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ON THE CALDERÒN PROBLEM IN PERIODIC CYLINDRICAL DOMAIN WITH PARTIAL DIRICHLET AND NEUMANN DATA

MOURAD CHOULLI, YAVAR KIAN, ERIC SOCCORSI

Abstract. We consider the Calderòn problem in an infinite cylindrical domain, whose cross section is a bounded domain of the plane. We prove log-log stability in the determination of the isotropic periodic conductivity coefficient from partial Dirichlet data and partial Neumann boundary observations of the solution.

1. Introduction

Let ω be a bounded domain of $\mathbb{R}^2$ which contains the origin, with a C^2 boundary. Set $\Omega := \mathbb{R} \times \omega$ and denote any point $x \in \Omega$ by $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' := (x_2, x_3) \in \omega$. Given $V \in L^\infty(\Omega)$, real-valued and 1-periodic with respect to x_1, i.e.

$$V(x_1 + 1, x') = V(x_1, x'),\ x' \in \omega,\ x_1 \in \mathbb{R}, \tag{1.1}$$

we consider the boundary value problem (BVP) with non-homogeneous Dirichlet data f,

$$\begin{cases} (-\Delta + V)u &= 0, \quad \text{in } \Omega, \\ u &= f, \quad \text{on } \Gamma := \partial\Omega = \mathbb{R} \times \partial\omega. \end{cases} \tag{1.2}$$

Next we fix $\xi_0 \in \mathbb{S}^1 := \{y \in \mathbb{R}^2;\ |y| = 1\}$ and define the ξ_0-shadowed (resp., ξ_0-illuminated) face of $\partial\omega$ as

$$\partial\omega_{\xi_0}^+ := \{x' \in \partial\omega;\ \nu'(x') \cdot \xi_0 \geqslant 0\} \text{ (resp. } \partial\omega_{\xi_0}^- := \{x' \in \partial\omega;\ \nu'(x') \cdot \xi_0 \leqslant 0\}), \tag{1.3}$$

where ν' is the outgoing unit normal vector to $\partial\omega$. Here and henceforth the symbol $\cdot$ denotes the Euclidian scalar product in $\mathbb{R}^k$, $k \geqslant 2$, and $|y| := (y \cdot y)^{\frac{1}{2}}$ for all $y \in \mathbb{R}^k$.

Then for any closed neighborhood G' of $\partial\omega_{\xi_0}^-$ in $\partial\omega$, we know from [CKS2, Theorem 1.1] that knowledge of the partial Dirichlet-to-Neumann (DN) map restricted to $G := \mathbb{R} \times G'$,

$$\Lambda_V : f \mapsto \partial_\nu u_{|G}, \tag{1.4}$$

uniquely and logarithmic-stably determines V. Here we used the usual notation $\partial_\nu u := \nabla u \cdot \nu$, where ∇ denotes the gradient operator with respect to $x \in \Omega$, u is the solution to (1.2), and

$$\nu(x_1, x') := (0, \nu'(x')),\ x = (x_1, x') \in \Gamma,$$

is the outward unit vector normal to Γ. Otherwise stated, the unknown potential V appearing in the first line of (1.2) can be stably recovered from boundary observation of the current flowing through G, upon probing the system (1.2) with non-homogeneous Dirichlet data.

Notice that in the above mentioned result, only the output, i.e. the measurement of the current flowing across Γ, is local (or partial) in the sense that it is performed on G and not on the whole boundary Γ, while the input, i.e. the Dirichlet data, remains global as it is possibly supported everywhere on Γ. Therefore, [CKS2, Theorem 1.1] claims logarithmic stability in the inverse problem of determining the electric potential V in the first line of (1.2) from the knowledge of partial Neumann, and full Dirichlet, data. In the present paper, we aim for the same type of result under the additional constraint that not only the Neumann data, but also the Dirichlet data, be partial. Namely, given an arbitrary closed neighborhood F' of $\partial\omega_\xi^+$ in $\partial\omega$, such that

$$F' \cap G' \neq \emptyset \text{ and } F' \cup G' = \partial\omega, \tag{1.5}$$

we seek stable identification of V by the input-restricted DN map (1.4) to Dirichlet data functions f supported in $F := \mathbb{R} \times F'$.

1.1. **State of the art.** Since the seminal paper [Ca] by Calderón, the electrical impedance tomography problem, or Calderón problem, of retrieving the conductivity from the knowledge of the DN map on the boundary of a bounded domain, has attracted many attention. If the conductivity coefficient is scalar, then the Liouville transform allows us to rewrite the Calderón problem into the inverse problem of determining the electric potential in Laplace operator, from boundary measurements. There is an extensive literature on the Calderón problem. For isotropic conductivities, a great deal of work has been spent to weaken the regularity assumption on the conductivity, in the study of the uniqueness issue, see e.g. [BT, HT]. In all the above mentioned papers, the full DN map are needed, i.e. lateral observations are performed on the whole boundary. The first uniqueness result from partial data for the Calderón problem, was obtained in dimension 3 or greater, by Bukhgeim and Uhlmann in [BU]. Their result, which requires that Dirichlet data be imposed on the whole boundary, and that Neumann data boundary be observed on slightly more than half of the boundary, was improved by Kenig, Sjöstrand and Uhlmann in [KSU], where both input and ouput data are measured on subsets of the boundary. In the two-dimensional case, Imanuvilov, Uhlmann and Yamamoto proved in [IUY1, IUY2] that the partial DN map uniquely determines the conductivity. We also mention that the special case of the Calderón problem in a bounded cylindrical domain of $\mathbb{R}^3$, was treated in [IY1].

The stability issue for the Calderón problem was addressed by Alessandrini in [Al]. He proved a log-type stability estimate with respect to the full DN map. Such a result, which is known to be optimal, see [Ma], degenerates to log-log stability with partial Neumann data, see [HW, CKS1]. In [CDR1, CDR2], Caro, Dos Santos Ferreira and Ruiz proved stability results of log-log type, corresponding to the uniqueness results of [KSU] in dimension 3 or greater. We refer to [BIY, NSa, Sa] for stability estimates associated with the two-dimensional Calderón problem, and we point out that both the electric and the magnetic potentials are stably determined by the partial DN map in [T].

Notice that all the above mentioned papers are concerned with the Calderón problem in a bounded domain. It turns out that there is only a small number of mathematical papers dealing with inverse coefficients problems in an unbounded domain. Several authors considered the problem of recovering coefficients in an unbounded domain from boundary measurements. Object identification in an infinite slab, was proved in [Ik, SW]. Unique determination of a compactly supported electric potential of the Laplace equation in an infinite slab, by partial DN map, is established in [LU]. This result was extended to the magnetic case in [KLU], and to bi-harmonic operators in [Y].

The stability issue in inverse coefficients problems stated in an infinite cylindrical waveguide is addressed in [BKS, CKS, CKS1, CS, KKS, Ki, KPS1, KPS2], and a log-log type stability estimate by partial Neumann data for the periodic electric potential of the Laplace equation can be found in [CKS2]. In the present paper, we are aiming for the same result as in [CKS2], where the full Dirichlet data is replaced by partial voltage. The proof the corresponding stability estimate relies on two different types of complex geometric optics (CGO) solutions to the quasi-periodic Laplace equation in $(0,1)\times\omega$, which are supported in F. These functions are built in Section 3 by means of a suitable Carleman estimate. This technique is inspired by [KSU], but, in contrast to [CDR1, CDR2], due to the quasi-periodic boundary conditions imposed on the CGO solution, we cannot apply the Carleman estimate of [KSU], here.

1.2. **Settings and main result.** We stick with the notations of [CKS2] and denote by C_ω the square root of the first eigenvalue of the Dirichlet Laplacian in $L^2(\omega)$, that is the largest of those positive constants c, such that the Poincaré inequality

$$\|\nabla' u\|_{L^2(\omega)} \geqslant c\|u\|_{L^2(\omega)},\ u \in H_0^1(\omega), \tag{1.6}$$

holds true. Here $\nabla' := (\partial_{x_2}, \partial_{x_3})$ stands for the gradient with respect to $x' = (x_2, x_3)$. This can be equivalently reformulated as

$$C_\omega := \sup\{c > 0 \text{ satisfying (1.6)}\}. \tag{1.7}$$

Next, for $M_- \in (0, C_\omega)$ and $M_+ \in [M_-, +\infty)$, we introduce the set $\mathscr{V}_\omega(M_\pm)$ of admissible unknown potentials in the same way as in [CKS2, Sect. 1.2]:

$$\mathscr{V}_\omega(M_\pm) := \{V \in L^\infty(\Omega;\mathbb{R}) \text{ satisfying (1.1)},\ \|V\|_{L^\infty(\Omega)} \leqslant M_+ \text{ and } \|\max(0,-V)\|_{L^\infty(\Omega)} \leqslant M_-\}. \tag{1.8}$$

Before stating the main result of this paper we need to define the DN map associated with the BVP (1.2) and $V \in \mathscr{V}_\omega(M_\pm)$. To this end, we introduce the Hilbert space $H_\Delta(\Omega) := \{u \in L^2(\Omega);\ \Delta u \in L^2(\Omega)\}$, endowed with the norm

$$\|u\|^2_{H_\Delta(\Omega)} := \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)},$$

and refer to [CKS2, Lemma 2.2] in order to extend the mapping

$$\mathcal{T}_0 u := u_{|\Gamma} \ (\text{resp.}, \mathcal{T}_1 u := \partial_\nu u_{|\Gamma}),\ u \in C_0^\infty(\overline{\Omega}),$$

into a continuous function $\mathcal{T}_0 : H_\Delta(\Omega) \to H^{-2}(\mathbb{R}; H^{-\frac{1}{2}}(\partial\omega))$ (resp., $\mathcal{T}_1 : H_\Delta(\Omega) \to H^{-2}(\mathbb{R}; H^{-\frac{3}{2}}(\partial\omega))$). Since $\mathcal{T}_0$ is one-to-one from $B := \{u \in L^2(\Omega);\ \Delta u = 0\}$ onto

$$\mathscr{H}(\Gamma) := \mathcal{T}_0 H_\Delta(\Omega) = \{\mathcal{T}_0 u;\ u \in H_\Delta(\Omega)\},$$

by [CKS2, Lemma 2.3], we put

$$\|f\|_{\mathscr{H}(\Gamma)} := \left\|\mathcal{T}_0^{-1} f\right\|_{H_\Delta(\Omega)} = \left\|\mathcal{T}_0^{-1} f\right\|_{L^2(\Omega)}, \tag{1.9}$$

where $\mathcal{T}_0^{-1}$ denotes the operator inverse to $\mathcal{T}_0 : B \to \mathscr{H}(\Gamma)$. Throughout this text, we consider Dirichlet data in $\mathscr{H}(\Gamma)$ which are supported in F, i.e. input functions belonging to

$$\mathscr{H}_c(F) := \{f \in \mathscr{H}(\Gamma);\ \text{supp} f \subset F\}.$$

To any $f \in \mathscr{H}_c(F)$, we associate the unique solution $u \in H_\Delta(\Omega)$ to (1.2), given by [CKS2, Proposition 1.1 (i)], and define the partial DN map associated with (1.2), as

$$\Lambda_V : f \in \mathscr{H}_c(F) \mapsto \mathcal{T}_1 u_{|G}. \tag{1.10}$$

Upon denoting by $\mathcal{B}(X_1, X_2)$, where X_j, $j = 1, 2$, are two arbitrary Banach spaces, the class of bounded operators $T : X_1 \to X_2$, we recall from [CKS2, Proposition 1.1 (ii)-(iii)] that

$$\Lambda_V \in \mathcal{B}(\mathscr{H}_c(F), H^{-2}(\mathbb{R}, H^{-\frac{3}{2}}(G'))) \text{ and } \Lambda_V - \Lambda_W \in \mathcal{B}(\mathscr{H}_c(F), L^2(G)),\ V,\ W \in \mathscr{V}_\omega(M_\pm). \tag{1.11}$$

The main result of this article, which claims that unknown potentials of $\mathscr{V}_\omega(M_\pm)$ are stably determined in the elementary cell $\check{\Omega} := (0,1) \times \omega$, by the partial DN map, is stated as follows.

