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# CYCLICITY IN THE HARMONIC DIRICHLET SPACE

E. ABAKUMOV, O. EL-FALLAH, K. KELLAY, AND T. RANSFORD

ABSTRACT. The harmonic Dirichlet space is the Hilbert space of functions  $f \in L^2(\mathbb{T})$  such that

$$\|f\|_{\mathcal{D}(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} (1 + |n|) |\widehat{f}(n)|^2 < \infty.$$

We give sufficient conditions for  $f$  to be cyclic in  $\mathcal{D}(\mathbb{T})$ , in other words, for  $\{\zeta^n f(\zeta) : n \geq 0\}$  to span a dense subspace of  $\mathcal{D}(\mathbb{T})$ .

## 1. INTRODUCTION

Let  $\mathcal{X}$  be a topological linear space of complex functions on the unit circle  $\mathbb{T}$  such that the shift operator  $S$ , given by

$$S(f)(\zeta) := \zeta f(\zeta), \quad f \in \mathcal{X},$$

is an isomorphism of  $\mathcal{X}$  onto itself.

A closed subspace  $\mathcal{M}$  of  $\mathcal{X}$  is called *invariant* if  $S(\mathcal{M}) \subset \mathcal{M}$ . It is said to be *1-invariant* (or *simply invariant*) for  $S$  if  $S(\mathcal{M}) \subsetneq \mathcal{M}$ , and it is called *2-invariant* (or *doubly invariant*) if  $S(\mathcal{M}) = \mathcal{M}$ . The latter condition is equivalent to the invariance of  $\mathcal{M}$  under multiplication by both  $\zeta$  and  $\bar{\zeta}$ .

Let  $\mathbb{Z}$  denote the integers and let  $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 0\}$ . Given  $f \in \mathcal{X}$ , we write

$$\begin{aligned} [f]_{\mathbb{N}} &:= \overline{\text{Span}}^{\mathcal{X}} \{z^n f : n \in \mathbb{N}\}, \\ [f]_{\mathbb{Z}} &:= \overline{\text{Span}}^{\mathcal{X}} \{z^n f : n \in \mathbb{Z}\}. \end{aligned}$$

A function  $f$  is said to be *1-invariant* if the space  $[f]_{\mathbb{N}}$  is 1-invariant for  $S$ . We say that a function  $f \in \mathcal{X}$  is *cyclic* (resp. *bicyclic*) for  $\mathcal{X}$  if  $[f]_{\mathbb{N}} = \mathcal{X}$  (resp.  $[f]_{\mathbb{Z}} = \mathcal{X}$ ).

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Let us begin with the classical case, namely  $\mathcal{X} = L^2(\mathbb{T})$ . By a well-known theorem of Wiener, the 2-invariant subspaces have the form

$$\mathcal{M} = \{f \in L^2(\mathbb{T}) : f = 0 \text{ a.e. on } \mathbb{T} \setminus \sigma\},$$

where  $\sigma$  is a Borel subset of  $\mathbb{T}$  (see e.g. [11, p.8, Theorem 1.2.1]). It follows from Szegő's infimum theorem that a function  $f$  is 1-invariant in  $L^2(\mathbb{T})$  if and only if  $\log |f| \in L^1(\mathbb{T})$  (see e.g. [10, p.12, Corollary 4]).

For  $\mathcal{X} = C^\infty(\mathbb{T})$ , Makarov [7] gave a complete description of the invariant subspaces of  $S$ . He also obtained the following characterization of 1-invariant functions of  $C^\infty(\mathbb{T})$ .

**Theorem** (Makarov [7, p.3]). *A function  $f$  is 1-invariant in  $C^\infty(\mathbb{T})$  if and only if  $\log |f| \in L^1(\mathbb{T})$ .*

The case  $\mathcal{X} = C^n(\mathbb{T})$  ( $n \geq 1$ ) is more complicated, and no characterization of 1-invariant functions is known ([7, p.3], see also [8, Theorem 1.3] or [9, Theorem 5]).

