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# CYCLICITY IN THE HARMONIC DIRICHLET SPACE 

E. ABAKUMOV, O. EL-FALLAH, K. KELLAY, AND T. RANSFORD

Abstract. The harmonic Dirichlet space is the Hilbert space of functions $f \in L^{2}(\mathbb{T})$ such that

$$
\|f\|_{\mathcal{D}(\mathbb{T})}^{2}:=\sum_{n \in \mathbb{Z}}(1+|n|)|\widehat{f}(n)|^{2}<\infty .
$$

We give sufficient conditions for $f$ to be cyclic in $\mathcal{D}(\mathbb{T})$, in other words, for $\left\{\zeta^{n} f(\zeta): n \geq 0\right\}$ to span a dense subspace of $\mathcal{D}(\mathbb{T})$.

## 1. Introduction

Let $\mathcal{X}$ be a topological linear space of complex functions on the unit circle $\mathbb{T}$ such that the shift operator $S$, given by

$$
S(f)(\zeta):=\zeta f(\zeta), \quad f \in \mathcal{X}
$$

is an isomorphism of $\mathcal{X}$ onto itself.
A closed subspace $\mathcal{M}$ of $\mathcal{X}$ is called invariant if $S(\mathcal{M}) \subset \mathcal{M}$. It is said to be 1-invariant (or simply invariant) for $S$ if $S(\mathcal{M}) \subsetneq \mathcal{M}$, and it is called 2-invariant (or doubly invariant) if $S(\mathcal{M})=\mathcal{M}$. The latter condition is equivalent to the invariance of $\mathcal{M}$ under multiplication by both $\zeta$ and $\bar{\zeta}$.

Let $\mathbb{Z}$ denote the integers and let $\mathbb{N}:=\{n \in \mathbb{Z}: n \geq 0\}$. Given $f \in \mathcal{X}$, we write

$$
\begin{aligned}
& {[f]_{\mathbb{N}}:=\overline{\operatorname{Span}}^{\mathcal{X}}\left\{z^{n} f: n \in \mathbb{N}\right\}} \\
& {[f]_{\mathbb{Z}}:=\overline{\operatorname{Span}}^{\mathcal{X}}\left\{z^{n} f: n \in \mathbb{Z}\right\}}
\end{aligned}
$$

A function $f$ is said to be 1 -invariant if the space $[f]_{\mathbb{N}}$ is 1 -invariant for $S$. We say that a function $f \in \mathcal{X}$ is cyclic (resp. bicyclic) for $\mathcal{X}$ if $[f]_{\mathbb{N}}=\mathcal{X}\left(\right.$ resp. $\left.[f]_{\mathbb{Z}}=\mathcal{X}\right)$.

[^0]Let us begin with the classical case, namely $\mathcal{X}=L^{2}(\mathbb{T})$. By a wellknown theorem of Wiener, the 2-invariant subspaces have the form

$$
\mathcal{M}=\left\{f \in L^{2}(\mathbb{T}): f=0 \text { a.e. on } \mathbb{T} \backslash \sigma\right\},
$$

where $\sigma$ is a Borel subset of $\mathbb{T}$ (see e.g. [11, p.8, Theorem 1.2.1]). It follows from Szegő's infimum theorem that a function $f$ is 1-invariant in $L^{2}(\mathbb{T})$ if and only if $\log |f| \in L^{1}(\mathbb{T})$ (see e.g. [10, p.12, Corollary 4]).

For $\mathcal{X}=\mathcal{C}^{\infty}(\mathbb{T})$, Makarov [7] gave a complete description of the invariant subspaces of $S$. He also obtained the following characterization of 1-invariant functions of $\mathcal{C}^{\infty}(\mathbb{T})$.
Theorem (Makarov [7, p.3]). A function $f$ is 1-invariant in $\mathcal{C}^{\infty}(\mathbb{T})$ if and only if $\log |f| \in L^{1}(\mathbb{T})$.

The case $\mathcal{X}=\mathcal{C}^{n}(\mathbb{T})(n \geq 1)$ is more complicated, and no characterization of 1-invariant functions is known ([7, p.3], see also [8, Theorem 1.3] or [9, Theorem 5]).

We shall focus our attention on the harmonic Dirichlet space $\mathcal{D}(\mathbb{T})$. This is the set of functions $f \in L^{2}(\mathbb{T})$ whose Fourier coefficients satisfy

$$
\mathcal{D}(f):=\sum_{n \in \mathbb{Z}}|\widehat{f}(n)|^{2}|n|<\infty .
$$

It becomes a Hilbert space if endowed with the norm $\|\cdot\|_{\mathcal{D}(\mathbb{T})}$, given by

$$
\|f\|_{\mathcal{D}(\mathbb{T})}^{2}:=\|f\|_{L^{2}(\mathbb{T})}^{2}+\mathcal{D}(f)=\sum_{n \in \mathbb{Z}}|\widehat{f}(n)|^{2}(1+|n|) .
$$

