



**HAL**  
open science

## A Dialectica-Like Approach to Tree Automata

Colin Riba

► **To cite this version:**

| Colin Riba. A Dialectica-Like Approach to Tree Automata. 2016. hal-01261183v1

**HAL Id: hal-01261183**

**<https://hal.science/hal-01261183v1>**

Preprint submitted on 24 Jan 2016 (v1), last revised 15 Oct 2019 (v10)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# A Dialectica-Like Approach to Tree Automata

Colin Riba

ENS de Lyon, Université de Lyon, LIP\*

colin.riba@ens-lyon.fr <http://perso.ens-lyon.fr/colin.riba/>

## Abstract

We propose a fibred monoidal closed category of alternating tree automata. Our notion is based on Dialectica-like categories, suggested by the specific logical form of the transitions of alternating automata. The basic monoidal closed structure gives a realizability interpretation of proofs of a first-order multiplicative linear logic as winning strategies in corresponding acceptance games.

Moreover, we show that the usual powerset operation translating an alternating automaton to an equivalent non-deterministic one satisfies the deduction rules of the '!' modality of linear logic. We thus get a deduction system for intuitionistic linear logic, which in particular gives deduction for minimal intuitionistic predicate logic via the Girard translation. Using a suitable negative translation based on the '?' modality, we can interpret proofs of minimal classical logic, and also get a weak form of completeness of our realizers wrt language inclusion.

## 1. Introduction

We propose a fibred monoidal closed category of alternating tree automata. Alternating tree automata (see e.g. [6, 23, 29]) are equivalent in expressive power to the Monadic Second-Order Logic on infinite trees (MSO). They are easily closed under complement, and together with the translation of alternating automata to non-deterministic ones (the *Simulation Theorem* [23]) this provides a convenient decomposition of the translation MSO formulas to automata (see e.g. [6, 29]), implying the decidability of MSO [25].

This papers shows that this decomposition corresponds to some extent to the decomposition of intuitionistic logic in linear logic [5]. The fibred symmetric monoidal closed structure allows to organize automata in a deduction system that we present below. Our model, building from [26], is based on games semantics, which provides a realizability interpretation of this deduction system.

We use Gödel's *Dialectica* interpretation (see e.g. [2, 18]) in two related ways. First, Dialectica can be seen as a constructive notion of prenex  $\exists\forall$ -formulas, on which we base the transition function of the internal implication of tree automata. This leads to our notion of tree automata presented in §2. Second, our notion of morphism (issued from [26]) is based on *zig-zag* strategies, which can be

represented using Dialectica-like categories (see e.g. [4, 8, 10]). This allows to conveniently describe the dependencies on inputs and tree directions.

**Monoidal Fibrations of Tree Automata.** Given an alternating tree automaton  $\mathcal{A}$  on alphabet  $\Sigma$  and a  $\Sigma$ -labeled tree  $t$ , one can ask whether  $t$  is accepted by  $\mathcal{A}$  or by  $\sim\mathcal{A}$ , the complement of  $\mathcal{A}$ . We write these two possibilities respectively as

$$\mathbf{1}; \mathbf{I} \vdash \mathcal{A}(t) \quad \text{and} \quad \mathbf{1}; \mathbf{I} \vdash \sim\mathcal{A}(t)$$

Acceptance of a tree by an automata can be defined *via* a two-player *acceptance game*, where the *Proponent*  $\mathbf{P}$  (also called *Automaton* or  $\exists$ loise) tries to force the execution of the automaton on a successful path, while its *Opponent*  $\mathbf{O}$  ( $\forall$ belard) tries to find a failing path (see e.g. [6, 24, 28]). Then  $\mathcal{A}$  accepts  $t$  iff  $\mathbf{P}$  has a winning strategy in this game. As shown in [26], using tools coming from game semantics and categorical logic, such strategies can be organized as *categories*. This means that strategies can be used as realizers for implicative statements of the form

$$\mathbf{1}; \mathcal{A}(t) \vdash \mathcal{B}(u)$$

satisfying the two following usual *Axiom* and *Cut* deduction rules

$$\frac{}{\mathbf{1}; \mathcal{A}(t) \vdash \mathcal{A}(t)} \quad \frac{\mathbf{1}; \mathcal{A}(t) \vdash \mathcal{B}(u) \quad \mathbf{1}; \mathcal{B}(u) \vdash \mathcal{C}(v)}{\mathbf{1}; \mathcal{A}(t) \vdash \mathcal{C}(v)}$$

They correspond to the fact that there are *identity* strategies, and that strategies can be composed.

Moreover, these categories are *fibred* in the sense that if  $\mathcal{A}$  and  $\mathcal{B}$  have the same input alphabet  $\Sigma$ , then we can form an implication with input alphabet  $\Sigma$ :

$$\Sigma; \mathcal{A} \vdash \mathcal{B}$$

On the other hand, there is a category of trees whose objects are alphabet and whose morphisms from  $\Sigma$  to  $\Gamma$  are  $(\Sigma \rightarrow \Gamma)$ -labeled trees, noted  $\Sigma \vdash t : \Gamma$ . (So that a tree  $\mathbf{1} \vdash t : \Sigma$  is essentially the same thing as a usual  $\Sigma$ -labeled tree.) These trees are incorporated in the deduction system *via* *Substitution* rules

$$\frac{\Gamma; \mathcal{A} \vdash \mathcal{B} \quad \Sigma \vdash t : \Gamma}{\Sigma; \mathcal{A}(t) \vdash \mathcal{B}(t)}$$

Substitution can of course be iterated, e.g. as in

$$\frac{\Sigma; \mathcal{A}(u) \vdash \mathcal{B}(v) \quad \Delta \vdash t : \Sigma}{\Delta; \mathcal{A}(u \circ t) \vdash \mathcal{B}(v \circ t)}$$

More precisely, for each alphabet  $\Sigma$  there is a category of games over  $\Sigma$ , on which tree morphisms act as functors. In particular, the general *Axiom* and *Cut* rules have the form

$$\frac{}{\Sigma; \mathcal{A}(t) \vdash \mathcal{A}(t)} \quad \frac{\Sigma; \mathcal{A}(t) \vdash \mathcal{B}(u) \quad \Sigma; \mathcal{B}(u) \vdash \mathcal{C}(v)}{\Sigma; \mathcal{A}(t) \vdash \mathcal{C}(v)}$$

The categories of [26] have a symmetric monoidal product  $\_ \otimes \_$ , allowing to form an automaton  $\mathcal{A} \otimes \mathcal{B}$  on  $\Sigma$  from automata  $\mathcal{A}$  and  $\mathcal{B}$  on  $\Sigma$ . This product has *unit automata*  $\mathbf{I}$  and satisfies the following

\* UMR 5668 CNRS ENS Lyon UCBL INRIA

rules (here and in the following, we leave implicit input trees when they are not relevant):

$$\frac{}{\Sigma; \mathcal{A} \otimes \mathbf{I} \dashv\vdash \mathcal{A}} \quad \frac{}{\Sigma; \mathcal{A} \otimes \mathcal{B} \dashv\vdash \mathcal{B} \otimes \mathcal{A}}$$

$$\frac{}{\Sigma; (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \dashv\vdash \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})}$$

$$\frac{\Sigma; \mathcal{A}_1 \vdash \mathcal{B}_1 \quad \Sigma; \mathcal{A}_2 \vdash \mathcal{B}_2}{\Sigma; \mathcal{A}_1 \otimes \mathcal{A}_2 \vdash \mathcal{B}_1 \otimes \mathcal{B}_2}$$

where  $\Sigma; \mathcal{A} \dashv\vdash \mathcal{B}$  stands for  $(\Sigma; \mathcal{A} \vdash \mathcal{B} \text{ and } \Sigma; \mathcal{B} \vdash \mathcal{A})$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are *complete* (in the sense that a move is always possible in acceptance games), then  $\mathcal{A} \otimes \mathcal{B}$  satisfies the expected relation

$$\mathbf{1}; \mathbf{I} \vdash (\mathcal{A} \otimes \mathcal{B})(t) \quad \text{iff} \quad (\mathbf{1}; \mathbf{I} \vdash \mathcal{A}(t) \text{ and } \mathbf{1}; \mathbf{I} \vdash \mathcal{B}(t))$$

Moreover, there are *falsity automata*  $\Sigma \vdash \perp$ , which accept no tree and satisfy the rules

$$\frac{\Sigma; \mathcal{A} \otimes \mathcal{B} \vdash \perp}{\Sigma; \mathcal{A} \vdash \sim \mathcal{B}} \quad \text{and} \quad \frac{\Sigma; \mathcal{A} \vdash \sim \mathcal{B}}{\Sigma; \mathcal{A} \otimes \mathcal{B} \vdash \perp}$$

Finally, the fibred categories of [26] are equipped with a categorical notation of existential quantification. This means that from an automaton  $\mathcal{A}$  on  $\Sigma \times \Gamma$ , we can form an automaton  $\exists_{\Gamma} \mathcal{A}$  on  $\Sigma$  satisfying the rules

$$\frac{\Sigma; \exists_{\Gamma} \mathcal{A} \vdash \mathcal{B}}{\Sigma \times \Gamma; \mathcal{A} \vdash \mathcal{B}(\pi)} \quad \text{and} \quad \frac{\Sigma \times \Gamma; \mathcal{A} \vdash \mathcal{B}(\pi)}{\Sigma; \exists_{\Gamma} \mathcal{A} \vdash \mathcal{B}}$$

where  $\Sigma \times \Gamma \vdash \pi : \Sigma$  is a projection. If  $\Sigma = \mathbf{1}$ , so that  $\Sigma \times \Gamma \simeq \Gamma$ , we can derive

$$\frac{}{\mathbf{1}; \mathcal{A}(t) \vdash \exists_{\Gamma} \mathcal{A}}$$

In the case of a *non-deterministic* automaton  $\mathcal{N}$ , we have the expected converse:

$$\mathbf{1}; \mathbf{I} \vdash \exists_{\Gamma} \mathcal{N} \quad \Longrightarrow \quad \mathbf{1}; \mathbf{I} \vdash \mathcal{N}(t) \quad \text{for some } t$$

Strategies provide a *realizability semantics* to this deduction system, in the sense that from a formal derivation of  $\Sigma; \mathcal{A} \vdash \mathcal{B}$  we can compute a strategy  $\sigma$  such that for every input tree  $t$  and every winning strategy  $\tau$  in the acceptance game for  $\mathbf{1}; \mathbf{I} \vdash \mathcal{A}(t)$ , the strategy  $\sigma \circ \tau$  is winning in the acceptance game for  $\mathbf{1}; \mathbf{I} \vdash \mathcal{B}(t)$ .

**Monoidal Closed Fibrations of Tree Automata.** In this paper, we extend [26] to monoidal *closed* fibrations.

A first consequence is that, instead of the Hilbert-like system above, we can now make deductions using sequents of the form

$$\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n \vdash \mathcal{B}$$

and interpreted in the old system as

$$\Sigma; \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n \vdash \mathcal{B}$$

The main consequence is the introduction of a *linear implication* connective on automata, satisfying

$$\frac{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{A} \vdash \mathcal{B}}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n \vdash \mathcal{A} \multimap \mathcal{B}}$$

and which is compatible with cut-elimination, in the sense that the following two derivations are interpreted by the same strategy

$$\frac{\frac{\Delta_1}{\Sigma; \mathcal{A} \vdash \mathcal{B}}}{\Sigma; \mathbf{I} \vdash \mathcal{A} \multimap \mathcal{B}} \quad \frac{\frac{\Delta_2}{\Sigma; \mathbf{I} \vdash \mathcal{A}} \quad \frac{}{\Sigma; \mathcal{B} \vdash \mathcal{B}}}{\Sigma; \mathcal{A} \multimap \mathcal{B} \vdash \mathcal{B}} \quad \vdots \quad \frac{\Delta_1[\Delta_2/\mathcal{A}]}{\Sigma; \mathbf{I} \vdash \mathcal{B}}$$

We thus obtain a system whose propositional connectives are  $\otimes, \multimap, \dashv\vdash, \sim, \perp$  and  $\perp$ . The rules for these connectives are direct adaptations of the usual rules of the multiplicative fragment of

intuitionistic linear logic (which can be found e.g. in [22]) obtained by adding alphabets to sequents with the following proviso: the premises and the conclusion of a propositional rule must have the same alphabet. The rules for existential quantification are now

$$\frac{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, \exists_{\Gamma} \mathcal{A} \vdash \mathcal{B}}{\Sigma \times \Gamma; \mathcal{A}_1(\pi), \dots, \mathcal{A}_n(\pi), \mathcal{A} \vdash \mathcal{B}(\pi)}$$

$$\frac{\Sigma \times \Gamma; \mathcal{A}_1(\pi), \dots, \mathcal{A}_n(\pi), \mathcal{A} \vdash \mathcal{B}(\pi)}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, \exists_{\Gamma} \mathcal{A} \vdash \mathcal{B}}$$

**Intuitionistic Linear Logic and Non-Determinization.** Finally, we show that when restricting to positional strategies and parity automata, the non-determinization construction  $!\mathcal{A}$  satisfies the deduction rules of the ‘!’ modality of linear logic:

$$\frac{\Sigma; !\mathcal{A}_1, \dots, !\mathcal{A}_n \vdash \mathcal{A}}{\Sigma; !\mathcal{A}_1, \dots, !\mathcal{A}_n \vdash !\mathcal{A}} \quad \frac{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, \mathcal{A} \vdash \mathcal{B}}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, !\mathcal{A} \vdash \mathcal{B}} \quad (1)$$

$$\frac{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n \vdash \mathcal{B}}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, !\mathcal{A} \vdash \mathcal{B}} \quad \frac{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, !\mathcal{A}, !\mathcal{A} \vdash \mathcal{B}}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n, !\mathcal{A} \vdash \mathcal{B}}$$

Unfortunately, positional strategies do not compose, so the above rules are not compatible with cut-elimination. But we can still interpret proofs of minimal intuitionistic logic (*i.e.* without the Ex-Falso rule) using Girard’s decomposition  $\mathcal{A} \Rightarrow \mathcal{B} := !\mathcal{A} \multimap \mathcal{B}$ .

**Weak Completeness and Minimal Classical Logic.** Our model is weakly complete in the sense that if  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ , then from the strategy witnessing  $\mathcal{L}(!\mathcal{A} \otimes !(\mathcal{B}^{\perp})) = \emptyset$ , we can effectively build a strategy realizing  $\mathcal{A} \Rightarrow ?\mathcal{B}$ , where  $?\mathcal{B} := (!(\mathcal{B}^{\perp}))^{\perp}$  and  $(-)^{\perp} := (-) \multimap \perp$ .

Automata of the form  $?\mathcal{A} = \mathcal{A}^{\perp} \Rightarrow \perp$  live in the target of a negative translation of minimal classical logic to minimal intuitionistic logic [27]. We can therefore interpret proofs from minimal classical logic *via* this translation, for instance Peirce’s law, whose translation is derivable:

$$\frac{}{\Sigma; \mathcal{A}_1, \dots, \mathcal{A}_n \vdash ((?\mathcal{A} \Rightarrow ?\mathcal{B}) \Rightarrow ?\mathcal{A}) \Rightarrow ?\mathcal{A}}$$

**Outline of the Paper.** We present our notion of tree automata in §2, and §3 gives a basic setting of games. In §4 we give a Dialectica-like presentation of zig-zag strategies, which lead in §5 to our Dialectica-like fibrations. We then discuss non-deterministic automata in §6.

A full version of this paper is available at <https://perso.ens-lyon.fr/colin.riba/papers/dialaut.pdf>.

We fix a finite non-empty set  $D$  of *tree directions*, and let  $p$  range over  $D^*$ . A  $\Sigma$ -labeled tree is a map  $D^* \rightarrow \Sigma$ .

## 2. Tree Automata

We present here our notion of tree automata. It allows to build a linear implication automaton  $\mathcal{A} \multimap \mathcal{B}$  from automata  $\mathcal{A}$  and  $\mathcal{B}$ .

Usually, an alternating tree automaton  $\mathcal{A}$  with state set  $Q_{\mathcal{A}}$  has transitions given by a function  $\delta_{\mathcal{A}}$  mapping a state  $q$  and an input letter  $a$  to an irredundant disjunctive normal form<sup>1</sup> over  $Q_{\mathcal{A}} \times D$  ([23], see also [29]). In particular, for some finite sets  $U, X$ :

$$\delta_{\mathcal{A}}(q, a) = \bigvee_{u \in U} \bigwedge_{x \in X} (q_{u,x}, d_{u,x}) \quad (2)$$

We therefore can see  $\delta_{\mathcal{A}}$  as being of the form

$$\delta_{\mathcal{A}}(q, a) : U \times X \longrightarrow Q_{\mathcal{A}} \times D$$

<sup>1</sup>That is, an element of the free distributive lattice over  $Q_{\mathcal{A}} \times D$  [14, §4.8].

This leads to acceptance games where the *Proponent* P (Eloise) plays from the  $\vee$ 's by choosing some  $u \in U$  and the *Opponent* O (Abelard) plays from the  $\wedge$ 's by choosing some  $x \in X$ .

Consider now another automaton  $\mathcal{B}$  with transitions given by  $\delta_{\mathcal{B}}(q_{\mathcal{B}}, b) : V \times Y \rightarrow Q_{\mathcal{B}} \times D$ . We discuss how to build a linear implication  $\mathcal{A} \multimap \mathcal{B}$  which runs  $\mathcal{A}$  and  $\mathcal{B}$  in parallel. Assume for now that both  $\mathcal{A}$  and  $\mathcal{B}$  have input alphabet  $\mathbf{1} = \{\bullet\}$  (the general case is presented in §5). We first stipulate that  $Q_{\mathcal{A} \multimap \mathcal{B}} := Q_{\mathcal{A}} \times Q_{\mathcal{B}}$ . We then want to see the transition function of  $\mathcal{A} \multimap \mathcal{B}$  as a form of implication:

$$\delta_{\mathcal{A} \multimap \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), \bullet) \equiv \delta_{\mathcal{A}}(q_{\mathcal{A}}, \bullet) \multimap \delta_{\mathcal{B}}(q_{\mathcal{B}}, \bullet)$$

This leads to a form of implication between  $\vee/\wedge$ -forms:

$$\bigvee_{u \in U} \bigwedge_{x \in X} (q_{u,x}, d_{u,x}) \multimap \bigvee_{v \in V} \bigwedge_{y \in Y} (q'_{v,y}, d'_{v,y})$$

We now follow the pattern of Gödel's Dialectica interpretation (see e.g. [2, 18]). It consists in Skolemization for a suitable (constructive) prenex form:

$$\begin{aligned} \bigvee_{u \in U} \bigwedge_{x \in X} (q_{u,x}, d_{u,x}) &\multimap \bigvee_{v \in V} \bigwedge_{y \in Y} (q'_{v,y}, d'_{v,y}) \\ &\equiv \\ \bigwedge_{u \in U} \bigvee_{v \in V} \bigwedge_{y \in Y} \bigvee_{x \in X} [(q_{u,x}, d_{u,x}) \multimap (q'_{v,y}, d'_{v,y})] \\ &\equiv \\ \bigvee_{f \in V^U} \bigwedge_{u \in U} \bigwedge_{y \in Y} \bigvee_{x \in X} [(q_{u,x}, d_{u,x}) \multimap (q'_{f(u),y}, d'_{f(u),y})] \\ &\equiv \\ \bigvee_f \bigvee_F \bigwedge_u \bigwedge_y [(q_{u,F(u),y}, d_{u,F(u),y}) \multimap (q'_{f(u),y}, d'_{f(u),y})] \end{aligned}$$

This would suggest  $\delta_{\mathcal{A} \multimap \mathcal{B}}$  (denoted  $\delta$  below) to be of the form:

$$\delta(q, \bullet) : (V^U \times X^{U \times Y}) \times (U \times Y) \rightarrow Q_{\mathcal{A} \multimap \mathcal{B}} \times D$$

In order for  $\mathcal{A} \multimap \mathcal{B}$  to run  $\mathcal{A}$  and  $\mathcal{B}$  in parallel along the input tree, we impose *D-synchronicity*, that is  $d_{u,F(u),y} = d'_{f(u),y}$ . In particular, the choice of the tree direction should be made explicit in the acceptance games induced by (2). Moreover, with *non-deterministic automata*, it is O (called then *Pathfinder*) who chooses the tree direction. This leads to transition functions of the form:

$$\delta(q, \bullet) : U \times X \rightarrow (D \rightarrow Q)$$

which we read as the  $\vee/\wedge$ -form

$$\bigvee_{u \in U} \bigwedge_{x \in X} \bigwedge_{d \in D} \delta(q, \bullet, u, x, d)$$

In the case of  $\mathcal{A} \multimap \mathcal{B}$ , by applying the Dialectica pattern to such  $\vee/\wedge$ -forms and imposing *D-synchronicity*, we can let

$$\delta_{\mathcal{A} \multimap \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), \bullet, (f, F), (u, y), d) := (q'_{\mathcal{A}}, q'_{\mathcal{B}})$$

where

$$q'_{\mathcal{A}} := \delta_{\mathcal{A}}(q_{\mathcal{A}}, \bullet, u, F(u, y, d), d) \text{ and } q'_{\mathcal{B}} := \delta_{\mathcal{B}}(q_{\mathcal{B}}, \bullet, f(u), y)$$

We therefore have arrived at the following notion.

