0\leq n\leq N} |\sum_{k=0}^n a_k W_k|^2$. By Theorem 9 and the hypothesis that $(W_n)_{n\in\mathbb{N}}$ is a Riesz sequence, we know that there exist $C, D > 0$ such that

$$D\sqrt{\sum_{k=0}^N |a_k|^2} \leq \mathbb{E}Z_N \leq \|Z_N\|_q \leq C\sqrt{\sum_{k=0}^N |a_k|^2}\,.$$

Hence, we infer that

$$\mathbb{P}\big(Z_N \geq \lambda D \sum_{k=0}^N |a_k|^2\big) \geq \Big((1-\lambda)\frac{D}{C}\Big)^{q/(q-1)}.$$

Since, $\sum_{n\in\mathbb{N}} |a_n|^2 = \infty$, the result follows. □

3. Analysis using decreasing filtration

We can also use decreasing filtrations to analyze our series. Recall that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $(Z_n)_{n\geq 0} \subset L^1(\Omega, \mathcal{A}, \mathbb{P})$ is a sequence of random variables such that $\mathbb{E}Z_n = 0$ for all n. Our object of study is the random series

$$\sum_{n=0}^\infty Z_n(\omega). \tag{29}$$

Suppose that we are given an decreasing filtration $(\mathcal{B}_n)_{n\in\mathbb{N}}$. Assume that $\mathcal{B}_0 = \mathcal{A}$. Let $\mathcal{B}_\infty := \bigcap_{n=0}^\infty \mathcal{B}_n$. We will suppose that

$$\forall n, \quad \mathbb{E}(Z_n|\mathcal{B}_\infty) = 0 \tag{30}$$

which implies $\mathbb{E}Z_n = 0$.

For $n \geq 0$, denote the partial sum

$$S_n = \sum_{k=0}^n Z_k.$$

For $N \geq 0$, define the maximal function

$$S_N^*(\omega) = \max_{0\leq n\leq N} |S_n(\omega)|.$$

The basic idea is to convert the random series into reverse martingales.

For every $n \in \mathbb{N} \cup \{\infty\}$, denote $\mathbb{E}^n := \mathbb{E}(\cdot|\mathcal{B}_n)$ and

$$\mathfrak{d}_n = \mathbb{E}_n - \mathbb{E}_{n+1}.$$

Let us state the useful properties of the operators $\mathfrak{d}_n$ in the following proposition. The following lemma is the same as Lemma 1.

Lemma 11. *Let $h \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $f, g \in L^2(\Omega, \mathcal{A}, \mathbb{P})$. The operators $\mathfrak{d}_n$ have the following properties:*

(1) *For and $n \geq 0$, $\mathcal{D}_n$ is $\mathcal{B}_n$-measurable and $\mathbb{E}_{n+1}\mathcal{D}_n f = 0$.*
(2) *For any distinct integers n and m, $\mathfrak{d}^n f$ and $\mathfrak{d}^m g$ are orthogonal.*
(3) *For any $N_1 < N_2$ we have*

$$\sum_{n=N_1}^{N_2} \|\mathfrak{d}_n f\|_2^2 = \|\mathbb{E}_{N_1} f - \mathbb{E}_{N_2+1} f\|_2^2.$$

The first assertion implies that for any sequence $(f_n) \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, $(\mathfrak{d}_n f_n)$ is a sequence of reverse martingale differences.

For any integral random variable such that $\mathbb{E}(Z|\mathcal{B}_\infty) = 0$, we may decompose it into martingale as follows. We have the following analogue of Lemma 2, whose proof is the same (it does not matter here that we are dealing with reverse martingale differences rather that martingale differences).

Lemma 12. *Let $Z \in L^p(\Omega, \mathcal{A}, \mathbb{P})$, $p \geq 1$ with $\mathbb{E}(Z|\mathcal{B}_\infty) = 0$. We have the decomposition*

$$Z = \sum_{n=0}^{\infty} \mathcal{D}_n Z\,, \tag{31}$$

where the series converges in L^p and $\mathbb{P}$-a.s. Moreover we have

$$\|Z\|_p \leq \max(1, \sqrt{p-1}) \Big(\sum_{n=0}^{\infty} \|\mathfrak{d}_n Z\|_p^{p'} \Big)^{1/p'}\,, \tag{32}$$

where $p' := \min(2, p)$ and, if $p > 1$,

$$\Big\| \Big(\sum_{n=0}^{\infty} |\mathfrak{d}_n Z|^2 \Big)^{1/2} \Big\|_p \leq C_p \|Z\|_p\,. \tag{33}$$

We call $\mathfrak{d}_n Z$ the *n-th order detail* of Z with respect to $(\mathcal{B}_n)$. Since $\mathcal{B}_n$ is decreasing, we say that $\mathfrak{d}_n Z$ is a detail of higher order than $\mathfrak{d}_m Z$ when $n < m$.