According to Douglas' formula [3], we have

$$
\mathcal{D}(f)=\frac{1}{4 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| .
$$

This can also be written

$$
\mathcal{D}(f)=\frac{1}{2 \pi} \int_{\mathbb{T}} \mathcal{D}_{\zeta}(f)|d \zeta|,
$$

where $\mathcal{D}_{\zeta}(f)$ is the so-called local Dirichlet integral of $f$ at $\zeta$, given by

$$
\mathcal{D}_{\zeta}(f):=\frac{1}{2 \pi} \int_{\mathbb{T}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right|, \quad \zeta \in \mathbb{T}
$$

As $\mathcal{D}(\mathbb{T})$ is dense in $L^{2}(\mathbb{T})$, if a function $f$ is cyclic for $\mathcal{D}(\mathbb{T})$, then it is also cyclic for $L^{2}(\mathbb{T})$. So by Szegő's theorem we have

$$
f \text { is cyclic for } \mathcal{D}(\mathbb{T}) \Rightarrow \int_{\mathbb{T}} \log |f(\zeta)||d \zeta|=-\infty
$$

In [12], Ross, Richter and Sundberg gave a complete characterization of the 2-invariant subspaces $\mathcal{M}$ of $\mathcal{D}(\mathbb{T})$ in terms of their zero
sets. In order to state their result, we need to introduce the notion of logarithmic capacity (see for instance $[6, \S 2.4]$ ).

The energy of a Borel probability measure $\mu$ on $\mathbb{T}$ is defined by

$$
I(\mu):=\iint_{\mathbb{T}^{2}} \log \frac{1}{\left|\zeta-\zeta^{\prime}\right|} d \mu(\zeta) d \mu\left(\zeta^{\prime}\right)=\sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^{2}}{n} .
$$

Then we define the logarithmic capacity of a Borel subset $E$ of $\mathbb{T}$ by

$$
c(E):=1 / \inf \{I(\mu): \mu \in \mathcal{P}(E)\},
$$

where $\mathcal{P}(E)$ denotes the set of probability measures supported on a compact subset of $E$. We say that a property holds quasi-everywhere (q.e.) if it holds everywhere outside a set of logarithmic capacity zero.

It is known that, if $f \in \mathcal{D}(\mathbb{T})$, then the radial limit of the Poisson integral of $f$ exists q.e. and is equal to $f$ a.e. (for more details we refer to [12] and the references therein). In the sequel, $f$ will denote this limit, and will therefore be defined q.e. on $\mathbb{T}$. We shall write $\mathcal{Z}(f)$ for the zero set of $f$, namely

$$
\mathcal{Z}(f):=\{\zeta \in \mathbb{T}: f(\zeta)=0\} .
$$

Note that this set is defined up to sets of logarithimic capacity zero.
Theorem (Richter-Ross-Sundberg [12]). $\mathcal{M}$ is a 2 -invariant subspace of $\mathcal{D}(\mathbb{T})$ if and only if there exists a measurable set $E \subset \mathbb{T}$ such that

$$
\mathcal{M}=\mathcal{D}_{E}:=\{f \in \mathcal{D}(\mathbb{T}): f \mid E=0 \text { q.e. }\} .
$$

Note that the problem of characterization of 1-invariant subspaces of $\mathcal{D}(\mathbb{T})$ remains open. It was proved in $[1,13]$ that, for each $n \in$ $\mathbb{N} \cup\{\infty\}$, there exists an invariant subspace $\mathcal{M}$ of $\mathcal{D}(\mathbb{T})$ such that $\operatorname{dim}(\mathcal{M} / S(\mathcal{M}))=n$. This suggests that the lattice of 1-invariant subspaces has a very complicated structure.

As a direct consequence of the Richter-Ross-Sundberg theorem, we obtain the following necessary conditions for cyclicity in $\mathcal{D}(\mathbb{T})$.

Theorem 1. If $f$ is cyclic for $\mathcal{D}(\mathbb{T})$, then

$$
\int_{\mathbb{T}} \log |f(\zeta)||d \zeta|=-\infty \quad \text { and } \quad c(\mathcal{Z}(f))=0
$$

Our goal in this paper is to give sufficient conditions for a function $f \in \mathcal{D}(\mathbb{T})$ to be cyclic.

For $\beta \in(0,1]$, we shall denote by $\operatorname{Lip}_{\beta}(\mathbb{T})$ the set of functions $f$ continuous on $\mathbb{T}$ such that

$$
\|f\|_{\operatorname{Lip}_{\beta}(\mathbb{T})}:=\|f\|_{\mathcal{C}(\mathbb{T})}+\sup _{\zeta, \zeta^{\prime} \in \mathbb{T}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|}{\left|\zeta-\zeta^{\prime}\right|^{\beta}}<\infty .
$$

For $\alpha \in(0,1)$, we set

$$
\mathcal{C}^{1+\alpha}(\mathbb{T}):=\left\{f \in C^{1}(\mathbb{T}): f^{\prime} \in \operatorname{Lip}_{\alpha}(\mathbb{T})\right\}
$$

Of course, if $f$ belongs to $\operatorname{Lip}_{\beta}(\mathbb{T})$ or $C^{1+\alpha}(\mathbb{T})$, then $\mathcal{Z}(f)$ is closed in $\mathbb{T}$. We shall establish the following result.
Theorem 2. Let $f \in \mathcal{D}(\mathbb{T})$ such that $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in(0,1)$. Suppose further that $\log |f| \notin L^{1}(\mathbb{T})$. Then $\left[f^{2}\right]_{\mathbb{N}}=\mathcal{D}_{\mathcal{Z}(f)}$.