Theorem 1.1. *Let $V_j \in \mathscr{V}_\omega(M_\pm)$, $j = 1, 2$, where $M_+ \in [M_-, +\infty)$, $M_- \in (0, C_\omega)$, and C_ω is defined by (1.7). Then, there exist two constants $C > 0$ and $\gamma_* > 0$, both of them depending only on ω, $M_\pm$, F', and G', such that the estimate*

$$\|V_1 - V_2\|_{H^{-1}(\check{\Omega})} \leqslant C\Phi\left(\|\Lambda_{V_1} - \Lambda_{V_2}\|\right), \tag{1.12}$$

holds with

$$\Phi(\gamma) := \begin{cases} \gamma & \text{if } \gamma \geqslant \gamma^*, \\ (\ln|\ln\gamma|)^{-1} & \text{if } \gamma \in (0, \gamma^*), \\ 0 & \text{if } \gamma = 0. \end{cases} \tag{1.13}$$

Here $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathscr{H}_c(F), L^2(G))$.

The statement of Theorem 1.1 remains valid for any periodic potential $V \in L^\infty(\Omega)$, provided 0 lies in the resolvent set of A_V, the self-adjoint realization in $L^2(\Omega)$ of the Dirichlet Laplacian $-\Delta + V$. In this case, the multiplicative constants C and γ_*, appearing in (1.12)-(1.13), depend on (the inverse of) the distance $d > 0$, between 0 and the spectrum of A_V. In the particular case where $V \in \mathscr{V}_\omega(M_\pm)$, with $M_- \in (0, C_\omega)$, we have $d \geqslant C_\omega - M_-$, and the implicit condition $d > 0$ imposed on V, can be replaced by the explicit one on the negative part of the potential, i.e. $\|\max(0,-V)\|_{L^\infty(\Omega)} \leqslant M_-$.

1.3. **Application to the Calderón Problem.** The inverse problem addressed in Subsection 1.2 is closely related to the periodic Calderón problem in Ω, i.e. the inverse problem of determining the conductivity coefficient a, obeying

$$a(x_1+1,x')=a(x_1,x'),\ x'\in\omega,\ x_1\in\mathbb{R}, \tag{1.14}$$

from partial boundary data of the BVP in the divergence form

$$\begin{cases} -\mathrm{div}(a\nabla u) &= 0, \quad \text{in } \Omega,\\ u &= f, \quad \text{on } \Gamma. \end{cases} \tag{1.15}$$

Let $\mathcal{T}_0$ denote the trace operator $u\mapsto u_{|\Gamma}$ on $H^1(\Omega)$. We equip the space $\mathscr{K}(\Gamma):=\mathcal{T}_0(H^1(\Omega))$ with the norm

$$\|f\|_{\mathscr{K}(\Gamma)}:=\inf\{\|u\|_{H^1(\Omega)}\,;\ \mathcal{T}_0u=f\},$$

and recall for any $a\in C^1(\overline{\Omega})$ satisfying the ellipticity condition

$$a(x)\geqslant a_*>0,\ x\in\Omega, \tag{1.16}$$

for some fixed positive constant a_*, that the BVP (1.15) admits a unique solution $u\in H^1(\Omega)$ for each $f\in\mathscr{K}(\Gamma)$. Moreover, the full DN map associated with (1.15), defined by $f\mapsto a\mathcal{T}_1u$, where $\mathcal{T}_1u:=\partial_\nu u_{|\Gamma}$, is a bounded operator from $\mathscr{K}(\Gamma)$ to $H^{-1}(\mathbb{R};H^{-\frac12}(\partial\omega))$. Here, we rather consider the partial DN map,

$$\Sigma_a: f\in\mathscr{K}(\Gamma)\cap a^{-\frac12}(\mathscr{H}_c(F))\mapsto a\mathcal{T}_1u_{|G}, \tag{1.17}$$

where $a^{-\frac12}(\mathscr{H}_c(F)):=\{a^{-\frac12}f;\ f\in\mathscr{H}_c(F)\}$.

Further, since the BVP (1.15) is brought by the Liouville transform into the form (1.2), with $V_a:=a^{-\frac12}\Delta a^{\frac12}$, then, with reference to (1.8), we impose that V_a be bounded in Ω and satisfies the following conditions

$$\|V_a\|_{L^\infty(\Omega)}\leqslant M_+ \text{ and } \|\max(0,-V_a)\|_{L^\infty(\Omega)}\leqslant M_-, \tag{1.18}$$

where $M_-\in(0,C_\omega)$ and $M_+\in[M_-,+\infty)$ are *a priori* arbitrarily fixed constants. Namely, we introduce the set of admissible conductivities, as

$$\mathscr{A}_\omega(a_*,M_\pm):=\left\{a\in C^1(\overline{\Omega};\mathbb{R}) \text{ satisfying } \Delta a\in L^\infty(\Omega),\ \|a\|_{W^{1,\infty}(\Omega)}\leqslant M_+, (1.14), (1.16), \text{and } (1.18)\right\}. \tag{1.19}$$

We check by standard computations that the condition (1.18) is automatically verified, provided the conductivity $a\in\mathscr{A}_\omega(a_*,M_\pm)$ is taken so small that $\|a\|^2_{W^{1,\infty}(\Omega)}+2a_*\|\Delta a\|_{L^\infty(\Omega)}\leqslant 4M_-a_*^2$, or even that

$$\|a\|_{W^{2,\infty}(\Omega)}\leqslant\frac{4M_-}{(4M_-+1)^{\frac12}+1}a_*,$$

in the particular case where $a\in W^{2,\infty}(\Omega)$.

The main result of this section claims stable determination of such admissible conductivities a, from the knowledge of Σ_a. It is stated as follows.

Corollary 1.2. *Fix $a_*>0$, and let $M_\pm$ be as in Theorem 1.1. Pick $a_j\in\mathscr{A}_\omega(a_*,M_\pm)$, for $j=1,2$, obeying*

$$a_1(x)=a_2(x),\ x\in\partial\Omega \tag{1.20}$$

and

$$\partial_\nu a_1(x)=\partial_\nu a_2(x),\ x\in F\cap G. \tag{1.21}$$

Then $\Sigma_{a_1}-\Sigma_{a_2}$ is extendable to a bounded operator from $a_1^{-\frac12}(\mathscr{H}_c(F))$ into $L^2(G)$. Moreover, there exists two constant $C>0$ and $\gamma_*>0$, both of them depending only on ω, $M_\pm$, a_*, F', and G', such that we have

$$\|a_1-a_2\|_{H^1(\check{\Omega})}\leqslant C\Phi\left(a_*^{-\frac12}\|\Sigma_{a_1}-\Sigma_{a_2}\|\right), \tag{1.22}$$

where Φ is the same as in Theorem 1.1. Here $\|\cdot\|$ denotes the usual operator norm in $\mathcal{B}(a_1^{-\frac12}(\mathscr{H}_c(F)),L^2(G))$.

1.4. **Floquet decomposition.** In this subsection, we reformulate the inverse problem presented in Subsection 1.2 into a family of inverse coefficients problems associated with the BVP

$$\left\{\begin{array}{rcll} (-\Delta+V)v &=& 0, & \text{in } \check{\Omega}:=(0,1)\times\omega,\\ v &=& g, & \text{on } \check{\Gamma}:=(0,1)\times\partial\omega,\\ v(1,\cdot)-e^{i\theta}v(0,\cdot) &=& 0, & \text{in } \omega,\\ \partial_{x_1}v(1,\cdot)-e^{i\theta}\partial_{x_1}v(0,\cdot) &=& 0, & \text{in } \omega, \end{array}\right. \tag{1.23}$$

for $\theta\in[0,2\pi)$, and suitable Dirichlet data g. This is by means of the Floquet-Bloch-Gel'fand (FBG) transform introduced in [CKS2, Section 3.1]. We stick with the notations of [CKS2, Section 3.1], and, for Y being either ω of $\partial\omega$, we denote by $\mathcal{U}$ the FBG transform from $L^2(\mathbb{R}\times Y)$ onto $\int_{(0,2\pi)}^{\oplus}L^2((0,1)\times Y)\frac{d\theta}{2\pi}$. That is to say, the FBG transform $\mathcal{U}$ maps $L^2(\Omega)$ onto $\int_{(0,2\pi)}^{\oplus}L^2(\check{\Omega})\frac{d\theta}{2\pi}$ if $Y=\omega$, and $L^2(\Gamma)$ onto $\int_{(0,2\pi)}^{\oplus}L^2(\check{\Gamma})\frac{d\theta}{2\pi}$ when $Y=\partial\omega$. We recall that the operator $\mathcal{U}$ is unitary in both cases. We start by introducing several functional spaces and trace operators that are needed by the analysis of the inverse problem associated with (1.23).

1.4.1. *Functional spaces and trace operators.* Fix $\theta\in[0,2\pi)$. With reference to [CKS, Section 6.1] or [CKS2, Section 3.1], we set for each $n\in\mathbb{N}\cup\{\infty\}$,

$$\mathcal{C}_\theta^n\left([0,1]\times\overline{\omega}\right):=\left\{u\in\mathcal{C}^n\left([0,1]\times\overline{\omega}\right);\ \partial_{x_1}^j u(1,\cdot)-e^{i\theta}\partial_{x_1}^j u(0,\cdot)=0 \text{ in } \omega,\ j\leqslant n\right\},$$

and for Y being either ω or $\partial\omega$, we put

$$H_\theta^s((0,1)\times Y):=\left\{u\in H^s((0,1)\times Y);\ \partial_{x_1}^j u(1,\cdot)-e^{i\theta}\partial_{x_1}^j u(0,\cdot)=0 \text{ in } \omega,\ j<s-\frac{1}{2}\right\} \text{ if } s>\frac{1}{2},$$

and

$$H_\theta^s((0,1)\times Y):=H^s((0,1)\times Y) \text{ if } s\in\left[0,\frac{1}{2}\right].$$

Further, we recall from [CKS2, Eq. (3.29)] that $\mathcal{U}H_\Delta(\Omega)=\int_{(0,2\pi)}^{\oplus}H_{\Delta,\theta}(\check{\Gamma})\frac{d\theta}{2\pi}$, where

$$H_{\Delta,\theta}(\check{\Omega}):=\{u\in L^2(\check{\Omega});\ \Delta u\in L^2(\check{\Omega}) \text{ and } u(1,\cdot)-e^{i\theta}u(0,\cdot)=\partial_{x_1}u(1,\cdot)-e^{i\theta}\partial_{x_1}u(0,\cdot)=0 \text{ in } \omega\}.$$

Moreover, the space $\mathcal{C}_\theta^\infty\left([0,2\pi]\times\overline{\omega}\right)$ is dense in $H_{\Delta,\theta}(\check{\Omega})$, and we have $\mathcal{U}\mathcal{T}_j\mathcal{U}^{-1}=\int_{(0,2\pi)}^{\oplus}\mathcal{T}_{j,\theta}\frac{d\theta}{2\pi}$ for $j=0,1$, where the linear bounded operator

$$\mathcal{T}_{j,\theta}:H_{\Delta,\theta}(\check{\Omega})\to H_\theta^{-2}(0,1,H^{-\frac{2j+1}{2}}(\partial\omega)),$$

fulfills $\mathcal{T}_{0,\theta}u=u_{|\check{\Gamma}}$ if $j=0$, and $\mathcal{T}_{1,\theta}u=\partial_\nu u_{|\check{\Gamma}}$ if $j=1$, provided $u\in\mathcal{C}_\theta^\infty\left([0,1]\times\overline{\omega}\right)$. Therefore, putting

$$\mathscr{H}_\theta(\check{\Gamma}):=\{\mathcal{T}_{0,\theta}u;\ u\in H_{\Delta,\theta}(\check{\Omega})\}, \text{ and } \mathscr{H}_{c,\theta}(\check{F}):=\{f\in\mathscr{H}_\theta(\check{\Gamma}),\ \text{supp } f\subset\check{F}\},$$

we get that $\mathcal{U}\mathscr{H}(\Gamma)=\int_{(0,2\pi)}^{\oplus}\mathscr{H}_\theta(\check{\Gamma})\frac{d\theta}{2\pi}$ and $\mathcal{U}\mathscr{H}_c(F)=\int_{(0,2\pi)}^{\oplus}\mathscr{H}_{c,\theta}(\check{F})\frac{d\theta}{2\pi}$. As in [CKS2, Eq. (3.30)], the space $\mathscr{H}_\theta(\check{\Gamma})$ is endowed with the norm $\|g\|_{\mathscr{H}_\theta(\check{\Gamma})}:=\|v_g\|_{L^2(\check{\Omega})}$, where v_g denotes the unique $L^2(\check{\Omega})$-solution to (1.23) with $V=0$, given by [CKS2, Proposition 3.2 (i)].