**Definition 2.1** (Tree Automata). *A tree automaton on alphabet  $\Sigma$  has the form*

$$\mathcal{A} = (Q, q^i, U, X, \delta, \Omega)$$

where  $Q$  is the finite set of states,  $q^i \in Q$  is the initial state,  $U$  and  $X$  are finite sets of resp. P and O-labels,  $\Omega \subseteq Q^\omega$  is the acceptance condition, and the transition function  $\delta$  has the form

$$\delta : Q \times \Sigma \rightarrow U \times X \rightarrow (D \rightarrow Q)$$

We suppose for simplicity that automata are *complete*, in the sense that  $U$  and  $X$  are always non-empty.

### 3. Games

Following [26], the morphisms of our monoidal closed categories of automata are based on a restriction of the linear arrow of *simple games* (see e.g. [1, 9]) between acceptance games.

**Simple Games.** Simple games are two-player games where the *Proponent* P ( $\exists$ loise) and the *Opponent* O ( $\forall$ belard) play in turn moves from a specific set, producing sequences of moves which may be subject to specified rules.

Formally, a simple game  $A$  has the form

$$A = (A_P, A_O, \xi_A, L_A)$$

where  $A_P$  and  $A_O$  are resp. the sets of P-moves and O-moves,  $\xi_A \in \{+, -\}$  is the *polarity* of  $A$ , and  $L_A \subseteq \wp_A^{\xi_A}$  is a non-empty prefix-closed set of *legal plays*, where the sets  $\wp_A^+$  and  $\wp_A^-$  of positive and negative plays are

$$\begin{aligned} \wp_A^+ &:= (A_P \cdot A_O)^* + (A_P \cdot A_O)^* \cdot A_P \\ \wp_A^- &:= (A_O \cdot A_P)^* + (A_O \cdot A_P)^* \cdot A_O \end{aligned}$$

So P starts in a positive game and O starts in a negative one. We let  $s, t, \dots$  range of over plays and  $m, n, \dots$  range over moves.

The *dual* of  $A$  is the game  $\bar{A} := (A_O, A_P, -\xi_A, L_A)$ . A game  $A$  is *full* if  $L_A = \wp_A^{\xi_A}$ . We write

$$A = (U, X)$$

to denote a full positive game with  $A_P := U$  and  $A_O := X$ .

A play is a P-play (resp. an O-play) if it is either empty or ends with a P-move (resp. an O-move). A *P-strategy*  $\sigma$  is a non-empty set of legal P-plays which is

**P-prefix-closed:** if  $s.t \in \sigma$  and  $s$  is a P-play then  $s \in \sigma$ , and

**P-deterministic:** if  $s.n \in \sigma$  and  $s.m \in \sigma$  then  $n = m$ .

**Games Over an Automaton.** An automaton

$$\mathcal{A} = (Q_{\mathcal{A}}, q_{\mathcal{A}}^i, U, X, \delta_{\mathcal{A}}, \Omega_{\mathcal{A}})$$

generates the positive full game

$$\wp(\mathcal{A}) = A = (U, X \times D)$$

**Linear Arrow Games.** Simple games form a category  $\mathbf{SG}$ , in which, given games  $A$  and  $B$  of the same polarity, the morphisms from  $A$  to  $B$  are P-strategies in the negative *linear arrow game*

$$A \multimap B = (B_P + A_O, B_O + A_P, -, L_{A \multimap B})$$

$L_{A \multimap B} \subseteq \wp_{A \multimap B}^-$  consists of those negative plays  $s$  such that  $s|A \in L_A$  and  $s|B \in L_B$ , where  $s|A$  is the restriction of  $s$  to  $A_P + A_O$ , and similarly for  $s|B$ .

Note that the polarity of moves in component  $B$  is preserved while the polarity of moves in  $A$  is reversed. The plays of  $A \multimap B$  start in component  $A$  iff  $A$  and  $B$  are both positive. Moreover, plays satisfy the *switching condition*: given  $s.m.n \in \wp_{A \multimap B}$ , with  $n \in (A \multimap B)_O$ , then  $m$  and  $n$  are in the same component (*i.e.* only P is allowed to switch between  $A$  and  $B$ ).

**The Hyland-Schalk Functor [11].** There is a faithful functor

$$\mathbf{HS} : \mathbf{SG} \rightarrow \mathbf{Rel}$$

mapping a simple game to its set of legal plays, and a strategy  $\sigma : A \multimap B$  to

$$\mathbf{HS}(\sigma) := \{(s|A, s|B) \mid s \in \sigma\} \subseteq L_A \times L_B$$

Hence strategies  $\sigma : A \multimap B$  can be represented as spans

$$\begin{array}{ccc} & \mathbf{HS}(\sigma) & \\ \swarrow & & \searrow \\ L_A & & L_B \end{array}$$

	$\mathcal{D}(\mathcal{A})$	$\multimap$	$\mathcal{D}(\mathcal{B})$	
O	$u$		$v$	P
P	$(x, d)$		$(y, d)$	O

**Figure 1.** A play in a synchronous  $\sigma : \mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$

In particular, the identity strategy  $\text{id}_A$  is the unique strategy such that  $\text{HS}(\text{id}_A) = L_A \times_{L_A} L_A$ , where  $L_A \times_{L_A} L_A$  is the pullback of the identity  $L_A \rightarrow L_A$  with itself in  $\mathbf{Set}$ .

**Synchronous and Zig-Zag Strategies.** The morphisms of our categories of automata are based on the combinatorics of a restriction of the linear arrow  $\mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$ , similar to that of [26].

Since we want monoidal *closed* categories, we need a restriction of  $\mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$  which can be *internalized* in a automaton  $\mathcal{A} \multimap \mathcal{B}$ , so that strategies on  $\mathcal{D}(\mathcal{A} \multimap \mathcal{B})$  correspond to strategies in that restriction of  $\mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$ . This is possible if we restrict to strategies on  $\mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$  which evaluate  $\mathcal{A}$  and  $\mathcal{B}$  in parallel, and *along the path of the input tree*. We call such strategies *D-synchronous*. They are formally defined as follows.

The plays  $s$  of  $\mathcal{D}(\mathcal{A})$ , have a *trace*  $\text{tr}_D(s) \in D^*$ , defined by composing restriction and projection:

$$\text{tr}_D(\varepsilon) := \varepsilon \quad \text{tr}_D(s.u) := \text{tr}_D(s) \quad \text{tr}_D(s.(x, d)) := \text{tr}_D(s).d$$

A strategy

$$\sigma : \mathcal{D}(\mathcal{A}) \multimap \mathcal{D}(\mathcal{B})$$

is *D-synchronous* if  $\text{tr}_D(s) = \text{tr}_D(t)$  for all  $(s, t) \in \text{HS}(\sigma)$ . This imposes the plays of  $\sigma$  to have the form depicted in Fig. 1.

Note that  $\sigma$  is *D-synchronous* iff the following commutes:

$$\begin{array}{ccc} & \text{HS}(\sigma) & \\ & \swarrow \quad \searrow & \\ L_A & & L_B \\ & \searrow \quad \swarrow & \\ & D^* & \end{array} \quad \begin{array}{c} \text{tr}_D \\ \text{tr}_D \end{array} \quad (3)$$

It follows that automata and *D-synchronous* strategies form a category **SAG**.

Much of what is required for the monoidal *closed* structure in our Dialectica-like approach actually follows from the specific *zig-zag* shape of the plays in Fig. 1. Given simple games  $A$  and  $B$ , a strategy  $\sigma : A \multimap B$  is a *zig-zag strategy* if for all play  $s \in \sigma$ , the restrictions  $s \upharpoonright A$  and  $s \upharpoonright B$  have the same length.

Zig-zag strategies can be characterized by diagrams similar to (3) (by taking lengths of plays instead of their traces). Since moreover the identity strategy on  $A$  is zig-zag, it follows that simple games and zig-zag strategies form a category.

Note that synchronous strategies are zig-zag strategies. A zig-zag strategy  $\sigma : A \multimap B$  always has to switch component (recall that O can never switch). In particular, if  $A$  and  $B$  are positive, then the plays of  $\sigma$  have the same shape as those of Fig. 1.

**Totality.** Given games  $A$  and  $B$ , a strategy  $s : A \multimap B$  is *total* if given  $s \in \sigma$ , if  $s.n$  is legal then  $s.n.m \in \sigma$  for some move  $m$ .

It is easy to see that if  $\sigma : A \multimap B$  and  $\tau : B \multimap C$  are both zig-zag and total, then  $\tau \circ \sigma$  is zig-zag and total.

Indeed, consider  $(s, t) \in \text{HS}(\tau \circ \sigma) = \text{HS}(\tau) \circ \text{HS}(\sigma)$ , and  $u$  such that  $(s, u) \in \text{HS}(\sigma)$  and  $(u, t) \in \text{HS}(\tau)$ . Given a legal  $(A \multimap C)_O$ -move  $m$  in (say) component  $A$ , since  $\sigma$  is zig-zag and total, there is some  $n$  such that  $(s.m, u.n) \in \text{HS}(\sigma)$ . Since  $n \in B_P \subseteq (B \multimap C)_O$ , and since  $\tau$  is zig-zag and total, there is some  $r \in C_P$  such that  $(u.n, t.r) \in \text{HS}(\tau)$ , from which it follows that  $(s.m, t.r) \in \text{HS}(\tau \circ \sigma)$ . The case of  $m \in C_O$  is similar.

Since identity strategies are total, it follows that simple games and total zig-zag strategies form a category.

**Winning.** Simple games can be equipped with *winning conditions*, which are infinite sequences of moves.

It is well-known (see e.g. [1, 9]), that total and winning strategies compose and form a category. The case of zig-zag strategies is particularly simple. Given  $(A, \mathcal{W}_A)$  and  $(B, \mathcal{W}_B)$ , a total zig-zag strategy  $\sigma : A \multimap B$  is *winning* if for all infinite sequences of moves  $\pi$  such that  $\pi(0). \dots . \pi(n) \in \sigma$  for infinitely many  $n$ ,  $\pi \upharpoonright A \in \mathcal{W}_A$  implies  $\pi \upharpoonright B \in \mathcal{W}_B$ .

## 4. A Dialectica-Like Interpretation of Zig-Zag Strategies

We now give a Dialectica-like presentation of total zig-zag strategies  $\sigma : A \multimap B$  for  $A$  and  $B$  positive full games. It relies on an instance of Dialectica called *simple self-dualization* in [12]. We will perform it in the topos of trees  $\mathcal{L}$ .

**Simple Self Dualization.** Given a category  $\mathbb{C}$ , its *simple self-dualization* is  $\mathbf{G}(\mathbb{C}) := \mathbb{C} \times \mathbb{C}^{\text{op}}$  (also written  $\mathbb{C}^d$  in [12]). Its objects are pairs  $U, X$  of objects of  $\mathbb{C}$ , and a morphism from  $(U, X)$  to  $(V, Y)$  is given by a pair of maps  $(f, F)$ , denoted

$$(f, F) : (U, X) \multimap (V, Y)$$

where  $f : U \rightarrow V$  and  $F : Y \rightarrow X$ . If  $\mathbb{C}$  is symmetric monoidal, then  $\mathbf{G}(\mathbb{C})$  is an instance of a *Girard* category, in the sense of de Paiva [4, 12].

Assume that  $\mathbb{C}$  is symmetric monoidal closed w.r.t.  $(\otimes, \mathbf{I})$ . Then  $\mathbf{G}(\mathbb{C})$  is symmetric monoidal closed w.r.t.

$$(U, X) \otimes_{\mathbf{G}} (V, Y) := (U \otimes V, X^V \otimes Y^U) \quad \text{with unit } (\mathbf{I}, \mathbf{I})$$

The linear exponentials are given by

$$(U, X) \multimap_{\mathbf{G}} (V, Y) := (V^U \times X^Y, U \times Y)$$

Assume now that the monoidal structure  $(\otimes, \mathbf{I}) = (\times, \mathbf{1})$  of  $\mathbb{C}$  is Cartesian. Then  $\mathbf{G}(\mathbb{C})$  can be equipped with a co-monad  $(T, \epsilon, \delta)$  where the action on objects of  $T$  is

$$T(U, X) := (U, X^U)$$

The co-Kleisli category  $\mathbf{D}(\mathbb{C}) := \mathbf{Kl}(T)$  is a Dialectica category in the sense of [4, 10]. Explicitly, its objects are pairs  $A = (U, X)$  of objects of  $\mathbb{C}$ , and a map from  $A$  to  $(V, Y)$  is a  $\mathbf{G}(\mathbb{C})$ -morphism  $(f, F)$  from  $TA$  to  $(V, Y)$ , that is

$$(f, F) : (U, X^U) \multimap (V, Y)$$

$\mathbf{D}(\mathbb{C})$  is symmetric monoidal closed w.r.t. the product

$$(U, X) \otimes (V, Y) := (U \times V, X \times Y) \quad \text{with unit } (\mathbf{1}, \mathbf{1})$$

Note that with  $A = (U, X)$  and  $B = (V, Y)$ ,

$$\begin{aligned} T(A \otimes B) &= (U \times V, (X \times Y)^{U \times V}) \\ &\simeq (U \times V, X^{U^V} \times Y^{V^U}) \\ &= TA \otimes_{\mathbf{G}} TB \end{aligned}$$

The linear exponentials of  $\mathbf{D}(\mathbb{C})$  are given by

$$(U, X) \multimap (V, Y) := (V^U \times X^{U \times Y}, U \times Y)$$

Note that  $A \multimap B \simeq TA \multimap_{\mathbf{G}} B$ , so the monoidal closure of  $\mathbf{D}(\mathbb{C})$  actually follows from that of  $\mathbf{G}(\mathbb{C})$ :

$$\begin{aligned} \mathbf{D}(\mathbb{C})[A \otimes B, C] &= \mathbf{G}(\mathbb{C})[T(A \otimes B), C] \\ &\simeq \mathbf{G}(\mathbb{C})[TA \otimes_{\mathbf{G}} TB, C] \\ &\simeq \mathbf{G}(\mathbb{C})[TA, TB \multimap_{\mathbf{G}} C] \\ &\simeq \mathbf{D}(\mathbb{C})[A, B \multimap C] \end{aligned}$$

**The Topos of Trees.** The topos of trees  $\mathcal{S}$  is the presheaf category over the order  $(\mathbb{N}, \leq)$  seen as a category, see e.g. [3].

An object  $X$  of  $\mathcal{S}$  is given by a family of sets  $(X_n)_{n \in \mathbb{N}}$  equipped with restriction maps  $r_n^X : X_{n+1} \rightarrow X_n$ . A morphism  $f$  from  $X$  to  $Y$  is a family of functions  $f_n : X_n \rightarrow Y_n$  compatible with restriction:  $r_n^Y \circ f_{n+1} = f_n \circ r_n^X$ .

As a topos,  $\mathcal{S}$  is Cartesian closed w.r.t. to the Cartesian product of presheaves, which is given by  $(X \times Y)_n := X_n \times Y_n$ . Exponentials are defined as usual for presheaves (see e.g. [20]) by

$$(X^Y)_n := \text{Nat}[\mathbb{N}[-, n] \times Y, X]$$

Explicitly,  $(X^Y)_n$  consists of sequences  $(\xi_k : Y_k \rightarrow X_k)_{k \leq n}$  which are compatible with  $r^X$  and  $r^Y$ . The restriction map of  $X^Y$  takes  $(\xi_k)_{k \leq n+1} \in (X^Y)_{n+1}$  to  $(\xi_k)_{k \leq n} \in (X^Y)_n$ .

We will use the functor  $\blacktriangleright : \mathcal{S} \rightarrow \mathcal{S}$  of [3]. On objects, it maps  $X$  to  $(\blacktriangleright(X_n))_{n \in \mathbb{N}}$  where  $\blacktriangleright(X_{n+1}) = X_n$  and  $\blacktriangleright(X_0) = \mathbf{1}$ , with  $\blacktriangleright(r_{n+1}^X) = r_n^X$  and  $\blacktriangleright(r_0^X) = \mathbf{1} : X_0 \rightarrow \mathbf{1}$ . On morphisms,  $\blacktriangleright(f_{n+1}) = f_n$  and  $\blacktriangleright(f_0) = \mathbf{1} : \mathbf{1} \rightarrow \mathbf{1}$ .

Define the family of maps  $\text{pred}^X : X \Rightarrow \blacktriangleright X$ , natural in  $X$ , as  $\text{pred}_0^X := \mathbf{1} : X_0 \rightarrow \mathbf{1}$  and  $\text{pred}_{n+1}^X := r_n^X$ .

The functor  $\blacktriangleright$  allows  $\mathcal{S}$  to be equipped with fixpoint operators  $\text{fix}^X : X^{\blacktriangleright X} \Rightarrow X$ , defined as

$$\text{fix}_n^X((f_m)_{m \leq n}) := (f_n \circ \dots \circ f_0)(\bullet)$$

The maps  $\text{fix}^X$  are natural in  $X$ . Given  $f : \blacktriangleright X \times Y \Rightarrow X$ , writing  $f^t : Y \Rightarrow X^{\blacktriangleright X}$  for the exponential transpose of  $f$ ,  $\text{fix}^X \circ f^t$  is the unique map  $h : Y \Rightarrow X$  satisfying  $f \circ \langle \text{pred}^X \circ h, \text{id}_Y \rangle = h$  (see [3, Thm. 2.4]).

Given a set  $M$ , write  $M^*$  for the object of  $\mathcal{S}$  with  $M_n^* := M^n$  and  $r_n(\overline{m}.m) := \overline{m}$ . Note that  $M^0 \simeq \mathbf{1}$  and that  $r_n$  is surjective.