Combining Theorems 1 and 2, we deduce
Corollary. Let $f \in \mathcal{D}(\mathbb{T})$ such that $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in(0,1)$. Then the following assertions are equivalent:
(1) $f^{2}$ is cyclic for $\mathcal{D}(\mathbb{T})$;
(2) $\log |f| \notin L^{1}(\mathbb{T})$ and $c(\mathcal{Z}(f))=0$.

A closed set $E \subset \mathbb{T}$ is said to be a Carleson set (and we write $E \in(C))$ if

$$
\int_{\mathbb{T}} \log \frac{1}{d(\zeta, E)}|d \zeta|<\infty
$$

For background information on Carleson sets, see e.g. [6, §4.4]. Note that, if $f \in \operatorname{Lip}_{\beta}(\mathbb{T})$ and $\mathcal{Z}(f) \notin(C)$, then $\log |f| \notin L^{1}(\mathbb{T})$.

It is known that $\operatorname{Lip}_{\beta}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ if and only if $\beta>1 / 2$. The inclusion $\operatorname{Lip}_{\beta}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ for $\beta>1 / 2$ can easily be obtained from Douglas' formula.

We shall establish the following theorem.
Theorem 3. Let $f \in \operatorname{Lip}_{\beta}(\mathbb{T})$, where $\beta \in\left(\frac{1}{2}, 1\right]$. If $\mathcal{Z}(f) \notin(C)$, then $[f]_{\mathbb{N}}=\mathcal{D}_{\mathcal{Z}(f)}$. If furthermore $c(\mathcal{Z}(f))=0$, then $f$ is cyclic for $\mathcal{D}(\mathbb{T})$.

## 2. Proof of Theorem 2

For the proof of Theorem 2, we shall need the following standard result.
Lemma 4. Let $f \in \mathcal{D}(\mathbb{T})$. The following assertions are equivalent:
(1) $[f]_{\mathbb{N}}=[f]_{\mathbb{Z}}$;
(2) $f \in[S f]_{\mathbb{N}}$;
(3) $\inf \left\{\|p f\|_{\mathcal{D}(\mathbb{T})}: p \in H^{\infty}\right.$, pf $\in \mathcal{D}(\mathbb{T})$ and $\left.p(0)=1\right\}=0$.

Proof. Since $S$ is invertible, (1) and (2) are equivalent.
If $f \in[S f]_{\mathbb{N}}$, then there is a sequence $\left(p_{n}\right)$ of polynomials such that $p_{n}(0)=1$ and $\left\|\left(1-p_{n}\right) f\right\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$. This proves that (2) implies (3).

Finally, suppose that (3) holds. Let $\left(p_{n}\right) \subset H^{\infty}$ be a sequence such that $p_{n}(0)=1, p_{n} f \in \mathcal{D}(\mathbb{T})$ and $\left\|p_{n} f\right\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$. Writing $p_{n}=1-z q_{n}$, by [12, Proposition 3.4] we have $z q_{n} f \in[S f]_{\mathbb{N}}$. Since $z q_{n} f$ converges to $f$, it follows that $f \in[S f]_{\mathbb{N}}$, so that (2) holds.

We shall also need the following result, which is a special case of a theorem due to Carleson-Jacobs-Havin-Shamoyan [2, Theorem 6.1].

Lemma 5. Let $F$ be an outer function on $\mathbb{D}$ that is continuous on $\overline{\mathbb{D}}$. If $|F| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, where $\alpha \in(0,1)$, then $F \in \operatorname{Lip}_{(1+\alpha) / 2}(\overline{\mathbb{D}})$. Furthermore, the Lipschitz constant associated to $F$ on $\overline{\mathbb{D}}$ depends only on the Lipschitz constants and bounds for the derivatives of $|F|$ on $\mathbb{T}$.
Proof of Theorem 2. Let $p_{\epsilon}$ be the outer function such that

$$
\left|p_{\epsilon}(\zeta)\right|=\frac{e^{-M_{\epsilon}}}{|f(\zeta)|+\epsilon} \quad \text { a.e. on } \mathbb{T}
$$

where the constant $M_{\epsilon}$ is chosen so that $p_{\epsilon}(0)=1$. Thus

$$
M_{\epsilon}=\int_{\mathbb{T}} \log \left(\frac{1}{|f(\zeta)|+\epsilon}\right) \frac{|d \zeta|}{2 \pi},
$$

and since $\log |f| \notin L^{1}(\mathbb{T})$, it follows that $M_{\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0^{+}$. We are going to prove that

$$
\lim _{\epsilon \rightarrow 0^{+}}\left\|p_{\epsilon} f^{2}\right\|_{\mathcal{D}(\mathbb{T})}=0
$$

If this holds, then by Lemma 4 we have $\left[f^{2}\right]_{\mathbb{N}}=\left[f^{2}\right]_{\mathbb{Z}}$, and since clearly $\mathcal{Z}\left(f^{2}\right)=\mathcal{Z}(f)$, we can apply the Richter-Ross-Sundberg theorem to obtain the desired result.