Theorem 13. *Let $(Z_n)_{n\in\mathbb{N}} \subset L^p(\Omega, \mathcal{A}, \mathbb{P})$, $p > 1$, be such that $\mathbb{E}_\infty(Z_n) = 0$ for every $n \in \mathbb{N}$. Then, for every $N \in \mathbb{N}$,*

$$\|S_N^*\|_p \leq 2K_p \Big(\sum_{k=1}^{N} \Big(\sum_{\ell=k}^{N} \|\mathfrak{d}_{\ell-k} Z_\ell\|_{p'}^{p'} \Big)^{1/p'} + \sum_{k=0}^{\infty} \Big(\sum_{\ell=0}^{N} \|\mathfrak{d}_{\ell+k} Z_\ell\|_{p'}^{p'} \Big)^{1/p'} \Big)\,. \tag{34}$$

Consequently, the series $\sum_{n\in\mathbb{N}} g_n$ converges $\mathbb{P}$-a.s. and in $L^p(\Omega, \mathcal{A}, \mathbb{P})$ under the following conditions

$$\sum_{k=0}^{\infty} \Big(\sum_{n=0}^{\infty} \|\mathfrak{d}_{n+k} Z_n\|_{p'}^{p'} \Big)^{1/p'} < \infty \tag{35}$$

and

$$\sum_{k=1}^{\infty}\Big(\sum_{n=0}^{\infty}\|\mathfrak{d}_n Z_{n+k}\|_{p'}^{p'}\Big)^{1/p'}<\infty\,. \tag{36}$$

If (Z_n) is adapted to $(\mathcal{B}_n)$, the condition (36) on the higher order details is trivially satisfied.

The proof being similar to that of Theorem 5, we leave it to the reader. Some similar arguments can be found [9].

In order to apply Theorem 13, it will be convenient to use the sufficient conditions in the next lemma.

Lemma 14. *Let $(Z_n)_{n\in\mathbb{N}}\subset L^p(\Omega,\mathcal{A},\mathbb{P})$, $p>1$, with $\mathbb{E}_\infty Z_n=0$ for all n. The condition* (36) *on the higher order details is satisfied if*

$$\sum_{\ell=0}^{\infty}2^{\ell(1-1/p)}\Big(\sum_{n=2^\ell}^{\infty}\|Z_n-\mathbb{E}_{n-2^\ell+1}Z_n\|_{p'}^{p'}\Big)^{1/p'}<\infty\,. \tag{37}$$

The condition (35) *on the lower order details is satisfied if*

$$\sum_{\ell=0}^{\infty}2^{\ell(1-1/p)}\Big(\sum_{n=0}^{\infty}\|\mathbb{E}_{n+2^\ell-1}Z_n\|_{p'}^{p'}\Big)^{1/p'}<\infty\,. \tag{38}$$

The proof is similar to that of Lemme 6 and we leave it to the reader.

Results similar to Theorems 7, 9 and 10 holds.

4. Convergence of ergodic series

Let $(X,\mathcal{B},\mu,T)$ be a measure-preserving dynamical system. By an ergodic series we mean a series of the form

$$\sum_{n=0}^{\infty}a_n f_n(T^n x) \tag{39}$$

where it is assumed that the f_n's are integrable with $\mathbb{E}f_n=0$ and that (a_n) is a sequence of numbers. The almost everywhere convergence of such series was studied in [9] where the martingale method was already used, and which is a motivation of our present study.

In this case, the natural filtration that we can use to analyze the series is the one defined by

$$\mathcal{B}_n=T^{-n}\mathcal{B}.$$

It is a decreasing filtration. Let L be the transfer operator associated to the dynamical system which can be defined by

$$\int f\cdot Lg\,d\mu=\int f\circ T\cdot g\,d\mu\qquad(\forall f\in L^\infty(\mu),\forall g\in L^1(\mu)).$$

Theorem 15. *Assume that $(X, \mathcal{B}, T, \mu)$ is an ergodic measure-preserving dynamical system. Let $(f_n)_{n\in\mathbb{N}} \subset L^p(\mu)$, $1 < p \leq \infty$. Suppose*

(H1) $\forall n \in \mathbb{N}$, $\lim_{m\to\infty} \|\mathbb{E}(f_n|T^{-m}\mathcal{B})\|_p = 0$;

(H2) $\sum_{i=0}^{\infty} 2^{\ell(1-1/p)} \sup_{n\geq 0} \|L^{2^\ell} f_n\|_p < \infty$.

Then for any complex sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ such that $\sum_{n=0}^{\infty} |a_n|^{p'} < \infty$, the ergodic series $\sum_{n=0}^{\infty} a_n f_n(T^n x)$ converges a.e., and in $L^p(X, \mathcal{B}, T, \mu)$ if $p < \infty$. Moreover, when $1 < p < \infty$, $\sup_{N\in\mathbb{N}} |\sum_{n=0}^{N} a_n f_n(T^n x)| \in L^p(X, \mathcal{B}, T, \mu)$ and when $p = \infty$, $\mathbb{E}(\exp(\beta \sup_{N\in\mathbb{N}} |\sum_{n=0}^{N} a_n f_n \circ T^n|^2)) < \infty$ for some $\beta > 0$.

Proof. Since $(f_n \circ T^n)$ is adapted to $(\mathcal{B}_n)$, we can apply Theorem 13 by just checking the condition (38) on the higher order details. The checking is easy and is a direct consequence of the fact that for $m \geq n$ we have

$$\mathbb{E}(f_n \circ T^n | T^{-m}\mathcal{B}) = (L^{m-n} f) \circ T^m.$$

Then, letting $Z_n = a_n f \circ T^n$, it suffices to notice

$$\|\mathbb{E}_{n+2^\ell-1} Z_n\|_p^{p'} = |a_n|^{p'} \|(L^{2^\ell-1} f_n) \circ T^m\|_p^{p'} = |a_n|^{p'} \|L^{2^\ell-1} f_n\|_p^{p'}.$$

□

When $p = \infty$, the condition $(H2)$ reads $\sum_{i=0}^{\infty} 2^\ell \sup_{n\geq 0} \|L^{2^\ell} f_n\|_\infty < \infty$. Under this last condition, the above theorem was proved in [9] and the exponential integrability of the maximal function was not mentioned in [9] but it follows from the obtained estimates there. If one is only interested in almost everywhere convergence but not in the exponential integrability, Theorem 15 relax the condition in [9] by requiring only L^p-integrability, $2 \leq p \leq \infty$ and weakening the exponent of 2^ℓ in (H2).