We shall focus our attention on the *harmonic Dirichlet space*  $\mathcal{D}(\mathbb{T})$ . This is the set of functions  $f \in L^2(\mathbb{T})$  whose Fourier coefficients satisfy

$$\mathcal{D}(f) := \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 |n| < \infty.$$

It becomes a Hilbert space if endowed with the norm  $\|\cdot\|_{\mathcal{D}(\mathbb{T})}$ , given by

$$\|f\|_{\mathcal{D}(\mathbb{T})}^2 := \|f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(f) = \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 (1 + |n|).$$

According to Douglas' formula [3], we have

$$\mathcal{D}(f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

This can also be written

$$\mathcal{D}(f) = \frac{1}{2\pi} \int_{\mathbb{T}} \mathcal{D}_\zeta(f) |d\zeta|,$$

where  $\mathcal{D}_\zeta(f)$  is the so-called local Dirichlet integral of  $f$  at  $\zeta$ , given by

$$\mathcal{D}_\zeta(f) := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'|, \quad \zeta \in \mathbb{T}.$$

As  $\mathcal{D}(\mathbb{T})$  is dense in  $L^2(\mathbb{T})$ , if a function  $f$  is cyclic for  $\mathcal{D}(\mathbb{T})$ , then it is also cyclic for  $L^2(\mathbb{T})$ . So by Szegő's theorem we have

$$f \text{ is cyclic for } \mathcal{D}(\mathbb{T}) \Rightarrow \int_{\mathbb{T}} \log |f(\zeta)| |d\zeta| = -\infty.$$

In [12], Ross, Richter and Sundberg gave a complete characterization of the 2-invariant subspaces  $\mathcal{M}$  of  $\mathcal{D}(\mathbb{T})$  in terms of their zero

sets. In order to state their result, we need to introduce the notion of logarithmic capacity (see for instance [6, §2.4]).

The *energy* of a Borel probability measure  $\mu$  on  $\mathbb{T}$  is defined by

$$I(\mu) := \iint_{\mathbb{T}^2} \log \frac{1}{|\zeta - \zeta'|} d\mu(\zeta) d\mu(\zeta') = \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n}.$$

Then we define the *logarithmic capacity* of a Borel subset  $E$  of  $\mathbb{T}$  by

$$c(E) := 1 / \inf\{I(\mu) : \mu \in \mathcal{P}(E)\},$$

where  $\mathcal{P}(E)$  denotes the set of probability measures supported on a compact subset of  $E$ . We say that a property holds *quasi-everywhere* (*q.e.*) if it holds everywhere outside a set of logarithmic capacity zero.

It is known that, if  $f \in \mathcal{D}(\mathbb{T})$ , then the radial limit of the Poisson integral of  $f$  exists q.e. and is equal to  $f$  a.e. (for more details we refer to [12] and the references therein). In the sequel,  $f$  will denote this limit, and will therefore be defined q.e. on  $\mathbb{T}$ . We shall write  $\mathcal{Z}(f)$  for the zero set of  $f$ , namely

$$\mathcal{Z}(f) := \{\zeta \in \mathbb{T} : f(\zeta) = 0\}.$$

Note that this set is defined up to sets of logarithmic capacity zero.

**Theorem** (Richter–Ross–Sundberg [12]).  *$\mathcal{M}$  is a 2-invariant subspace of  $\mathcal{D}(\mathbb{T})$  if and only if there exists a measurable set  $E \subset \mathbb{T}$  such that*

$$\mathcal{M} = \mathcal{D}_E := \{f \in \mathcal{D}(\mathbb{T}) : f|_E = 0 \text{ q.e.}\}.$$

Note that the problem of characterization of 1-invariant subspaces of  $\mathcal{D}(\mathbb{T})$  remains open. It was proved in [1, 13] that, for each  $n \in \mathbb{N} \cup \{\infty\}$ , there exists an invariant subspace  $\mathcal{M}$  of  $\mathcal{D}(\mathbb{T})$  such that  $\dim(\mathcal{M}/S(\mathcal{M})) = n$ . This suggests that the lattice of 1-invariant subspaces has a very complicated structure.