We have

$$
\left\|p_{\epsilon} f^{2}\right\|_{\mathcal{D}(\mathbb{T})}^{2}=\left\|p_{\epsilon} f^{2}\right\|_{L^{2}(\mathbb{T})}^{2}+\mathcal{D}\left(p_{\epsilon} f^{2}\right) .
$$

For the first term, we have

$$
\left\|p_{\epsilon} f^{2}\right\|_{L^{2}(\mathbb{T})}^{2}=\int_{\mathbb{T}} \frac{e^{-2 M_{\epsilon}}|f|^{4}}{(|f|+\epsilon)^{2}} \frac{|d \zeta|}{2 \pi} \leq e^{-2 M_{\epsilon}}\left\|f^{2}\right\|_{L^{2}(\mathbb{T})}^{2} \rightarrow 0 \quad \text { as } \epsilon \rightarrow 0^{+} .
$$

The second term we estimate using Douglas' formula, namely

$$
\mathcal{D}\left(p_{\epsilon} f^{2}\right)=\frac{1}{4 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\left|\left(p_{\epsilon} f^{2}\right)(\zeta)-\left(p_{\epsilon} f^{2}\right)\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| .
$$

Let

$$
\Gamma:=\left\{\left(\zeta, \zeta^{\prime}\right) \in \mathbb{T}^{2}:\left|f\left(\zeta^{\prime}\right)\right| \leq|f(\zeta)|\right\}
$$

Then, by symmetry,

$$
\mathcal{D}\left(p_{\epsilon} f\right)=2 \frac{1}{4 \pi^{2}} \iint_{\Gamma} \frac{\left|\left(p_{\epsilon} f\right)(\zeta)-\left(p_{\epsilon} f\right)\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| .
$$

Now, for all $\zeta, \zeta^{\prime} \in \mathbb{T}$, we have

$$
\begin{aligned}
& \left|\left(p_{\epsilon} f^{2}\right)(\zeta)-\left(p_{\epsilon} f^{2}\right)\left(\zeta^{\prime}\right)\right|^{2} \\
& \quad=\left|p_{\epsilon}(\zeta)\left(f^{2}(\zeta)-f^{2}\left(\zeta^{\prime}\right)\right)+f^{2}\left(\zeta^{\prime}\right)\left(p_{\epsilon}(\zeta)-p_{\epsilon}\left(\zeta^{\prime}\right)\right)\right|^{2} \\
& \quad \leq 2\left|p_{\epsilon}(\zeta)\right|^{2}\left|f^{2}(\zeta)-f^{2}\left(\zeta^{\prime}\right)\right|^{2}+2\left|f^{2}\left(\zeta^{\prime}\right)\right|^{2}\left|p_{\epsilon}(\zeta)-p_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2} .
\end{aligned}
$$

Hence

$$
\iint_{\Gamma} \frac{\left|\left(p_{\epsilon} f^{2}\right)(\zeta)-\left(p_{\epsilon} f^{2}\right)\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \leq 2 A_{\epsilon}+2 B_{\epsilon}
$$

where

$$
A_{\epsilon}:=\iint_{\Gamma}\left|p_{\epsilon}(\zeta)\right|^{2} \frac{\left|f^{2}(\zeta)-f^{2}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta|
$$

and

$$
B_{\epsilon}:=\iint_{\Gamma}\left|f^{2}\left(\zeta^{\prime}\right)\right|^{2} \frac{\left|p_{\epsilon}(\zeta)-p_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| .
$$

We estimate $A_{\epsilon}$ directly as follows:

$$
\begin{aligned}
A_{\epsilon} & =e^{-2 M_{\epsilon}} \iint_{\Gamma} \frac{\left|f(\zeta)+f\left(\zeta^{\prime}\right)\right|^{2}}{(|f(\zeta)|+\epsilon)^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \\
& \leq 4 e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \\
& \leq 4 e^{-2 M_{\epsilon}} 4 \pi^{2} \mathcal{D}(f) .
\end{aligned}
$$

Hence $A_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^{+}$.
To estimate $B_{\epsilon}$, we consider the outer function $F_{\epsilon}$ such that

$$
\left|F_{\epsilon}(\zeta)\right|=|f(\zeta)|+\epsilon \quad \text { a.e. on } \mathbb{T} \text {. }
$$