**A Dialectica-Like Interpretation of Zig-Zag Strategies.** Consider a positive full game  $A = (U, X)$ . There is a bijection

$$\partial = \langle \partial_U, \partial_X \rangle : \wp_A^{\text{even}} \longrightarrow \bigcup_{n \in \mathbb{N}} (U^n \times X^n)$$

with  $\partial(\varepsilon) = (\bullet, \bullet)$  and  $\partial(s.u.x) = (\partial_U(s).u, \partial_X(s).x)$ .

Consider now another positive full game  $B = (V, Y)$  and let  $\sigma : A \multimap B$  be a total zig-zag strategy. By induction on  $n \in \mathbb{N}$ , it is easy to see that for all  $(\overline{u}, \overline{y}) \in U^n \times Y^n$ , there is a unique  $(s, t) \in \text{HS}(\sigma)$  such that  $\overline{u} = \partial_U(s)$  and  $\overline{y} = \partial_Y(t)$ .

The property vacuously holds for  $n = 0$ . Assuming it for  $n$ , given  $(\overline{u}.u, \overline{y}.y) \in U^{n+1} \times Y^{n+1}$ , by induction hypothesis, there is a unique  $(s, t) \in \text{HS}(\sigma)$  such that  $\overline{u} = \partial_U(s)$  and  $\overline{y} = \partial_Y(t)$ . Now, since  $\sigma$  is total and zig-zag, there is a unique  $v \in V$  such that  $(s.u, t.v) \in \text{HS}(\sigma)$ . Similarly, there is a unique  $x \in X$  such that  $(s.u, t.v.y) \in \text{HS}(\sigma)$ , and the property follows.

Furthermore, since  $\overline{u}.u$  and  $\overline{y}$  uniquely determine  $\overline{v} = \partial_V(t)$  and  $v$ , and since  $\overline{u}.u$  and  $\overline{y}.y$  uniquely determine  $\overline{x} = \partial_X(s)$  and  $x$ , we obtain functions

$$\begin{aligned} f_{n+1} &: U^{n+1} \times Y^n \longrightarrow V^{n+1} \\ F_{n+1} &: U^{n+1} \times Y^{n+1} \longrightarrow X^{n+1} \end{aligned}$$

It follows that  $\sigma$  uniquely determine a  $\mathbf{G}(\mathcal{S})$ -morphism

$$(f, F) : (U^*, X^{*U^*}) \dashrightarrow (V^{*\blacktriangleright Y^*}, Y^*)$$

Conversely, each  $(f, F)$  uniquely determine a total zig-zag strategy  $\sigma$ , with, for all  $\overline{u}.u \in U^{n+1}$ , and all  $\overline{y} \in Y^n$ ,

$$(\partial^{-1}(\overline{u}, \overline{x}).u, \partial^{-1}(\overline{v}, \overline{y}).v) \in \text{HS}(\sigma)$$

where  $\overline{v}.v = f_{n+1}(\overline{u}.u, \overline{y})$  and  $\overline{x} = F_n(\overline{u}, \overline{y})$ ; and for all  $y$ ,

$$(\partial^{-1}(\overline{u}, \overline{x}).u.x, \partial^{-1}(\overline{v}, \overline{y}).v.y) \in \text{HS}(\sigma)$$

where  $\overline{x}.x = F_{n+1}(\overline{u}.u, \overline{y}.y)$ .

We therefore have shown:

**Proposition 4.1.** Given positive full games  $A = (U, X)$  and  $B = (V, Y)$ , total zig-zag strategies  $\sigma : A \multimap B$  are in 1-1 correspondence with  $\mathbf{G}(\mathcal{S})$ -morphisms

$$(f, F) : (U^*, X^{*U^*}) \dashrightarrow (V^{*\blacktriangleright Y^*}, Y^*)$$

Note that given  $(\overline{u}, \overline{x}, \overline{v}, \overline{y}) \in (U \times X \times V \times Y)^n$ , we have  $((\overline{u}, \overline{x}), (\overline{v}, \overline{y})) \in \text{HS}(\sigma)$  if and only if  $\overline{v} = f_n(\overline{u}, \blacktriangleright(\overline{y}))$  and  $\overline{x} = F_n(\overline{u}, \overline{y})$ . Here, we have written  $((\overline{u}, \overline{x}), (\overline{v}, \overline{y})) \in \text{HS}(\sigma)$  for  $(\partial^{-1}(\overline{u}, \overline{x}), \partial^{-1}(\overline{v}, \overline{y})) \in \text{HS}(\sigma)$ . We adopt the same convention in the following.

Consider positive full games  $A = (U, X)$ ,  $B = (V, Y)$  and  $C = (W, Z)$ , and  $\mathbf{G}(\mathcal{S})$ -morphisms

$$\begin{aligned} (f, F) &: (U^*, X^{*U^*}) \dashrightarrow (V^{*\blacktriangleright Y^*}, Y^*) \\ (g, G) &: (V^*, Y^{*V^*}) \dashrightarrow (W^{*\blacktriangleright Z^*}, Z^*) \end{aligned}$$

We want to define their composite

$$(h, H) : (U^*, X^{*U^*}) \dashrightarrow (W^{*\blacktriangleright Z^*}, Z^*)$$

Write  $\sigma$  and  $\tau$  for the total zig-zag strategies corresponding to resp.  $(f, F)$  and  $(g, G)$ . Then the relational composite

$$\text{HS}(\tau \circ \sigma) = \text{HS}(\tau) \circ \text{HS}(\sigma)$$

must be such that  $((\overline{u}, \overline{x}), (\overline{w}, \overline{z})) \in \text{HS}(\tau) \circ \text{HS}(\sigma)$  if and only if there are  $(\overline{v}, \overline{y})$  such that

$$((\overline{u}, \overline{x}), (\overline{v}, \overline{y})) \in \text{HS}(\sigma) \quad \text{and} \quad ((\overline{v}, \overline{y}), (\overline{w}, \overline{z})) \in \text{HS}(\tau)$$

But this is possible iff the following equations are satisfied:

$$\begin{aligned} \overline{v} &= f_n(\overline{u}, \blacktriangleright(\overline{y})) & \overline{w} &= g_n(\overline{v}, \blacktriangleright(\overline{z})) \\ \overline{x} &= F_n(\overline{u}, \overline{y}) & \overline{y} &= G_n(\overline{v}, \overline{z}) \end{aligned}$$

The derived equation

$$\overline{y} = G_n(f_n(\overline{u}, \blacktriangleright(\overline{y})), \overline{z})$$

uniquely defines  $\overline{y}$  from  $\overline{u}$  and  $\overline{z}$  as

$$\overline{y} = y(\overline{u}, \overline{z}) = \text{fix}_n^Y(\lambda y. G_n(f_n(\overline{u}, y), \overline{z}))$$

(We have here tacitly used the fact that  $\xi \in (M^{*\blacktriangleright M^*})_n$  is completely determined by its last component  $\xi_n$ .) Now, since  $\blacktriangleright(y(\overline{u}, \overline{z})) = y(\blacktriangleright \overline{u}, \blacktriangleright \overline{z})$ , we can define

$$\begin{aligned} h_{n+1}(\overline{u}.u, \overline{z}) &:= g_{n+1}(f_{n+1}(\overline{u}.u, y(\overline{u}, \overline{z})), \overline{z}) \\ H_{n+1}(\overline{u}.u, \overline{z}.z) &:= F_{n+1}(\overline{u}.u, y(\overline{u}.u, \overline{z}.z)) \end{aligned}$$

More generally, given  $\mathbf{G}(\mathcal{S})$ -objects  $(U, X)$ ,  $(V, Y)$ ,  $(W, Z)$ , and  $\mathbf{G}(\mathcal{S})$ -morphisms

$$\begin{aligned} (f, F) &: (U, X^U) \dashrightarrow (V^{\blacktriangleright Y}, Y) \\ (g, G) &: (V, Y^V) \dashrightarrow (W^{\blacktriangleright Z}, Z) \end{aligned}$$

we can define their composite

$$(g, G) \circ (f, F) = (h, H) : (U, X^U) \dashrightarrow (W^{\blacktriangleright Z}, Z)$$

as, modulo exponential transpose and again using the internal  $\lambda$ -calculus of  $\mathcal{S}$ :

$$\begin{aligned} h(u, z) &:= g(f(u, y(\blacktriangleright u, z)), z) \\ H(z, u) &:= F(u, y(u, z)) \end{aligned}$$

where  $y(u, z) := \text{fix}^Y(\lambda y. G(f(u, y), z))$

It is possible to directly check that this composition is associative and preserves identities. We can actually do better: the operation

$$(-)^{\blacktriangleright} : (U, X) \longmapsto (U^{\blacktriangleright X}, X)$$

is the action on objects of a functor part of a monad, and the composition of  $\mathbf{G}(\mathcal{S})$ -morphisms

$$\begin{aligned} (f, F) &: TA \dashrightarrow B^{\blacktriangleright} \\ (g, G) &: TB \dashrightarrow C^{\blacktriangleright} \end{aligned}$$

can be described by a distributive law of  $T$  over  $(-)^{\blacktriangleright}$ .

**Distributive Laws.** A distributive law  $\lambda$  of a co-monad  $G$  over a monad  $T$  on a category  $\mathbb{C}$  is given by a natural transformation

$$\lambda : G \circ T \Longrightarrow T \circ G$$

subject to some coherence conditions, which can be found e.g. in [7]. These coherence conditions ensure that we can define a category  $\mathbf{Kl}(\lambda)$ , whose objects are the objects of  $\mathbb{C}$ , and whose morphisms are given by  $\mathbf{Kl}(\lambda)[A, B] := \mathbb{C}[GA, TB]$ .

In our case, the co-monad is the co-monad  $T$  of [4], and the monad  $(-)^{\blacktriangleright} = ((-)^{\blacktriangleright}, \eta, \mu)$  is given by

$$\begin{aligned} (f_\eta, F_\eta) & : (U, X) \quad \dashrightarrow \quad (U^{\blacktriangleright X}, X) \\ (f_\mu, F_\mu) & : (U^{\blacktriangleright X} \times \blacktriangleright^X X, X) \quad \dashrightarrow \quad (U^{\blacktriangleright X}, X) \end{aligned}$$

where  $F_\eta = F_\mu = \text{id}_X$ ,  $f_\eta(u, x) = u$  and  $f_\mu(h, x) = h(x, x)$ .

Given a  $\mathbf{G}(\mathcal{S})$ -object  $A = (U, X)$ , define the  $\mathbf{G}(\mathcal{S})$ -map

$$\begin{aligned} \lambda_A = (f^\lambda, F^\lambda) & : (U^{\blacktriangleright X}, X^{\blacktriangleright X}) \quad \dashrightarrow \quad (U^{\blacktriangleright(X^U)}, X^U) \\ \text{where } f^\lambda & : U^{\blacktriangleright X} \times \blacktriangleright(X^U) \quad \longrightarrow \quad U \\ F^\lambda & : U^{\blacktriangleright X} \times X^U \quad \longrightarrow \quad X \end{aligned}$$

as follows. Let  $f_0^\lambda(\theta_0, \bullet) := \theta_0$ . Given  $\xi \in (X^U)_n$ ,  $\theta \in (U^{\blacktriangleright X})_n$  and  $\theta' \in (U^{\blacktriangleright X})_{n+1}$ ,

$$\begin{aligned} F_n^\lambda(\theta, \xi) & := \text{fix}_n^X(\xi \circ \theta) \\ f_{n+1}^\lambda(\theta', \xi) & := \theta'_{n+1}(\text{fix}_n^X(\xi \circ r_n(\theta'))) \\ & = \theta'_{n+1}(F_n(r_n(\theta'), \xi)) \end{aligned}$$

**Proposition 4.2.** *The family of maps  $\lambda_A : T(A^{\blacktriangleright}) \dashrightarrow (TA)^{\blacktriangleright}$  forms a distributive law.*

We write  $\mathbf{DZ}_{\mathcal{S}}$  for  $\mathbf{Kl}(\lambda)$  and  $\mathbf{DZ}$  for the full subcategory of  $\mathbf{DZ}_{\mathcal{S}}$  whose objects are of the form  $(U^*, X^*)$ .

**Corollary 4.3.** *The category of positive full games and total zig-zag strategies is equivalent to the category  $\mathbf{DZ}$ .*

**Symmetric Monoidal Structure.** The categories  $\mathbf{DZ}_{\mathcal{S}}$  and  $\mathbf{DZ}$  inherit the monoidal structure of  $\mathbf{D}(\mathcal{S})$ , which is given by the monoidal product  $\otimes$  of  $\mathbf{G}(\mathcal{S})$ . It acts on  $\mathbf{DZ}_{\mathcal{S}}$ -maps as follows: Given  $(U_i, X_i)$ ,  $(V_i, Y_i)$  for  $i \in \{1, 2\}$  and

$$(f_i, F_i) : (U_i, X_i^{U_i}) \dashrightarrow (V_i^{\blacktriangleright Y_i}, Y_i)$$

let  $(f_1, F_2) \otimes (f_2, F_2)$  be  $(h, H)$  where

$$\begin{aligned} h & : (U_1 \times U_2) \times \blacktriangleright(Y_1 \times Y_2) \quad \dashrightarrow \quad (V_1 \times V_2) \\ H & : (U_1 \times U_2) \times (Y_1 \times Y_2) \quad \dashrightarrow \quad (X_1 \times X_2) \end{aligned}$$

are defined as follows. First, note that  $\blacktriangleright(Y_1 \times Y_2) = \blacktriangleright Y_1 \times \blacktriangleright Y_2$ . Then let

$$\begin{aligned} h((u_1, u_2), (y_1, y_2)) & := (f_1(u_1, y_1), f_2(u_2, y_2)) \\ H((u_1, u_2), (y_1, y_2)) & := (F_1(u_1, y_1), F_2(u_2, y_2)) \end{aligned}$$

The bifunctionality of  $\_ \otimes \_$  follows from the fact that

$$\text{fix}^{X \times Y}(\lambda(x, y) \cdot (f(x), g(x))) = (\text{fix}^X(f), \text{fix}^Y(g))$$

**Proposition 4.4.** *The category  $\mathbf{DZ}_{\mathcal{S}}$  is symmetric monoidal.*

Since  $(M \times N)^* \simeq M^* \times N^*$ , we also get

**Corollary 4.5.** *The category  $\mathbf{DZ}$  is symmetric monoidal.*

**Symmetric Monoidal Closed Structure of  $\mathbf{DZ}$ .** For the monoidal closed structure of  $\mathbf{DZ}$ , first note that

$$\begin{aligned} \mathbf{DZ}[A \otimes B, C] & = \mathbf{G}(\mathcal{S})[T(A \otimes B), C^{\blacktriangleright}] \\ & \simeq \mathbf{G}(\mathcal{S})[TA \otimes_{\mathbf{G}} TB, C^{\blacktriangleright}] \\ & \simeq \mathbf{G}(\mathcal{S})[TA, TB \dashv_{\mathbf{G}} C^{\blacktriangleright}] \end{aligned}$$

Hence, given  $A = (U^*, X^*)$  and  $B = (V^*, Y^*)$ , we are looking for a linear exponent  $(A \dashv_{\mathbf{DZ}} B) = (W^*, Z^*)$  such that we have

$(A \dashv_{\mathbf{DZ}} B)^{\blacktriangleright} \simeq (TA \dashv_{\mathbf{G}} B^{\blacktriangleright})$ , that is (modulo some  $\mathcal{S}$ -isos)

$$(W^{\blacktriangleright Z}, Z^*) \simeq (V^{\blacktriangleright U^*} \times \blacktriangleright Y^* \times X^{*(U \times Y)^*}, (U \times Y)^*)$$

The obvious choice for  $Z$  is to take  $U \times Y$ . For  $W$ , it is natural to look for  $W = W_1 \times W_2$ , so that (modulo some  $\mathcal{S}$ -isos)

$$W^{\blacktriangleright Z^*} \simeq W_1^{\blacktriangleright U^*} \times \blacktriangleright Y^* \times W_2^{\blacktriangleright (U \times Y)^*}$$

We can actually take  $W_1 := V^U$  and  $W_2 := X^{U \times Y}$  since, in  $\mathcal{S}$ ,

$$N^{*M^*} \simeq (N^M)^{* \blacktriangleright M^*} \quad (4)$$

Indeed, first note that  $\xi \in (N^{*M^*})_n$  is completely determined by  $\xi_n : M^n \rightarrow N^n$ . Moreover, a map  $(\blacktriangleright M^*)_{n+1} \rightarrow (N^M)^{n+1}$  is just an  $M^n$ -indexed family of sequences of maps  $(M \rightarrow N)^{n+1}$ . Also, given  $\xi \in (N^{*M^*})_{n+1}$ , the map  $\xi_{n+1} : M^{n+1} \rightarrow N^{n+1}$  is uniquely determined by

$$\xi_{n+1}(\overline{m}, m) := \xi_n(\overline{m}).\xi_{\overline{m}}(m)$$

where  $\xi_{\overline{m}}$  is an  $M^n$ -indexed family of maps  $M \rightarrow N$ .

Then, the  $\mathcal{S}$ -isomorphism takes  $\xi \in (N^{*M^*})_n$  to the family  $(\theta_k)_{k \leq n} \in (N^M)^{* \blacktriangleright M^*}$  defined by induction on  $k$  as:

$$\begin{aligned} \theta_0 & := \mathbf{1} \\ \theta_{k+1} & := (\overline{m} \mapsto \theta_k(\blacktriangleright \overline{m}).\xi_{\overline{m}}) \end{aligned}$$

We thus get the following.

**Proposition 4.6.** *The category  $\mathbf{DZ}$  is symmetric monoidal closed, with linear exponent*

$$(U^*, X^*) \dashv_{\mathbf{DZ}} (V^*, Y^*) := ((V^U \times X^{U \times Y})^*, (U \times Y)^*)$$

## 5. Dialectica-like Categories of Tree Automata

We describe our Dialectica-like categories  $\text{DialAut}$  of tree automata, starting from  $\mathbf{DZ}$ . This involves two steps: we first deal with  $D$ -synchronicity, and then handle the fibred structure. We finally discuss the logical connectives presented in §1.

Since we will not make use of  $\mathbf{DZ}_{\mathcal{S}}$  anymore, from now on we leave implicit the  $(-)^*$  in the denotation of  $\mathbf{DZ}$ -objects.

### 5.1 $D$ -Synchronicity in $\mathbf{DZ}$

We express  $D$ -synchronicity in  $\mathbf{DZ}$  using a monad. Given automata  $\mathcal{A}$  and  $\mathcal{B}$ , a  $D$ -synchronous total strategy  $\wp(\mathcal{A}) \dashv \wp(\mathcal{B})$  generates via Prop. 4.1 a  $\mathbf{DZ}$ -morphism

$$(f, F) : (U, X \times D) \longrightarrow (V, Y \times D)$$

where  $F$  is the identity on  $D$ . We can therefore restrict to  $\mathbf{DZ}$ -morphisms of the form

$$(U, X) \longrightarrow (V, Y \times D) \quad (5)$$

Now, note that in  $\mathbf{DZ}$ ,

$$(V, Y \times D) = (V, Y) \otimes (\mathbf{1}, D)$$

The object  $D = (\mathbf{1}, D)$  is a commutative monoid in  $\mathbf{DZ}$ , and we can use the Kleisli category of the monad of *monoid indexing* with  $D$  (see [11, 12], which actually use the co-monad of comonoid indexing).