As a direct consequence of the Richter–Ross–Sundberg theorem, we obtain the following necessary conditions for cyclicity in  $\mathcal{D}(\mathbb{T})$ .

**Theorem 1.** *If  $f$  is cyclic for  $\mathcal{D}(\mathbb{T})$ , then*

$$\int_{\mathbb{T}} \log |f(\zeta)| |d\zeta| = -\infty \quad \text{and} \quad c(\mathcal{Z}(f)) = 0.$$

Our goal in this paper is to give sufficient conditions for a function  $f \in \mathcal{D}(\mathbb{T})$  to be cyclic.

For  $\beta \in (0, 1]$ , we shall denote by  $\text{Lip}_\beta(\mathbb{T})$  the set of functions  $f$  continuous on  $\mathbb{T}$  such that

$$\|f\|_{\text{Lip}_\beta(\mathbb{T})} := \|f\|_{C(\mathbb{T})} + \sup_{\zeta, \zeta' \in \mathbb{T}} \frac{|f(\zeta) - f(\zeta')|}{|\zeta - \zeta'|^\beta} < \infty.$$

For  $\alpha \in (0, 1)$ , we set

$$\mathcal{C}^{1+\alpha}(\mathbb{T}) := \{f \in C^1(\mathbb{T}) : f' \in \text{Lip}_\alpha(\mathbb{T})\}.$$

Of course, if  $f$  belongs to  $\text{Lip}_\beta(\mathbb{T})$  or  $C^{1+\alpha}(\mathbb{T})$ , then  $\mathcal{Z}(f)$  is closed in  $\mathbb{T}$ .

We shall establish the following result.

**Theorem 2.** *Let  $f \in \mathcal{D}(\mathbb{T})$  such that  $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$ , where  $\alpha \in (0, 1)$ . Suppose further that  $\log |f| \notin L^1(\mathbb{T})$ . Then  $[f^2]_{\mathbb{N}} = \mathcal{D}_{\mathcal{Z}(f)}$ .*

Combining Theorems 1 and 2, we deduce

**Corollary.** *Let  $f \in \mathcal{D}(\mathbb{T})$  such that  $|f| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$ , where  $\alpha \in (0, 1)$ . Then the following assertions are equivalent:*

- (1)  $f^2$  is cyclic for  $\mathcal{D}(\mathbb{T})$ ;
- (2)  $\log |f| \notin L^1(\mathbb{T})$  and  $c(\mathcal{Z}(f)) = 0$ .

A closed set  $E \subset \mathbb{T}$  is said to be a *Carleson set* (and we write  $E \in (C)$ ) if

$$\int_{\mathbb{T}} \log \frac{1}{d(\zeta, E)} |d\zeta| < \infty.$$

For background information on Carleson sets, see e.g. [6, §4.4]. Note that, if  $f \in \text{Lip}_\beta(\mathbb{T})$  and  $\mathcal{Z}(f) \notin (C)$ , then  $\log |f| \notin L^1(\mathbb{T})$ .

It is known that  $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$  if and only if  $\beta > 1/2$ . The inclusion  $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$  for  $\beta > 1/2$  can easily be obtained from Douglas' formula.

We shall establish the following theorem.

**Theorem 3.** *Let  $f \in \text{Lip}_\beta(\mathbb{T})$ , where  $\beta \in (\frac{1}{2}, 1]$ . If  $\mathcal{Z}(f) \notin (C)$ , then  $[f]_{\mathbb{N}} = \mathcal{D}_{\mathcal{Z}(f)}$ . If furthermore  $c(\mathcal{Z}(f)) = 0$ , then  $f$  is cyclic for  $\mathcal{D}(\mathbb{T})$ .*

## 2. PROOF OF THEOREM 2

For the proof of Theorem 2, we shall need the following standard result.