By Lemma 5 , since $\left|F_{\epsilon}\right| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$, we have $F_{\epsilon} \in \operatorname{Lip}_{(1+\alpha) / 2}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$ and there exists a positive constant $D$, depending only on $|f|$, such that $\mathcal{D}\left(F_{\epsilon}\right) \leq D$ for all $\epsilon \in(0,1)$. We then have

$$
\begin{aligned}
B_{\epsilon} & =\iint_{\Gamma} \frac{e^{-2 M_{\epsilon}\left|f^{2}\left(\zeta^{\prime}\right)\right|^{2}}}{(|f(\zeta)|+\epsilon)^{2}\left(\left|f\left(\zeta^{\prime}\right)\right|+\epsilon\right)^{2}} \frac{\left|1 / p_{\epsilon}(\zeta)-1 / p_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \\
& \leq e^{-2 M_{\epsilon}} \iint_{\Gamma} \frac{\left|F_{\epsilon}(\zeta)-F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \\
& \leq e^{-2 M_{\epsilon}} 4 \pi^{2} \mathcal{D}\left(F_{\epsilon}\right) .
\end{aligned}
$$

Thus $B_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^{+}$. This completes the proof of Theorem 2.

## 3. Proof of Theorem 3

To prove Theorem 3, we shall need the following additional lemma.
Lemma 6. Let $f \in \operatorname{Lip}_{\beta}(\mathbb{T})$, where $\beta>1 / 2$. Then, for $\eta \in\left(0, \frac{2 \beta-1}{2 \beta}\right)$, we have

$$
\iint_{\mathbb{T}^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2-2 \eta}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta|<+\infty
$$

Proof. Since $\beta>1 / 2(1-\eta)$, we get

$$
\iint_{\mathbb{T}^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2-2 \eta}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta| \lesssim \iint_{\mathbb{T}^{2}} \frac{\left|d \zeta^{\prime}\right||d \zeta|}{\left|\zeta-\zeta^{\prime}\right|^{2(1-(1-\eta) \beta)}}<\infty
$$

Note that here, and in what follows, we write $A \lesssim B$ to mean that there is an absolute constant $C$ such that $A \leq C B$.
Proof of Theorem 3. By [12], it suffices to prove that $[f]_{\mathbb{N}}$ is 2-invariant, which is equivalent to proving that $f \in[S f]_{\mathbb{N}}$.

Let $\epsilon, \gamma>0$, where $\gamma$ will be taken small. Let $E$ be a closed subset of $\mathcal{Z}(f)$ such that $|E|=0$ and $E \notin(C)$. Let $p_{\epsilon}$ be the outer function satisfying

$$
\left|p_{\epsilon}(\zeta)\right|=\frac{e^{-M_{\epsilon}}}{\left(d(\zeta, E)^{\gamma}+\epsilon\right)^{1 / 2}} \quad \text { a.e. on } \mathbb{T},
$$

where the constant $M_{\epsilon}$ is chosen so that $p_{\epsilon}(0)=1$. Thus

$$
M_{\epsilon}:=\frac{1}{2} \int_{\mathbb{T}} \log \left(\frac{1}{d(\zeta, E)^{\gamma}+\epsilon}\right) \frac{|d \zeta|}{2 \pi},
$$

and since $E \notin(C)$, it follows that $M_{\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0^{+}$. By Lemma 4, it suffices to prove that

$$
\lim _{\epsilon \rightarrow 0^{+}}\left\|p_{\epsilon} f\right\|_{\mathcal{D}(\mathbb{T})}=0
$$

Now

$$
\left\|p_{\epsilon} f\right\|_{\mathcal{D}(\mathbb{T})}^{2}=\left\|p_{\epsilon} f\right\|_{L^{2}(\mathbb{T})}^{2}+\mathcal{D}\left(p_{\epsilon} f\right) .
$$

For the first term, we have

$$
\left\|p_{\epsilon} f\right\|_{L^{2}(\mathbb{T})}^{2} \lesssim e^{-2 M_{\epsilon}} \int_{\mathbb{T}} \frac{d(\zeta, E)^{2 \beta}}{d(\zeta, E)^{\gamma}}|d \zeta| .
$$

Thus $\left\|p_{\epsilon} f\right\|_{L^{2}(\mathbb{T})} \rightarrow 0$ as $\epsilon \rightarrow 0^{+}$, provided that $\gamma<2 \beta$.
For the second term we again use Douglas' formula, namely

$$
\mathcal{D}\left(p_{\epsilon} f\right)=\frac{1}{4 \pi^{2}} \iint_{\mathbb{T}^{2}} \frac{\left|\left(p_{\epsilon} f\right)(\zeta)-\left(p_{\epsilon} f\right)\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\left|d \zeta^{\prime}\right||d \zeta|
$$

Let

$$
\Gamma:=\left\{\left(\zeta, \zeta^{\prime}\right) \in \mathbb{T}^{2}: d\left(\zeta^{\prime}, E\right) \leq d(\zeta, E)\right\} .
$$