**Monoid Indexing [11, 12].** Let  $(\mathbb{C}, \otimes, \mathbf{I})$  be a symmetric monoidal category and let  $\mathbf{Mon}(\mathbb{C})$  be its category of commutative monoids. Its objects are objects  $M$  of  $\mathbb{C}$  equipped with structure maps

$$\mathbf{I} \xrightarrow{u} M \xleftarrow{m} M \otimes M$$

subject to some coherence conditions (see e.g. [22]). A morphism from  $(M, u, m)$  to  $(N, u', m')$  is a  $\mathbb{C}$ -morphism  $M \rightarrow N$  which commutes with the structure maps.

Given a commutative monoid  $(M, u, m)$  in  $\mathbb{C}$ , define the monad  $M = (M, \eta, \mu)$  as follows. The functor  $M$  acts on objects by tensoring with  $M$  on the right and on morphisms by

$$(f : A \rightarrow B) \mapsto (f \otimes \text{id}_M : A \otimes M \rightarrow B \otimes M)$$

The natural maps  $\eta$  and  $\mu$  are given by

$$\begin{aligned} \eta_A &= (e, \text{id}_A) : K \otimes A \rightarrow A \\ \mu_A &= (m, \text{id}_A) : K \otimes K \otimes A \rightarrow K \otimes A \end{aligned}$$

(we leave here implicit the coherence isos of  $\mathbb{C}$ ).

The structure maps of the commutative monoid  $D = (\mathbf{1}, D)$  of  $\mathbf{DZ}$  are given, modulo  $\mathcal{S}$ -isomorphism, by

$$\begin{aligned} u &= (\mathbf{1}, \mathbf{1}) : (\mathbf{1}, \mathbf{1}) \mapsto (\mathbf{1}, D) \\ m &= (\mathbf{1}, \langle \text{id}, \text{id} \rangle) : (\mathbf{1}, D \times D) \mapsto (\mathbf{1}, D) \end{aligned}$$

We let  $\mathbf{DZ}_D$  be the Kleisli category  $\mathbf{Kl}(D)$ .

**Symmetric Monoidal Closed Structure.** The monoidal product  $\otimes$  of  $\mathbf{DZ}_D$  acts on objects as the monoidal product  $\otimes$  of  $\mathbf{DZ}$ . It acts on maps as follows: Given  $(U_i, X_i), (V_i, Y_i)$  for  $i \in \{1, 2\}$  and

$$(f_i, F_i) : (U_i, X_i^{U_i}) \mapsto (V_i^{\blacktriangleright(Y_i \times D)}, Y_i \times D)$$

let  $(f_1, F_2) \otimes (f_2, F_2)$  be  $(h, H)$  where

$$\begin{aligned} h &: (U_1 \times U_2) \times \blacktriangleright(Y_1 \times Y_2 \times D) \mapsto (V_1 \times V_2) \\ H &: (U_1 \times U_2) \times (Y_1 \times Y_2 \times D) \mapsto (X_1 \times X_2) \end{aligned}$$

are defined explicitly as

$$\begin{aligned} h((u_1, u_2), (y_1, y_2, p)) &:= (f_1(u_1, y_1, p), f_2(u_2, y_2, p)) \\ H((u_1, u_2), (y_1, y_2, p)) &:= (F_1(u_1, y_1, p), F_2(u_2, y_2, p)) \end{aligned}$$

Concerning monoidal closure, since

$$\begin{aligned} \mathbf{DZ}_D[A \otimes B, C] &= \mathbf{DZ}[A \otimes B, C \otimes D] \\ &\simeq \mathbf{DZ}[A, B \multimap_{\mathbf{DZ}} C \otimes D] \end{aligned}$$

we should have  $(A \multimap_{\mathbf{DZ}_D} B) \otimes D \simeq (A \multimap_{\mathbf{DZ}} B \otimes D)$ . This leads to  $((U, X) \multimap_{\mathbf{DZ}_D} (V, Y)) = (W, Z)$  with

$$(W, Z \times D) \simeq (V^U \times X^{U \times Y \times D}, U \times Y \times D)$$

We therefore let

$$(U, X) \multimap_{\mathbf{DZ}_D} (V, Y) := (V^U \times X^{U \times Y \times D}, U \times Y)$$

**Proposition 5.1.**  $\mathbf{DZ}_D$  is symmetric monoidal closed.

## 5.2 Indexed Structure

We discussed some Dialectica-like categories in §4 in order to give a presentation of total zig-zag strategies. In full generality, these Dialectica-like constructions are based on *fibrations* (see e.g. [13]), which allow to handle some logic on top of the combinatorics of the morphisms. We will use this framework to handle input alphabets and a form of quantification.

Quantification in usual Dialectica-like categories can be added along a pattern similar to the *simple fibration*  $s : s(\mathbb{B}) \rightarrow \mathbb{B}$  over a Cartesian base category  $\mathbb{B}$  (see e.g. [8, 10]). Recall (from e.g. [13]) that  $s(\mathbb{B})$  has pairs  $(I, X)$  of  $\mathbb{B}$ -objects as objects, with maps  $(I, X) \rightarrow (J, Y)$  given by a pairs of  $\mathbb{B}$ -maps  $f_0 : I \rightarrow J$  and  $f : I \times X \rightarrow Y$ . The functor  $s : s(\mathbb{B}) \rightarrow \mathbb{B}$  is the first projection. We would like to use as base the category  $\mathbf{DZ}_D$  and its monoidal product, which is not Cartesian. However, it is well-known (1) that the fibre category of  $s(\mathbb{B})$  over say  $I$ , is the co-Kleisli category of a co-monad whose functor is  $I \times (-)$ , and (2) that commutative comonoids form a Cartesian category. *Comonoid indexing* [11, 12], allows to get a fibration whose fiber over  $K$ , for  $K$  a commutative comonoid, is the co-Kleisli category for  $K \otimes (-)$ .

**The Base Category  $\mathbf{T}$ .** Let  $\mathbf{T}$  be the category whose objects are finite sets and whose morphisms are given by

$$\mathbf{T}[\Sigma, \Gamma] := \mathcal{S}[\Sigma \times \blacktriangleright D, \Gamma] \simeq \mathbf{DZ}_D[(\Sigma, \mathbf{1}), (\Gamma, \mathbf{1})]$$

Explicitly, the  $\mathbf{T}$ -composite  $t' \circ t$  of  $t : \Sigma \rightarrow \Gamma$  and  $t' : \Gamma \rightarrow \Lambda$  is

$$(t' \circ t)(\bar{a}, p) := u(t(\bar{a}, p), p)$$

Note that a function  $t : D^* \rightarrow (\Sigma \rightarrow \Gamma)$  induces the  $\mathbf{T}$ -map  $t : \Sigma \rightarrow \Gamma$  inductively defined as  $t_0 := \mathbf{1}$  and  $t_{n+1} := \lambda(\bar{a}. a, p). t_n(\bar{a}, \blacktriangleright p). t(a, p)$ . Also, a  $\Sigma$ -labeled tree  $t$  corresponds to an  $\mathcal{S}$ -map  $\blacktriangleright D \rightarrow \Sigma$ , hence to a  $\mathbf{T}$ -map  $\dot{t} : \mathbf{1} \rightarrow \Sigma$ .

**Comonoid Indexing [11, 12].** This is dual to monoid indexing used above. Let  $\mathbf{Comon}(\mathbb{C})$  be the category of commutative comonoids on a symmetric monoidal category  $\mathbb{C}$ . Its objects are objects  $K$  of  $\mathbb{C}$  equipped with structure maps

$$\mathbf{I} \xleftarrow{e} K \xrightarrow{m} K \otimes K$$

subject to some condition dual to those of  $\mathbf{Mon}(\mathbb{C})$ . A morphism from  $(K, e, m)$  to  $(L, e', m')$  is also a  $\mathbb{C}$ -morphism  $K \rightarrow L$  compatible with the structure maps. It is well-known (see e.g. [22, Cor. 18, §6.5]) that  $\mathbf{Comon}(\mathbb{C})$  is Cartesian.

Given a commutative comonoid  $(K, e, m)$  in  $\mathbb{C}$ , define a comonad  $K = (K, \epsilon, \delta)$  whose functor  $K$  acts on objects by tensoring on the left and on morphisms by

$$(f : A \rightarrow B) \mapsto (\text{id}_K \otimes f : K \otimes A \rightarrow K \otimes B)$$

The natural maps  $\epsilon$  and  $\delta$  are given by

$$\begin{aligned} \epsilon_A &= (e, \text{id}_A) : K \otimes A \rightarrow A \\ \delta_A &= (m, \text{id}_A) : K \otimes A \rightarrow K \otimes K \otimes A \end{aligned}$$

Now, any comonoid morphism  $u : K \rightarrow L$  induces a functor  $u^* : \mathbf{Kl}(L) \rightarrow \mathbf{Kl}(K)$  acting as the identity on objects and taking  $f : L \otimes A \rightarrow B$  to  $f \circ (u \otimes \text{id}_A) : K \otimes A \rightarrow B$ . It readily follows that  $\text{id}_K^* = \text{id}_{\mathbf{Kl}(K)}$  and that  $(u \circ v)^* = v^* \circ u^*$ . In other words, we have a functor  $\text{CI}(\mathbb{C}) : \mathbf{Comon}(\mathbb{C})^{\text{op}} \rightarrow \mathbf{Cat}$ . Its Grothendieck completion  $s_{\text{CI}}(\mathbb{C}) := \int \text{CI}(\mathbb{C})$  (see e.g. [13]) is the category whose objects are pairs  $(K, A)$  of an object  $K$  of  $\mathbf{Comon}(\mathbb{C})$  and an object  $A$  of  $\mathbb{C}$ , and whose morphisms from  $(K, A)$  to  $(L, B)$  are pairs  $(u, f)$  where  $u : K \rightarrow L$  is a comonoid morphism and  $f : K \otimes A \rightarrow B$ . As usual,  $s_{\text{CI}}(\mathbb{C})$  is fibred over  $\mathbf{Comon}(\mathbb{C})$  via the first projection. Its is a *split* fibration since  $\text{CI}(\mathbb{C})$  is strict.

**The Fibration  $\text{DialZ}$ .** The fibred category  $\text{DialZ}$  is a full subcategory of  $s_{\text{CI}}(\mathbf{DZ}_D)$ . It consists in the fibers of  $s_{\text{CI}}(\mathbf{DZ}_D)$  over commutative comonoids of the form  $\Sigma = (\Sigma, \mathbf{1})$  with structure maps given, modulo  $\mathcal{S}$ -isomorphism, by

$$\begin{aligned} e_\Sigma &= (\mathbf{1}, \mathbf{1}) : (\Sigma, \mathbf{1}) \mapsto (\mathbf{1}, D) \\ \delta_\Sigma &= (\langle \text{id}, \text{id} \rangle, \mathbf{1}) : (\Sigma, \mathbf{1}) \mapsto (\Sigma \times \Sigma, D) \end{aligned}$$

Note that  $\Sigma \otimes (U, V) \simeq (\Sigma \times U, V)$  in  $\mathbf{DZ}_D$ .

Explicitly, the objects of  $\text{DialZ}$  are triples  $(\Sigma, U, X)$  of finite non-empty sets. Its morphisms from  $(\Sigma, U, X)$  to  $(\Gamma, V, Y)$  are triples  $(t, f, F)$  consisting of an  $\mathcal{S}$ -map  $t : \Sigma \times \blacktriangleright D \rightarrow \Gamma$ , together with a  $\mathbf{DZ}_D$ -map  $(f, F) : (\Sigma \times U, X) \rightarrow (V, Y)$ , that is

$$(f, F) : (\Sigma \times U, X^{\Sigma \times U}) \mapsto (V^{\blacktriangleright(Y \times D)}, Y \times D)$$

Note that  $\mathbf{Comon}(\mathbf{DZ}_D)$ -morphisms  $\Gamma \rightarrow \Sigma$  are in 1-to-1 with  $\mathbf{T}$ -maps  $\Gamma \rightarrow \Sigma$ , so that  $\text{DialZ}$  is fibred over  $\mathbf{T}$ .

We write  $\text{DialZ}_\Sigma$  for the fibre of  $\text{DialZ}$  over  $\Sigma$ . Its objects have the form  $(\Sigma, U, X)$ , and a  $\text{DialZ}_\Sigma$ -morphism from  $(\Sigma, U, X)$  to  $(\Sigma, V, Y)$  is a  $\mathbf{DZ}_D$ -map  $(f, F) : (\Sigma \times U, X) \rightarrow (V, Y)$ .

**Symmetric Monoidal Closed Structure.** The fibers of  $\text{DialZ}$  are symmetric monoidal closed. The monoidal product  $\otimes$  acts as the



monoidal product of  $\mathbf{DZ}_D$  on objects (that is as that of  $\mathbf{DZ}$ ). On maps, given  $(U_i, X_i), (V_i, Y_i)$  for  $i \in \{1, 2\}$  and

$$(f_i, F_i) : (\Sigma \times U_i, X_i^{U_i}) \dashrightarrow (V_i^{\blacktriangleright(Y_i \times D)}, Y_i \times D)$$

let  $(f_1, F_2) \otimes (f_2, F_2)$  be  $(h, H)$  where

$$\begin{aligned} h(a, \bar{u}, (\bar{y}, p)) &:= (f_1(a, u_1, y_1, p), f_2(a, u_2, y_2, p)) \\ H(a, \bar{u}, (\bar{y}, p)) &:= (F_1(a, u_1, y_1, p), F_2(a, u_2, y_2, p)) \end{aligned}$$

Substitution functors are strong monoidal, with structure maps given by the structure maps  $(e_\Sigma, d_\Sigma)$  of  $\Sigma = (\Sigma, \mathbf{1})$ .

Monoidal closure follows from

$$\begin{aligned} \text{DialZ}_\Sigma[A \otimes B, C] &= \mathbf{DZ}_D[\Sigma \otimes A \otimes B, C] \\ &\simeq \mathbf{DZ}_D[\Sigma \otimes A, B \multimap_{\mathbf{DZ}_D} C] \end{aligned}$$

### 5.3 The Fibered Category DialAut

Consider an automaton on  $\Sigma$ , say

$$\mathcal{A} = (Q_{\mathcal{A}}, q_{\mathcal{A}}^i, U, X, \delta_{\mathcal{A}}, \Omega_{\mathcal{A}})$$

We have seen that the sets  $U$  and  $X$  of P and O-labels generate a positive full game  $(U, X)$ , and that  $\mathcal{A}$  can be seen as inducing the DialZ-object  $(\Sigma, U, X)$ . In order to deal with acceptance, we have handle its transition function  $\delta_{\mathcal{A}}$ .

We see the evaluation of  $\mathcal{A}$  on a  $\Sigma$ -labeled tree  $t$  as producing an  $\mathcal{S}$ -map (we here explicit the  $(-)^*$ )

$$\alpha(t) : U^* \times X^* \times D^* \longrightarrow Q_{\mathcal{A}}^*$$

We define  $\alpha(t)_n$  by induction on  $n$  as follows. First, let  $\alpha(t)_0 := \mathbf{1}$ . Assume now that  $\alpha(t)_n(\bar{u}, \bar{x}, p) = \bar{q}_{\mathcal{A}}$ . Let  $q_{\mathcal{A}}$  be the last element of  $\bar{q}_{\mathcal{A}}$ , with the convention that  $q_{\mathcal{A}} := q_{\mathcal{A}}^i$  if  $n = 0$ . Then let

$$\alpha(t)_{n+1}(\bar{u}.u, \bar{x}.x, p.d) := \bar{q}_{\mathcal{A}}.\delta_{\mathcal{A}}(q_{\mathcal{A}}, t(p), u, x, d)$$

More generally, a  $\mathbf{T}$ -morphism  $t : \Gamma \rightarrow \Sigma$  generates a map

$$\alpha(t) : \Gamma \times U \times X \times D \longrightarrow Q_{\mathcal{A}}$$

defined similarly, but with

$$\alpha(t)_{n+1}(\bar{a}.a, \bar{u}.u, \bar{x}.x, p.d) := \bar{q}_{\mathcal{A}}.\delta_{\mathcal{A}}(q_{\mathcal{A}}, t(\bar{a}.a, p), u, x, d)$$

This leads to the category DialAut. Its objects are tuples  $(\Sigma, Q_{\mathcal{A}}, U, X, \alpha)$  where  $\Sigma, Q_{\mathcal{A}}, U, X$  are finite non-empty sets and  $\alpha : \Sigma \times U \times X \times D \rightarrow Q_{\mathcal{A}}$  is an  $\mathcal{S}$ -map. Morphisms from  $(\Sigma, Q_{\mathcal{A}}, U, X, \alpha)$  to  $(\Gamma, Q_{\mathcal{B}}, V, Y, \beta)$  are DialZ-maps from  $(\Sigma, U, X)$  to  $(\Gamma, V, Y)$ .

If  $t$  is a  $\mathbf{T}$ -map  $\Gamma \rightarrow \Sigma$ , and  $\mathcal{A}$  is as above, we write  $\mathcal{A}(t)$  for the DialAut-object  $(\Gamma, Q_{\mathcal{A}}, U, X, \alpha(t))$ , and simply  $\mathcal{A}$  for  $\mathcal{A}(\text{id}_\Sigma)$ .

**Fibred Structure.** DialAut is fibred over  $\mathbf{T}$ . The lifting of  $A = (\Sigma, Q_{\mathcal{A}}, U, X, \alpha)$  along  $t : \Gamma \rightarrow \Sigma$  is given by  $t^*(A) := (\Gamma, Q_{\mathcal{A}}, U, X, t^*(\alpha))$  with  $t^*(\alpha)(b, u, x, p) = \alpha(t(b, \blacktriangleright p), u, x, p)$ .

The Cartesian map  $\bar{t}(A) : t^*(A) \rightarrow A$  is induced by lifting along  $t$  in DialZ (that is, in  $\int \text{CI}(\mathbf{DZ})$ ) so that  $\bar{t}(A) := (t, \text{id})$  where  $\text{id}$  is the DialZ $_{\Gamma}$ -identity. This leads to substitution functors  $t^*$  whose actions on maps are given by their actions in DialZ.

**Morphisms as Strategies** We identify fibre DialAut-maps with strategies. Let  $A = (\Lambda, Q_{\mathcal{A}}, U, X, \alpha)$  and  $B = (\Lambda, Q_{\mathcal{B}}, V, Y, \beta)$  be DialAut $_{\Lambda}$ -objects, and consider a DialAut $_{\Lambda}$ -morphism

$$(f, F) : (\Lambda \times U, X^{\Lambda \times U}) \dashrightarrow (V^{\blacktriangleright(Y \times D)}, Y \times D)$$

We extend  $(f, F)$  to a  $\mathbf{DZ}$ -map

$$\begin{aligned} (\tilde{f}, \tilde{F}) &: (\Lambda \times U, (X \times D)^{\Lambda \times U}) \dashrightarrow \\ &((\Lambda \times V)^{\blacktriangleright(Y \times D)}, Y \times D) \end{aligned}$$

where  $\tilde{f}$  is the identity on  $\Lambda$  and  $\tilde{F}$  is the identity on  $D$ . Prop. 4.1 gives a total  $D$ -synchronous strategy  $\sigma : \mathfrak{D}_{\Lambda}(A) \multimap \mathfrak{D}_{\Lambda}(B)$  where  $\mathfrak{D}_{\Lambda}(A)$  is the positive full game  $(\Lambda \times U, X \times D)$ , and similarly for  $\mathfrak{D}_{\Lambda}(B)$ . We identify  $(f, F)$  with  $\sigma$ .