In the study of dynamical systems, the decay of $L^n f$ (measured by L^∞-norm or the Hölder norm) was extensively studied for regular functions f like Hölder functions. The above theorem shows that for the almost everywhere convergence of the ergodic series, weaker regularity would be sufficient and that there is an interest to study the decay of $L^n f$ measured by L^2-norm. Several situations where such a decay is measured, for unbounded functionals, may be found for instance in [5], section 3.2 or in [8].

To conclude the section, let us show that the condition (H2) in Theorem 15 is sharp. We shall use the notation $L_0(x) = 1$, $L_1(x) = \max(1, \log x)$ and $L_m(x) = L_1 \circ \cdot \circ L_1(x)$ (where L_1 appears m times).

Let T be the measure preserving transformation on $([0,1), \mathcal{B}([0,1)), \lambda)$ (λ being the Lebesgue measure) defines by $Tx = 2x \mod 1$. Then $\|L^n g\|_1 \to 0$ for every $g \in L^1([0,1), \mathcal{B}([0,1)), \lambda)$ such that $\int g(x)dx = 0$.

Proposition 16. *Consider the dynamics $([0,1), \mathcal{B}([0,1)), \lambda, T)$ where λ is the Lebesque measure and $Tx = 2x \mod 1$. For every $m \in \mathbb{N}$, there exist $(a_n)_{n\in\mathbb{N}} \in \ell^2$ and $f \in L^p([0,1), \mathcal{B}([0,1)), \lambda)$ for every $1 \leq p < \infty$ with $\int f(x)dx = 0$, such*

that $\|L^{2^n} f\|_2 = O(\frac{2^{-n/2}}{L_m(2^n)})$ *and that the series* $\sum_{n\in\mathbb{N}} a_n f \circ T^n$ *diverges almost everywhere.*

The proposition follows from Proposition 18 in the next section and the fact that for $Tx = \{2x\}$ and for $f \in L^2([0,1), \mathcal{B}([0,1)), \lambda)$ with $\int f(x)d(x)dx = 0$, we have $\|L^n f\|_2 = \omega_2(f, 2^{-n})$ (Theorem 2 in [8]), where ω_2 is the L^2-modulus of continuity.

5. Convergence of dilated series

We want to apply our general results in Section 2 to the study of the following series, called dilated series,

$$\sum_{k=0}^{\infty} a_k f(n_k x), \tag{40}$$

where we assume $f \in L^p([0,1], \mathcal{B}, \lambda)$ for some $p > 1$ (λ being the Lebesgue measure and $\mathcal{B}$ the Borel σ-algebra), $(a_k)_{k\in\mathbb{N}} \in \ell^{p'}(\mathbb{N})$ ($p' := \min(2, p)$), and $(n_k)_{k\in\mathbb{N}}$ an increasing sequence of positive integers.

There is an extensive literature on the topic for $p = 2$, which is our main concern here. Let us mention the surveys [11, (1966)] by Gaposhkin and [2, (2009)] by Berkes and Weber. For more recent results, one may refer to Weber [22], see also the references therein.

5.1. Lacunary dilated series.

Before stating our next result we need a definition.

Definition 2. *For every* $f \in L^p([0,1], \mathcal{B}, \lambda)$*, we define its* L^p*-modulus of continuity* $\omega_p(\cdot, f)$ *by*

$$\omega_p(\delta, f) := \sup_{0\le h\le\delta} \|\tau_h f - f\|_p \qquad \forall \delta \in [0,1]$$

where $\tau_h f(x) := f(x+h)$.

Theorem 17. *Let* $f \in L^p([0,1], \mathcal{B}, \lambda)$*,* $1 < p \le \infty$*, be such that*

$$\sum_{n\in\mathbb{N}} \frac{\omega_p(2^{-n}, f)}{n^{1/p}} < \infty. \tag{41}$$

Let $(n_k)_{k\in\mathbb{N}}$ *be a Hadamard lacunary sequence of positive integers. Then for every* $(a_k)_{k\in\mathbb{N}} \in \ell^{p'}(\mathbb{N})$*, the series* $\sum_k a_k f(n_k \cdot)$ *converges* λ*-a.s. and in* L^p*. Moreover, if* $1 < p < \infty$*, we have*

$$\sup_{N\in\mathbb{N}} |\sum_{k}^{N} a_k f(n_k \cdot)| \in L^p([0,1], \mathcal{B}, \lambda);$$

and if $p = \infty$, there exists $\gamma > 0$ such that

$$\mathbb{E} \exp\Big(\gamma \sup_{N\in\mathbb{N}} |\sum_{k}^{N} a_k f(n_k\cdot)|^2\Big) < \infty. \tag{42}$$

Remarks. If f is a trigonometric polynomial, the condition (41) holds with $p = \infty$. In this case Kuelbs-Woyczyński [17, Corollary 4.1] proved (42) for all $\gamma > 0$.