Arguing as in the proof of Theorem 2, we have

$$
\begin{aligned}
\mathcal{D}\left(p_{\epsilon} f\right) \lesssim & \iint_{\Gamma}\left|p_{\epsilon}(\zeta)\right|^{2} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}|d \zeta|\left|d \zeta^{\prime}\right| \\
& +\iint_{\Gamma}\left|f\left(\zeta^{\prime}\right)\right|^{2} \frac{\left|p(\zeta)-p\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}|d \zeta|\left|d \zeta^{\prime}\right| \\
= & A_{\epsilon}+B_{\epsilon}, \text { say. }
\end{aligned}
$$

To estimate $A_{\epsilon}$, let $\eta \in(0,(2 \beta-1) /(2 \beta))$ and let $\gamma<2 \beta \eta$. Then

$$
\begin{aligned}
A_{\epsilon} & =e^{-2 M_{\epsilon}} \iint_{\Gamma}\left(\frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2 \eta}}{d(\zeta, E)^{\gamma}+\epsilon}\right)\left(\frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2-2 \eta}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\right)|d \zeta|\left|d \zeta^{\prime}\right| \\
& \leq 2 C e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} \frac{d(\zeta, E)^{2 \beta \eta}}{d(\zeta, E)^{\gamma}}\left(\frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2-2 \eta}}{\left|\zeta-\zeta^{\prime}\right|^{2}}\right)|d \zeta|\left|d \zeta^{\prime}\right| \\
& \leq 2 C^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} \frac{\left|f(\zeta)-f\left(\zeta^{\prime}\right)\right|^{2-2 \eta}}{\left|\zeta-\zeta^{\prime}\right|^{2}}|d \zeta|\left|d \zeta^{\prime}\right|,
\end{aligned}
$$

where $C$ depends only on $f$, and the double integral is finite, thanks to Lemma 6. Hence $A_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^{+}$.

To estimate $B_{\epsilon}$, we introduce the outer function $F_{\epsilon}$ satisfying

$$
\left|F_{\epsilon}(\zeta)\right|=d(\zeta, E)^{\gamma}+\epsilon \quad \text { a.e. on } \mathbb{T} \text {. }
$$

By the Carleson-Richter-Sundberg formula [6, Theorem 7.4.2], we have

$$
\mathcal{D}_{\zeta}\left(F_{\epsilon}\right)=\int_{\mathbb{T}} \frac{\left|F_{\epsilon}(\zeta)\right|^{2}-\left|F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}-2\left|F_{\epsilon}\left(\zeta^{\prime}\right)\right| \log \left|F_{\epsilon}(\zeta) / F_{\epsilon}\left(\zeta^{\prime}\right)\right|}{\left|\zeta-\zeta^{\prime}\right|^{2}} \frac{\left|d \zeta^{\prime}\right|}{2 \pi} .
$$

Therefore

$$
\begin{aligned}
& B_{\epsilon} \lesssim e^{-2 M_{\epsilon}} \iint_{\Gamma} \frac{d\left(\zeta^{\prime}, E\right)^{2 \beta}}{\left(d(\zeta, E)^{\gamma}+\epsilon\right)\left(d\left(\zeta^{\prime}, E\right)^{\gamma}+\epsilon\right)} \frac{\left|F_{\epsilon}(\zeta)-F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}}|d \zeta|\left|d \zeta^{\prime}\right| \\
& \lesssim e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} \frac{\left|F_{\epsilon}(\zeta)-F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}}{\left|\zeta-\zeta^{\prime}\right|^{2}} d\left(\zeta^{\prime}, E\right)^{2(\beta-\gamma)}|d \zeta|\left|d \zeta^{\prime}\right| \\
& \lesssim e^{-2 M_{\epsilon}} \iint_{\mathbb{T}} \mathcal{D}_{\zeta}\left(F_{\epsilon}\right) d(\zeta, E)^{2(\beta-\gamma)}|d \zeta| \\
& \lesssim e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}}\left(\frac{\left|F_{\epsilon}(\zeta)\right|^{2}-\left|F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}-2\left|F_{\epsilon}\left(\zeta^{\prime}\right)\right| \log \left|F_{\epsilon}(\zeta) / F_{\epsilon}\left(\zeta^{\prime}\right)\right|}{\left|\zeta-\zeta^{\prime}\right|^{2}}\right) \\
& \quad \times\left(d(\zeta, E)^{2(\beta-\gamma)}+d\left(\zeta^{\prime}, E\right)^{2(\beta-\gamma)}\right)|d \zeta|\left|d \zeta^{\prime}\right| .
\end{aligned}
$$

Exchanging the roles of $\zeta$ and $\zeta^{\prime}$, and taking the average, we obtain

$$
\begin{aligned}
B_{\epsilon} \lesssim e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} & \left(\frac{\left(\left|F_{\epsilon}(\zeta)\right|^{2}-\left|F_{\epsilon}\left(\zeta^{\prime}\right)\right|^{2}\right) \log \left|F_{\epsilon}(\zeta) / F_{\epsilon}\left(\zeta^{\prime}\right)\right|}{\left|\zeta-\zeta^{\prime}\right|^{2}}\right) \\
& \times\left(d(\zeta, E)^{2(\beta-\gamma)}+d\left(\zeta^{\prime}, E\right)^{2(\beta-\gamma)}\right)|d \zeta|\left|d \zeta^{\prime}\right| .
\end{aligned}
$$