**Winning.** Given automata  $\mathcal{A}$  and  $\mathcal{B}$  generating the  $A$  and  $B$  above, let  $\mathcal{W}_{\mathcal{A}}$  be the set of infinite sequences of moves  $\pi$  such that  $\pi(0) \cdots \pi(n) \in \wp_{\mathcal{A}}^+$  for all  $n$  and such that the projection of  $\pi$  on  $(\Lambda \times U \times X \times D)^{\omega}$  composed with  $\alpha$  belongs to  $\Omega_{\mathcal{A}}$ . Define  $\mathcal{W}_{\mathcal{B}}$  similarly. We then say that  $(f, F)$  as above is *winning* if  $\sigma$  is winning.

Since winning total strategies form a category, it follows from Cor. 4.3 that winning maps compose. Moreover, winning is preserved by substitution functors  $t^*$ , since, writing  $t^*(\sigma)$  for the strategy corresponding to  $t^*(f, F)$ , all infinite sequences induced from  $t^*(\sigma)$  are induced from  $\sigma$ .

**The Fibration Aut.** Let Aut be the subcategory of DialAut whose objects are of the form  $\mathcal{A} = \mathcal{A}(\text{id})$ , and whose morphisms are DialAut-maps  $(t, f, F)$  where  $t : \Gamma \rightarrow \Sigma$  is induced by a  $\mathbf{FinSet}$ -function  $\psi : \Gamma \rightarrow \Sigma$ . Then Aut is fibred over  $\mathbf{FinSet}$ . With the same notations as above, the lifting of  $\mathcal{A}$  along  $\psi : \Gamma \rightarrow \Sigma$  is the automaton  $\mathcal{A}[\psi]$  on  $\Gamma$  whose transition function is given by  $\delta_{\mathcal{A}[\psi]}(q, b, u, x, d) := \delta_{\mathcal{A}}(q, \psi(b), u, x, d)$ . Let  $\text{Aut}_{\Sigma}^{\mathbf{W}}$  be the subcategory of  $\text{Aut}_{\Sigma}$  restricted to winning maps.

**Acceptance.** We now define when an automaton accepts an input tree. First, define the *unit automaton*  $\mathbf{I}$  on  $\Sigma$  as  $(\mathbf{1}, \bullet, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}^{\omega})$ . (where, as usual, we use  $\mathbf{1}$  to denote both the singleton set and any map to it). Note that all  $\mathbf{T}$ -maps  $\mathbf{1} \rightarrow \Sigma$  generate the same DialAut $_{\mathbf{1}}$ -object  $\mathbf{I}(t) = (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$ .

Then  $\mathcal{A}$  *accepts*  $t$  if there is a winning DialAut $_{\mathbf{1}}$ -map from  $\mathbf{I}(t)$  to  $\mathcal{A}(t)$ . We let  $\mathcal{L}(\mathcal{A})$ , the *language* of  $\mathcal{A}$ , be the set of  $\Sigma$ -labeled trees accepted by  $\mathcal{A}$ .

Note that the positive full game generated by  $\mathbf{I}$  on alphabet  $\mathbf{1}$  is of the form  $(\mathbf{1}, \mathbf{1})$ . It follows that  $\mathcal{A}$  accepts  $t$  iff  $\mathbf{P}$  has winning strategy in  $\mathfrak{D}_{\mathbf{1}}(\mathcal{A}(t))$  with winning condition generated from  $\alpha(t)$ .

**Correctness of DialAut.** Our categories DialAut $_{(-)}$  are correct in the following sense:

**Proposition 5.2.** *Assume given  $\mathcal{A}$  and  $\mathcal{B}$  as above, where  $\mathcal{B}$  has also input alphabet  $\Sigma$ .*

*If there is a winning map  $(f, F) : \mathcal{A} \rightarrow \mathcal{B}$  then  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ .*

### 5.4 Logical Connectives

**Symmetric Monoidal Closed Structure of DialAut.** The fibred symmetric monoidal closed structure of DialZ extends to DialAut.

Consider DialAut $_{\Sigma}$  objects  $A = (\Sigma, Q_{\mathcal{A}}, U, X, \alpha)$  and  $B = (\Sigma, Q_{\mathcal{B}}, V, Y, \beta)$ . Following the structure in DialZ, let

$$\begin{aligned} A \otimes B &:= (\Sigma, Q, U \times V, X \times Y, \alpha \sqcap \beta) \\ A \multimap B &:= (\Sigma, Q, V^U \times X^{U \times Y \times D}, U \times Y, \alpha \sqcup \beta) \end{aligned}$$

where  $Q := Q_{\mathcal{A}} \times Q_{\mathcal{B}}$ , and  $\alpha \sqcap \beta$  and  $\alpha \sqcup \beta$  are the Cartesian products of  $\alpha$  and  $\beta$  precomposed respectively with projections and with evaluation maps.

**Proposition 5.3.** *The fibration DialAut is symmetric monoidal closed.*

**Propositional Connectives on Automata.** Consider automata  $\mathcal{A}$  and  $\mathcal{B}$  on  $\Sigma$ . Using the same notations as above, we now define

$$\begin{aligned} \mathcal{A} \otimes \mathcal{B} &:= (Q, q, U \times V, X \times Y, \delta_{\mathcal{A} \otimes \mathcal{B}}, \Omega_{\mathcal{A} \otimes \mathcal{B}}) \\ \mathcal{A} \multimap \mathcal{B} &:= (Q, q, W, Z, \delta_{\mathcal{A} \multimap \mathcal{B}}, \Omega_{\mathcal{A} \multimap \mathcal{B}}) \end{aligned}$$

where  $Q := Q_{\mathcal{A}} \times Q_{\mathcal{B}}$ ,  $q := (q_{\mathcal{A}}^i, q_{\mathcal{B}}^i)$ ,  $W := V^U \times X^{U \times Y \times D}$  and  $Z := U \times Y$ . First, let  $\Omega_{\mathcal{A} \otimes \mathcal{B}}$  consist of the infinite sequences  $\rho$  such that  $\rho|_{Q_{\mathcal{A}}} \in \Omega_{\mathcal{A}}$  and  $\rho|_{Q_{\mathcal{B}}} \in \Omega_{\mathcal{B}}$ . Similarly,  $\Omega_{\mathcal{A} \multimap \mathcal{B}}$  consists of the  $\rho$  such that  $\rho|_{Q_{\mathcal{A}}} \in \Omega_{\mathcal{A}}$  implies  $\rho|_{Q_{\mathcal{B}}} \in \Omega_{\mathcal{B}}$ .

For the transition function of  $\mathcal{A} \otimes \mathcal{B}$ , let

$$\delta_{\mathcal{A} \otimes \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), a, (u, v), (x, y), d) := (q'_{\mathcal{A}}, q'_{\mathcal{B}})$$

where  $\delta_{\mathcal{A}}(q_{\mathcal{A}}, a, u, x, d) = q'_{\mathcal{A}}$  and  $\delta_{\mathcal{B}}(q_{\mathcal{B}}, a, v, y, d) = q'_{\mathcal{B}}$ .

For  $\mathcal{A} \multimap \mathcal{B}$ , let

$$\delta_{\mathcal{A} \multimap \mathcal{B}}((q_{\mathcal{A}}, q_{\mathcal{B}}), a, (f, F), (u, y), d) := (q'_{\mathcal{A}}, q'_{\mathcal{B}})$$

if  $\delta_{\mathcal{A}}(q_{\mathcal{A}}, a, u, F(u, y, d), d) = q'_{\mathcal{A}}$  and  $\delta_{\mathcal{B}}(q_{\mathcal{B}}, a, f u, y, d) = q'_{\mathcal{B}}$ .

Note that the  $\text{DialAut}_{\Sigma}$ -objects  $\mathcal{A} \multimap \mathcal{B}$ , build either in  $\text{DialAut}_{\Sigma}$  from automata  $\mathcal{A}$  and  $\mathcal{B}$  or from automaton  $\mathcal{A} \multimap \mathcal{B}$  coincide (and similarly for  $\otimes$ ).

In order to define  $\mathcal{A} \otimes \mathcal{B}$  and  $\mathcal{A} \multimap \mathcal{B}$  when  $\mathcal{A}$  and  $\mathcal{B}$  have different input alphabets, say  $\Sigma$  and  $\Gamma$ , first lift  $\mathcal{A}$  and  $\mathcal{B}$  to  $\mathcal{A}[\pi]$  and  $\mathcal{B}[\pi']$  in  $\text{Aut}_{\Sigma \times \Gamma}$  and then do the construction. In particular, referring to the notations of §1,  $\mathcal{A}(t) \multimap \mathcal{B}(u)$  is formed as  $(\mathcal{A}[\pi] \multimap \mathcal{B}[\pi'])((t, u))$  (and similarly for  $\otimes$ ).

**Proposition 5.4.** *The fibration  $\text{Aut}$  is symmetric monoidal closed.*

There are winning fibre projection maps, say  $(f, F) : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$

$$(f, F) : (U \times V, (X \times Y)^{U \times V}) \dashrightarrow (U^{\blacktriangleright(X \times D)}, X \times D)$$

where  $F$  is the identity on  $X$  and plays any  $y \in Y$  (recall that  $Y$  is required to be non-empty), and  $f$  is the identity on  $U$ .

**Proposition 5.5.**  $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$ .

**Complementation and Falsity.** The *falsity automaton*  $\perp$  on  $\Sigma$  is  $(\mathbb{B}, \text{ff}, D, \mathbf{1}, \delta_{\perp}, \Omega_{\perp})$  where  $\Omega_{\perp}$  is the set of sequences containing infinitely many  $\text{tt}$ 's, and the transition function  $\delta_{\perp}$  is defined as follows: let  $\delta_{\perp}(\text{tt}, -, d', \bullet, d) := \text{tt}$ , and

$$\delta_{\perp}(\text{ff}, -, d', \bullet, d) := \begin{cases} \text{ff} & \text{if } d = d' \\ \text{tt} & \text{otherwise} \end{cases}$$

Note that  $\perp$  accepts no tree since in an acceptance game,  $\text{O}$  can always play the same  $d$  as  $\text{P}$ . Given an automaton  $\mathcal{A}$  on  $\Sigma$ , let  $\mathcal{A}^{\perp} := \mathcal{A} \multimap \perp$ . The automaton  $\mathcal{A}^{\perp}$  can be seen as

$$(Q_{\mathcal{A}} \times \mathbb{B}, (q_{\mathcal{A}}, \text{ff}), D^U \times X^{U \times D}, \delta_{\mathcal{A}^{\perp}}, \Omega_{\mathcal{A}^{\perp}} \multimap \perp)$$

with  $\delta_{\mathcal{A}^{\perp}}(a, (q_{\mathcal{A}}, \text{ff}), (f, F), u, d) = (q'_{\mathcal{A}}, \text{b})$  where  $\text{b} = \text{ff}$  iff  $f(u) = d$ , and  $\delta_{\mathcal{A}^{\perp}}(a, (q_{\mathcal{A}}, \text{tt}), (f, F), u, d) = (q'_{\mathcal{A}}, \text{tt})$ , where  $q'_{\mathcal{A}} := \delta_{\mathcal{A}}(a, q_{\mathcal{A}}, u, F(u, d), d)$ . Hence  $\text{O}$  loses as soon as he does not follow the direction proposed by  $\text{P}$  via  $f$ . Thanks to the determinacy of Borel games [21], we get:

**Proposition 5.6.** *If  $\Omega_{\mathcal{A}}$  is Borel, then  $t \in \mathcal{L}(\mathcal{A}^{\perp})$  iff  $t \notin \mathcal{L}(\mathcal{A})$ .*

**Existential Quantification.** A fibration  $\text{p} : \mathbb{E} \rightarrow \mathbb{B}$  has existential quantifications (also called simple coproducts [13]) when the weakening functors  $\pi^* : \mathbb{E}_J \rightarrow \mathbb{E}_{I \times J}$  (induced by the  $\mathbb{B}$ -projections  $\pi : I \times J \rightarrow I$ ) have left adjoints  $\coprod_{I, J} : \mathbb{E}_{I \times J} \rightarrow \mathbb{E}_I$  satisfying some coherence conditions, called the *Beck-Chevalley* conditions, insuring that the adjunction  $\coprod_{I, J} \dashv \pi^*$  is preserved by substitution (see e.g. [13]). This allows to interpret the existential rules of §1.

The simple fibration  $\text{s} : \text{s}(\mathbb{B}) \rightarrow \mathbb{B}$  has simple coproducts (see e.g. [13, Prop. 1.9.3]). They are induced by

$$\coprod_{I, J} (I \times J, X) := (I, J \times X)$$

So, for  $\mathbb{C}$  symmetric monoidal,  $\text{s}_{\text{CI}} : \text{s}_{\text{CI}}(\mathbb{C}) \rightarrow \mathbf{Comon}(\mathbb{C})$  has coproducts induced, recalling that  $\mathbf{Comon}(\mathbb{C})$  is Cartesian, by

$$\coprod_{I, J} (I \times J, X) := (I, J \otimes X)$$

This leads in  $\text{DialZ}$  to  $\coprod_{\Sigma, \Gamma} (\Sigma \times \Gamma, U, X) := (\Sigma, \Gamma \times U, X)$ .

In  $\text{DialAut}$ , the action on objects of  $\coprod_{\Sigma, \Gamma}$  is given by

$$\coprod_{\Sigma, \Gamma} (\Sigma \times \Gamma, Q_{\mathcal{A}}, U, X, \alpha) := (\Sigma, Q_{\mathcal{A}}, \Gamma \times U, X, \coprod_{\Sigma, \Gamma} \alpha)$$

where  $\coprod_{\Sigma, \Gamma} \alpha(a, (b, u), x, d) := \alpha((a, b), u, x, d)$ . Its action on maps is inherited from  $\text{s}_{\text{CI}}(\mathbf{DZ}_D)$ . It follows that the Beck-

Chevalley is reduced to, for  $t : \Lambda \rightarrow \Sigma$ ,

$$\coprod_{\Lambda, \Gamma} (t \times \text{id})^*(A) = t^*(\coprod_{\Sigma, \Gamma} A)$$

which follows from the fact that  $\coprod_{\Lambda, \Gamma} (t \times \text{id})^*(\alpha) = t^*(\coprod_{\Sigma, \Gamma} \alpha)$ .

Given an automaton  $\mathcal{A}$  on  $\Sigma \times \Gamma$ , we let

$$\exists_{\Gamma} \mathcal{A} := (Q_{\mathcal{A}}, q_{\mathcal{A}}, (\Gamma \times U), X, \delta_{\exists_{\Gamma} \mathcal{A}}, \Omega_{\mathcal{A}})$$

where  $\delta_{\exists_{\Gamma} \mathcal{A}}(q_{\mathcal{A}}, a, (b, u), x, d) := \delta_{\mathcal{A}}(q_{\mathcal{A}}, (a, b), u, x, d)$ , so that, as  $\text{Aut}$ -objects,  $\coprod_{\Sigma, \Gamma} \mathcal{A} = \exists_{\Gamma} \mathcal{A}$ .  $\exists_{\Gamma} \mathcal{A}$  is essentially the *lifted projection* of [26].

**Proposition 5.7.** *The fibrations  $\text{DialAut}$  and  $\text{Aut}$  have existential quantifications.*

## 6. Non-Deterministic Automata

**Regular and Parity Automata.** An automaton  $\mathcal{A}$  is  $(\omega)$ -regular if  $\Omega_{\mathcal{A}}$  is an  $\omega$ -regular set (see e.g. [6, 24, 28]). *Parity* automata are  $\omega$ -regular automata  $\mathcal{A}$  such that  $\Omega_{\mathcal{A}}$  is generated from a map  $c : Q_{\mathcal{A}} \rightarrow \mathbb{N}$  as the set of infinite sequences  $q = (q_n)_n$  such that the least  $k$  occurring infinitely often in the sequence  $c \circ q$  is even. It is well-known (see e.g. [6, 24, 28]) that every  $\Omega$ -regular language  $L$  can be recognized by a deterministic  $\omega$ -word parity automaton  $(\Pi(L), q_{\Pi(L)}^i, \delta_{\Pi(L)}, c_{\Pi(L)})$ .

Given a regular tree automata  $\mathcal{A}$  on  $\Sigma$ , let

$$\mathcal{A}^{\dagger} := (Q_{\mathcal{A}} \times \Pi(L), (q_{\mathcal{A}}^i, q_{\Pi(L)}^i), U, X, \delta_{\mathcal{A}^{\dagger}}, \Omega_{\mathcal{A}^{\dagger}})$$

where  $L = \Omega_{\mathcal{A}}, \Omega_{\mathcal{A}^{\dagger}}$  is generated from  $c_{\Pi(L)}$  (via second projection) and the transition function  $\delta_{\mathcal{A}^{\dagger}}$  is defined as

$$\delta_{\mathcal{A}^{\dagger}}((q_{\mathcal{A}}, q_{\Pi(L)}), a, u, x, d) := (q'_{\mathcal{A}}, \delta_{\Pi(L)}(q_{\Pi(L)}, q'_{\mathcal{A}}))$$

with  $q'_{\mathcal{A}} := \delta_{\mathcal{A}}(q_{\mathcal{A}}, a, u, x, d)$ . Note that  $\mathcal{A} \simeq \mathcal{A}^{\dagger}$  in  $\text{Aut}_{\Sigma}^{\text{W}}$ .

**Positionality.** Consider  $\text{DialAut}_{\Sigma}$  objects  $A$  and  $B$ . An even-length play  $s$  on  $\partial_{\Sigma}(A)$  can be mapped (via the  $\partial^{-1}$  map of §4) to some  $(\bar{a}, \bar{u}, \bar{x}, p) \in \Sigma^n \times U^n \times X^n \times D^n$ , which in turn can be mapped via  $\alpha$  to some  $(\bar{q}_{\mathcal{A}}, p)$ . Let  $q_{\mathcal{A}}$  be the last element of  $\bar{q}_{\mathcal{A}}$  (with  $q_{\mathcal{A}} = q_{\mathcal{A}}^i$  if  $n = 0$ ). We then let the *position* of  $s$  be  $\text{pos}(s) := (p, q_{\mathcal{A}})$ . Given a  $4n$ -length play  $s$  on  $\partial_{\Sigma}(A) \multimap \partial_{\Sigma}(B)$ , we let  $\text{pos}(s) := (\text{pos}(s|A), \text{pos}(s|B))$ . We extend  $\text{pos}$  to arbitrary plays by taking the position of the longest prefix.

We say that a strategy  $\sigma$  is *positional* if it agrees on plays with the same position, i.e. if  $s.m \in \sigma, t.m' \in \sigma$  with  $\text{pos}(s) = \text{pos}(t)$  implies  $m = m'$ . It is well-known (see e.g. [6, 24, 28]) that if  $\mathcal{A}$  is a parity automaton, then either  $\text{P}$  or  $\text{O}$  has a winning positional strategy in  $\partial_1(\mathcal{A}(t))$ .

Consider now parity automata  $\mathcal{A}_1, \dots, \mathcal{A}_n$  and  $\mathcal{B}$ . then the acceptance condition of  $\mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_n \multimap \mathcal{B}$  is a disjunction of parity conditions, also called a *Rabin* condition. It has been shown in [15–17, 30] that if  $\text{P}$  has a winning strategy  $\sigma$  in such a game, then he has a winning *positional* strategy, which according to [30] is recursive in  $\sigma$ .