Let us mention some previous results closely related to Theorem 17 with $p = 2$. Gaposhkin [13] (see also Theorem 2.4.2 in [11] and the footnote there), proved that if $\sum_{n\in\mathbb{N}} \omega_2(2^{-n}, f) < \infty$, then the series $\sum_{k\in\mathbb{N}} a_k f(n_k x)$ converges almost everywhere for every $(a_n)_{n\in\mathbb{N}} \in \ell^2$. Here the lacunarity of (n_k) is not needed. But under the assumption of lacunarity, we gain a factor $\sqrt{n}$ in (41). Moreover, in [12], Gaposhkin proved that if f is defined by a lacunary Fourier series with $\omega_2(f, 2^{-n}) = O(n^{-\gamma})$ for some $\gamma > 1/2$, the same conclusion holds, and the condition $\gamma > \frac{1}{2}$ is sharp by Proposition 18, quoted from Gaposhkin [12]. Proposition 18 also proves the sharpness of (41) in Theorem 17.

To conclude our discussion let us mention that our proof makes use of dyadic martingales while the proof of Gaposhkin is based on Lebesgue's differentiation theorem. It is well-known that these two objects are linked (see for instance [14]). In our context such a link appears in the proof of Lemma 20, see the remark after it.

Proposition 18. (Gaposhkin [12, Theorem 3]) For every $m \in \mathbb{N}$, there exists f which is in $L^p([0,1], \mathcal{B}, \lambda)$ for every $1 \le p < \infty$, such that $\omega_2(f, 2^{-n}) = O(\frac{1}{\sqrt{n} L_m(n)})$ and a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2$ such that the series $\sum_{n\in\mathbb{N}} a_n f(2^n x)$ diverges almost everywhere.

Proof. Fix $m \in \mathbb{N}$. Define

$$f(x) := \sum_{k=1}^{\infty} \frac{\sin(2^k \cdot 2\pi x)}{k \prod_{i=0}^{m} L_i(k)}; \quad a_n := \frac{1}{\sqrt{n \prod_{i=0}^{m-1} L_i(n) L_m(n)}} \ (\forall n \ge 1).$$

Since f is lacunary and in L^2, it is in every L^p by Theorem 8.20, Chap V, vol I (page 215) of [24]. Then, the rest of the proof is exactly as in Gaposhkin [12]. □

Now we show that the condition $(a_n)_{n\in\mathbb{N}} \in \ell^2$ in Theorem 17 may be necessary.

Theorem 19. *Let $f \in L^p([0,1], \mathcal{B}, \lambda)$, $2 < p \le \infty$, satisfying (41). Let $(n_k)_{k\in\mathbb{N}}$ be a Hadamard-lacunary sequence of positive integers. Suppose that $(f(n_k\cdot))_{k\in\mathbb{N}}$ is a Riesz system. Then, for every sequence $(a_n)_{n\in\mathbb{N}}$, with $\sum_{n\in\mathbb{N}} |a_n|^2 = \infty$, the series $\sum_{k\in\mathbb{N}} a_k f(n_k\cdot)$ is not a.e. convergent.*

5.2. **Proofs of Theorem 17 and Theorem 19.** As for several results on the topic (see e.g. Theorem 2.1 of [2]), dyadic martingales were used. We also use the dyadic filtration.

For every $n \in \mathbb{N}$, denote by $\mathcal{F}_n$ the σ-algebra generated the family $\mathfrak{I}_n$ of the intervals $I_{n,k} := [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $0 \leq k \leq 2^n - 1$. We will choose our filtration $(\mathcal{A}_k)$ to be $(\mathcal{F}_{m_k})$ for some suitable increasing sequence of integers (m_k).

For every increasing sequence $(m_k)_{k\in\mathbb{N}}$, we do have $\bigvee_{k\in\mathbb{N}} \mathcal{F}_{m_k} = \mathcal{B}$. For every $f \in L^p([0,1], \mathcal{B}, \lambda)$, $p \geq 1$, $(\mathbb{E}(f|\mathcal{F}_{m_k})_{k\in\mathbb{N}}$ converges to f λ-a.s. and in $L^p([0,1], \mathcal{B}, \lambda)$.

In the next lemma, which would be known to specialists, we control the rate of approximation of $f \in L^p$ by $\mathbb{E}(f|\mathcal{F}_n)$, by using the L^p-modulus of continuity of f.

We convention that our functions are periodically extended to the whole line $\mathbb{R}$.

Lemma 20. *Let $f \in L^p([0,1], \mathcal{B}, \lambda)$, $1 \leq p \leq \infty$, and $n \in \mathbb{N}$. We have*

$$\|f - \mathbb{E}(f|\mathcal{F}_n)\|_p \leq 2\Omega_p(2^{-n}, f)\,. \tag{43}$$

Remark. When, $p = 2$, the lemma improves Lemma 2.1 in [2]. Notice that in the proof we make use of $\sum_{k=0}^{2^n-1} m_{I_{n,k}}(\tau_{x-2^{-n}k} f)\mathbf{1}_{I_{n,k}}(x) = 2^n \int_x^{x+\frac{1}{2^n}} f(u)du$, which is related to Lebesgue's differentiation theorem.

Proof. We give the proof when $p < \infty$, the case $p = \infty$ being obvious. For simplicity, for any interval I we write

$$m_I(f) = \frac{1}{|I|}\int_I f(x)dx$$

where $|I|$ denotes the length of I. Let $n \in \mathbb{N}$. We have, for λ-a.e. $x \in [0,1]$,

$$\mathbb{E}(f|\mathcal{F}_n)(x) = \sum_{k=0}^{2^n-1} m_{I_{n,k}}(f)\mathbf{1}_{I_{n,k}}(x)\,.$$