1.4.2. *Inverse fibered problems.* Let $V\in\mathscr{V}_\omega(M_\pm)$, where $M_\pm$ are as in Theorem 1.1. Then, for any f be in $\mathscr{H}_c(F)$, u is the $H_\Delta(\Omega)$-solution to (1.2), if and only if, for almost every $\theta\in[0,2\pi)$, $(\mathcal{U}u)_\theta$ is the $H_\Delta(\check{\Omega})$-solution to (1.23), associated with $g=(\mathcal{U}f)_\theta\in\mathscr{H}_{c,\theta}(\check{F})$. The corresponding partial DN map, defined by $\Lambda_{V,\theta}:g\in\mathscr{H}_{c,\theta}(\check{F})\mapsto\mathcal{T}_{1,\theta}v_{|\check{G}}$, where v is the unique $H_\Delta(\check{\Omega})$-solution to (1.23), is a bounded operator from $\mathscr{H}_{c,\theta}(\check{F})$ into $H_\theta^{-2}(0,1;H^{-\frac{3}{2}}(G'))$, and we have

$$\mathcal{U}\Lambda_V\mathcal{U}^{-1}=\int_{(0,2\pi)}^{\oplus}\Lambda_{V,\theta}\frac{d\theta}{2\pi}, \tag{1.24}$$

according to [CKS2, Proposition 7.1]. Further, if V_1 and V_2 are two potentials lying in $\mathscr{V}_\omega(M_\pm)$, then $\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta} \in \mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}), L^2(\check{G}))$, for each $\theta \in [0, 2\pi)$, by (1.11) and (1.24). Moreover, $\Lambda_{V_1} - \Lambda_{V_2}$ being unitarily equivalent to the family of partial DN maps $\{\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta},\ \theta \in [0, 2\pi)\}$, it holds true that

$$\|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F), L^2(G))} = \sup_{\theta\in[0,2\pi)} \|\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta}\|_{\mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}), L^2(\check{G}))}. \tag{1.25}$$

Therefore, it is clear from (1.25) that Theorem 1.1 is a byproduct of the following statement.

Theorem 1.3. *Let $M_\pm$ and V_j, $j = 1, 2$, be as in Theorem 1.1. Fix $\theta \in [0, 2\pi)$. Then, there exist two constants $C_\theta > 0$ and $\gamma_{\theta,*} > 0$, both of them depending only on ω, $M_\pm$, F', and G', such that we have*

$$\|V_1 - V_2\|_{H^{-1}(\check{\Omega})} \leqslant C_\theta \Phi_\theta \left(\|\Lambda_{V_1,\theta} - \Lambda_{V_2,\theta}\|\right). \tag{1.26}$$

Here, Φ_θ is the function defined in Theorem 1.1, upon substituting $\gamma_{\theta,*}$ for γ in (1.13)*, and $\|\cdot\|$ denotes the usual norm in $\mathcal{B}(\mathscr{H}_{c,\theta}(\check{F}), L^2(\check{G}))$.*

We notice that the constants C_θ and $\gamma_{\theta,*}$ of Theorem 1.3, may possibly depend on θ. Nevertheless, we infer from (1.25) that this is no longer the case for C and γ, appearing in the stability estimate (1.12) of Theorem 1.1, as we can choose $C = C_\theta$ and $\gamma = \gamma_{\theta,*}$ for any arbitrary $\theta \in [0, 2\pi)$. Therefore, we may completely leave aside the question of how C_θ and $\gamma_{\theta,*}$ depend on θ. For this reason, we shall not specify the possible dependence with respect to θ of the various constants appearing in the remaining part of this text. Finally, we stress out that the function Φ_θ does actually depend on θ through the constant γ_θ, as it is obtained by substituting γ_θ for θ in the definition (1.13).

1.5. **Outline.** The remaining part of this text is organized as follows. In Sections 2 and 3, we build the two different types of (CGO) solutions to the BVP (1.23), for $\theta \in [0, 2\pi)$, needed for the proof of Theorem 1.3, which is presented in Section 4. Section 5 contains the proof of Corollary 1.2. Finally, a suitable characterization of the space $H_{\Delta,\theta}$, $\theta \in [0, 2\pi)$, used in Section 3, is derived in Section A in appendix.

2. Complex geometric optics solutions

In this section we build CGO solutions to the system

$$\begin{cases} (-\Delta + V)u &= 0, & \text{in } \check{\Omega}, \\ u(1,\cdot) &= e^{i\theta} u(0,\cdot), & \text{on } \omega, \\ \partial_{x_1} u(1,\cdot) &= e^{i\theta} \partial_{x_1} u(0,\cdot), & \text{on } \omega, \end{cases} \tag{2.27}$$

where $\theta \in [0, 2\pi)$ and the real valued potential $V \in L^\infty(\check{\Omega})$ are arbitrarily fixed. More precisely, we seek a sufficiently rich set of solutions u_ζ to (2.27), parametrized by

$$\zeta \in \mathcal{Z}_\theta := \{\zeta \in i(\theta + 2\pi\mathbb{Z}) \times \mathbb{C}^2;\ |\Re\zeta| = |\Im\zeta| \text{ and } \Re\zeta \cdot \Im\zeta = 0\}, \tag{2.28}$$

of the form

$$u_\zeta(x) = (1 + v_\zeta(x))\, e^{\zeta\cdot x},\ x \in \check{\Omega}, \tag{2.29}$$

where the behavior of the function v_ζ with respect to ζ, is prescribed in a sense we shall specify further. Notice from definition (2.28) that for all $\zeta \in \mathcal{Z}_\theta$, we have

$$\Delta e^{\zeta\cdot x} = 0,\ x \in \Omega. \tag{2.30}$$

Actually, the analysis carried out in this text requires two different types of CGO solutions to (2.27), denoted generically by u_{ζ_1} and u_{ζ_2}, with different features we shall make precise below. The corresponding parameters ζ_1 and ζ_2 are the same as in [CKS2, section 4]. Namely, given $k \in \mathbb{Z}$ and $\eta \in \mathbb{R}^2 \setminus \{0\}$, we pick $\xi \in \mathbb{S}^1$ such that $\xi \cdot \eta = 0$, and we set

$$\ell = \ell(k, \eta, r, \theta) := \begin{cases} (\theta + 2\pi([r] + 1)) \left(1, -2\pi k \frac{\eta}{|\eta|^2}\right), & \text{if } k \text{ is even}, \\ \left(\theta + 2\pi \left([r] + \frac{3}{2}\right)\right) \left(1, -2\pi k \frac{\eta}{|\eta|^2}\right), & \text{if } k \text{ is odd}, \end{cases} \tag{2.31}$$

for each $r > 0$, in such a way that $\ell \cdot (2\pi k, \eta) = \ell' \cdot \xi = 0$. Here $[r]$ stands for the integer part of r, that is the unique integer fulfilling $[r] \leqslant r < [r] + 1$, and we used the notation $\ell = (\ell_1, \ell') \in \mathbb{R} \times \mathbb{R}^2$. Next, we introduce

$$\tau = \tau(k, \eta, r, \theta) := \sqrt{\frac{|\eta|^2}{4} + \pi^2 k^2 + |\ell|^2}, \tag{2.32}$$

and notice that

$$2\pi r < \tau \leqslant \frac{|(2\pi k, \eta)|}{2} + 4\pi(r+1)\left(1 + \frac{|2\pi k|}{|\eta|}\right). \tag{2.33}$$

Thus, putting

$$\zeta_1 := \left(i\pi k, -\tau\xi + i\frac{\eta}{2}\right) + i\ell \text{ and } \zeta_2 := \left(-i\pi k, \tau\xi - i\frac{\eta}{2}\right) + i\ell, \tag{2.34}$$

it is easy to see that

$$\zeta_1, \zeta_2 \in \mathcal{Z}_\theta \text{ satisfy the equation } \zeta_1 + \overline{\zeta_2} = i(2\pi k, \eta). \tag{2.35}$$

2.1. **Second order smooth CGO solutions.** The first type of CGO solutions u_{ζ_1} we need are smooth CGO functions, in the sense that the remainder term v_{ζ_1} appearing in (2.29) is taken in

$$H^2_{per}(\check{\Omega}) := \{f \in H^2(\check{\Omega});\ f(1,\cdot) = f(0,\cdot) \text{ and } \partial_{x_1} f(1,\cdot) = \partial_{x_1} f(0,\cdot) \text{ in } \omega\}.$$

The existence of such functions was already established in [CKS2, Lemma 4.1], as follows.

Lemma 2.1. *Let $V \in \mathscr{V}_\omega(M_\pm)$. Pick $\theta \in [0, 2\pi)$, $k \in \mathbb{Z}$, and $\eta \in \mathbb{R}^2 \setminus \{0\}$, and choose $\xi \in \mathbb{S}^1$ such that $\xi \cdot \eta = 0$. Let ζ_1 be given by* (2.34)*, as a function of τ, defined in* (2.31)-(2.32)*. Then, one can find $\tau_1 > 0$ so that for all $\tau \geqslant \tau_1$, there exists $v_{\zeta_1} \in H^2_{per}(\check{\Omega})$ such that the function u_{ζ_1}, defined by* (2.29) *with $\zeta = \zeta_1$, is solution to* (2.27)*. Moreover, for every $s \in [0, 2]$, the estimate*

$$\|v_{\zeta_1}\|_{H^s(\check{\Omega})} \leqslant C_s \tau^{s-1}, \tag{2.36}$$

holds uniformly in $\tau \geqslant \tau_1$, for some constant $C_s > 0$, depending only on s, ω, and $M_\pm$.

The second type of test functions needed for the proof of Theorem 1.1 are CGO solutions to (2.27), vanishing on a suitable subset of the boundary $(0, 1) \times \partial\omega$. They are described in Subsection 2.2.

2.2. **CGO solutions vanishing on a sub-part of the boundary.** For $\xi \in \mathbb{S}^1$ and $\varepsilon > 0$, we set

$$\partial\omega^+_{\varepsilon,\xi} := \{x' \in \partial\omega;\ \xi \cdot \nu'(x') > \varepsilon\} \text{ and } \partial\omega^-_{\varepsilon,\xi} := \{x' \in \partial\omega;\ \xi \cdot \nu'(x') \leqslant \varepsilon\}, \tag{2.37}$$

and we write $\check{\Gamma}^\pm_{\varepsilon,\xi}$ instead of $(0, 1) \times \partial\omega^\pm_{\varepsilon,\xi}$, in the sequel.

Bearing in mind that F' is a closed neighborhood in $\partial\omega$, of the subset $\partial\omega^+_{\xi_0}$ defined by (1.3), we pick $\varepsilon > 0$ so small that

$$\forall \xi \in \mathbb{S}^1,\ (|\xi - \xi_0| \leqslant \varepsilon) \Longrightarrow \left(\partial\omega^-_{\varepsilon,-\xi} \subset F'\right). \tag{2.38}$$

In this subsection, we aim for building solutions $u \in \mathcal{H}_{\Delta,\theta}(\check{\Omega})$ of the form (2.29) with $\zeta = \zeta_2$, where ζ_2 is given by (2.31)-(2.32) and (2.34), to the BVP

$$\begin{cases} (-\Delta + V)u &= 0, & \text{in } \check{\Omega}, \\ u(1,\cdot) &= e^{i\theta} u(0,\cdot), & \text{on } \omega, \\ \partial_{x_1} u(1,\cdot) &= e^{i\theta} \partial_{x_1} u(0,\cdot), & \text{on } \omega, \\ u &= 0, & \text{on } \check{\Gamma}^+_{\frac{\varepsilon}{2},-\xi}. \end{cases} \tag{2.39}$$

The result we have in mind is as follows.