**Non-Deterministic Automata.** An automaton  $\mathcal{A}$  is said to be *non-deterministic* if  $X = \mathbf{1}$ . Note that  $\mathbf{I}$  and  $\perp$  are non-deterministic, and that if  $\mathcal{A}$  and  $\mathcal{B}$  are non-deterministic, then so is  $\mathcal{A} \otimes \mathcal{B}$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are non-deterministic, then a fibre map  $(f, F) : \mathcal{A} \rightarrow \mathcal{B}$  is uniquely determined by  $f$ , which moreover can be seen as a map  $\Sigma \times U \times \blacktriangleright D \rightarrow V$ . In particular, the monoidal product  $\otimes$  is Cartesian (with unit  $\mathbf{I}$ ) on non-deterministic automata.

**Non-Determinization.** Given a parity automaton  $\mathcal{A}$ , by adapting the construction of [29], we let

$$! \mathcal{A} := (Q_{! \mathcal{A}}, q_{! \mathcal{A}}^i, U^{Q_{\mathcal{A}}}, \mathbf{1}, \delta_{! \mathcal{A}}, \Omega_{! \mathcal{A}})$$

where  $Q_{!A} := \mathcal{P}(Q_A \times Q_A)$ ,  $q_{!A}^i := \{(q_A^i, q_A^i)\}$  and the transition function  $\delta_{!A}$  is defined as follows: Given  $a \in \Sigma$ ,  $f \in U^{Q_A}$ ,  $d \in D$  and  $S = \{(-, q_1), \dots, (-, q_n)\} \in Q_{!A}$ , let

$$\delta_{!A}(S, a, f, \bullet, d) := T_1 \cup \dots \cup T_n$$

where, for each  $k \in \{1, \dots, n\}$ ,

$$T_k := \{(q_k, q) \mid \exists x \in X. q = \delta_A(q_k, a, f(q_k), x, d)\}$$

Let a trace in an infinite sequence  $(S_n)_n \in Q_{!A}^\omega$  be a sequence  $(q_n)_n$  such that for all  $n$ ,  $(q_n, q_{n+1}) \in S_{n+1}$ . We let  $\Omega_{!A}$  be the set of sequences  $(S_n)_n$  whose traces all belong to  $\Omega_A$ . Note that  $\Omega_{!A}$  is  $\omega$ -regular since  $\Omega_A$  is  $\omega$ -regular.

If  $\mathcal{A}$  is a regular automaton, we let  $!A := !(A^\dagger)$ .

**Interpretation of the  $!$  Rules.** We now discuss the interpretation of the rules (1) of §1. The first rule (called *Promotion*) follows from:

**Proposition 6.1.** *Given a regular non-deterministic automaton  $\mathcal{N}$  and a regular automaton  $\mathcal{A}$ , if there is a winning map  $\sigma : \mathcal{N} \rightarrow \mathcal{A}$  then there is a winning map  $\tau : \mathcal{N} \rightarrow !\mathcal{A}$ .*

Prop. 6.1 relies on the existence of positional winning strategies in Rabin games. The second rule (called *Deriliction*) is given by:

**Proposition 6.2.** *If  $\mathcal{A}$  is regular, there is a winning  $\eta_{\mathcal{A}} : !\mathcal{A} \rightarrow \mathcal{A}$ .*

**Corollary 6.3.**  $\mathcal{L}(\mathcal{A}) = \mathcal{L}(!\mathcal{A})$  for a regular  $\mathcal{A}$ .

The last two rules (*Weakening* and *Contraction*) follow from the fact that  $!A$  is non-deterministic.

**Weak Completeness.** Let  $\pi : \Sigma \times \Gamma \rightarrow \Sigma$  be the **Set**-projection. Given a non-deterministic  $\mathcal{N}$ , a  $\text{DialAut}_1$ -map  $\mathbf{I}(t) \rightarrow (\exists_{\Sigma} \mathcal{N})(t)$  is given by an  $\mathcal{S}$ -map  $f : \blacktriangleright D \rightarrow \Gamma \times U$ , hence by a pair  $\langle t', g \rangle$  where  $g$  gives a  $\text{DialAut}_1$ -map  $\mathbf{I}(\langle t, t' \rangle) \rightarrow \mathcal{N}(\langle t, t' \rangle)$ .

**Proposition 6.4.**  $\mathcal{L}(\exists_{\Gamma} \mathcal{N}) = \pi(\mathcal{L}(\mathcal{N}))$  for  $\mathcal{N}$  non-deterministic.

**Proposition 6.5** (Weak Completeness). *Given automata  $\mathcal{A}$  and  $\mathcal{B}$  on  $\Sigma$ , if  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$  then there is an effective winning  $\text{Aut}_{\Sigma}$ -map  $!A \rightarrow !(B^\perp)^\perp$ .*

## 7. Conclusion

We proposed fibered monoidal closed categories of tree automata. They handle their basic constructs (closure under Boolean operations and equivalence with non-deterministic automata). Our model is based on games, which provide a realizability semantics for tree automata. Further work will include the interpretation of deduction systems for MSO.

## References

[1] S. Abramsky. Semantics of Interaction. In A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, volume 14 of *Publications of the Newton Institute*, page 1. Cambridge University Press, 1997. 3, 4

[2] J. Avigad and S. Feferman. Gödel’s functional (“Dialectica”) interpretation. In S. Buss, editor, *Handbook Proof Theory*, volume 137 of *Studies in Logic and the Foundations of Mathematics*, pages 337–405. Elsevier, 1998. 1, 3

[3] L. Birkedal, R. E. Møgelberg, J. Schwinghammer, and K. Støvring. First steps in synthetic guarded domain theory: step-indexing in the topos of trees. *Logical Methods in Computer Science*, 8(4), 2012. 5

[4] V. de Paiva. The Dialectica categories. Technical Report 213, University of Cambridge Computer Laboratory, January 1991. 1, 4, 6

[5] J.-Y. Girard. Linear Logic. *Theoretical Computer Science*, 50:1–102, 1987. 1

[6] E. Grädel, W. Thomas, and T. Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research*, volume 2500 of *LNCS*, 2002. Springer. 1, 9

[7] R. Harmer, M. Hyland, and P.-A. Mellès. Categorical combinatorics for innocent strategies. In *LICS 2007*, pages 379–388, 2007. 6, 11

[8] P. J. W. Hofstra. The dialectica monad and its cousins. In M. Makkai and B. Hart, editors, *Models, Logics, and Higher-dimensional Categories: A Tribute to the Work of Mihály Makkai*, CRM proceedings & lecture notes. American Mathematical Society, 2011. 1, 7

[9] J. M. E. Hyland. Game Semantics. In A. M. Pitts and P. Dybjer, editors, *Semantics and Logics of Computation*, volume 14 of *Publications of the Newton Institute*, page 131. Cambridge University Press, 1997. 3, 4

[10] J. M. E. Hyland. Proof theory in the abstract. *Ann. Pure Appl. Logic*, 114(1-3):43–78, 2002. 1, 4, 7

[11] J. M. E. Hyland and A. Schalk. Abstract Games for Linear Logic. *Electr. Notes Theor. Comput. Sci.*, 29:127–150, 1999. 3, 6, 7, 15

[12] J. M. E. Hyland and A. Schalk. Glueing and orthogonality for models of linear logic. *Theoretical Computer Science*, 294(1/2):183–231, 2003. 4, 6, 7

[13] B. Jacobs. *Categorical Logic and Type Theory*. Studies in logic and the foundations of mathematics. Elsevier, 2001. 7, 9

[14] P. Johnstone. *Stone Spaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986. 2

[15] C. S. Jutla. Determinization and Memoryless Winning Strategies. *Inf. Comput.*, 133(2):117–134, 1997. 9, 17

[16] K. Klarlund and D. Kozen. Rabin Measures. *Chicago J. Theor. Comput. Sci.*, 1995, 1995.

[17] N. Klarlund. Progress measures, immediate determinacy, and a subset construction for tree automata. *Annals of Pure and Applied Logic*, 69(2-3):243–268, 1994. 9, 17

[18] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Springer, 2008. 1, 3

[19] S. Mac Lane. *Categories for the Working Mathematician*. Springer, 2nd edition, 1998. 15

[20] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer, 1992. 5

[21] D. A. Martin. Borel Determinacy. *The Annals of Mathematics, Second Series*, 102(2):363–371, 1975. 9, 16

[22] P.-A. Mellès. Categorical semantics of linear logic. In *Interactive models of computation and program behaviour*, volume 27 of *Panoramas et Synthèses*. SMF, 2009. 2, 6, 7, 15

[23] D. E. Muller and P. E. Schupp. Simulating Alternating Tree Automata by Nondeterministic Automata: New Results and New Proofs of the Theorems of Rabin, McNaughton and Safra. *Theor. Comput. Sci.*, 141(1&2):69–107, 1995. 1, 2

[24] D. Perrin and J.-É. Pin. *Infinite Words: Automata, Semigroups, Logic and Games*. Pure and Applied Mathematics. Elsevier, 2004. 1, 9, 17

[25] M. O. Rabin. Decidability of Second-Order Theories and Automata on Infinite Trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969. 1

[26] C. Riba. Fibrations of tree automata. In *TLCA*, volume 38 of *LIPICs*, pages 302–316. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2015. 1, 2, 3, 4, 9, 15

[27] T. Streicher. Direct Semantics is Continuation Semantics, Linearly. Unpublished notes. 2

[28] W. Thomas. Languages, Automata, and Logic. In G. Rozenberg and A. Salomaa, editors, *Handbook of Formal Languages*, volume III, pages 389–455. Springer, 1997. 1, 9

[29] I. Walukiewicz. Monadic second-order logic on tree-like structures. *Theor. Comput. Sci.*, 275(1-2):311–346, 2002. 1, 2, 9, 16

[30] Z. Zielonka. Infinite games on finitely coloured graphs with applications to automata on infinite trees. *Theoretical Computer Science*, 200(1-2):135–183, 1998. 9, 17

## A. Proofs of Section 4

### A.1 Prop. 4.2: $\lambda$ is a Distributive Law

We have to check that  $\lambda : T((-)^\blacktriangleright) \rightarrow (T-)^{\blacktriangleright}$  is natural and that the following four coherence diagrams commute (see e.g. [7]):

$$\begin{array}{ccccc}
 & & (TA)^{\blacktriangleright} & & \\
 & \nearrow^{\lambda_A} & & \searrow^{(\delta_A)^{\blacktriangleright}} & \\
 T(A^{\blacktriangleright}) & & & & (TTA)^{\blacktriangleright} \\
 & \searrow_{\delta_A^{\blacktriangleright}} & & \nearrow_{\lambda_{TA}} & \\
 & & TT(A^{\blacktriangleright}) & \xrightarrow{T\lambda_A} & T((TA)^{\blacktriangleright})
 \end{array} \tag{6}$$

$$\begin{array}{ccccc}
 & & T(A^{\blacktriangleright}) & & \\
 & \nearrow^{T(\mu_A)} & & \searrow^{\lambda_A} & \\
 T(A^{\blacktriangleright\blacktriangleright}) & & & & (TA)^{\blacktriangleright\blacktriangleright} \\
 & \searrow_{\lambda_A^{\blacktriangleright}} & & \nearrow_{\mu_{TA}} & \\
 & & (T(A^{\blacktriangleright}))^{\blacktriangleright} & \xrightarrow{(\lambda_A)^{\blacktriangleright}} & (TA)^{\blacktriangleright\blacktriangleright}
 \end{array} \tag{7}$$

$$\begin{array}{ccccc}
 & & (TA)^{\blacktriangleright} & & \\
 & \nearrow^{\lambda_A} & & \searrow^{(\epsilon_A)^{\blacktriangleright}} & \\
 T(A^{\blacktriangleright}) & & & & A^{\blacktriangleright} \\
 & \xrightarrow{\epsilon_A^{\blacktriangleright}} & & & 
 \end{array} \tag{8}$$

$$\begin{array}{ccccc}
 & & T(A^{\blacktriangleright}) & & \\
 & \nearrow^{T(\eta_A)} & & \searrow^{\lambda_A} & \\
 TA & & & & (TA)^{\blacktriangleright} \\
 & \xrightarrow{\eta_{TA}} & & & 
 \end{array} \tag{9}$$

Here,  $T$  is the comonad  $T = (T, \epsilon, \delta)$  and  $(-)^{\blacktriangleright}$  is the monad  $((-)^{\blacktriangleright}, \eta, \mu)$  on  $\mathbf{G}(\mathcal{S})$ . First recall the definition of the functors  $T$  and  $(-)^{\blacktriangleright}$ :

$$\begin{array}{ccc}
 (f, F) : (U, X) \rightrightarrows (V, Y) & \xrightarrow{T} & (f, \lambda h.F \circ h \circ f) : (U, X^U) \rightrightarrows (V, Y^V) \\
 (f, F) : (U, X) \rightrightarrows (V, Y) & \xrightarrow{(-)^{\blacktriangleright}} & (\lambda h.f \circ h \circ \blacktriangleright F, F) : (U^{\blacktriangleright X}, X) \rightrightarrows (V^{\blacktriangleright Y}, Y)
 \end{array}$$

and of the maps  $\eta$  and  $\mu$ :

$$\begin{array}{ccc}
 (f_\eta, F_\eta) & : & (U, X) \rightrightarrows (U^{\blacktriangleright X}, X) \\
 (f_\mu, F_\mu) & : & (U^{\blacktriangleright X} \times^{\blacktriangleright X} X, X) \rightrightarrows (U^{\blacktriangleright X}, X)
 \end{array}$$

where  $F_\eta = F_\mu = \text{id}_X$ ,  $f_\eta(u, x) = u$  and  $f_\mu(h, x) = h(x, x)$ .

Moreover, the maps  $\epsilon$  and  $\delta$  are given by

$$\begin{array}{ccc}
 (f_\epsilon, F_\epsilon) & : & (U, X^U) \rightrightarrows (U, X) \\
 (f_\delta, F_\delta) & : & (U, X^{U \times U}) \rightrightarrows (U, X^U)
 \end{array}$$

where  $f_\epsilon = f_\delta = \text{id}_U$ ,  $F_\epsilon(u, x) = x$  and  $F_\delta(h, u) = h(u, u)$ .

It is routine to check that  $((-)^{\blacktriangleright}, \eta, \mu)$  is a monad on  $\mathbf{G}(\mathcal{S})$ .

**Lemma A.1.**  $(-)^{\blacktriangleright} = ((-)^{\blacktriangleright}, \eta, \mu)$  is monad on  $\mathbf{G}(\mathcal{S})$ .

We now check in turn the required diagrams.

**Lemma A.2.**  $\lambda$  is natural, that is, given  $(g, G) : A \rightrightarrows B$ , we have

$$\begin{array}{ccc}
 T(A^{\blacktriangleright}) & \xrightarrow{T((g, G)^{\blacktriangleright})} & T(B^{\blacktriangleright}) \\
 \lambda_A \downarrow & & \downarrow \lambda_B \\
 (TA)^{\blacktriangleright} & \xrightarrow{(T(g, G))^{\blacktriangleright}} & (TB)^{\blacktriangleright}
 \end{array}$$

*Proof.* Let  $A = (U, X)$  and  $B = (V, Y)$ , and consider  $(g, G) : (U, X) \dashrightarrow (V, Y)$ . Note that

$$\begin{aligned} (g, G)^\blacktriangleright &= (\lambda h.gh \blacktriangleright G, G) &: (U^{\blacktriangleright X}, X) &\dashrightarrow (V^{\blacktriangleright Y}, Y) \\ T((g, G)^\blacktriangleright) &= (\lambda h.gh \blacktriangleright G, \lambda h.Gh(\lambda h.gh \blacktriangleright G)) &: (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) &\dashrightarrow (V^{\blacktriangleright Y}, Y^{V^{\blacktriangleright Y}}) \\ T(g, G) &= (g, \lambda h.Ghg) &: (U, X^U) &\dashrightarrow (V, Y^V) \\ (T(g, G))^\blacktriangleright &= (\lambda h.gh \blacktriangleright (\lambda h.Ghg), \lambda h.Ghg) &: (U^{\blacktriangleright (X^U)}, X^U) &\dashrightarrow (V^{\blacktriangleright (Y^V)}, Y^V) \end{aligned}$$

We have to show that

$$(T(g, G))^\blacktriangleright \circ \lambda_A = \lambda_B \circ T((g, G)^\blacktriangleright)$$

that is

$$(\lambda h.gh \blacktriangleright (\lambda h.Ghg)) \circ f^{\lambda_A} = f^{\lambda_B} \circ (\lambda h.gh \blacktriangleright G) \quad \text{and} \quad F^{\lambda_A} \circ (\lambda h.Ghg) = \lambda h.Gh(\lambda h.gh \blacktriangleright G) \circ F^{\lambda_B}$$

For the first equation, which has type  $U^{\blacktriangleright X} \rightarrow V^{\blacktriangleright (Y^V)}$ , given  $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$  and  $\xi_n \in (Y^V)_n$ , one has to show the following (where some  $\circ$  are replaced by juxtaposition)

$$((\lambda h.g_{n+1}h \blacktriangleright (\lambda h.G_{n+1}hg_{n+1})) \circ f_{n+1}^{\lambda_A})(\theta_{n+1})(\xi_n) = (f_{n+1}^{\lambda_B} \circ (\lambda h.g_{n+1}h \blacktriangleright G_{n+1}))(\theta_{n+1})(\xi_n)$$

that is

$$((\lambda h.g_{n+1} \circ h \circ (\lambda h.G_nhg_n)) (f_{n+1}^{\lambda_A}(\theta_{n+1}))) (\xi_n) = (f_{n+1}^{\lambda_B}((\lambda h.g_{n+1} \circ h \circ G_n)(\theta_{n+1}))) (\xi_n)$$

that is

$$(g_{n+1} \circ (f_{n+1}^{\lambda_A}(\theta_{n+1})) \circ (\lambda h.G_nhg_n)) (\xi_n) = (f_{n+1}^{\lambda_B}(g_{n+1}\theta_{n+1}G_n)) (\xi_n)$$

that is

$$g_{n+1}(f_{n+1}^{\lambda_A}(\theta_{n+1}))((\lambda h.G_nhg_n)\xi_n) = f_{n+1}^{\lambda_B}(g_{n+1}\theta_{n+1}G_n, \xi_n)$$

that is

$$g_{n+1}(f_{n+1}^{\lambda_A}(\theta_{n+1}, G_n\xi_ng_n)) = f_{n+1}^{\lambda_B}(g_{n+1}\theta_{n+1}G_n, \xi_n)$$

that is

$$g_{n+1} \circ \theta_{n+1} \circ \text{fix}_n(G_n\xi_n\theta_n) = g_{n+1} \circ \theta_{n+1} \circ G_n \circ \text{fix}_n(\xi_ng_n\theta_nG_{n-1})$$

which is easily seen to hold, when unfolding the fixpoints, thanks to associativity of composition.