Since $f(x) = \sum_{k=0}^{2^n-1} f(x)\mathbf{1}_{I_{n,k}}(x)$, we can write

$$f(x) - \mathbb{E}(f|\mathcal{F}_n)(x) = \varphi_n(x) + \psi_n(x)$$

where

$$\varphi_n(x) := \sum_{k=0}^{2^n-1} [f(x) - m_{I_{n,k}}(\tau_{x-2^{-n}k} f)]\mathbf{1}_{I_{n,k}}(x)$$

$$\psi_n(x) := \sum_{k=0}^{2^n-1} [m_{I_{n,k}}(\tau_{x-2^{-n}k} f - m_{I_{n,k}}(f))]\mathbf{1}_{I_{n,k}}(x).$$

Using that $\frac{f(x)}{2^n} = f(x)\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} du$ and using the Hölder inequality in the following integral with respect to du, we get that

$$\begin{aligned}
\|\varphi_n\|_p^p &= \sum_{k=0}^{2^n-1}\int_{I_{n,k}} |f(x)-\tau_{x-2^{-n}k}f(u)|^p du \\
&= \sum_{k=0}^{2^n-1}\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \Big|2^n\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \big(f(x)-f(u+x-k2^{-n})\big)\,du\Big|^p dx \\
&\le 2^n\sum_{k=0}^{2^n-1}\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \Big[\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \big|f(x)-f(u+x-k2^{-n})\big|^p\,du\Big]dx \\
&= 2^{np}\sum_{k=0}^{2^n-1}\int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \Big(2^{n(1-p)}\int_0^{\frac{1}{2^n}} \big|f(x)-f(x+v)\big|^p\,dv\Big)dx \\
&= 2^n\int_0^{2^{-n}} dv\int_0^1 |f(x)-f(x+v)|^p dx \\
&\le \Omega_p^p(2^{-n},f).
\end{aligned}$$

By a similar computation, we infer that

$$\|\psi\|_p^p \le \Omega_p^p(2^{-n},f)\,,$$

and the result follows. □

We will also need the following lemma.

Lemma 21. *Let $f\in L^p([0,1],\mathcal{B},\lambda)$, $1\le p\le\infty$, with $\int_0^1 f(u)du=0$. For any integers $n\in\mathbb{N}$ and $m\ge 1$, we have*

$$\|\mathbb{E}(f(m\cdot)|\mathcal{F}_n)\|_p \le \frac{2^n}{m}\|f\|_p. \tag{44}$$

Proof. Assume that $p<\infty$. Since $\int_0^1 f(u)du=0$, the integral of f over any interval of integral length is equal to zero. So that for any $a<b$, we have

$$\int_a^b f(x)dx = \int_a^{a+[b-a]} f(x)dx + \int_{a+[b-a]}^{a+[b-a]+\{b-a\}} f(x)dx = \int_a^{a+\{b-a\}} f(x)dx$$

where $[x]$ and $\{x\}$ denote respectively the integral part and the fractional part of a real number x. It follows that

$$\Big|\int_{\frac{km}{2^n}}^{\frac{(k+1)m}{2^n}} f(u)\,du\Big|^p \le \Big(\int_0^1 |f(u)|\,du\Big)^p \le \|f\|_p^p.$$

Hence,

$$\|\mathbb{E}(f(m\cdot)|\mathcal{F}_n)\|_p^p = \frac{2^{n(p-1)}}{m^p}\sum_{k=0}^{2^n-1}\Big|\int_{\frac{km}{2^n}}^{\frac{(k+1)m}{2^n}} f(u)\,du\Big|^p \le \frac{2^{np}}{m^p}\|f\|_p^p\,.$$

The proof when $p = \infty$ follows similarly. □

Remark. Notice that if $m \equiv \ell \bmod 2^n$, with $0 \le \ell \le 2^n - 1$, we actually have $\|\mathbb{E}(f(m\cdot)|\mathcal{F}_n)\|_2 \le \frac{\sqrt{\ell 2^n}}{m}\|f\|_2$. We can replace the norm $\|\cdot\|_2$ by the norm $\|\cdot\|_1$.

Proof of Theorem 17. It is easy to see that there exists an integer $r \ge 1$, such that in each interval $[2^\ell, 2^{\ell+1} - 1]$, there are at most r terms from the sequence $(n_k)_{k\in\mathbb{N}}$. Splitting our series into r series we may and do assume that $r = 1$.

For every $k \in \mathbb{N}$, define $m_k := [\log_2 n_k]$, in other words $2^{m_k} \le n_k < 2^{m_k+1}$. The sequence $(m_k)_{k\in\mathbb{N}}$ is strictly increasing and tends to the infinity. We are going to apply Theorem 5 to

$$Z_k = a_k f(n_k\cdot), \quad \mathcal{A}_k = \mathcal{F}_{m_k}.$$

By Lemma 6, it suffices to check that

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=0}^{\infty} a_k^{p'}\|f(n_k\cdot) - \mathbb{E}(f(n_k\cdot|\mathcal{F}_{m_{k+2^l}})\|_p^{p'}\Big)^{1/p'} < \infty\,, \tag{45}$$

and that

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=0}^{\infty} a_k^{p'}\|\mathbb{E}(f_{n_k}\cdot)|\mathcal{F}_{m_{k-2^\ell}})\|_p^{p'}\Big)^{1/p'} < \infty\,. \tag{46}$$

Using (43), we see that (45) holds as soon as

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=0}^{\infty} a_k^{p'}\,\omega_p^{p'}(\frac{n_k}{2^{m_{k+2^\ell}}}, f)\Big)^{1/p'} < \infty\,.$$

Notice that as function of δ, $\omega_2(\delta, f)$ is non-decreasing. Then (45) holds as soon as

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sup_{k\in\mathbb{N}}\omega_p^{p'}(\frac{2n_k}{n_{k+2^\ell}}, f)\Big)^{1/p'} \le \sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\omega_p^{p'}(\frac{2}{q^{2^\ell}}, f)\Big)^{1/p'} < \infty\,,$$

which is equivalent to (41), using once more the monotony of $\omega_p(\cdot, f)$.