Proposition 2.2. *Let V, θ, k, η, ξ, and τ, be the same as in Lemma 2.1, and let ζ_2 be defined by* (2.34)*. Then, one can find $\tau_2 > 0$, so that for each $\tau \geqslant \tau_2$, there exists $v_{\zeta_2} \in \mathcal{H}_{\Delta,0}(\check{\Omega})$ such that the function u_{ζ_2}, defined by* (2.29) *with $\zeta = \zeta_2$, is a $\mathcal{H}_{\Delta,\theta}(\check{\Omega})$-solution to* (2.39)*. Moreover, v_{ζ_2} satisfies*

$$\|v_{\zeta_2}\|_{L^2(\check{\Omega})} \leqslant C\tau^{-\frac{1}{2}},\ \tau \geqslant \tau_2, \tag{2.40}$$

for some constant $C > 0$ depending only on ω, M_+ and F'.

The proof of Proposition 2.2 is postponed to Section 3.

We notice from the last line in (2.39), that the solution u_{ζ_2} given by Proposition 2.2 for $\tau \geqslant \tau_2$, verifies $\mathcal{T}_{0,\theta} u_{\zeta_2} = 0$ on $\check{\Gamma} \setminus \check{F}$, as we have $\check{\Gamma} \setminus \check{F} \subset \check{\Gamma} \setminus \check{\Gamma}^-_{\varepsilon,-\xi} \subset \check{\Gamma}^+_{\frac{\varepsilon}{2},-\xi}$. Therefore, it holds true that

$$\mathcal{T}_{0,\theta} u_{\zeta_2} \in \mathscr{H}_{c,\theta}(\check{F}),\ \tau \geqslant \tau_2. \tag{2.41}$$

Further, we have

$$\|\mathcal{T}_{0,\theta} u_{\zeta_2}\|_{\mathscr{H}_\theta(\check{\Gamma})} \leqslant C\left(\|u_{\zeta_2}\|_{L^2(\check{\Omega})} + \|V u_{\zeta_2}\|_{L^2(\check{\Omega})}\right) \leqslant C(1+M_+)\|u_{\zeta_2}\|_{L^2(\check{\Omega})}, \tag{2.42}$$

for some positive constant $C = C(\omega)$, by the continuity of $\mathcal{T}_{0,\theta} : H_{\Delta,\theta}(\check{\Omega}) \to \mathscr{H}_\theta(\check{\Gamma})$ and the first line of (2.39). As

$$\zeta_2 \cdot x - \tau\xi \cdot x' \in i\mathbb{R},\ x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2, \tag{2.43}$$

by (2.31)-(2.32) and (2.34), we infer from the identity $u_{\zeta_2} = e^{\zeta_2 \cdot x}(1 + v_{\zeta_2})$, that

$$\|u_{\zeta_2}\|_{L^2(\check{\Omega})} \leqslant \|e^{\zeta_2 \cdot x}\|_{L^2(\check{\Omega})} + \|e^{\zeta_2 \cdot x} v_{\zeta_2}\|_{L^2(\check{\Omega})} \leqslant \|e^{\tau\xi \cdot x'}\|_{L^2(\check{\Omega})} + \|e^{\tau\xi \cdot x'} v_{\zeta_2}\|_{L^2(\check{\Omega})},$$

and hence that

$$\|u_{\zeta_2}\|_{L^2(\check{\Omega})} \leqslant e^{c_\omega \tau}\left(\mathrm{meas}(\omega) + \|v_{\zeta_2}\|_{L^2(\check{\Omega})}\right),$$

where $c_\omega := \sup\{|x'|,\ x' \in \omega\}$ and $\mathrm{meas}(\omega)$ denotes the two-dimensional Lebesgue measure of ω. From this, (2.40) and (2.42), it then follows that

$$\|\mathcal{T}_{0,\theta} u_{\zeta_2}\|_{\mathscr{H}_\theta(\check{\Gamma})} \leqslant C e^{c_\omega \tau},\ \tau \geqslant \tau_2, \tag{2.44}$$

where the constant $C > 0$, depends only on ω, M_+, and F'.

3. Proof of Proposition 2.2

The proof of Proposition 2.2 is by means of two technical results, given in Subsection 3.1. The first one is a Carleman estimate for the quasi-periodic Laplace operator $-\Delta + V$ in $L^2(\check{\Omega})$, which was inspired by [BU, Lemma 2.1]. The second one is an existence result for the BVP (1.23) with non zero source term.

3.1. Two useful tools.

We start by recalling the following Carleman inequality, borrowed from [CKS2, Corollary 5.2].

Lemma 3.1. *For $M > 0$ arbitrarily fixed, let $V \in L^\infty(\Omega)$ satisfy* (1.1) *and $\|V\|_{L^\infty(\Omega)} \leqslant M$. Then, there exist two constants $C > 0$ and $\tau_0 > 0$, both of them depending only on ω and M, such that for all $\xi \in \mathbb{S}^1$, all $\theta \in [0, 2\pi)$, and all $w \in C^2_\theta([0,1] \times \overline{\omega})$ obeying $w_{|\check{\Gamma}} = 0$, we have*

$$\begin{aligned} & C\|e^{-\tau\xi \cdot x'} w\|^2_{L^2(\check{\Omega})} + \tau\|e^{-\tau\xi \cdot x'}(\xi \cdot \nu')^{\frac{1}{2}} \partial_\nu w\|^2_{L^2(\check{\Gamma}^+_\xi)} \\ \leqslant\ & \|e^{-\tau\xi \cdot x'}(-\Delta + V) w\|^2_{L^2(\check{\Omega})} + \tau\|e^{-\tau\xi \cdot x'}|\xi \cdot \nu'|^{\frac{1}{2}} \partial_\nu w\|^2_{L^2(\check{\Gamma}^-_\xi)}, \end{aligned} \tag{3.45}$$

provided $\tau \geqslant \tau_0$.

Here and henceforth, we write $\check{\Gamma}^\pm_\xi$ instead of $(0,1) \times \partial\omega^\pm_\xi$.

Armed with Lemma 3.1, we turn now to establishing the following existence result for the BVP (1.23) with non zero source term.

Lemma 3.2. *For $M>0$, let $V\in L^\infty(\check{\Omega})$ be real valued and such that $\|V\|_{L^\infty(\check{\Omega})}\leqslant M$, and let $\theta\in[0,2\pi)$, $\xi\in\mathbb{S}^1$, $f\in L^2(\check{\Omega})$, and $g\in L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi;\ |\xi\cdot\nu'|^{-\frac{1}{2}}dx)$. Then, for every $\tau\in[\tau_0,+\infty)$, where τ_0 is the same as in Lemma 3.1, there exists $v\in H_{\Delta,\theta}(\check{\Omega})$ fulfilling*

$$\left\{\begin{array}{rcll}(-\Delta+V)v&=&f,&\text{in }\check{\Omega},\\ v&=&g,&\text{on }\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi.\end{array}\right.\tag{3.46}$$

Moreover, v satisfies the estimate

$$\|e^{-\tau\xi\cdot x'}v\|_{L^2(\check{\Omega})}\leqslant C\left(\tau^{-1}\|e^{-\tau\xi\cdot x'}f\|_{L^2(\check{\Omega})}+\tau^{-\frac{1}{2}}\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\right),\tag{3.47}$$

for some constant $C>0$, depending only on ω and M.

Proof. We denote by $C^2_{0,\theta}([0,1]\times\overline{\omega})$ the set of $C^2_\theta([0,1]\times\overline{\omega})$-functions vanishing on the boundary $\check{\Gamma}$, i.e.

$$C^2_{0,\theta}([0,1]\times\overline{\omega}):=\left\{w\in C^2_\theta\left([0,1]\times\overline{\omega}\right);\ w_{|\check{\Gamma}}=0\right\}.$$

By substituting $(-\xi)$ for ξ in the Carleman estimate (3.45), we get for all $w\in C^2_{0,\theta}([0,1]\times\overline{\omega})$, that

$$\begin{aligned}&\tau\left\|e^{\tau\xi\cdot x'}w\right\|_{L^2(\check{\Omega})}+\tau^{\frac{1}{2}}\left\|e^{\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\check{\Gamma}^-_\xi)}\\ \leqslant\ &C\left(\left\|e^{\tau\xi\cdot x'}(-\Delta+V)w\right\|_{L^2(\check{\Omega})}+\left\|e^{\tau\xi\cdot x'}(\tau\xi\cdot\nu')^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\check{\Gamma}^+_\xi)}\right).\end{aligned}\tag{3.48}$$

Put $\mathcal{M}_\theta:=\varrho\left(C^2_{0,\theta}([0,1]\times\overline{\omega})\right)=\left\{\varrho(w);w\in C^2_{0,\theta}\left([0,1]\times\overline{\omega}\right)\right\}$, where $\varrho(w):=\left((-\Delta+V)w,\partial_\nu w_{|\check{\Gamma}^+_\xi}\right)$. Since ϱ is one-to-one from $C^2_{0,\theta}\left([0,1]\times\overline{\omega}\right)$ onto $\mathcal{M}_\theta$, according to (3.48), then the antilinear form

$$\Upsilon:\varrho(w)\mapsto\langle f,w\rangle_{L^2(\check{\Omega})}-\langle g,\partial_\nu w\rangle_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)},\tag{3.49}$$

is well defined on $\mathcal{M}_\theta$, regarded as a subspace of $\mathscr{L}_{\tau,\frac{1}{2}}:=L^2(\check{\Omega};e^{\tau\xi\cdot x'}dx)\times L^2(\check{\Gamma}^+_\xi;e^{\tau\xi\cdot x'}|\tau\xi\cdot\nu'(x')|^{\frac{1}{2}}dx)$. Moreover, for each $w\in C^2_{0,\theta}([0,1]\times\overline{\omega})$, we have

$$\begin{aligned}|\Upsilon\left(\varrho(w)\right)|\ \leqslant\ &\left\|e^{-\tau\xi\cdot x'}f\right\|_{L^2(\check{\Omega})}\left\|e^{\tau\xi\cdot x'}w\right\|_{L^2(\check{\Omega})}+\left\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\right\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\left\|e^{\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\check{\Gamma}^-_\xi)}\\ \leqslant\ &\left(\tau^{-1}\left\|e^{-\tau\xi\cdot x'}f\right\|_{L^2(\check{\Omega})}+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\right\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\right)\\ &\times\left(\tau\left\|e^{\tau\xi\cdot x'}w\right\|_{L^2(\check{\Omega})}+\tau^{\frac{1}{2}}\left\|e^{\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\check{\Gamma}^-_\xi)}\right)\\ \leqslant\ &C\left(\tau^{-1}\left\|e^{-\tau\xi\cdot x'}f\right\|_{L^2(\check{\Omega})}+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\right\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\right)\\ &\times\left(\left\|e^{\tau\xi\cdot x'}(-\Delta+V)w\right\|_{L^2(\check{\Omega})}+\left\|e^{\tau\xi\cdot x'}|\tau\xi\cdot\nu'|^{\frac{1}{2}}\partial_\nu w\right\|_{L^2(\check{\Gamma}^+_\xi)}\right),\end{aligned}$$

by (3.48), and hence

$$|\Upsilon\left(\varrho(w)\right)|\leqslant C\left(\tau^{-1}\left\|e^{-\tau\xi\cdot x'}f\right\|_{L^2(\check{\Omega})}+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\right\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\right)\|\varrho(w)\|_{\mathscr{L}_{\tau,\frac{1}{2}}},$$

where C is the same constant as in (3.48). Thus, by Hahn Banach's theorem, Υ extends to an antilinear form on $\mathscr{L}_{\tau,\frac{1}{2}}$, still denoted by Υ, satisfying

$$\|\Upsilon\|\leqslant C\left(\tau^{-1}\left\|e^{-\tau\xi\cdot x'}f\right\|_{L^2(\check{\Omega})}+\tau^{-\frac{1}{2}}\left\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{-\frac{1}{2}}g\right\|_{L^2(\check{\Gamma}^-_\xi\setminus\check{\Gamma}^+_\xi)}\right).\tag{3.50}$$