Thus

$$
\begin{equation*}
B_{\epsilon} \lesssim e^{-2 M_{\epsilon}} \iint_{\mathbb{T}^{2}} \frac{\delta^{\gamma}-\delta^{\prime \gamma}}{\left|\zeta-\zeta^{\prime}\right|^{2}} \log \left(\frac{\delta^{\gamma}+\epsilon}{\delta^{\prime \gamma}+\epsilon}\right)\left(\delta^{2(\beta-\gamma)}+\delta^{\prime 2(\beta-\gamma)}\right)|d \zeta|\left|d \zeta^{\prime}\right|, \tag{3.1}
\end{equation*}
$$

where $\delta:=d(\zeta, E)$ and $\delta^{\prime}:=d\left(\zeta^{\prime}, E\right)$.

Let $\left(I_{j}\right)$ be the connected components of $\mathbb{T} \backslash E$, and set

$$
N_{E}(t):=2 \sum_{j} 1_{\left\{\left|I_{j}\right|>2 t\right\}}, \quad 0<t<1 .
$$

Then, for every measurable function $\Omega:[0, \pi] \rightarrow \mathbb{R}^{+}$, we have

$$
\int_{\mathbb{T}} \Omega(d(\zeta, E))|d \zeta|=\int_{0}^{\pi} \Omega(t) N_{E}(t) d t
$$

Using similar ideas to those in $[4,5]$, we obtain

$$
\begin{aligned}
J & :=\iint_{\mathbb{T}^{2}} \frac{\delta^{\gamma}-\delta^{\prime \gamma}}{\left|\zeta-\zeta^{\prime}\right|^{2}} \log \left(\frac{\delta^{\gamma}+\epsilon}{\delta^{\prime \gamma}+\epsilon}\right)\left(\delta^{2(\beta-\gamma)}+\delta^{\prime 2(\beta-\gamma)}\right)|d \zeta|\left|d \zeta^{\prime}\right| \\
\lesssim & \int_{0}^{\pi} \int_{0}^{\pi} \frac{\left((s+t)^{\gamma}-t^{\gamma}\right)}{s^{2}} \log \left(\frac{(s+t)^{\gamma}+\epsilon}{t^{\gamma}+\epsilon}\right)(t+s)^{2(\beta-\gamma)} N_{E}(t) d s d t \\
\lesssim & \int_{0}^{\pi} \int_{0}^{t} \frac{\left((s+t)^{\gamma}-t^{\gamma}\right) \log \left[(s+t)^{\gamma} / t^{\gamma}\right]}{s^{2}}(t+s)^{2(\beta-\gamma)} N_{E}(t) d s d t \\
& +\int_{0}^{\pi} \int_{t}^{\pi} \frac{\left((s+t)^{\gamma}-t^{\gamma}\right)}{s^{2}} \log \left[1 /\left(t^{\gamma}+\epsilon\right)\right](t+s)^{2(\beta-\gamma)} d s N_{E}(t) d t \\
= & J_{1}+J_{2},
\end{aligned}
$$

where

$$
\begin{aligned}
J_{1} & \lesssim \int_{0}^{\pi} t^{2 \beta-\gamma-1} \int_{0}^{1} \frac{\left((1+x)^{\gamma}-1\right)}{x^{2}} \log (1+x) d x N_{E}(t) d t \\
& \lesssim \int_{0}^{\pi} t^{2 \beta-\gamma-1} N_{E}(t) d t=O(1)
\end{aligned}
$$

and

$$
\begin{aligned}
J_{2} & \lesssim \int_{0}^{\pi} t^{2 \beta-\gamma-1} \log \left[1 /\left(t^{\gamma}+\epsilon\right)\right] \int_{1}^{\pi / t} \frac{\left.(1+x)^{\gamma}-1\right)}{s^{2}}(1+x)^{2(\beta-\gamma)} d x N_{E}(t) d t \\
& \lesssim \int_{0}^{\pi} \log \left[1 /\left(t^{\gamma}+\epsilon\right)\right] N_{E}(t) d t \\
& \lesssim \int_{\mathbb{T}}\left|\log \left(d(\zeta, E)^{\gamma}+\epsilon\right)\right||d \zeta|=2 M_{\epsilon} .
\end{aligned}
$$

Thus $J=O\left(M_{\epsilon}\right)$. Combining this with the estimate (3.1), we get

$$
B_{\epsilon} \lesssim M_{\epsilon} e^{-2 M_{\epsilon}} .
$$

Hence $B_{\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0^{+}$. This completes the proof of Theorem 3 .