It remains to prove (46). Using (44), we see that (46) holds as soon as

$$\sum_{\ell=0}^{\infty} 2^{\ell(1-1/p)}\Big(\sum_{k=2^\ell}^{\infty} a_k^{p'}\,\frac{2^{p'm_{k+1-2^\ell}}}{n_k^{p'}}\Big)^{1/p'} < \infty\,.$$

To conclude, just notice that

$$\frac{2^{p'm_{k+1-2^\ell}}}{n_k^{p'}} \le 2^{p'}\frac{n_{k+1-2^\ell}^{p'}}{n_k^{p'}} \le \frac{2^{p'}}{q^{p'2^\ell}}\,.$$

□

5.3. **Lacunary Davenport series.** As an example we apply Theorem 17 and Theorem 19 to Davenport series. For $\lambda > 0$?we consider the function

$$f_\lambda(x) = \sum_{m\geq 1} \frac{\sin(2\pi m x)}{m^\lambda} \qquad \forall x \in [0,1]\,.$$

It is everywhere defined and continuous except at $x = 0$ and belongs to $L^2([0,1])$. It is Hölder continuous when $\lambda > 1$. When $\lambda = 1$, we have $f_1(x) = -\{x\}$ (where $\{x\} = x - [x] - \frac{1}{2}$), hence $\omega(\delta, f_1) = O(\delta^{1/p})$, pour tout $p > 1$.

The study of the regularity of f_λ for $0 < \lambda < 1$ needs more care (notice that when $1/2 \leq \lambda < 1$ one may use the Hausdorf-Young theorem).

Let $0 < \lambda < 1$. By formula (2.3) p. 186 and formula (1.18) p. 77 of Zygmund [24, Vol. I], there exists $(c_n)_{n\geq 1}$ with $|c_n| \leq C_\lambda n^{-1-\lambda}$ for some $C_\lambda > 0$ and $\kappa_\lambda \in \mathbb{R}$, such that for every $x \in [0,1)$,

$$f_\lambda(x) = \kappa_\lambda \mathrm{Im}\big((1 - \mathrm{e}^{2i\pi x})^{\lambda-1}\big) + \sum_{n\geq 1} c_n \sin(2\pi n x) := g_\lambda(x) + h_\lambda(x)\,.$$

It is not difficult to see that h_λ is Hölder continuous and that

$$\omega_p(\delta, g_\lambda) = O(\delta^{\lambda - \frac{p-1}{p}})\,, \tag{47}$$

for every $p < (1-\lambda)^{-1}$.

These regularity properties allow us to apply Theorem 17. We shall see right now that when $\lambda > \frac{1}{2}$, it is also possible to apply Theorem 19. In fact, for any $\lambda > 1/2$, Wintner proved that $\{f_\lambda(nx)\}$ is complete in $L^2([0,1])$. When $\lambda > 1$, Hedenmalm, Lindqvist and Seip proved that $\{f_\lambda(nx)\}$ is a Riesz basis, a fortiori $\{f_\lambda(n_k x)\}_{n_k\geq 1}$ is a Riesz sequence for any sequence $\{n_k\}$. When $1/2 < \lambda \leq 1$, Brémont proved that $\{f_\lambda(n_k x)\}_{n_k\geq 1}$ is a Riesz sequence for any Hadamard lacunary sequence $\{n_k\}$.

Theorem 22. *Let $\lambda > 1/2$. Suppose that $\{n_k\} \subset \mathbb{N}$ is lacunary in the sense of Hadamard. Then for every sequence $(a_k)_{k\in\mathbb{N}}$ the following are equivalent*

(i) The series $\sum_{k=1}^\infty a_k f_\lambda(n_k x)$ converges almost everywhere;
(ii) $\sum_{k=1}^\infty |a_k|^2 < \infty$.

Moreover, if any of the above holds then, when $1/2 < \lambda \leq 1$, for every $p \leq \frac{1}{1-\lambda}$, $\sup_{n\in\mathbb{N}} |\sum_{k=1}^N a_k f_\lambda(n_k x)| \in L^p([0,1))$; and; when $\lambda > 1$, there exists $\beta > 0$ such that $\int_0^1 \mathrm{e}^{\beta \sup_{n\in\mathbb{N}} |\sum_{k=1}^N a_k f_\lambda(n_k x)|^2}\, dx < \infty$.

Remark. The fact that $(ii) \Rightarrow (i)$ follows from Gaposhkin [12], but we provide here integrability properties of the maximal functions. The Theorem says that when $\sum_{k=1}^\infty |a_k|^2 = \infty$ then the series $\sum_{k=1}^\infty a_k f_\lambda(n_k x)$ does not converge almost everywhere. Hence, one may wonder whether we have almost everywhere divergence. For a large class of sequences $(a_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ a positive answer follows from Theorem 2.4.14 of [13]. Finally, notice that when $1/2 < \lambda \leq 1$, $\sum |a_n|^2 < \infty$

is not sufficient for $\sum a_n f_\lambda(nx)$ to converge almost everywhere. But when $\lambda > 1$, $\sum |a_n|^2 < \infty$ is sufficient, as a consequence of Carleson theorem.