**Lemma 4.** *Let  $f \in \mathcal{D}(\mathbb{T})$ . The following assertions are equivalent:*

- (1)  $[f]_{\mathbb{N}} = [f]_{\mathbb{Z}}$ ;
- (2)  $f \in [Sf]_{\mathbb{N}}$ ;
- (3)  $\inf\{\|pf\|_{\mathcal{D}(\mathbb{T})} : p \in H^\infty, pf \in \mathcal{D}(\mathbb{T}) \text{ and } p(0) = 1\} = 0$ .

*Proof.* Since  $S$  is invertible, (1) and (2) are equivalent.

If  $f \in [Sf]_{\mathbb{N}}$ , then there is a sequence  $(p_n)$  of polynomials such that  $p_n(0) = 1$  and  $\|(1 - p_n)f\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$ . This proves that (2) implies (3).

Finally, suppose that (3) holds. Let  $(p_n) \subset H^\infty$  be a sequence such that  $p_n(0) = 1$ ,  $p_n f \in \mathcal{D}(\mathbb{T})$  and  $\|p_n f\|_{\mathcal{D}(\mathbb{T})} \rightarrow 0$ . Writing  $p_n = 1 - zq_n$ , by [12, Proposition 3.4] we have  $zq_n f \in [Sf]_{\mathbb{N}}$ . Since  $zq_n f$  converges to  $f$ , it follows that  $f \in [Sf]_{\mathbb{N}}$ , so that (2) holds.  $\square$

We shall also need the following result, which is a special case of a theorem due to Carleson–Jacobs–Havin–Shamoyan [2, Theorem 6.1].

**Lemma 5.** *Let  $F$  be an outer function on  $\mathbb{D}$  that is continuous on  $\overline{\mathbb{D}}$ . If  $|F| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$ , where  $\alpha \in (0, 1)$ , then  $F \in \text{Lip}_{(1+\alpha)/2}(\overline{\mathbb{D}})$ . Furthermore, the Lipschitz constant associated to  $F$  on  $\overline{\mathbb{D}}$  depends only on the Lipschitz constants and bounds for the derivatives of  $|F|$  on  $\mathbb{T}$ .*

*Proof of Theorem 2.* Let  $p_\epsilon$  be the outer function such that

$$|p_\epsilon(\zeta)| = \frac{e^{-M_\epsilon}}{|f(\zeta)| + \epsilon} \quad \text{a.e. on } \mathbb{T},$$

where the constant  $M_\epsilon$  is chosen so that  $p_\epsilon(0) = 1$ . Thus

$$M_\epsilon = \int_{\mathbb{T}} \log \left( \frac{1}{|f(\zeta)| + \epsilon} \right) \frac{|d\zeta|}{2\pi},$$

and since  $\log |f| \notin L^1(\mathbb{T})$ , it follows that  $M_\epsilon \rightarrow \infty$  as  $\epsilon \rightarrow 0^+$ . We are going to prove that

$$\lim_{\epsilon \rightarrow 0^+} \|p_\epsilon f^2\|_{\mathcal{D}(\mathbb{T})} = 0.$$

If this holds, then by Lemma 4 we have  $[f^2]_{\mathbb{N}} = [f^2]_{\mathbb{Z}}$ , and since clearly  $\mathcal{Z}(f^2) = \mathcal{Z}(f)$ , we can apply the Richter–Ross–Sundberg theorem to obtain the desired result.

We have

$$\|p_\epsilon f^2\|_{\mathcal{D}(\mathbb{T})}^2 = \|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(p_\epsilon f^2).$$

For the first term, we have

$$\|p_\epsilon f^2\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} \frac{e^{-2M_\epsilon} |f|^4 |d\zeta|}{(|f| + \epsilon)^2} \leq e^{-2M_\epsilon} \|f^2\|_{L^2(\mathbb{T})}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0^+.$$

The second term we estimate using Douglas' formula, namely

$$\mathcal{D}(p_\epsilon f^2) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Let

$$\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : |f(\zeta')| \leq |f(\zeta)|\}.$$