The second equation, of type  $Y^V \rightarrow X^{U^{\blacktriangleright X}}$ , amounts, for  $\xi_n \in (Y^V)_n$  and  $\theta_n \in (U^{\blacktriangleright X})_n$ , to the following (where some  $\circ$  are replaced by juxtaposition)

$$F_n^{\lambda_A}(G_n\xi_ng_n, \theta_n) = ((\lambda h.Gh(\lambda h.gh \blacktriangleright G))(F_n^{\lambda_B}(\xi_n)))(\theta_n)$$

that is

$$F_n^{\lambda_A}(G_n\xi_ng_n, \theta_n) = (G_n \circ (F_n^{\lambda_B}(\xi_n)) \circ (\lambda h.g_nh \blacktriangleright G_n))(\theta_n)$$

that is

$$F_n^{\lambda_A}(G_n\xi_ng_n, \theta_n) = G_n(F_n^{\lambda_B}(\xi_n)((\lambda h.g_nh \blacktriangleright G_n)(\theta_n)))$$

that is

$$F_n^{\lambda_A}(G_n\xi_ng_n, \theta_n) = G_n(F_n^{\lambda_B}(\xi_n, g_n\theta_n \blacktriangleright G_n))$$

which also holds thanks to associativity of composition (when unfolding the fixpoints).  $\square$

**Lemma A.3.** *Diagram (6) commutes.*

*Proof.* Let  $A = (U, X)$ , so that

$$T(A^\blacktriangleright) = T(U^{\blacktriangleright X}, X) = (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \quad \text{and} \quad (TA)^\blacktriangleright = (U, X^U)^\blacktriangleright = (U^{\blacktriangleright (X^U)}, X^U)$$

The diagram has type

$$T(A^\blacktriangleright) \dashrightarrow (TTA)^\blacktriangleright = (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \dashrightarrow (U^{\blacktriangleright (X^{U \times U})}, X^{U \times U})$$

Moreover,

$$\begin{aligned} (\delta_A)^\blacktriangleright &= (\text{id}_U, \lambda hu.h(u, u))^\blacktriangleright = (\lambda h.h \blacktriangleright (\lambda hu.h(u, u)), \lambda hu.h(u, u)) \\ T\lambda_A &= T(f^{\lambda_A}, F^{\lambda_A}) = (f^{\lambda_A}, \lambda h.F^{\lambda_A}h.f^{\lambda_A}) \end{aligned}$$

We have to check the following two equations:

$$f_{\delta_A} \circ f^{\lambda_A} = f^{\lambda_{TA}} \circ f_{T\lambda_A} \circ f_{\delta_A} \quad \text{and} \quad F^{\lambda_A} \circ F_{\delta_A} = F_{\delta_A} \circ F_{T\lambda_A} \circ F^{\lambda_{TA}}$$

The first one, of type  $U^{\blacktriangleright X} \rightarrow U^{\blacktriangleright (X^{U \times U})}$ , amounts, for  $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$  and  $\xi_{n+1} \in X_{n+1}^{U \times U}$ , to the following

$$((\lambda h.h \blacktriangleright (\lambda hu.h(u, u))) \circ f_{n+1}^{\lambda_A})(\theta_{n+1})(\xi_{n+1}) = (f_{n+1}^{\lambda_{TA}} f_{n+1}^{\lambda_A})(\theta_{n+1})(\xi_{n+1})$$

that is

$$(f_{n+1}^{\lambda_A}(\theta_{n+1}) \circ \blacktriangleright (\lambda hu.h(u, u)))(\xi_{n+1}) = f_{n+1}^{\lambda_{TA}}(f_{n+1}^{\lambda_A}(\theta_{n+1}), \xi_n)$$

that is

$$f_{n+1}^{\lambda A}(\theta_{n+1}, \lambda u. \xi_n(u, u)) = f_{n+1}^{\lambda A}(\theta_{n+1}, \text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)))$$

Write

$$l_n := f_{n+1}^{\lambda A}(\theta_{n+1}, \lambda u. \xi_n(u, u)) \quad \text{and} \quad r_n := f_{n+1}^{\lambda A}(\theta_{n+1}, \text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)))$$

The proof is then by induction on  $n$ . In the base case  $n = 0$ , both sides unfold to  $\theta_1(\bullet)$ . For the induction step, assuming the property for  $r_n = l_n$ , we show  $l_{n+1} = r_{n+1}$ .

First, note that Note that

$$\begin{aligned} \text{fix}_{n+1}^U(\lambda u. \xi_{n+1}(u, u) \circ \theta_{n+1}) &= \text{fix}_{n+1}^U(\lambda x. \xi_{n+1}(\theta_{n+1}(x), \theta_{n+1}(x))) \\ &= (\lambda x. \xi_{n+1}(\theta_{n+1}(x), \theta_{n+1}(x)))(\text{fix}_n^U(\lambda x. \xi_n(\theta_n(x), \theta_n(x)))) \\ &= (\lambda u. \xi_{n+1}(u, u))(\theta_{n+1}(\text{fix}_n^U((\lambda u. \xi_n(u, u)) \circ \theta_n))) \\ &= \xi_{n+1}(l_n, l_n) \end{aligned}$$

so that

$$l_{n+1} = \theta_{n+2}(\xi_{n+1}(l_n, l_n))$$

On the other hand, note that

$$\begin{aligned} \text{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\lambda A}(\theta_{n+1})) &= \xi_{n+1}(f_{n+1}^{\lambda A}(\theta_{n+1}, \text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)))) \\ &= \xi_{n+1}(r_n) \end{aligned}$$

and so in particular

$$\begin{aligned} r_n &= \theta_{n+1}(\text{fix}_n(\text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)) \circ \theta_n)) \\ &= \theta_{n+1}(\text{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)) \end{aligned}$$

We thus have

$$\begin{aligned} r_{n+1} &= \theta_{n+2}(\text{fix}_{n+1}(\text{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\lambda A}(\theta_{n+1})) \circ \theta_{n+1})) \\ &= \theta_{n+2}(\text{fix}_{n+1}(\xi_{n+1}(r_n) \circ \theta_{n+1})) \\ &= \theta_{n+2}(\xi_{n+1}(r_n)(\theta_{n+1}(\text{fix}_n(\xi_n(r_{n-1}) \circ \theta_n)))) \\ &= \theta_{n+2}(\xi_{n+1}(r_n)(r_n)) \end{aligned}$$

and we conclude by induction hypothesis.

The second equation, of type  $X^{U \times U} \rightarrow X^{U \blacktriangleright X}$ , amounts, for  $\xi_n \in (X^{U \times U})_n$  and  $\theta_n \in (U^{\blacktriangleright X})_n$ , to the following:

$$F_n^{\lambda A} \circ (\lambda h u. h(u, u))(\xi_n)(\theta_n) = ((\lambda h k. h(k, k)) \circ (\lambda h. F_n^{\lambda A} h f_n^{\lambda A}) \circ F_n^{\lambda T A})(\xi_n)(\theta_n)$$

that is

$$F_n^{\lambda A}((\lambda h u. h(u, u))\xi_n, \theta_n) = ((\lambda h k. h(k, k))((\lambda h. F_n^{\lambda A} h f_n^{\lambda A})(F_n^{\lambda T A}(\xi_n))))(\theta_n)$$

that is

$$F_n^{\lambda A}(\lambda u. \xi_n(u, u), \theta_n) = ((\lambda h k. h(k, k))((F_n^{\lambda A} \circ F_n^{\lambda T A}(\xi_n) \circ f_n^{\lambda A}))) (\theta_n)$$

that is

$$F_n^{\lambda A}(\lambda u. \xi_n(u, u), \theta_n) = (\lambda k. (F_n^{\lambda A} \circ F_n^{\lambda T A}(\xi_n) \circ f_n^{\lambda A})(k, k))\theta_n$$

that is

$$F_n^{\lambda A}(\lambda u. \xi_n(u, u), \theta_n) = (F_n^{\lambda A} \circ F_n^{\lambda T A}(\xi_n) \circ f_n^{\lambda A})(\theta_n)(\theta_n)$$

that is

$$F_n^{\lambda A}(\lambda u. \xi_n(u, u), \theta_n) = F_n^{\lambda A}(F_n^{\lambda T A}(\xi_n, f_n^{\lambda A}(\theta_n)), \theta_n)$$

Reasoning as for the first equation, write

$$l_n := F_n^{\lambda A}(\lambda u. \xi_n(u, u), \theta_n) \quad \text{and} \quad r_n := F_n^{\lambda A}(F_n^{\lambda T A}(\xi_n, f_n^{\lambda A}(\theta_n)), \theta_n)$$

with

$$\begin{aligned} l_{n+1} &= \text{fix}_{n+1}((\lambda u. \xi_{n+1}(u, u)) \circ \theta_{n+1}) \\ &= \xi_{n+1}(\theta_{n+1}(l_n), \theta_{n+1}(l_n)) \end{aligned}$$

and on the other hand

$$\begin{aligned} F_{n+1}^{\lambda T A}(\xi_{n+1}, f_{n+1}^{\lambda A}(\theta_{n+1})) &= \text{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\lambda A}(\theta_{n+1})) \\ &= \xi_{n+1}(f_{n+1}^{\lambda A}(\theta_{n+1}, \text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)))) \\ &= \xi_{n+1}(\theta_{n+1}(F_n^{\lambda A}(\text{fix}_n^{X^U}(\xi_n \circ f_n^{\lambda A}(\theta_n)), \theta_n))) \\ &= \xi_{n+1}(\theta_{n+1}(F_n^{\lambda A}(F_n^{\lambda T A}(\xi_n, f_n^{\lambda A}(\theta_n)), \theta_n))) \\ &= \xi_{n+1}(\theta_{n+1}(r_n)) \end{aligned}$$

We thus have

$$\begin{aligned} r_{n+1} &= \text{fix}_{n+1}(\text{fix}_{n+1}^{X^U}(\xi_{n+1} \circ f_{n+1}^{\lambda A}(\theta_{n+1})) \circ \theta_{n+1}) \\ &= \text{fix}_{n+1}(\xi_{n+1}(\theta_{n+1}(r_n)) \circ \theta_{n+1}) \\ &= \xi_{n+1}(\theta_{n+1}(r_n), \theta_{n+1}(\text{fix}_n(\xi_n(\theta_n(r_{n-1})) \circ \theta_n))) \\ &= \xi_{n+1}(\theta_{n+1}(r_n), \theta_{n+1}(r_n)) \end{aligned}$$

and we conclude by induction hypothesis.  $\square$

**Lemma A.4.** *Diagram (7) commutes.*

*Proof.* Let  $A = (U, X)$  so that the diagram has type

$$T(A^{\blacktriangleright}) \dashrightarrow (TA)^{\blacktriangleright} = (U^{\blacktriangleright X \times \blacktriangleright X}, X^{U^{\blacktriangleright X \times \blacktriangleright X}}) \dashrightarrow (U^{\blacktriangleright (X^U)}, X^U)$$

Note that

$$\begin{aligned} T(\mu_A) &= T(\lambda h x . h(x, x), \text{id}_X) = (\lambda h x . h(x, x), \lambda k . (k \circ \lambda h x . h(x, x))) \\ (\lambda_A)^{\blacktriangleright} &= (f^{\lambda_A}, F^{\lambda_A})^{\blacktriangleright} = (\lambda h . f^{\lambda_A} \circ h \circ \blacktriangleright F^{\lambda_A}, F^{\lambda_A}) \end{aligned}$$

We have to check the following two equations:

$$f^{\lambda_A} \circ f_{T\mu_A} = f_{\mu_{TA}} \circ f_{(\lambda_A)^{\blacktriangleright}} \circ f^{\lambda_A} \quad \text{and} \quad F_{T\mu_A} \circ F^{\lambda_A} = F^{\lambda_A} \circ F_{(\lambda_A)^{\blacktriangleright}} \circ F_{\mu_{TA}}$$

The first equation, of type  $U^{\blacktriangleright X \times \blacktriangleright X} \rightarrow U^{\blacktriangleright (X^U)}$ , amounts, for  $\theta_{n+1} \in (U^{\blacktriangleright X \times \blacktriangleright X})_{n+1}$  and  $\xi_n \in (X^U)_n$ , to the following:

$$(f_{n+1}^{\lambda_A} \circ (\lambda h x . h(x, x)))(\theta_{n+1})(\xi_n) = ((\lambda h k . h(k, k)) \circ (\lambda h . f_{n+1}^{\lambda_A} h \blacktriangleright F_{n+1}^{\lambda_A}) \circ f_{n+1}^{\lambda_A})(\theta_{n+1})(\xi_n)$$

that is

$$f_{n+1}^{\lambda_A}(\lambda x . \theta_{n+1}(x, x), \xi_n) = ((\lambda h k . h(k, k)) \circ (\lambda h . f_{n+1}^{\lambda_A} h F_{n+1}^{\lambda_A}) \circ f_{n+1}^{\lambda_A})(\theta_{n+1})(\xi_n)$$

that is

$$f_{n+1}^{\lambda_A}(\lambda x . \theta_{n+1}(x, x), \xi_n) = (\lambda h k . h(k, k))(f_{n+1}^{\lambda_A} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}) \circ F_n^{\lambda_A})(\xi_n)$$

that is

$$f_{n+1}^{\lambda_A}(\lambda x . \theta_{n+1}(x, x), \xi_n) = (f_{n+1}^{\lambda_A} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}) \circ F_n^{\lambda_A})(\xi_n)(\xi_n)$$

that is

$$f_{n+1}^{\lambda_A}(\lambda x . \theta_{n+1}(x, x), \xi_n) = f_{n+1}^{\lambda_A}(f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)), \xi_n)$$

Let

$$l_n := f_{n+1}^{\lambda_A}(\lambda x . \theta_{n+1}(x, x), \xi_n) \quad \text{and} \quad r_n := f_{n+1}^{\lambda_A}(f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)), \xi_n)$$

Note that for all  $n$  we have

$$\begin{aligned} l_{n+1} &= (\lambda x . \theta_{n+2}(x, x)) \text{fix}_{n+1}(\xi_{n+1} \circ \lambda x . \theta_{n+1}(x, x)) \\ &= (\lambda x . \theta_{n+2}(x, x))((\lambda x . \xi_{n+1}(\theta_{n+1}(x, x))) \text{fix}_n(\xi_n \circ \lambda x . \theta_n(x, x))) \\ &= \theta_{n+2}(\xi_{n+1}(l_n), \xi_{n+1}(l_n)) \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{n+1} &= f_{n+2}^{\lambda_A}(f_{n+2}^{\lambda_A}(\theta_{n+2}, F_{n+1}^{\lambda_A}(\xi_{n+1})), \xi_{n+1}) \\ &= f_{n+2}^{\lambda_A}(\theta_{n+2}, F_{n+1}^{\lambda_A}(\xi_{n+1}))(\text{fix}_{n+1}^X(\xi_{n+1} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)))) \\ &= \theta_{n+2}(\text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}), \text{fix}_{n+1}^X(\xi_{n+1} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)))) \end{aligned}$$

So we show by induction on  $n$  that

$$\xi_{n+1}(r_n) = \text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}) = \text{fix}_{n+1}^X(\xi_{n+1} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)))$$

The base case is trivial. For the induction step, on the one hand we have

$$\begin{aligned} \text{fix}_{n+2}(F_{n+2}^{\lambda_A}(\xi_{n+2}) \circ \theta_{n+2}) &= F_{n+2}^{\lambda_A}(\xi_{n+2}, \theta_{n+2}(\text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}))) \\ &= \xi_{n+2}(\theta_{n+2}(\text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}), F_{n+1}^{\lambda_A}(\xi_{n+1}), \theta_{n+1}(\text{fix}_n(F_n^{\lambda_A}(\xi_n) \circ \theta_n)))) \\ &= \xi_{n+2}(\theta_{n+2}(\text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}), \text{fix}_{n+1}(F_{n+1}^{\lambda_A}(\xi_{n+1}) \circ \theta_{n+1}))) \end{aligned}$$

and we conclude by induction hypothesis, and on the other hand

$$\begin{aligned} \text{fix}_{n+2}^X(\xi_{n+2} \circ f_{n+2}^{\lambda_A}(\theta_{n+2}, F_{n+1}^{\lambda_A}(\xi_{n+1}))) &= \xi_{n+2} \circ f_{n+2}^{\lambda_A}(\theta_{n+2}, F_{n+1}^{\lambda_A}(\xi_{n+1}))(\text{fix}_{n+1}^X(\xi_{n+1} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n)))) \\ &= \xi_{n+2}(\theta_{n+2}(\text{fix}_n(F_n^{\lambda_A}(\xi_n) \circ \theta_n), \text{fix}_{n+1}^X(\xi_{n+1} \circ f_{n+1}^{\lambda_A}(\theta_{n+1}, F_n^{\lambda_A}(\xi_n))))) \end{aligned}$$

and we also conclude by induction hypothesis.

The second equation, of type  $X^U \rightarrow X^{U^{\blacktriangleright X \times \blacktriangleright X}}$ , amounts, for  $\xi_n \in (X^U)_n$  and  $\theta_n \in (U^{\blacktriangleright X \times \blacktriangleright X})_n$ , to the following

$$((\lambda k . (k \circ \lambda h x . h(x, x))) \circ F_n^{\lambda_A})(\xi_n)(\theta_n) = (F_n^{\lambda_A} \circ F_n^{\lambda_A})(\xi_n)(\theta_n)$$

that is

$$(F_n^{\lambda_A}(\xi_n) \circ \lambda h x . h(x, x))(\theta_n) = F_n^{\lambda_A}(F_n^{\lambda_A}(\xi_n), \theta_n)$$

that is

$$F_n^{\lambda_A}(\xi_n, \lambda x . \theta_n(x, x)) = F_n^{\lambda_A}(F_n^{\lambda_A}(\xi_n), \theta_n)$$

This is dealt-with similarly to (but in a much simpler way then) the first equation.  $\square$

**Lemma A.5.** *Diagram (8) commutes.*

*Proof.* Let  $A = (U, X)$ , so that the diagram has type

$$T(A^\blacktriangleright) \dashrightarrow A^\blacktriangleright = (U^{\blacktriangleright X}, X^{U^{\blacktriangleright X}}) \dashrightarrow (U^{\blacktriangleright X}, X)$$

Note that

$$(\epsilon_A)^\blacktriangleright = (\text{id}_U, \lambda x u. x)^\blacktriangleright = (\lambda h. (h \circ \blacktriangleright (\lambda x u. x)), \lambda x u. x)$$

We have to show

$$\lambda h. (h \circ \blacktriangleright (\lambda x u. x)) \circ f^{\lambda A} = \text{id}_{U^{\blacktriangleright X}} \quad \text{and} \quad F^{\lambda A} \circ \lambda x u. x = \lambda x u. x$$

For the first equation, given  $\theta_{n+1} \in (U^{\blacktriangleright X})_{n+1}$ , we have to show

$$f_{n+1}^{\lambda A}(\theta_{n+1}) \circ \blacktriangleright (\lambda x u. x) = \theta_{n+1}$$

The result is trivial since the left-hand side unfolds to

$$\lambda \blacktriangleright x. f_{n+1}^{\lambda A}(\theta_{n+1}, \lambda \dots x) = \lambda \blacktriangleright x. \theta_{n+1}(\text{fix}_n(\lambda \dots x)) = \lambda \blacktriangleright x. \theta_{n+1}(x)$$

The second equation is simpler and omitted.  $\square$

**Lemma A.6.** *Diagram (9) commutes.*

*Proof.* Let  $A = (U, X)$ , so that the diragram has type

$$TA \dashrightarrow (TA)^\blacktriangleright = (U, X^U) \dashrightarrow (U^{\blacktriangleright (X^U)}, X^U)$$

Note that

$$T(\eta_A) = T(\lambda u x. u, \text{id}_X) = (\lambda u x. u, \lambda h. h \circ (\lambda u x. u))$$

We have to show

$$f^{\lambda A} \circ (\lambda u x. u) = \lambda u x. u \quad \text{and} \quad (\lambda h. h \circ (\lambda u x. u)) \circ F^{\lambda A} = \text{id}_{X^U}$$

For the first equation, given  $u \in U_{n+1}$  and  $\xi_n \in (X^U)_n$ , we have to show

$$f_{n+1}^{\lambda A}(\lambda x. u, \xi_n) = u$$

which is trivial. For the second equation, given  $\xi_n \in X_n$  and  $u \in U_n$  we have to show

$$F^{\lambda A}(\xi_n, \lambda x. u) = \xi_n(u)$$

which is also trivial.  $\square$

## A.2 The Category $\mathbf{DZ}$ is Symmetric Monoidal Closed

**Proposition A.7** (Prop. 4.6). *The category  $\mathbf{DZ}$  is symmetric monoidal closed.*

We only detail monoidal closure. We rely on Prop. 4.1 and on the faithfulness of  $\text{HS} : \mathbf{SG} \rightarrow \mathbf{Rel}$  (see [11], but also Lemma 4.6 in the Appendix of the long version of [26], available at <https://perso.ens-lyon.fr/colin.riba/papers/fibaut.pdf>). We omit the  $(-)^*$  in writing objects of  $\mathbf{DZ}$ .