Proposition 23. *Let $0 < \lambda \leq 1/2$ and $p < (1-\lambda)^{-1}$. For every Hadamard lacunary sequence $(n_k)_{k\in\mathbb{N}}$ and every $(a_k)_{k\in\mathbb{N}} \in \ell^p$ the series $\sum_{k\in\mathbb{N}} a_k f_\lambda(n_k x)$ converges almost everywhere. Moreover, $\sup_{n\in\mathbb{N}} |\sum_{k=1}^N a_k f_\lambda(n_k x)| \in L^p([0,1))$.*

6. Convergence of lacunary series with respect to Riesz products

The classical Riesz products in harmonic analysis are defined as follows [24]. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $\lambda_{n+1} \geq 3\lambda_n$ for all $n \geq 0$ and $(c_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $|c_n| \leq 1$ for all $n \geq 0$. Then we can define a Borel probability measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, denoted by

$$\mu_c = \prod_{n=0}^{\infty} (1 + \mathrm{Re}\ c_n e^{2\pi i \lambda_n t}). \tag{48}$$

Actually the partial products of the above infinite product are positive trigonometric polynomials which converge to a measure μ_c in the weak-* topology of $C(\mathbb{T})^*$. Suppose that we are given a sequence of Borel functions f_n on $\mathbb{T}$, say bounded. A problem associated to Riesz products is the μ_c-a.e. convergence of the following lacunary series

$$\sum_{n=0}^{\infty} a_n \big(f_n(\lambda_n x) - \mathbb{E}_{\mu_c} f_n(\lambda_n \cdot)\big). \tag{49}$$

It was independently proved in [7] and [20] that when $f_n(x) = e^{2\pi i x}$, the series (49) converges μ_c-a.e. if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. For more general functions f_n, such results are not known. But under the conditions that λ_n divises λ_{n+1} for all n and that f_n are analytic or more precisely

$$\exists \rho \in (0,1), \quad \sup_{j\in\mathbb{Z}} \rho^{-|j|} \sup_{n\geq 0} |\widehat{f_n}(j)| < \infty,$$

Peyriere [19] proved that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ implies the μ_c-a.e. convergence of the series (49). By using Theorem 13, we can improve Peyrière's result as follows

Theorem 24. *Assume that $\sup_{n\in\mathbb{N}} |c_n| < 1$. Let $(f_n)_{n\in\mathbb{N}}$ be functions on $[0,1]$, such that there exists $C > 0$ and $\epsilon > 0$ such that*

$$\sup_{n\geq 0} \omega(t, f_n) \leq \frac{C}{|\log t|^{1/2+\epsilon}},$$

Then, for every $(a_n)_{n\in\mathbb{N}} \subset \ell^2$, the series $\sum_{n=0}^{\infty} a_n \big(f_n(\lambda_n x) - \mathbb{E}_{\mu_c} f_n(\lambda_n \cdot)$ converges μ_c-a.e.

We emphasize that we keep the divisibility condition on $\{\lambda_n\}$. Otherwise, more efforts are needed and for the moment we don't succeed.

We can do little more than Riesz products. Actually the above result holds not only for Riesz products but also for non-homogeneous equilibrium states studied in [10]. We shall now consider this situation. The proof of the Theorem will be given at the end of the paper.

Let us recall the definition taken from [10]. Let $\{S_n\}_{n\geq 1}$ be a sequence of finite sets of discrete topology. Assume that $\ell_n := \text{card } S_n \geq 2$ for all $n \geq 1$. Consider the infinite product space $X := \prod_{n=1}^{\infty} S_n$ equipped with the product topology. A compatible metric on X may be defined as

$$d(x;y) = \frac{1}{\ell_1 \ell_2 \cdots \ell_n}$$

where $n = n(x,y) = \sup\{j \geq 1 : x_i = y_i, \forall i = 1, 2 \cdots, j\}$ (with convention $\sup \emptyset = 0$). Let $A = \{A_n\}_{n\geq 1}$ be a sequence of matrices such that $A_n \in M_{S_N, S_{n+1}}$, meaning that the rows of A are indexed by S_n and the columns by S_{n+1}. Suppose the entries of A_n are 0 or 1. Such a matrix is called an incidence matrix. We define a subspace X_A of X by

$$X_A = \{x = (x_n) \in X : \forall n \geq 1,\ A_n(x_n, x_{n+1}) = 1\}.$$

We call X_A a non-homogeneous symbolic space. We always suppose that there exists an integer $M \geq 0$ such that

$$\forall n \geq 1, \quad \prod_{j=n}^{n+M} A_j > 0$$

($A > 0$ means that the entries of A are all strictly positive). In this case, X_A is said to be transitive. If all entries of every A_n are equal to 1, we have $X_A = X$. We call X the full symbolic space.

A sequence $G = \{g_n\}_{n\geq 1}$ of non-negative functions defined on X_A is called a sequence of potentials if for any ≥ 1, $g_n(x)$ does not depend on the first $n-1$ coordinates of x (so, we sometimes write $g_n(x) = g_n(x_n, x_{n+1}, \cdots)$). It is said to be normalized if for any $n \geq 1$,

$$\sum_{y_n : A_n(y_n, x_{n+1}) = 1} g_n(y_n, x_{n+1}, \cdots) = 1 \quad (\forall x = (x_n) \in X_A).$$

For $n \geq 1$, let

$$G_n(x) = \prod_{j=1}^{n} g_n(x).$$

Then define a sequence of averaging operators $P_n : C(X_A) \to C(X_A)$, where $C(X_A)$ is the space of all continuous functions on X_A, by

$$P_n f(x) = \sum_{y_1, \cdots, y_n} G_n(y_1, \cdots, y_n, x_{n+1}, \cdots) f(y_1, \cdots, y_n, x_{n+1}, \cdots)$$

where the sum is taken over all sequences $(y_1, \cdots, y_n)$ such that

$$A_1(y_1, y_2) = \cdots = A_{n-1}(y_{n-1}, y_n) = A_n(y_n, x_{n+1}) = 1.$$

It is easy to check that P_n is positive and $P_n 1 = 1$. Therefore, the adjoint operator $P_n^* : M^+(X_A) \to M^+(X_A)$ admits fixed points, where $M^+(X_A)$ is the space of all Borel probability measures on X_A. A measure $\mu \in M^+(X_A)$ is called a *(non-homogeneous) equilibrium state* associated to $G = \{g_n\}$ if $P_n^* \mu = \mu$ for all $n \geq 1$.