Then, by symmetry,

$$\mathcal{D}(p_\epsilon f) = 2 \frac{1}{4\pi^2} \iint_{\Gamma} \frac{|(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Now, for all  $\zeta, \zeta' \in \mathbb{T}$ , we have

$$\begin{aligned} & |(p_\epsilon f^2)(\zeta) - (p_\epsilon f^2)(\zeta')|^2 \\ &= |p_\epsilon(\zeta)(f^2(\zeta) - f^2(\zeta')) + f^2(\zeta')(p_\epsilon(\zeta) - p_\epsilon(\zeta'))|^2 \\ &\leq 2|p_\epsilon(\zeta)|^2 |f^2(\zeta) - f^2(\zeta')|^2 + 2|f^2(\zeta')|^2 |p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2. \end{aligned}$$

Hence

$$\iint_{\Gamma} \frac{|(p_{\epsilon}f^2)(\zeta) - (p_{\epsilon}f^2)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \leq 2A_{\epsilon} + 2B_{\epsilon},$$

where

$$A_{\epsilon} := \iint_{\Gamma} |p_{\epsilon}(\zeta)|^2 \frac{|f^2(\zeta) - f^2(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|$$

and

$$B_{\epsilon} := \iint_{\Gamma} |f^2(\zeta')|^2 \frac{|p_{\epsilon}(\zeta) - p_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

We estimate  $A_{\epsilon}$  directly as follows:

$$\begin{aligned} A_{\epsilon} &= e^{-2M_{\epsilon}} \iint_{\Gamma} \frac{|f(\zeta) + f(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq 4e^{-2M_{\epsilon}} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq 4e^{-2M_{\epsilon}} 4\pi^2 \mathcal{D}(f). \end{aligned}$$

Hence  $A_{\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ .

To estimate  $B_{\epsilon}$ , we consider the outer function  $F_{\epsilon}$  such that

$$|F_{\epsilon}(\zeta)| = |f(\zeta)| + \epsilon \quad \text{a.e. on } \mathbb{T}.$$

By Lemma 5, since  $|F_{\epsilon}| \in \mathcal{C}^{1+\alpha}(\mathbb{T})$ , we have  $F_{\epsilon} \in \text{Lip}_{(1+\alpha)/2}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T})$  and there exists a positive constant  $D$ , depending only on  $|f|$ , such that  $\mathcal{D}(F_{\epsilon}) \leq D$  for all  $\epsilon \in (0, 1)$ . We then have

$$\begin{aligned} B_{\epsilon} &= \iint_{\Gamma} \frac{e^{-2M_{\epsilon}} |f^2(\zeta')|^2}{(|f(\zeta)| + \epsilon)^2 (|f(\zeta')| + \epsilon)^2} \frac{|1/p_{\epsilon}(\zeta) - 1/p_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq e^{-2M_{\epsilon}} \iint_{\Gamma} \frac{|F_{\epsilon}(\zeta) - F_{\epsilon}(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \\ &\leq e^{-2M_{\epsilon}} 4\pi^2 \mathcal{D}(F_{\epsilon}). \end{aligned}$$

Thus  $B_{\epsilon} \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ . This completes the proof of Theorem 2.  $\square$

### 3. PROOF OF THEOREM 3

To prove Theorem 3, we shall need the following additional lemma.

**Lemma 6.** *Let  $f \in \text{Lip}_{\beta}(\mathbb{T})$ , where  $\beta > 1/2$ . Then, for  $\eta \in (0, \frac{2\beta-1}{2\beta})$ , we have*

$$\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| < +\infty.$$

*Proof.* Since  $\beta > 1/2(1 - \eta)$ , we get

$$\iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta| \lesssim \iint_{\mathbb{T}^2} \frac{|d\zeta'| |d\zeta|}{|\zeta - \zeta'|^{2(1-(1-\eta)\beta)}} < \infty. \quad \square$$

Note that here, and in what follows, we write  $A \lesssim B$  to mean that there is an absolute constant  $C$  such that  $A \leq CB$ .

*Proof of Theorem 3.* By [12], it suffices to prove that  $[f]_{\mathbb{N}}$  is 2-invariant, which is equivalent to proving that  $f \in [Sf]_{\mathbb{N}}$ .