## 4. Concluding remarks

1. In order to produce cyclic functions for $\mathcal{D}(\mathbb{T})$ using Theorem 3, we need to construct closed subsets $E \subset \mathbb{T}$ such that $E \notin(C)$ and $c(E)=0$. An easy example can be given by countable sets. Indeed, taking $E_{\beta}:=\left\{e^{i /(\log n)^{\beta}}: n \geq 2\right\}$ with $\beta \leq 1$ provides such an example. Using Cantor-type sets, it is also possible to construct perfect sets $E$ such that $E \notin(C)$ and $c(E)=0$.
2. One can consider weighted harmonic Dirichlet spaces instead of the classical harmonic Dirichlet space. More precisely, given $\alpha \in[0,1)$, the weighted harmonic Dirichlet space $\mathcal{D}_{\alpha}(\mathbb{T})$ is the space of functions $f \in L^{2}(\mathbb{T})$ such that

$$
\|f\|_{\mathcal{D}_{\alpha}(\mathbb{T})}^{2}:=\sum_{n \in \mathbb{Z}}|\widehat{f}(n)|^{2}(1+|n|)^{1-\alpha}<\infty .
$$

We define the $\alpha$-capacity of a Borel subset $E \subset \mathbb{T}$ by

$$
c_{\alpha}(E)=1 / \inf \left\{I_{\alpha}(\mu): \mu \in \mathcal{P}(E)\right\},
$$

where $\mathcal{P}(E)$ is the set of all probability measures supported on a compact subset of $E$ and $I_{\alpha}(\mu):=\sum_{n \geq 1}|\widehat{\mu}(n)|^{2} / n^{1-\alpha}$ is the $\alpha$-energy of $\mu$. We say that a property holds $c_{\alpha}$-quasi-everywhere if it holds everywhere outside a set of $c_{\alpha}$-capacity zero.

It is well known that $\operatorname{Lip}_{\beta}(\mathbb{T}) \subset \mathcal{D}_{\alpha}(\mathbb{T})$ if and only $\beta>(1-\alpha) / 2$. Theorem 3 may be extended to show that, if $f \in \operatorname{Lip}_{\beta}(\mathbb{T})$, where $\beta \in$ $((1-\alpha) / 2,1]$, and if $\mathcal{Z}(f) \notin(C)$, then

$$
[f]_{\mathbb{N}}=\left\{g \in \mathcal{D}(\mathbb{T}):\left.g\right|_{\mathcal{Z}(f)}=0 c_{\alpha} \text {-quasi-everywhere }\right\}
$$

3. One can equally well consider the holomorphic Dirichlet space, namely $\mathcal{D}:=\{f \in \mathcal{D}(\mathbb{T}): \widehat{f}(n)=0(n<0)\}$. Here too the problem of characterizing the cyclic functions is still open. For more on this topic, see e.g. [6, Chapter 9].

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## References

[1] A. Aleman, S. Richter, W. Ross, Bergman spaces on disconnected domains, Canad. J. Math. 48 (1996), no. 2, 225-243.
[2] J. E. Brennan, Approximation in the mean by polynomials on non-Carathéodory domains, Ark. Mat. 15 (1977), no. 1, 117-168.
[3] J. Douglas, Solution of the problem of Plateau, Trans. Amer. Math. Soc. 33 (1931), no. 1, 263-321.
[4] O. El-Fallah, K. Kellay, T. Ransford, On the Brown-Shields conjecture for cyclicity in the Dirichlet space, Adv. Math. 222 (2009), no. 6, 2196-2214.
[5] O. El-Fallah, K. Kellay, T. Ransford, Cantor sets and cyclicity in weighted Dirichlet spaces, J. Math. Anal. Appl. 372 (2010), no. 2, 565-573.
[6] O. El-Fallah, K. Kellay, J. Mashreghi, T. Ransford, A Primer on the Dirichlet Space, Cambridge University Press, Cambridge, 2014.
[7] N. G. Makarov, Invariant subspaces of the space $\mathcal{C}^{\infty}$, Mat. Sb. (N.S.) $119(161)$ (1982), no. 1, 1-31, 160.
[8] N. G. Makarov, Sets of simple invariance. Investigations on linear operators and the theory of functions, X, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 107 (1982), 104-135, 230.
[9] N. G. Makarov, Sets of 1-invariance and 1-invariant subspaces (smooth functions), Dokl. Akad. Nauk SSSR 262 (1982), no. 5, 1072-1075.
[10] N. K. Nikolskii, Treatise on the Shift Operator, Springer, Berlin, 1986.
[11] N. K. Nikolski. Operators, Functions, and Systems: An Easy Reading, Vol. 1: Hardy, Hankel, and Toeplitz, Amer. Math. Soc., Providence RI, 2002.
[12] S. Richter, W. Ross, C. Sundberg, Hyperinvariant subspaces of the harmonic Dirichlet space, J. Reine Angew. Math. 448 (1994), 1-26.
[13] W. Ross, Invariant subspaces of the harmonic Dirichlet space with large codimension, Proc. Amer. Math. Soc. 124 (1996), no. 6, 1841-1846.

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