Therefore, there exists $(v,\tilde{g})$ in $\mathscr{L}_{-\tau,-\frac{1}{2}} := L^2(\check{\Omega}; e^{-\tau\xi\cdot x'}dx) \times L^2(\check{\Gamma}_\xi^+; e^{-\tau\xi\cdot x'}|\tau\xi\cdot\nu'(x')|^{-\frac{1}{2}}dx)$, the dual space to $\mathscr{L}_{\tau,\frac{1}{2}}$ with pivot space $L^2(\check{\Omega}) \times L^2(\check{\Gamma}_\xi^+)$, such that

$$\Upsilon\left(\varrho(w)\right) = \langle (v,\tilde{g}), \varrho(w)\rangle_{L^2(\check{\Omega})\times L^2(\check{\Gamma}_\xi^+)} = \langle v, (-\Delta+V)w\rangle_{L^2(\check{\Omega})} + \langle \tilde{g}, \partial_\nu w\rangle_{L^2(\check{\Gamma}_\xi^+)},\ w\in C^2_{0,\theta}([0,1]\times\overline{\omega}). \quad (3.51)$$

This and (3.49) yield that

$$\langle v, (-\Delta+V)w\rangle_{L^2(\check{\Omega})} + \langle \tilde{g}, \partial_\nu w\rangle_{L^2(\check{\Gamma}_\xi^+)} = \langle f, w\rangle_{L^2(\check{\Omega})} - \langle g, \partial_\nu w\rangle_{L^2(\check{\Gamma}_\xi^-\setminus\check{\Gamma}_\xi^+)},\ w\in C^2_{0,\theta}([0,1]\times\overline{\omega}). \quad (3.52)$$

Taking $w\in C_0^\infty(\check{\Omega})\subset C^2_{0,\theta}([0,1]\times\overline{\omega})$ in (3.52), we get that $(-\Delta+V)v=f$ in the distributional sense on $\check{\Omega}$. Moreover, since $f\in L^2(\check{\Omega}; e^{-\tau\xi\cdot x'}dx)$ and $L^2(\check{\Omega}; e^{-\tau\xi\cdot x'}dx)\subset L^2(\check{\Omega})$, then the identity $(-\Delta+V)v=f$ holds in $L^2(\check{\Omega})$. This entails that $v\in H_\Delta(\check{\Omega})$.

We turn now to proving that $v\in H_{\Delta,\theta}(\check{\Omega})$. With reference to Lemma A.1 in Appendix, it suffices to show that

$$\sum_{k\in\mathbb{Z}} \left\|(\Delta' - (\theta+2k\pi)^2)\hat{v}_{k,\theta}\right\|^2_{L^2(\omega)} < \infty, \quad (3.53)$$

where

$$\hat{v}_{k,\theta}(x') := \langle v(\cdot,x'), \varphi_{k,\theta}\rangle_{L^2(0,1)},\ x'\in\omega,\ k\in\mathbb{Z}, \quad (3.54)$$

and

$$\varphi_{k,\theta}(x_1) := e^{i(\theta+2k\pi)x_1},\ x_1\in(0,1),\ k\in\mathbb{Z}. \quad (3.55)$$

To do that, we fix $k\in\mathbb{Z}$, pick $\chi\in C_0^\infty(\omega)$, and apply (3.52) with $w(x)=\varphi_{k,\theta}(x_1)\chi(x')$, getting

$$\langle v, \varphi_{k,\theta}\left(-\Delta' + (\theta+2k\pi)^2\right)\chi\rangle_{L^2(\check{\Omega})} = \langle h, \varphi_{k,\theta}\chi\rangle_{L^2(\check{\Omega})}, \quad (3.56)$$

with $h := f - Vv$. Next, by Fubini's theorem, we have

$$\begin{aligned}\langle v, \varphi_{k,\theta}\left(-\Delta' + (\theta+2k\pi)^2\right)\chi\rangle_{L^2(\check{\Omega})} &= \int_\omega \left(\int_0^1 v(x_1,x')\overline{\varphi_{k,\theta}(x_1)}dx_1\right)\left(-\Delta' + (\theta+2k\pi)^2\right)\overline{\chi(x')}dx' \\ &= \langle \hat{v}_{k,\theta}, \left(-\Delta' + (\theta+2k\pi)^2\right)\chi\rangle_{L^2(\omega)} \\ &= \langle \left(-\Delta' + (\theta+2k\pi)^2\right)\hat{v}_{k,\theta}, \chi\rangle_{(C_0^\infty)'(\omega), C_0^\infty(\omega)}, \end{aligned} \quad (3.57)$$

and similarly

$$\langle h, \varphi_{k,\theta}\chi\rangle_{L^2(\check{\Omega})} = \langle \hat{h}_{k,\theta}, \chi\rangle_{(C_0^\infty)'(\omega), C_0^\infty(\omega)}. \quad (3.58)$$

Putting (3.56)–(3.58) together, we find that

$$\left(-\Delta' + (\theta+2k\pi)^2\right)\hat{v}_{k,\theta} = \hat{h}_{k,\theta},\ k\in\mathbb{Z}. \quad (3.59)$$

Since $\{\varphi_{k,\theta},\ k\in\mathbb{Z}\}$ is an Hibertian basis of $L^2(0,1)$ and $L^2(\check{\Omega}) = L^2(0,1;L^2(\omega))$, then $w\mapsto\{\hat{w}_{k,\theta},\ k\in\mathbb{Z}\}$ is a unitary transform of $L^2(\check{\Omega})$ onto $\bigoplus_{k\in\mathbb{Z}} L^2(\omega)$. From this, (3.59), and the fact that $h\in L^2(\check{\Omega})$, it then follows that

$$\sum_{k\in\mathbb{Z}} \left\|\left(-\Delta' + (\theta+2k\pi)^2\right)\hat{v}_{k,\theta}\right\|^2_{L^2(\omega)} = \sum_{k\in\mathbb{Z}} \|\hat{h}_{k,\theta}\|^2_{L^2(\omega)} = \|h\|^2_{L^2(\check{\Omega})},$$

which entails (3.53).

Further, as $v\in H_{\Delta,\theta}(\check{\Omega})$, we infer from the Green formula that

$$\langle v, (-\Delta+V)w\rangle_{L^2(\check{\Omega})} = \langle (-\Delta+V)v, w\rangle_{L^2(\check{\Omega})} - \langle v, \partial_\nu w\rangle_{L^2(\check{\Gamma})},\ w\in C^2_{0,\theta}([0,1]\times\overline{\omega}). \quad (3.60)$$

Bearing in mind hat $(-\Delta+V)v=f$, it follows from (3.52) and (3.60) that

$$\langle v, \partial_\nu w\rangle_{L^2(\check{\Gamma})} = \langle g, \partial_\nu w\rangle_{L^2(\check{\Gamma}_\xi^-\setminus\check{\Gamma}_\xi^+)} + \langle \tilde{g}, \partial_\nu w\rangle_{L^2(\check{\Gamma}_\xi^+)},\ w\in C^2_{0,\theta}([0,1]\times\overline{\omega}). \quad (3.61)$$

Since w is arbitrary in $C^2_{0,\theta}([0,1]\times\overline{\omega})$, and $\check{\Gamma} = \left(\check{\Gamma}_\xi^- \setminus \check{\Gamma}_\xi^+\right)\cup\check{\Gamma}_\xi^+$, we deduce from (3.61) that $v=g$ on $\check{\Gamma}_\xi^-\setminus\check{\Gamma}_\xi^+$, and $v=\tilde{g}$ on $\check{\Gamma}_\xi^+$.

Last, upon plugging the identity $v = \tilde{g}$ on $\check{\Gamma}_{\xi}^{+}$, into (3.51), we find that

$$\Upsilon\left(\varrho(w)\right) = \langle v, (-\Delta + V)w\rangle_{L^2(\check{\Omega})} + \langle v, \partial_\nu w\rangle_{L^2(\check{\Gamma}_{\xi}^{+})} = \left\langle \left(v, v_{|\check{\Gamma}_{\xi}^{+}}\right), \varrho(w)\right\rangle_{L^2(\check{\Omega})\times L^2(\check{\Gamma}_{\xi}^{+})}, \ w \in C_{0,\theta}^2([0,1]\times\overline{\omega}),$$

which may be equivalently rewritten as

$$\Upsilon\left(h\right) = \left\langle \left(v, v_{|\check{\Gamma}_{\xi}^{+}}\right), h\right\rangle_{\mathscr{L}_{-\tau,-\frac{1}{2}},\mathscr{L}_{\tau,\frac{1}{2}}}, \ h \in \mathcal{M}_\theta.$$

As a consequence we have $\|\Upsilon\| = \left\|\left(v, v_{|\check{\Gamma}_{\xi}^{+}}\right)\right\|_{\mathscr{L}_{-\tau,-\frac{1}{2}}} = \|e^{-\tau\xi\cdot x'}v\|_{L^2(\check{\Omega})} + \|e^{-\tau\xi\cdot x'}|\tau\xi\cdot\nu'|^{-\frac{1}{2}}\partial_\nu v\|_{L^2(\check{\Gamma}_{\xi}^{+})}$, and hence

$$\|e^{-\tau\xi\cdot x'}v\|_{L^2(\check{\Omega})} \leqslant \|\Upsilon\|.$$

Putting this together with (3.50), we end up getting (3.47). □

Armed with Lemma 3.2 we are in position to complete the proof of Proposition 2.2.

3.2. **Completion of the proof.** Let us first notice that $u_{\zeta_2} = e^{\zeta_2\cdot x}(1 + v_{\zeta_2})$ is a solution to (2.39) if and only if $v := e^{\zeta_2\cdot x}v_{\zeta_2}$ is a solution to the BVP

$$\left\{\begin{array}{rcll} (-\Delta + V)v &=& f, & \text{in } \check{\Omega},\\ v(1,\cdot) &=& e^{i\theta}v(0,\cdot), & \text{on } \omega,\\ \partial_{x_1}v(1,\cdot) &=& e^{i\theta}\partial_{x_1}v(0,\cdot), & \text{on } \omega,\\ v &=& -e^{\zeta_2\cdot x}, & \text{on } \check{\Gamma}_{\frac{\varepsilon}{2},-\xi}^{+}, \end{array}\right. \tag{3.62}$$

with $f(x) := -V(x)e^{\zeta_2\cdot x}$.

Next, we take ψ in $C_0^\infty(\mathbb{R}^2)$ such that supp $\psi \cap \partial\omega \subset \partial\omega_{\frac{\varepsilon}{3},-\xi}^{+}$ and $\psi(x') = 1$ for $x' \in \partial\omega_{\frac{\varepsilon}{2},-\xi}^{+}$. Since $g(x) := -e^{\zeta_2\cdot x}\psi(x')$ vanishes for every $x \in \check{\Gamma}_{\frac{\varepsilon}{3},-\xi}^{-}$, then we see that $|\xi\cdot\nu'|^{-\frac{1}{2}}g \in L^2(\check{\Gamma}_{\xi}^{-}\setminus\check{\Gamma}_{\xi}^{+})$. Therefore, we may apply Lemma 3.2 for the above defined functions f and g. We get that (3.46) admits a solution $v \in H_{\Delta,\theta}(\check{\Omega})$, which is obviously a solution to (3.62) as well.

Further, remembering (2.44), we find by combining the identity $v_{\zeta_2} = e^{-\zeta_2\cdot x}v$ with the estimate (3.47), that

$$\begin{aligned} \|v_{\zeta_2}\|_{L^2(\check{\Omega})} = \|e^{-\tau\xi\cdot x'}v\|_{L^2(\check{\Omega})} &\leqslant C\left(\tau^{-1}\|V\|_{L^2(\check{\Omega})} + \tau^{-\frac{1}{2}}\||\xi\cdot\nu'|^{-\frac{1}{2}}\psi\|_{L^2(\check{\Gamma}_{\xi}^{-})}\right)\\ &\leqslant C\tau^{-\frac{1}{2}}\left(\tau^{-\frac{1}{2}}M\text{meas}(\omega) + \varepsilon^{-\frac{1}{2}}\|\psi\|_{L^2(\check{\Gamma}_{\xi}^{-})}\right). \end{aligned}$$

This yields (2.40) and terminates the proof of Proposition 2.2.

4. Proof of Theorem 1.3

The derivation of Theorem 1.3 follows the same path as the proof of [CKS2, Theorem 3.3], the only difference being the introduction of the CGO solutions described by Proposition 2.2. The main technical task here is to establish the estimate (4.65) stated in Lemma 4.1, below, which is similar to [CKS2, Lemma 6.1].