Let  $\epsilon, \gamma > 0$ , where  $\gamma$  will be taken small. Let  $E$  be a closed subset of  $\mathcal{Z}(f)$  such that  $|E| = 0$  and  $E \notin (C)$ . Let  $p_\epsilon$  be the outer function satisfying

$$|p_\epsilon(\zeta)| = \frac{e^{-M_\epsilon}}{(d(\zeta, E)^\gamma + \epsilon)^{1/2}} \quad \text{a.e. on } \mathbb{T},$$

where the constant  $M_\epsilon$  is chosen so that  $p_\epsilon(0) = 1$ . Thus

$$M_\epsilon := \frac{1}{2} \int_{\mathbb{T}} \log \left( \frac{1}{d(\zeta, E)^\gamma + \epsilon} \right) \frac{|d\zeta|}{2\pi},$$

and since  $E \notin (C)$ , it follows that  $M_\epsilon \rightarrow \infty$  as  $\epsilon \rightarrow 0^+$ . By Lemma 4, it suffices to prove that

$$\lim_{\epsilon \rightarrow 0^+} \|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})} = 0.$$

Now

$$\|p_\epsilon f\|_{\mathcal{D}(\mathbb{T})}^2 = \|p_\epsilon f\|_{L^2(\mathbb{T})}^2 + \mathcal{D}(p_\epsilon f).$$

For the first term, we have

$$\|p_\epsilon f\|_{L^2(\mathbb{T})}^2 \lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \frac{d(\zeta, E)^{2\beta}}{d(\zeta, E)^\gamma} |d\zeta|.$$

Thus  $\|p_\epsilon f\|_{L^2(\mathbb{T})} \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ , provided that  $\gamma < 2\beta$ .

For the second term we again use Douglas' formula, namely

$$\mathcal{D}(p_\epsilon f) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} \frac{|(p_\epsilon f)(\zeta) - (p_\epsilon f)(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta'| |d\zeta|.$$

Let

$$\Gamma := \{(\zeta, \zeta') \in \mathbb{T}^2 : d(\zeta', E) \leq d(\zeta, E)\}.$$

Arguing as in the proof of Theorem 2, we have

$$\begin{aligned} \mathcal{D}(p_\epsilon f) &\lesssim \iint_{\Gamma} |p_\epsilon(\zeta)|^2 \frac{|f(\zeta) - f(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &\quad + \iint_{\Gamma} |f(\zeta')|^2 \frac{|p_\epsilon(\zeta) - p_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &= A_\epsilon + B_\epsilon, \text{ say.} \end{aligned}$$



To estimate  $A_\epsilon$ , let  $\eta \in (0, (2\beta - 1)/(2\beta))$  and let  $\gamma < 2\beta\eta$ . Then

$$\begin{aligned} A_\epsilon &= e^{-2M_\epsilon} \iint_{\Gamma} \left( \frac{|f(\zeta) - f(\zeta')|^{2\eta}}{d(\zeta, E)^\gamma + \epsilon} \right) \left( \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} \right) |d\zeta| |d\zeta'| \\ &\leq 2C e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{d(\zeta, E)^{2\beta\eta}}{d(\zeta, E)^\gamma} \left( \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} \right) |d\zeta| |d\zeta'| \\ &\leq 2C^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{|f(\zeta) - f(\zeta')|^{2-2\eta}}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'|, \end{aligned}$$

where  $C$  depends only on  $f$ , and the double integral is finite, thanks to Lemma 6. Hence  $A_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ .

To estimate  $B_\epsilon$ , we introduce the outer function  $F_\epsilon$  satisfying

$$|F_\epsilon(\zeta)| = d(\zeta, E)^\gamma + \epsilon \quad \text{a.e. on } \mathbb{T}.$$

By the Carleson–Richter–Sundberg formula [6, Theorem 7.4.2], we have

$$\mathcal{D}_\zeta(F_\epsilon) = \int_{\mathbb{T}} \frac{|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2 - 2|F_\epsilon(\zeta')| \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \frac{|d\zeta'|}{2\pi}.$$

Therefore

$$\begin{aligned} B_\epsilon &\lesssim e^{-2M_\epsilon} \iint_{\Gamma} \frac{d(\zeta', E)^{2\beta}}{(d(\zeta, E)^\gamma + \epsilon)(d(\zeta', E)^\gamma + \epsilon)} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{|F_\epsilon(\zeta) - F_\epsilon(\zeta')|^2}{|\zeta - \zeta'|^2} d(\zeta', E)^{2(\beta-\gamma)} |d\zeta| |d\zeta'| \\ &\lesssim e^{-2M_\epsilon} \int_{\mathbb{T}} \mathcal{D}_\zeta(F_\epsilon) d(\zeta, E)^{2(\beta-\gamma)} |d\zeta| \\ &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \left( \frac{(|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2) \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \right) \\ &\quad \times \left( d(\zeta, E)^{2(\beta-\gamma)} + d(\zeta', E)^{2(\beta-\gamma)} \right) |d\zeta| |d\zeta'|. \end{aligned}$$

Exchanging the roles of  $\zeta$  and  $\zeta'$ , and taking the average, we obtain

$$\begin{aligned} B_\epsilon &\lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \left( \frac{(|F_\epsilon(\zeta)|^2 - |F_\epsilon(\zeta')|^2) \log |F_\epsilon(\zeta)/F_\epsilon(\zeta')|}{|\zeta - \zeta'|^2} \right) \\ &\quad \times \left( d(\zeta, E)^{2(\beta-\gamma)} + d(\zeta', E)^{2(\beta-\gamma)} \right) |d\zeta| |d\zeta'|. \end{aligned}$$

Thus

$$B_\epsilon \lesssim e^{-2M_\epsilon} \iint_{\mathbb{T}^2} \frac{\delta^\gamma - \delta'^\gamma}{|\zeta - \zeta'|^2} \log \left( \frac{\delta^\gamma + \epsilon}{\delta'^\gamma + \epsilon} \right) (\delta^{2(\beta-\gamma)} + \delta'^{2(\beta-\gamma)}) |d\zeta| |d\zeta'|, \quad (3.1)$$

where  $\delta := d(\zeta, E)$  and  $\delta' := d(\zeta', E)$ .

Let  $(I_j)$  be the connected components of  $\mathbb{T} \setminus E$ , and set

$$N_E(t) := 2 \sum_j 1_{\{|I_j| > 2t\}}, \quad 0 < t < 1.$$

Then, for every measurable function  $\Omega : [0, \pi] \rightarrow \mathbb{R}^+$ , we have

$$\int_{\mathbb{T}} \Omega(d(\zeta, E)) |d\zeta| = \int_0^\pi \Omega(t) N_E(t) dt.$$

Using similar ideas to those in [4, 5], we obtain

$$\begin{aligned} J &:= \iint_{\mathbb{T}^2} \frac{\delta^\gamma - \delta'^\gamma}{|\zeta - \zeta'|^2} \log\left(\frac{\delta^\gamma + \epsilon}{\delta'^\gamma + \epsilon}\right) (\delta^{2(\beta-\gamma)} + \delta'^{2(\beta-\gamma)}) |d\zeta| |d\zeta'| \\ &\lesssim \int_0^\pi \int_0^\pi \frac{((s+t)^\gamma - t^\gamma)}{s^2} \log\left(\frac{(s+t)^\gamma + \epsilon}{t^\gamma + \epsilon}\right) (t+s)^{2(\beta-\gamma)} N_E(t) ds dt \\ &\lesssim \int_0^\pi \int_0^t \frac{((s+t)^\gamma - t^\gamma) \log[(s+t)^\gamma/t^\gamma]}{s^2} (t+s)^{2(\beta-\gamma)} N_E(t) ds dt \\ &\quad + \int_0^\pi \int_t^\pi \frac{((s+t)^\gamma - t^\gamma)}{s^2} \log[1/(t^\gamma + \epsilon)] (t+s)^{2(\beta-\gamma)} ds N_E(t) dt \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &\lesssim \int_0^\pi t^{2\beta-\gamma-1} \int_0^1 \frac{((1+x)^\gamma - 1)}{x^2} \log(1+x) dx N_E(t) dt \\ &\lesssim \int_0^\pi t^{2\beta-\gamma-1} N_E(t) dt = O(1), \end{aligned}$$