Theorem 25. *[[10]] Let $G = \{g_n\}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A.*

(a) *The set of all equilibrium states associated to G is a non-empty convex compact set.*

(b) *There is a unique equilibrium state if and only if for any $f \in C(X_A)$, $P_n f(x)$ converges uniformly (in x) to a constant as $n \to \infty$.*

(c) *There is a unique equilibrium state if*

$$\inf_{n \geq 1} \inf_{x \in X_A} g_n(x) > 0; \quad \sup \left\{ \frac{G_n(x)}{G_n(y)} : x_1 = y_1, \cdots, x_n = y_n \right\} < \infty. \tag{50}$$

(c) *Under the condition in (c), there exist constants D_1 and D_2 such that*

$$D_1 G_n(x) \leq \mu(I_n(x)) \leq D_2 G_n(x)$$

for all $x \in X_A$ and all $n \geq 1$, where $I_n(x) = \{y \in X_A : y_j = x_j \forall 1 \leq j \leq n\}$.

For a function f defined on X_A and for $n \geq 1$, we define the n-th variation of f by

$$\mathrm{var}_n(f) = \sup\{|f(x) - f(y)| : x_1 = y_1, \cdots, x_n = y_n\}.$$

A careful inspection of the proof of Theorem 4 of [10] gives the next theorem, which corresponds essentially to the case where $\alpha = 1 + \epsilon$ for $\epsilon > 0$.

Theorem 26. *Let $\{g_n\}$ be a normalized sequence of potentials defined on a transitive symbolic space X_A. Suppose there are constants $A > 0$ and $\alpha > 0$ such that for every $m > n > 1$,*

$$\mathrm{var}_m(\log g_n) \leq \frac{A}{(m-n)^\alpha}. \tag{51}$$

Then there is a unique equilibrium state μ.

Let $\{f_n\}$ be a sequence of functions such that f_n depend only upon $x_{n+1}, x_{n+2}, \cdots$. Assume moreover that there exists $B > 0$ such that for every $m > n > 1$,

$$\|f_n\|_\infty \leq B; \quad \mathrm{var}_m(f_n) \leq \frac{B}{(m-n)^\alpha}.$$

Then there exists $C > 0$ such that for every $m > n > 1$,

$$\|P_m f_n\|_\infty \leq C \frac{(\log(1 + m - n))^{1+\alpha}}{(m-n)^\alpha}. \tag{52}$$

In particular, if $\alpha > 1/2$, the series

$$\sum_{n=1}^{\infty} a_n \left(f_n(x) - \int f_n d\mu \right) \tag{53}$$

converges μ-a.e. if $\sum_{n=1}^{\infty} |a_n|^2 < \infty$.

Proof. The condition (51) $\{g_n\}$ ensures that condition (50) holds, hence the uniqueness of the equilibrium state, by the Theorem 25.

The fact that (52) holds may be proved as in the proof of Theorem 4 of [10], see pages 111-112 there.

Assume that $\alpha > 1/2$. Let $\mathcal{B}_n$ be the σ-field generated by the coordinate functions $x \mapsto x_j$ $(j = n+1, n+2, \cdots)$. Then our series (53) is adapted to the decreasing filtration $\{\mathcal{B}_n\}$. Then we will apply Theorem 13. By Lemma 14, what we have to check is just the condition (37), because the condition (38) is trivially satisfied by adapted series.

Notice that $\mathbb{E}(f|\mathcal{B}_n) = P_n f$. So, taking $Z_n = a_n f_n$, we infer that for every $n, \ell \geq 0$,

$$\|E_{n+2^\ell-1} Z_n\|_2 \leq |a_n| (\|P_{n+2^\ell-1} f_n\|_2 \leq C \frac{|a_n| \ell^{1+\alpha}}{2^{\alpha \ell}}$$

for some $C > 0$. Then

$$\sum_{\ell=0}^{\infty} 2^{\ell/2} \Big(\sum_{n=0}^{\infty} \|\mathbb{E}_{n+2^\ell-1} Z_n\|_2^2 \Big)^{1/2} \leq C \sum_{\ell=0}^{\infty} \frac{\ell^{1+\alpha}}{2^{\ell(\alpha-1/2)}} \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{1/2} < \infty.$$

Thus the condition (35) is verified. □

Proof of Theorem 24. We can identify the full symbolic space X with the circle $\mathbb{T}$ by the map from X to $\mathbb{T}$:

$$(x_n) \mapsto \sum_{n=1}^{\infty} \frac{x_n}{\ell_1 \cdots \ell_n}.$$

Then the Riesz product is nothing but the equilibrium state associated to

$$\forall n \geq 0, \quad g_{n+1}(x) = \ell_{n+1}^{-1} (1 + \mathrm{Re} c_n e^{2\pi i \lambda_n x})$$

where $\ell_{n+1} = \lambda_{n+1}/\lambda_n$. The assumption (51) is easily verified, using that $\sup_{n \in \mathbb{N}} |c_n| < 1$. □

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