To this end, we start by setting the working parameter ε, appearing in (2.38). More precisely, since G' is a closed neighborhood of $\partial\omega_{\xi_0}^{-}$, we assume upon possibly shortening $\varepsilon > 0$, that

$$\forall\xi \in \mathbb{S}^1, \ (|\xi - \xi_0| \leqslant \varepsilon) \Longrightarrow \left(\partial\omega_{\varepsilon,-\xi}^{-} \subset F' \text{ and } \partial\omega_{\varepsilon,\xi}^{-} \subset G'\right). \tag{4.63}$$

Next, we refer to (2.31)-(2.32) and choose $r_* > 0$ so large that

$$(r \geqslant r_*) \Longrightarrow (\tau \geqslant \max\{\tau_j, \ j = 0,1,2\}), \tag{4.64}$$

where τ_0, τ_1, and τ_2, are the same as in Lemma 3.1, Lemma 2.1, and Proposition 2.2, respectively. Thus, putting

$$V(x) := \begin{cases} (V_2 - V_1)(x) & \text{if } x \in \check{\Omega} \\ 0, & \text{if } x \in (0,1) \times (\mathbb{R}^2 \setminus \omega), \end{cases}$$

we have the following result.

Lemma 4.1. *Let ε be defined by* (4.63) *and let r_* be as in* (4.64)*. Then the estimate*

$$\left| \int_{(0,1)\times\mathbb{R}^2} V(x_1, x') e^{-i(2\pi k x_1 + \eta\cdot x')} dx_1 d'_x \right|^2 \leqslant C \left(\frac{1}{\tau} + e^{C'\tau} \left\| \Lambda_{V_2,\theta} - \Lambda_{V_1,\theta} \right\|^2 \right), \tag{4.65}$$

holds for all $r \geqslant r_*$, all $\xi \in \mathbb{S}^1$ such that $|\xi - \xi_0| \leqslant \varepsilon$, all $\eta \in \mathbb{R}^2 \setminus \{0\}$ satisfying $\xi \cdot \eta = 0$, and all $k \in \mathbb{Z}$. Here the positive constants C and C' depend only on ω, $M_\pm$, F', G', and ξ_0.

Proof. We define ζ_j, $j = 1, 2$, as in (2.34) and, we denote by

$$u_{\zeta_1}(x) = (1 + v_{\zeta_1}(x)) e^{\zeta_1 \cdot x}, \ x \in \check{\Omega}, \tag{4.66}$$

the $\mathcal{H}^2_\theta(\check{\Omega})$-solution to (2.27) associated with $V = V_1$, which is given by Lemma 2.1. Similarly, we denote by

$$u_{\zeta_2}(x) = (1 + v_{\zeta_2}(x)) e^{\zeta_2 \cdot x}, \ x \in \check{\Omega}, \tag{4.67}$$

the $\mathcal{H}_{\Delta,\theta}(\check{\Omega})$-solution to (2.39), where V_2 is substituted for V, defined by Proposition 2.2.

Then we notice that if w_1 solves the BVP

$$\begin{cases} (-\Delta + V_1) w_1 &= 0, & \text{in } \check{\Omega}, \\ w_1 &= \mathcal{T}_{0,\theta} u_{\zeta_2}, & \text{on } \check{\Gamma}, \\ w_1(1,\cdot) - e^{i\theta} w_1(0,\cdot) &= 0, & \text{on } \omega, \\ \partial_{x_1} w_1(1,\cdot) - e^{i\theta} \partial_{x_1} w_1(0,\cdot) &= 0, & \text{on } \omega, \end{cases} \tag{4.68}$$

then $u := w_1 - u_{\zeta_2}$ is solution to the system

$$\begin{cases} (-\Delta + V_1) u &= V u_{\zeta_2}, & \text{in } \check{\Omega}, \\ u &= 0, & \text{on } \check{\Gamma}, \\ u(1,\cdot) - e^{i\theta} u(0,\cdot) &= 0, & \text{on } \omega, \\ \partial_{x_1} u(1,\cdot) - e^{i\theta} \partial_{x_1} u(0,\cdot) &= 0, & \text{on } \omega. \end{cases} \tag{4.69}$$

Thus, taking into account that $(-\Delta + V_1) u_{\zeta_1} = 0$ in $\check{\Omega}$, we deduce from (4.69) and the Green formula that

$$\int_{\check{\Omega}} V u_{\zeta_2} \overline{u_{\zeta_1}} dx = \int_{\check{\Omega}} (-\Delta + V_1) u \overline{u_{\zeta_1}} dx = \int_{\check{\Gamma}} (\partial_\nu u) \overline{u_{\zeta_1}} d\sigma(x),$$

which can be equivalently rewritten with the help of (2.37), as

$$\int_{\check{\Omega}} V u_{\zeta_2} \overline{u_{\zeta_1}} dx = \int_{\Gamma^+_{\xi,\varepsilon}} (\partial_\nu u) \overline{u_{\zeta_1}} d\sigma(x) + \int_{\Gamma^-_{\xi,\varepsilon}} (\partial_\nu u) \overline{u_{\zeta_1}} d\sigma(x). \tag{4.70}$$

Further, we infer from (4.66), the estimate (2.36) with $s = 1$, and the continuity of the trace from $H^1(\check{\Omega})$ into $L^2(\check{\Gamma})$, that

$$\begin{aligned} \left| \int_{\Gamma^\pm_{\xi,\varepsilon}} (\partial_\nu u) \overline{u_{\zeta_1}} d\sigma(x) \right| &\leqslant \int_{\Gamma^\pm_{\xi,\varepsilon}} \left| (\partial_\nu u) e^{-\tau \xi \cdot x'} (1 + v_{\zeta_1}(x)) \right| d\sigma(x') dx_1 \\ &\leqslant C \| e^{-\tau \xi \cdot x'} \partial_\nu u \|_{L^2(\Gamma^\pm_{\xi,\varepsilon})}, \end{aligned} \tag{4.71}$$

where C is another positive constant depending only on ω and M_+. Moreover, we have

$$\varepsilon \| e^{-\tau \xi \cdot x'} \partial_\nu u \|^2_{L^2(\Gamma^+_{\xi,\varepsilon})} \leqslant \| e^{-\tau \xi \cdot x'} (\xi \cdot \nu')^{\frac{1}{2}} \partial_\nu u \|^2_{L^2(\Gamma^+_{\xi,\varepsilon})} \leqslant \| e^{-\tau \xi \cdot x'} (\xi \cdot \nu')^{\frac{1}{2}} \partial_\nu u \|^2_{L^2(\Gamma^+_{\xi})},$$

from the very definition of $\Gamma^+_{\xi,\varepsilon}$ and the imbedding $\Gamma^+_{\xi,\varepsilon} \subset \Gamma^+_\xi$. Therefore, applying the Carleman estimate of Lemma 3.1 to the solution u of (4.69), which is possible since $\tau \geqslant \tau_0$, we get that

$$\begin{aligned} \tau\varepsilon\|e^{-\tau\xi\cdot x'}\partial_\nu u\|^2_{L^2(\Gamma^+_{\xi,\varepsilon})} &\leqslant \|e^{-\tau\xi\cdot x'}(-\Delta+V_1)u\|^2_{L^2(\check{\Omega})} + \tau\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac12}\partial_\nu u\|^2_{L^2(\Gamma^-_\xi)} \\ &\leqslant \|e^{-\tau\xi\cdot x'}Vu_{\zeta_2}\|^2_{L^2(\check{\Omega})} + \tau\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac12}\partial_\nu u\|^2_{L^2(\Gamma^-_\xi)}. \end{aligned} \tag{4.72}$$

Here we used the fact, arising from the density of $\{u \in \mathcal{C}^2_\theta([0,1]\times\overline{\omega}),\ u_{|\check{\Gamma}} = 0\}$ in $\{u \in \mathcal{H}^2_\theta(\check{\Omega}),\ u_{|\check{\Gamma}} = 0\}$, that the Carleman estimate (3.45) is still valid for $u \in \mathcal{H}^2_\theta(\check{\Omega})$ obeying $u_{|\check{\Gamma}} = 0$.

Next, as

$$e^{-\tau\xi\cdot x'}Vu_{\zeta_2}(x) = e^{-\tau\xi\cdot x'}Ve^{\zeta_2\cdot x}(1+v_{\zeta_2}(x)) = e^{-i\left(\pi k x_1 + \frac{\eta\cdot x'}{2} - \ell\cdot x\right)}V(1+v_{\zeta_2}(x)),\ x \in \check{\Omega},$$

by (2.34) and (4.67), we have $|e^{-\tau\xi\cdot x'}Vu_{\zeta_2}(x)| = |V(x)||1+v_{\zeta_2}(x)|$, according to (2.31), so it follows from (2.40) that

$$\|e^{-\tau\xi\cdot x'}Vu_{\zeta_2}\|_{L^2(\check{\Omega})} \leqslant M_+\left(|\omega| + \frac{C}{\tau}\right), \tag{4.73}$$

with $C = C(\omega, M_+, F') > 0$. Further, bearing in mind that $\partial\omega^-_\xi \subset \partial\omega^-_{\xi,\varepsilon}$ and $|\xi\cdot\nu'| \leqslant 1$ on $\partial\omega^-_{\xi,\varepsilon}$, we get that

$$\|e^{-\tau\xi\cdot x'}|\xi\cdot\nu'|^{\frac12}\partial_\nu u\|_{L^2(\Gamma^-_\xi)} \leqslant \|e^{-\tau\xi\cdot x'}\partial_\nu u\|_{L^2(\Gamma^-_{\xi,\varepsilon})}.$$

This and (4.72)-(4.73) yield the estimate

$$\|e^{-\tau\xi\cdot x'}\partial_\nu u\|^2_{L^2(\Gamma^+_{\xi,\varepsilon})} \leqslant \frac{C}{\varepsilon}\left(\frac{1}{\tau} + \|e^{-\tau\xi\cdot x'}\partial_\nu u\|^2_{L^2(\Gamma^-_{\xi,\varepsilon})}\right),$$

for some constant $C = C(\omega, M_+, F') > 0$. We infer from this and (4.70)-(4.71), that

$$\left|\int_{\check{\Omega}} Vu_{\zeta_2}\overline{u_{\zeta_1}}dx\right| \leqslant C\left(\frac{1}{\tau} + \|e^{-\tau\xi\cdot x'}\partial_\nu u\|^2_{L^2(\Gamma^-_{\xi,\varepsilon})}\right)^{\frac12}, \tag{4.74}$$

where C is a positive constant C depending on ω, M_+, F', and G'.