and

$$\begin{aligned} J_2 &\lesssim \int_0^\pi t^{2\beta-\gamma-1} \log[1/(t^\gamma + \epsilon)] \int_1^{\pi/t} \frac{(1+x)^\gamma - 1}{s^2} (1+x)^{2(\beta-\gamma)} dx N_E(t) dt \\ &\lesssim \int_0^\pi \log[1/(t^\gamma + \epsilon)] N_E(t) dt \\ &\lesssim \int_{\mathbb{T}} |\log(d(\zeta, E)^\gamma + \epsilon)| |d\zeta| = 2M_\epsilon. \end{aligned}$$

Thus  $J = O(M_\epsilon)$ . Combining this with the estimate (3.1), we get

$$B_\epsilon \lesssim M_\epsilon e^{-2M_\epsilon}.$$

Hence  $B_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0^+$ . This completes the proof of Theorem 3.  $\square$

## 4. CONCLUDING REMARKS

**1.** In order to produce cyclic functions for  $\mathcal{D}(\mathbb{T})$  using Theorem 3, we need to construct closed subsets  $E \subset \mathbb{T}$  such that  $E \notin (C)$  and  $c(E) = 0$ . An easy example can be given by countable sets. Indeed, taking  $E_\beta := \{e^{i/(\log n)^\beta} : n \geq 2\}$  with  $\beta \leq 1$  provides such an example. Using Cantor-type sets, it is also possible to construct perfect sets  $E$  such that  $E \notin (C)$  and  $c(E) = 0$ .

**2.** One can consider weighted harmonic Dirichlet spaces instead of the classical harmonic Dirichlet space. More precisely, given  $\alpha \in [0, 1)$ , the *weighted harmonic Dirichlet space*  $\mathcal{D}_\alpha(\mathbb{T})$  is the space of functions  $f \in L^2(\mathbb{T})$  such that

$$\|f\|_{\mathcal{D}_\alpha(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} |\widehat{f}(n)|^2 (1 + |n|)^{1-\alpha} < \infty.$$

We define the  $\alpha$ -*capacity* of a Borel subset  $E \subset \mathbb{T}$  by

$$c_\alpha(E) = 1/\inf\{I_\alpha(\mu) : \mu \in \mathcal{P}(E)\},$$

where  $\mathcal{P}(E)$  is the set of all probability measures supported on a compact subset of  $E$  and  $I_\alpha(\mu) := \sum_{n \geq 1} |\widehat{\mu}(n)|^2/n^{1-\alpha}$  is the  $\alpha$ -*energy* of  $\mu$ . We say that a property holds  $c_\alpha$ -*quasi-everywhere* if it holds everywhere outside a set of  $c_\alpha$ -capacity zero.

It is well known that  $\text{Lip}_\beta(\mathbb{T}) \subset \mathcal{D}_\alpha(\mathbb{T})$  if and only  $\beta > (1 - \alpha)/2$ . Theorem 3 may be extended to show that, if  $f \in \text{Lip}_\beta(\mathbb{T})$ , where  $\beta \in ((1 - \alpha)/2, 1]$ , and if  $\mathcal{Z}(f) \notin (C)$ , then

$$[f]_{\mathbb{N}} = \{g \in \mathcal{D}(\mathbb{T}) : g|_{\mathcal{Z}(f)} = 0 \text{ } c_\alpha\text{-quasi-everywhere}\}.$$

**3.** One can equally well consider the holomorphic Dirichlet space, namely  $\mathcal{D} := \{f \in \mathcal{D}(\mathbb{T}) : \widehat{f}(n) = 0 \text{ } (n < 0)\}$ . Here too the problem of characterizing the cyclic functions is still open. For more on this topic, see e.g. [6, Chapter 9].

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