Recall from e.g. [22] that a symmetric monoidal category  $\mathbb{C}$  is *closed* if for every object  $A$ , the functor  $A \otimes (-)$  has a right adjoint  $(-)^A$ . Since  $A \otimes (-)$  is already a functor, according to [19, Thm. IV.1.2] it is sufficient to show that for every object  $C$  there is an object  $A^B$  and map

$$\text{eval}_C : A \otimes C^A \longrightarrow C$$

such that for every  $f : A \otimes B \rightarrow C$  there is a unique  $h : B \rightarrow C^A$  such that

$$\begin{array}{ccc} A \otimes C^A & \xrightarrow{\text{eval}_C} & C \\ \uparrow \text{id}_A \otimes h & \nearrow f & \\ A \otimes B & & \end{array}$$

*Proof of Prop. 4.6.* Let  $A = (U, X)$ ,  $B = (V, Y)$  and  $C = (W, Z)$ . Recall that  $A \multimap_{\mathbf{DZ}} C = (W^U \times X^{U \times Z}, U \times Z)$ . We define the total zig-zag strategy  $\text{eval}_C : A \otimes (A \multimap_{\mathbf{DZ}} C) \multimap C$  as follows:

	$A \otimes (A \multimap_{\mathbf{DZ}} C)$	$\xrightarrow{\text{eval}_C}$	$C$	
O	$(u, (f, F))$		$f(u)$	P
			$z$	O
P	$(F(u, z), (u, z))$			



Given any  $\tau' : B \multimap (A \multimap_{\text{DZ}} C)$ , the composition  $\text{eval}_C \circ (\text{id}_A \otimes \tau')$  is given by:

	$A \otimes B$	$\xrightarrow{\text{id}_A \otimes \tau'}$	$A \otimes (A \multimap_{\text{DZ}} C)$	$\xrightarrow{\text{eval}_C}$	$C$	
O	$(u, v)$		$(u, (f', F'))$		$f'(u)$ $z$	P O
P	$(F'(u, z), y')$		$(F'(u, z), (u, z))$			

It follows that  $\text{eval}_C \circ (\text{id}_A \otimes \tau') = \text{eval}_C \circ (\text{id}_A \otimes \tau'')$  implies  $\tau' = \tau''$ .

We show this by induction on pairs of even-length plays  $(s, t) \in \wp_A^{\text{even}} \times \wp_{A \multimap_{\text{DZ}} C}^{\text{even}}$ . Assume toward a contradiction that for some such  $(s, t) \in \text{HS}(\tau') \cap \text{HS}(\tau'')$ , for some  $v \in V$  we have  $(s.v, t.(f', F')) \in \text{HS}(\tau')$  and  $(s.v, t.(f'', F'')) \in \text{HS}(\tau'')$  with  $f' \neq f''$ . Then for some  $u \in U$ , we have say  $f'(u) \neq f''(u)$ . Then, for some  $r$  we have

$$\text{eval}_C \circ (\text{id}_A \otimes \tau') \ni r.(u, v).f'(u) \neq r.(u, v).f''(u) \in \text{eval}_C \circ (\text{id}_A \otimes \tau'')$$

Hence a contradiction. The case of  $F' \neq F''$  is dealt-with similarly.

Fix now some total zig-zag  $\sigma : A \otimes B \multimap C$ .

We define  $\tau : B \multimap (A \multimap_{\text{DZ}} C)$  by induction on plays. To each  $(s, t) \in \text{HS}(\tau)$ , with  $s$  and  $t$  even-length, we associate  $(s', t') \in \text{HS}(\sigma)$ , with  $s'$  and  $t'$  of the same length, and such that, for  $(\bar{v}, \bar{y}) = \partial(s)$  and  $(\bar{f}, \bar{F}), (\bar{u}, \bar{z}) = \partial(t)$ , we have  $\partial(s') = ((\bar{u}, \bar{v}), (\bar{F}(\bar{u}, \bar{z}), \bar{y}))$  and  $\partial(t') = (\bar{f}(\bar{u}), \bar{z})$ , where we take the pointwise application of sequences of functions.

For the base case, we put  $(\varepsilon, \varepsilon) \in \text{HS}(\tau)$ , and associate it to  $(\varepsilon, \varepsilon) \in \text{HS}(\sigma)$ .

Assume now  $(s, t) \in \text{HS}(\tau)$ , associated to  $(s', t') \in \text{HS}(\sigma)$ . For each  $v \in V$ , we define the functions  $f_v : U \rightarrow W$  and  $F_v : U \times Z \rightarrow X$  as follows: given  $u \in U$ , let  $w$  such that  $(s'.(u, v), t'.w) \in \text{HS}(\sigma)$ , and for each  $z \in Z$ , let  $x$  and  $y_{u,z}$  such that  $(s'.(u, v).(x, y_{u,z}), t'.w.z) \in \text{HS}(\sigma)$ . We then let  $f_v(u) := w$  and  $F_v(u, z) := x$ . We now let  $(s.v.y_{u,z}, t.(f_v, F_v).(u, z)) \in \text{HS}(\tau)$ , and associate it to  $(s'.(u, v).(x, y_{u,z}), t'.w.z) = (s'.(u, v).(F_v(u, z), y_{u,z}), t'.f_v(u).z)$  so that the invariant is satisfied.

This concludes the definition of  $\tau$ .

It then follows from the invariant that we indeed have  $\text{eval}_C \circ \text{id}_A \otimes \tau = \sigma$ .

First note that the map  $(s, t) \in \text{HS}(\tau) \mapsto (s', t') \in \text{HS}(\sigma)$  is surjective. The property then follows from the fact that  $(s, t) \in \text{HS}(\tau)$  iff  $(s', t') \in \text{HS}(\text{eval}_C \circ \text{id}_A \otimes \tau)$ . This is shown by induction on pairs of plays  $(s, t) \in \wp_B^{\text{even}} \times \wp_{A \multimap_{\text{DZ}} C}^{\text{even}}$ . The base case is trivial. For the induction step, given such  $(s.v.y_{u,z}, t.(f_v, F_v).(u, z))$ , we have  $(s.v.y_{u,z}, t.(f_v, F_v).(u, z)) \in \text{HS}(\tau)$  if and only if  $(s'.(u, v).(F_v(u, z), y_{u,z}), t'.f_v(u).z) \in \text{HS}(\text{eval}_C \circ \text{id}_A \otimes \tau)$ .

This concludes the proof of Prop. 4.6  $\square$

## B. Proofs of Section 5

**Proposition B.1** (Prop. 5.2). *Assume given  $\mathcal{A}$  and  $\mathcal{B}$  as above, where  $\mathcal{B}$  has also input alphabet  $\Sigma$ .*

*If there is a winning map  $\mathcal{A} \rightarrow \mathcal{B}$  then  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ .*

*Proof.* Write  $\sigma$  for the strategy corresponding to the winning map. Let  $t \in \mathcal{L}(\mathcal{A})$ , with winning strategy  $\tau : \mathbf{I}(t) \rightarrow \mathcal{A}(t)$ . Then  $t^* \sigma$  is a winning DialAut<sub>1</sub>-map from  $(\mathbf{1}, Q_{\mathcal{A}}, U, X, t^*(\alpha^{\mathcal{A}}))$  to  $(\mathbf{1}, Q_{\mathcal{B}}, V, Y, t^*(\alpha^{\mathcal{B}}))$ . But

$$\begin{aligned} t^*(\alpha^{\mathcal{A}})(\bullet, u, x, p) &= \alpha^{\mathcal{A}}(t(\bullet, \blacktriangleright p), u, x, p) \\ &= \alpha(t)(\bullet, u, x, p) \end{aligned}$$

The same holds for  $\mathcal{B}$ , and it follows that  $t^* \sigma \circ \tau$  is winning from  $\mathbf{I}(t)$  to  $\mathcal{B}(t)$ . Hence  $t \in \mathcal{L}(\mathcal{B})$ .  $\square$

**Proposition B.2** (Prop. 5.5).  $\mathcal{L}(\mathcal{A} \otimes \mathcal{B}) = \mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{B})$ .

*Proof.* The inclusion  $(\subseteq)$  follows using the projections  $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A}$  and  $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B}$ .

For the other direction, tensor  $\sigma : \mathbf{I}(t) \rightarrow \mathcal{A}(t)$  with  $\tau : \mathbf{I}(t) \rightarrow \mathcal{B}(t)$  and then precompose with a monoidal unit map.  $\square$

**Proposition B.3** (Prop. 5.6). *If  $\Omega_{\mathcal{A}}$  is Borel, then  $t \in \mathcal{L}(\mathcal{A}^\perp)$  iff  $t \notin \mathcal{L}(\mathcal{A})$ .*

*Proof.* The argument is an adaptation of [29]. By Martin's Theorem [21], it is equivalent to show that P wins the acceptance game of  $\mathcal{A}^\perp$  on  $t$  iff O wins that of  $\mathcal{A}$  on  $t$ , where, using the notions of §3, an O-strategy is just a P-strategy on the dual game.

For  $(\Rightarrow)$ , assuming given a winning P-strat  $\sigma$  on  $\mathcal{A}(t) \multimap \perp$ , we build a winning O-strat  $\tau$  in  $\mathcal{A}(t)$ . The strategy  $\tau$  is build by induction on plays. To each play  $t$  of  $\tau$ , we associate a play  $s$  of  $\sigma$  such that if  $t$  leads to state  $q_{\mathcal{A}}$ , then  $s$  leads to state  $(q_{\mathcal{A}}, \mathbb{ff})$ . In the base case, both  $t$  and  $s$  are the empty plays, and the invariant is respected. For the induction step, assume that P plays  $u$  from  $t$  in  $\mathcal{A}(t)$ . Let  $(f, F)$  be the move of  $\sigma$  from  $s$ . We then let  $\tau$  answer the pair  $(F(u, f(u)), f(u))$  from  $s.u$ , and  $\mathcal{A}(t)$  goes to state  $q'_{\mathcal{A}}$ . In  $\mathcal{A}(t) \multimap \perp$ , we let O play the pair  $(f(u), u)$ . Then  $\mathcal{A}(t) \multimap \perp$  goes to state  $(q'_{\mathcal{A}}, \mathbb{ff})$  and the invariant is respected. Since  $\sigma$  is winning and  $\mathcal{A}(t) \multimap \perp$  stays in states of the form  $(-, \mathbb{ff})$  the infinite sequence of states produced in  $\mathcal{A}(t)$  is rejecting, as required.

For the conversion direction, assuming given a winning O-strat  $\tau$  on  $\mathcal{A}(t) \multimap \perp$ , we build a winning P-strat  $\sigma$  in  $\mathcal{A}(t) \multimap \perp$ . The strategy  $\sigma$  is build by induction on plays as long as  $\mathcal{A}(t) \multimap \perp$  stays in states of the form  $(-, \mathbb{ff})$  (if it switches to  $(-, \mathbb{tt})$  the P trivially wins). So to each play  $s$  of  $\sigma$  which leads to state  $(q_{\mathcal{A}}, \mathbb{ff})$ , we associate a play  $t$  of  $\tau$  which leads to state  $q_{\mathcal{A}}$ . The base case is trivial. For the induction step, we build  $(f, F)$  from  $\sigma$  as follows: to each  $u$ ,  $\sigma$  associates (from  $t$ ) a pair  $(x, d)$ . We let  $F(u, \_) := d$  and  $f(u) := x$ . Assume then that from  $s.(f, F)$ , O plays some  $(u, d)$ . If  $d \neq f(u)$  then we are done. Otherwise,  $\mathcal{A}(t) \multimap \perp$  switches to  $(q'_{\mathcal{A}}, d)$ . We then let P play  $u$  from  $t$ , so that

by construction  $\tau$  answers  $(F(u, \_), d)$ , and  $\mathcal{A}(t)$  goes to state  $q'_A$ . But then, since  $\tau$  is winning for O, the sequence of  $\mathcal{A}$ -states is rejecting, so that P wins in  $\mathcal{A}(t) \multimap \perp$ , as required.  $\square$

### C. Proofs of Section 6

**Proposition C.1.** *Given a regular non-deterministic automaton  $\mathcal{N}$  and a regular automaton  $\mathcal{A}$ , if there is a winning map  $\mathcal{N} \rightarrow \mathcal{A}$  then there is a winning map  $\mathcal{N} \rightarrow !\mathcal{A}$ .*

*Proof.* Note that we can assume the automata to be parity automata, and thanks to [15–17, 30], that there is a positional winning P-strategy  $\sigma : \mathcal{N} \multimap \mathcal{A}$ .

We build a winning P-strategy  $\tau$  by induction on plays as follows. To each play  $t$  of  $\tau$  with position  $(p, q_{\mathcal{N}}, S)$ ,  $S = \{(-, q_1), \dots, (-, q_n)\}$ , we associate a set  $\{s_1, \dots, s_n\}$  of plays of  $\sigma$ , with  $s_i$  of position  $(p, q_{\mathcal{N}}, q_i)$ . The case of the initial position is trivial. For the inductive step, let O play from  $t$  some  $(a, y)$  in component  $\mathcal{N}$  of  $\mathcal{N} \rightarrow !\mathcal{A}$ . For  $i \in \mathbb{N}$ , let  $u_i$  be the answer of  $\sigma$  from  $s_i(a, y)$ . This defines a partial map  $Q_{\mathcal{A}} \rightarrow U$  mapping  $q_i$  to  $u_i$  that we extend to a total map  $h$ , and we let  $\tau$  play  $s.(a, y).h$ . Then if O answers by  $t.(a, y).h.d$ , the position in  $\mathcal{N} \rightarrow !\mathcal{A}$  becomes  $(p, d, q'_{\mathcal{N}}, S')$  where  $S' = \{(-, q'_1), \dots, (-, q'_m)\}$ . Now, each  $q'_j$  is  $\delta_{\mathcal{A}}(q_{i_j}, a, u_{i_j}, x_j, d)$  for some  $i_j$  and some  $x_j$  (note that there might be several such  $i_j$  and  $x_j$ , but we select one). For each  $j$ , we let O play  $s_{i_j}.(a, y).u_{i_j}.(x_j, d)$  in the game  $\mathcal{N} \rightarrow \mathcal{A}$ . Since  $\mathcal{N}$  is non-deterministic, it goes to the same  $q'_{\mathcal{N}}$  for all  $j$ .

In an infinite play of  $\tau$  played like this, all traces of the  $Q_{!\mathcal{A}}$  component are sequences of states produced by  $\sigma$ . Since moreover sequences of states on  $\mathcal{N}$  agree, it follows that  $\tau$  is winning since  $\sigma$  is winning.  $\square$

**Proposition C.2** (Prop. 6.2). *If  $\mathcal{A}$  is regular, there is a winning  $\eta_{\mathcal{A}} : !\mathcal{A} \rightarrow \mathcal{A}$ .*

*Proof.* Note that we can assume  $\mathcal{A}$  to be a parity automaton. We define  $\text{HS}(\eta)$  by induction on plays as follows, with the following invariant: for each  $(s, t) \in \text{HS}(\eta)$ , with  $s, t$  of even length, writing  $q$  for the state of  $t$  and  $S$  for the state of  $s$ , we have  $q \in S \upharpoonright 2$ .

The base case is trivial. Let  $(s, t) \in \text{HS}(\eta)$  with  $s$  and  $t$  even-length, and with  $t$  in state  $q$  and  $s$  in state  $S$ . Given an O-move  $(a, h)$ , we let  $(s.(a, h), t.h(q)) \in \text{HS}(\eta)$ , and for all  $(x, d)$  we further let  $(s.(a, h).(\bullet, d), t.h(q).(x, d)) \in \text{HS}(\eta)$ . Then the invariant is insured by definition of  $!\mathcal{A}$ .

The strategy  $\tau$  is winning since the sequence of states produced in  $\mathcal{A}$  is a trace in the sequence of states produced in  $!\mathcal{A}$ .  $\square$

**Proposition C.3** (Weak Completeness – Prop. 6.5). *Given automata  $\mathcal{A}$  and  $\mathcal{B}$  on  $\Sigma$ , if  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$  then there is an effective winning  $\text{Aut}_{\Sigma}$ -map  $!\mathcal{A} \rightarrow !(B^{\perp})^{\perp}$ .*

*Proof.* By Prop. 5.6, Cor. 6.3, Prop. 5.5 and Prop. 6.4, if  $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$ , we have

$$\mathcal{L}(\exists_{\Sigma}(!\mathcal{A} \otimes !(B^{\perp}))) = \emptyset$$

Hence, again using Prop. 5.6,  $\mathcal{L}(\exists_{\Sigma}((!\mathcal{A} \otimes !(B^{\perp}))^{\perp}))^{\perp}$  contains the unique  $\mathbf{1}$ -labelled tree  $\mathbf{1} : D^* \rightarrow \mathbf{1}$ .

Since the game of  $\exists_{\Sigma}((!\mathcal{A} \otimes !(B^{\perp}))^{\perp})(t)$  is regular it follows that P has a regular winning strategy (see e.g. [24, Thm. IV.4.9]).

We thus get an effective winning  $\text{DialAut}_1$ -map

$$(\exists_{\Sigma}(!\mathcal{A} \otimes !(B^{\perp}))) (\mathbf{1}) \longrightarrow \perp (\mathbf{1})$$

hence an effective winning  $\text{DialAut}_1$ -map

$$\exists_{\Sigma}((!\mathcal{A} \otimes !(B^{\perp}))) \longrightarrow \perp$$

This map is lifted by Prop. 5.7 to an effective winning  $\text{DialAut}_{\Sigma}$ -map

$$!\mathcal{A} \otimes !(B^{\perp}) \longrightarrow \perp$$

hence by Prop. 5.3 to an effective winning  $\text{DialAut}_{\Sigma}$ -map

$$!\mathcal{A} \longrightarrow !(B^{\perp}) \multimap \perp$$

$\square$

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Tree Automata</b>	<b>2</b>
<b>3</b>	<b>Games</b>	<b>3</b>
<b>4</b>	<b>A Dialectica-Like Interpretation of Zig-Zag Strategies</b>	<b>4</b>
<b>5</b>	<b>Dialectica-like Categories of Tree Automata</b>	<b>6</b>
5.1	<i>D</i> -Synchronicity in <b>DZ</b> . . . . .	6
5.2	Indexed Structure . . . . .	7
5.3	The Fibered Category <b>DialAut</b> . . . . .	8
5.4	Logical Connectives . . . . .	8
<b>6</b>	<b>Non-Deterministic Automata</b>	<b>9</b>
<b>7</b>	<b>Conclusion</b>	<b>10</b>
<b>A</b>	<b>Proofs of Section 4</b>	<b>11</b>
A.1	Prop. 4.2: $\lambda$ is a Distributive Law . . . . .	11
A.2	The Category <b>DZ</b> is Symmetric Monoidal Closed . . . . .	15
<b>B</b>	<b>Proofs of Section 5</b>	<b>16</b>
<b>C</b>	<b>Proofs of Section 6</b>	<b>17</b>