On the other hand, with reference to (2.34)-(2.35), (4.66), and (4.67), we find through direct calculation that

$$\int_{\check{\Omega}} Vu_{\zeta_2}\overline{u_{\zeta_1}}dx = \int_{(0,1)\times\mathbb{R}^2} e^{-i(2\pi k x_1 + \eta\cdot x')}V(x_1,x')dx_1dx' + \int_{\check{\Omega}} R(x)dx, \tag{4.75}$$

where

$$R(x) := e^{-i(2\pi k x_1 + \eta\cdot x')}V(x)\left(v_{\zeta_2}(x) + \overline{v_{\zeta_1}(x)} + v_{\zeta_2}(x)\overline{v_{\zeta_1}(x)}\right),\ x = (x_1,x') \in \check{\Omega}.$$

Since $\|v_{\zeta_1}\|_{L^2(\check{\Omega})}$ (resp., $\|v_{\zeta_2}\|_{L^2(\check{\Omega})}$) is bounded, up to some multiplicative constant, from above by τ^{-1} (resp., $\tau^{-\frac12}$), according to (2.36) (resp., (2.40)), we obtain that

$$\left|\int_{\check{\Omega}} R(x)dx\right| \leqslant M_+\left(|\omega|^{\frac12}\left(\|v_{\zeta_1}\|_{L^2(\check{\Omega})} + \|v_{\zeta_2}\|_{L^2(\check{\Omega})}\right) + \|v_{\zeta_1}\|_{L^2(\check{\Omega})}\|v_{\zeta_2}\|_{L^2(\check{\Omega})}\right) \leqslant C\tau^{-\frac12},$$

where C is independent of τ. It follows from this and (4.74)-(4.75) that

$$\begin{aligned} \left|\int_{(0,1)\times\mathbb{R}^2} e^{-i(2\pi k x_1 + \eta\cdot x')}V(x)dx\right|^2 &\leqslant C\left(\frac{1}{\tau} + \|e^{-\tau\xi\cdot x'}\partial_\nu u\|^2_{L^2(\Gamma^-_{\xi,\varepsilon})}\right) \\ &\leqslant C\left(\frac{1}{\tau} + e^{c_\omega\tau}\|\partial_\nu u\|^2_{L^2(\Gamma^-_{\xi,\varepsilon})}\right), \end{aligned} \tag{4.76}$$

where we recall that $c_\omega := \max\{|x'|; x' \in \overline{\omega}\}$, and $C = C(\omega, M_\pm, F', G') > 0$. Finally, upon recalling that $u = w_1 - u_{\zeta_2}$, where w_1 is solution to (4.68) and u_{ζ_2} satisfies (2.39) with $V = V_2$, we see that

$$\partial_\nu u = (\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta})f,\ f = \mathcal{T}_{0,\theta}u_{\zeta_2}.$$

Since $\partial\omega_{\xi,\varepsilon}^- \subset G'$, by (4.63), we have $\|\partial_\nu u\|_{L^2(\Gamma_{\xi,\varepsilon}^-)} \leqslant \|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\| \, \|\mathcal{T}_{0,\theta} u_{\zeta_2}\|_{\mathscr{H}_\theta(\check{\Gamma})}$, and hence

$$\|\partial_\nu u\|_{L^2(\Gamma_{\xi,\varepsilon}^-)} \leqslant C \, \|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\| \, \|u_{\zeta_2}\|_{H_\Delta(\check{\Omega})} \leqslant C e^{c_\omega \tau} \, \|\Lambda_{V_2,\theta} - \Lambda_{V_1,\theta}\| \, ,$$

by (2.44), where $C = C(\omega, M_+, F') > 0$. This and (4.76) entail (4.65). □

Now the result of Theorem 1.3 follows from Lemma 4.1 by arguing as in the derivation (see [CKS2, Section 6]) of [CKS2, Theorem 3.3] from [CKS2, Lemma 6.1].

5. Proof of Corollary 1.2

It is well known that if u is a solution to (1.15), then the Liouville transform of u, $v := a^{\frac{1}{2}} u$, solves the following BVP

$$\begin{cases} (-\Delta + V_a) v &= 0, & \text{in } \Omega, \\ v &= a^{\frac{1}{2}} f, & \text{on } \Gamma, \end{cases}$$

where we recall that $V_a := a^{-\frac{1}{2}} \Delta a^{\frac{1}{2}}$. Moreover, standard computations yield that

$$\Sigma_a f = a^{\frac{1}{2}} \Lambda_{V_a} a^{\frac{1}{2}} f - a^{\frac{1}{2}} \left(\partial_\nu a^{\frac{1}{2}} \right) f, \ f \in \mathscr{K}(\Gamma) \cap a_1^{-\frac{1}{2}} (\mathscr{H}_c(F)),$$

where Λ_{V_a} are Σ_a are defined by (1.10) and (1.17), respectively. From this and (1.20)-(1.21), it then follows for every $f \in \mathscr{K}(\Gamma) \cap a_1^{-\frac{1}{2}} (\mathscr{H}_c(F))$, that

$$\Sigma_{a_j} f = a_1^{\frac{1}{2}} \Lambda_{V_j} a_1^{\frac{1}{2}} f - a_1^{\frac{1}{2}} \left(\partial_\nu a_1^{\frac{1}{2}} \right) f_{|G}, \ j = 1, 2,$$

where, for simplicity, V_j stands for V_{a_j}. As a consequence we have

$$(\Sigma_{a_1} - \Sigma_{a_2}) f = a_1^{\frac{1}{2}} (\Lambda_{V_1} - \Lambda_{V_2}) a_1^{\frac{1}{2}} f, \ f \in \mathscr{K}(\Gamma) \cap a_1^{-\frac{1}{2}} (\mathscr{H}_c(F)). \tag{5.77}$$

Since $a_j \in \mathscr{A}_\omega(c_*, M_\pm)$, $j = 1, 2$, it is clear that $V_j \in \mathscr{V}_\omega(M_\pm)$, whence $\Lambda_{V_1} - \Lambda_{V_2} \in \mathcal{B}(\mathscr{H}_c(F), L^2(G))$, by (1.11). Thus, $(\Sigma_{a_1} - \Sigma_{a_2}) f \in L^2(G)$, by (5.77), and we have the the estimate

$$\|(\Sigma_{a_1} - \Sigma_{a_2}) f\|_{L^2(G)} \leqslant \|a_1\|_{L^\infty(\Gamma)}^{\frac{1}{2}} \, \|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F), L^2(G))} \left\| a_1^{\frac{1}{2}} f \right\|_{\mathscr{H}(\Gamma)}, \ f \in \mathscr{K}(\Gamma) \cap a_1^{-\frac{1}{2}} (\mathscr{H}_c(F)).$$

Since the space $a_1^{-\frac{1}{2}} \mathscr{H}_c(F)$ is endowed with the norm $\|a_1^{\frac{1}{2}} \cdot\|_{\mathscr{H}(\Gamma)}$, this entails that $\Sigma_{a_1} - \Sigma_{a_2}$ can be extended to a bounded operator from $a_1^{-\frac{1}{2}} \mathscr{H}_c(F)$ into $L^2(G)$, fulfilling

$$\|\Sigma_{a_1} - \Sigma_{a_2}\| \leqslant \|a_1\|_{L^\infty(\Gamma)}^{\frac{1}{2}} \, \|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F), L^2(G))} \, .$$

Moreover, we notice from (1.16) and (5.77) that

$$\|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F), L^2(G))} \leqslant a_*^{-\frac{1}{2}} \, \|\Sigma_{a_1} - \Sigma_{a_2}\| \, . \tag{5.78}$$

Having established (5.78), we turn now to proving the stability estimate (1.22). To this purpose, we put $\alpha := a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}}$, get through basic computations that

$$\Delta \alpha = \Delta a_1^{\frac{1}{2}} - \Delta a_2^{\frac{1}{2}} = a_1^{\frac{1}{2}} V_1 - a_2^{\frac{1}{2}} V_2 = \alpha V_1 + a_2^{\frac{1}{2}} (V_1 - V_2),$$

and then deduce from (1.14) and (1.20)-(1.21), that

$$\begin{cases} (-\Delta + V_1) \alpha &= -a_2^{\frac{1}{2}} (V_1 - V_2), & \text{in } \check{\Omega}, \\ \alpha(1, \cdot) - \alpha(0, \cdot) &= 0, & \text{in } \omega, \\ \partial_{x_1} \alpha(1, \cdot) - \partial_{x_1} \alpha(0, \cdot) &= 0, & \text{in } \omega, \\ \alpha &= 0, & \text{on } \check{\Gamma}. \end{cases} \tag{5.79}$$

Since $a_1 \in \mathscr{A}_\omega(c_*, M_\pm)$, then (5.79) admits a unique solution

$$\alpha \in H^1_{0,\text{per}}(\check{\Omega}) := \{ v \in H^1(\check{\Omega}); \ v_{|\check{\Gamma}} = 0 \text{ and } v(1, \cdot) - v(0, \cdot) = 0 \text{ in } \omega \},$$

by the standard variational theory. Moreover, there exists a constant $C = C(\omega, a_*, M_\pm) > 0$, such that we have

$$\|\alpha\|_{H^1(\check{\Omega})} \leqslant C \left\| a_2^{\frac{1}{2}} (V_2 - V_1) \right\|_{(H^1_{0,\mathrm{per}}(\check{\Omega}))'} \leqslant C \|V_1 - V_2\|_{(H^1_{0,\mathrm{per}}(\check{\Omega}))'} . \tag{5.80}$$

Here, $(H^1_{0,\mathrm{per}}(\check{\Omega}))'$ denotes the dual space to $H^1_{0,\mathrm{per}}(\check{\Omega})$.

Next, arguing as in the derivation of Theorem 1.1, we find that

$$\|V_1 - V_2\|_{(H^1_{0,\mathrm{per}}(\check{\Omega}))'} \leqslant C\Phi \left(\|\Lambda_{V_1} - \Lambda_{V_2}\|_{\mathcal{B}(\mathscr{H}_c(F), L^2(G))} \right),$$

so we deduce from (5.78) and (5.80) that

$$\left\| a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}} \right\|_{H^1(\check{\Omega})} \leqslant C\Phi \left(a_*^{-\frac{1}{2}} \|\Sigma_{a_1} - \Sigma_{a_2}\| \right).$$

Here we used the fact that Φ is a non-decreasing function on $[0, +\infty)$. Now, the desired result follows from this upon noticing that

$$\begin{aligned} \|a_1 - a_2\|_{H^1(\check{\Omega})} &= \left\| (a_1^{\frac{1}{2}} + a_2^{\frac{1}{2}})(a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}}) \right\|_{H^1(\check{\Omega})} \\ &\leqslant a_*^{-\frac{1}{2}} \left(\|a_1\|_{W^{1,\infty}(\check{\Omega})} + \|a_2\|_{W^{1,\infty}(\check{\Omega})} \right) \left\| a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}} \right\|_{H^1(\check{\Omega})} \\ &\leqslant 2a_*^{-\frac{1}{2}} M_+ \left\| a_1^{\frac{1}{2}} - a_2^{\frac{1}{2}} \right\|_{H^1(\check{\Omega})} . \end{aligned}$$

Appendix A. Characterizing the space $H_{\Delta,\theta}(\check{\Omega})$

In this appendix we establish that any function $v \in H_\Delta(\check{\Omega})$ satisfying the condition (3.53) for some fixed $\theta \in [0, 2\pi)$, actually belongs to $H_\Delta(\check{\Omega})$.

As a warm up, we fix $\theta \in [0, 2\pi)$ and notice for each $v \in H_\Delta(\check{\Omega})$, that

$$\left(v \in H_{\Delta,\theta}(\check{\Omega}) \right) \Longleftrightarrow \left(\langle \Delta v, w \rangle_{L^2(\check{\Omega})} = \langle v, \Delta w \rangle_{L^2(\check{\Omega})},\ w \in H^2_\theta(0,1; H^2_0(\omega)) \right), \tag{A.81}$$

where $H^2_0(\omega)$ is the closure of $C^\infty_0(\omega)$ for the topology of $H^2(\omega)$. This can be seen from the the following identity

$$\begin{aligned} & \langle \Delta v, w \rangle_{L^2(\check{\Omega})} - \langle v, \Delta w \rangle_{L^2(\check{\Omega})} \\ = \ & \langle \partial_{x_1} v(1,\cdot), w(1,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} - \langle \partial_{x_1} v(0,\cdot), w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} \\ & - \left(\langle u(1,\cdot), \partial_{x_1} w(1,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} - \langle v(0,\cdot), \partial_{x_1} w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} \right), \end{aligned}$$

which holds true for any $v \in H_\Delta(\check{\Omega})$ and $w \in H^2(0,1; H^2_0(\omega))$, and leads to

$$\begin{aligned} & \langle \Delta v, w \rangle_{L^2(\check{\Omega})} - \langle v, \Delta w \rangle_{L^2(\check{\Omega})} \\ = \ & \langle e^{-i\theta} \partial_{x_1} v(1,\cdot) - \partial_{x_1} v(0,\cdot), w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} - \langle e^{-i\theta} v(1,\cdot) - v(0,\cdot), \partial_{x_1} w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)}, \end{aligned}$$

as soon as $w \in H^2_\theta(0,1; H^2_0(\omega))$. Therefore, the right hand side of (A.81) may be equivalently reformulated as

$$\begin{aligned} & \langle \partial_{x_1} v(1,\cdot) - e^{i\theta} \partial_{x_1} v(0,\cdot), w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} \\ - \ & \langle v(1,\cdot) - e^{i\theta} v(0,\cdot), \partial_{x_1} w(0,\cdot) \rangle_{H^{-2}(\omega), H^2_0(\omega)} = 0,\ w \in H^2_\theta(0,1; H^2_0(\omega)). \end{aligned} \tag{A.82}$$

Taking $w(x) = e^{i\theta x_1} \sin(2\pi x_1) \psi(x')$, where ψ is arbitrary in $H^2_0(\omega)$, in (A.82), we get that

$$\langle v(1,\cdot) - e^{i\theta} v(0,\cdot), \psi \rangle_{H^{-2}(\omega), H^2_0(\omega)} = 0,$$

and hence that $v(1,\cdot) - e^{i\theta} v(0,\cdot) = 0$. Doing the same with $w(x) = e^{i\theta x_1} \psi(x')$, we find that

$$\langle \partial_{x_1} v(1,\cdot) - e^{i\theta} \partial_{x_1} v(0,\cdot), \psi \rangle_{H^{-2}(\omega), H^2_0(\omega)} = 0,$$

which yields $\partial_{x_1} v(1,\cdot) - e^{i\theta}\partial_{x_1} v(0,\cdot) = 0$, and proves (A.81).

Armed with (A.81) we may now establish the main result of this appendix.

Lemma A.1. *Fix $\theta \in [0, 2\pi)$, and let $v \in H_\Delta(\check{\Omega})$ satisfy the condition* (3.53)*. Then v belongs to $H_{\Delta,\theta}(\check{\Omega})$.*

Proof. Since $v \in H_\Delta(\check{\Omega})$ satisfies the condition (3.53), then both identities

$$v = \sum_{k\in\mathbb{Z}} \hat{v}_{k,\theta}\varphi_{k,\theta} \text{ and } \Delta v = \sum_{k\in\mathbb{Z}} \left(\Delta' - (\theta + 2k\pi)^2\right) \hat{v}_{k,\theta}\varphi_{k,\theta}, \tag{A.83}$$

hold in $L^2(\check{\Omega})$, where $\hat{v}_{k,\theta}$ and $\varphi_{k,\theta}$ are defined by (3.54) and (3.55), respectively. Moreover, as $L^2(\check{\Omega}) = L^2(0,1;L^2(\omega))$ and $\{\varphi_{k,\theta},\ k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(0,1)$, it follows from (A.83) that

$$\sum_{k\in\mathbb{Z}} \|\hat{v}_{k,\theta}\|^2_{L^2(\omega)} = \|v\|^2_{L^2(\check{\Omega})} \text{ and } \sum_{k\in\mathbb{Z}} \| \left(\Delta' - (\theta + 2k\pi)^2\right) \hat{v}_{k,\theta}\|^2_{L^2(\omega)} = \|\Delta v\|^2_{L^2(\check{\Omega})}. \tag{A.84}$$

Similarly, for $w \in H^2_\theta(0,1;H^2_0(\omega))$, it holds true that

$$w = \sum_{k\in\mathbb{Z}} \hat{w}_{k,\theta}\varphi_{k,\theta} \text{ and } \Delta w = \sum_{k\in\mathbb{Z}} \left(\Delta' - (\theta + 2k\pi)^2\right) \hat{w}_{k,\theta}\varphi_{k,\theta}, \tag{A.85}$$

in $L^2(\check{\Omega})$, and we have the energy estimates $\sum_{k\in\mathbb{Z}} \|\hat{w}_{k,\theta}\|^2_{H^2(\omega)} = \|w\|^2_{L^2(0,1;H^2(\omega))}$, $\sum_{k\in\mathbb{Z}} (\theta+2k\pi)^4 \|\hat{w}_{k,\theta}\|^2_{L^2(\omega)} = \|\partial^2_{x_1} w\|^2_{L^2(\check{\Omega})}$, and $\sum_{k\in\mathbb{Z}} \|\Delta'\hat{w}_{k,\theta}\|^2_{L^2(\omega)} = \|\Delta' w\|^2_{L^2(\check{\Omega})}$. It follows from this and (A.83)–(A.85), that

$$\begin{aligned}
\langle \Delta v, w\rangle_{L^2(\check{\Omega})} &= \sum_{k\in\mathbb{Z}} \left\langle \left(\Delta' - (\theta + 2k\pi)^2\right) \hat{v}_{k,\theta}, \hat{w}_{k,\theta}\right\rangle_{L^2(\omega)} \\
&= \sum_{k\in\mathbb{Z}} \langle \Delta'\hat{v}_{k,\theta}, \hat{w}_{k,\theta}\rangle_{H^{-2}(\omega),H^2_0(\omega)} - \sum_{k\in\mathbb{Z}} \left\langle (\theta + 2k\pi)^2 \hat{v}_{k,\theta}, \hat{w}_{k,\theta}\right\rangle_{H^{-2}(\omega),H^2_0(\omega)} \\
&= \sum_{k\in\mathbb{Z}} \langle \hat{v}_{k,\theta}, \Delta'\hat{w}_{k,\theta}\rangle_{L^2(\omega)} - \sum_{k\in\mathbb{Z}} \left\langle \hat{v}_{k,\theta}, (\theta + 2k\pi)^2 \hat{w}_{k,\theta}\right\rangle_{L^2(\omega)} \\
&= \sum_{k\in\mathbb{Z}} \left\langle \hat{v}_{k,\theta}, \left(\Delta' - (\theta + 2k\pi)^2\right) \hat{w}_{k,\theta}\right\rangle_{L^2(\omega)} \\
&= \langle v, \Delta w\rangle_{L^2(\check{\Omega})}.
\end{aligned}$$

This and (A.81) yield the desired result. □

Remark A.2. The assumption $v \in H_\Delta(\check{\Omega})$ in Lemma A.1 can be weakened to $v \in L^2(\check{\Omega})$, as an $L^2(\check{\Omega})$-function satisfying condition (3.53) is automatically in $H_\Delta(\check{\Omega})$. Moreover, it is not hard to see that the result of Lemma A.1 can be improved significantly in order to provide the following characterization of the space $H_{\Delta,\theta}(\check{\Omega})$:

$$H_{\Delta,\theta}(\check{\Omega}) = \left\{ v \in L^2(\check{\Omega}),\ v \text{ satisfies (3.53)} \right\}.$$

References

[Al] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal., **27** (1-3) (1988), 153-172.

[BKS] M. Bellassoued, Y. Kian, E. Soccorsi, *An inverse stability result for non compactly supported potentials by one arbitrary lateral Neumann observation*, preprint, arXiv:1507.06864.

[BIY] E. Blasten, O. Imanuvilov, M. Yamamoto, *Stability and uniqueness for a two-dimensional inverse boundary value problem for less regular potentials*, Inverse Prob. Imaging, **9** (3) (2015), 709–723.

[BT] R. Brown, R. Torres, *Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L^p, $p > 2n$.*, J. Fourier Anal. Appl., **9** (2003), no. 6, 563–574.

[B] A. Bukhgeim, *Recovering the potential from Cauchy data in two dimensions*, J. Inverse Ill-Posed Probl., **16** (2008), 19-34.

[BU] A. L. Bukhgeim, G. Uhlmann, *Recovering a potential from partial Cauchy data*, Comm. PDE, **27** (3-4) (2002), 653-668.

[Ca] A. P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Rio de Janeiro, Sociedade Brasileira de Matematica, (1980), 65-73.

[CDR1] P. Caro, D. Dos Santos Ferreira, A. Ruiz, *Stability estimates for the Radon transform with restricted data and applications*, Advances in Math., **267** (2014), 523-564.

[CDR2] P. Caro, D. Dos Santos Ferreira, A. Ruiz, *Stability estimates for the Calderón problem with partial data*, preprint, arXiv:1405.1217.

[Ch] M. Choulli, *Une introduction aux problèmes inverses elliptiques et paraboliques*, Mathématiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.

[CKS] M. Choulli, Y. Kian, E. Soccorsi, *Stable determination of time-dependent scalar potential from boundary measurements in a periodic quantum waveguide*, SIAM Jour. Math. Analysis, **47** (2015), no 6, 4536-4558.

[CKS1] M. Choulli, Y. Kian, E. Soccorsi, *Double logarithmic stability estimate in the identification of a scalar potential by a partial elliptic Dirichlet-to-Neumann map*, Bulletin of the South Ural State University, Ser. Mathematical Modelling, Programming and Computer Software (SUSU MMCS), **8** (2015), no 3, 78-95.

[CKS2] M. Choulli, Y. Kian, E. Soccorsi, *Stable determination of periodic potential in cylindrical domain by partial elliptic Dirichlet-to-Neumann map*, preprint.

[CS] M. Choulli, E. Soccorsi, *Recovering the twisting function in a twisted waveguide from the DN map*, J. Spec. Theory, **5** (2015), 295-329.

[HT] B. Haberman, D. Tataru, *Uniqueness in Calderón's problem with Lipschitz conductivities*, Duke Math. Journal, **162** (3) (2013), 497-516.

[HW] H. Heck and J. N. Wang, *Stability estimate for the inverse boundary value problem by partial Cauchy data*, Inverse Problems, **22** (2006), 1787-1797.

[IY1] O. Imanuvilov and M. Yamamoto, *Inverse boundary value problem for the Schrödinger equation in a cylindrical domain by partial boundary data*, Inverse Problems, **29** (2013), 045002.

[IUY1] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *The Calderón problem with partial data in two dimensions*, Journal American Math. Society, **23** (2010), 655-691.

[IUY2] O. Imanuvilov, G. Uhlmann, M. Yamamoto, *Partial Cauchy data for general second order elliptic operators in two dimensions*, Publ. Research Institute Math. Sci., **48** (2012), 971-1055.

[Ik] M. Ikehata, *Inverse conductivity problem in the infinite slab*, Inverse Problems, **17** (2001), 437-454.

[KKS] O. Kavian, Y. Kian, E. Soccorsi, *Uniqueness and stability results for an inverse spectral problem in a periodic waveguide*, Jour. Math. Pures Appl., **104** (6) (2015), 1160-1189.

[KSU] C.E. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderon problem with partial data*, Ann. of Math., **165** (2007), 567-591.

[Ki] Y. Kian, *Stability of the determination of a coefficient for wave equations in an infinite waveguide*, Inverse Probl. Imaging, **8** (3) (2014), 713-732.

[KPS1] Y. Kian, Q. S. Phan, E. Soccorsi, *Carleman estimate for infinite cylindrical quantum domains and application to inverse problems*, Inverse Problems, **30** (5) (2014), 055016.

[KPS2] Y. Kian, Q. S. Phan, E. Soccorsi, *Hölder stable determination of a quantum scalar potential in unbounded cylindrical domains*, Jour. Math. Anal. Appl., **426** (1) (2015), 194-210.

[KV] R. Kohn, M. Vogelius,*Determining conductivity by boundary measurements*, Comm. Pure Appl. Math., **37** (1984), 289-298.

[KLU] K. Krupchyk, M. Lassas, G. Uhlmann, *Inverse Problems with Partial Data for a Magnetic Schrödinger Operator in an Infinite Slab or Bounded Domain*, Comm. Math. Phys., **312** (2012), 87-126.

[LU] X. Li, G. Uhlmann, *Inverse Problems on a Slab*, Inverse Problems and Imaging, **4** (2010), 449–462.

[LM1] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod (1968).

[Ma] N. Mandache, *Exponential instability in an inverse problem for the Schördinger equation*, Inverse Problems, **17** (5) (2001), 1435-1444.

[NS] A. Nachman, B. Street, *Reconstruction in the Calderón problem with partial data*, Comm. PDE, **35** (2010), 375-390.

[NSa] R.G. Novikov, M. Santacesaria, *A global stability estimate for the Gel'fand-Calderón inverse problem in two dimensions*, J. Inverse IllPosed Probl., **18** (2010), 765-785.

[SW] M. Salo, J. N. Wang, *Complex spherical waves and inverse problems in unbounded domains*, Inverse Problems, **22** (2006), 2299-2309.

[Sa] M. Santacesaria, *New global stability estimates for the Calderón problem in two dimensions*, J. Inst. Math. Jussieu, **12** (2013), 553-569.

[SU] J. Sylvester, G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), 153-169.

[T] L. Tzou, *Stability Estimate for the coefficients of magnetic Schrödinger equation from full and partial boundary measurements*, Commun. PDE, **11** (2008), 1911-1952.

[V] S. Vessella, *A continuous dependence result in the analytic continuation problem*, Forum math., **11** (1999), no. 6, 695-703.

[Y] Y. Yang, *Determining the first order perturbation of a bi-harmonic operator on bounded and unbounded domains from partial data*, J. Differential Equations, **257** (2014), 3607-3639.

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