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# On Lattices of Regular Sets of Natural Integers Closed under Decrementation

Patrick Cégielski\*<sup>†</sup>    Serge Grigorieff\*<sup>‡</sup>    Irène Guessarian\*<sup>‡§</sup>

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## Abstract

We consider lattices of regular sets of non negative integers, i.e. of sets definable in Presburger arithmetic. We prove that if such a lattice is closed under decrement then it is also closed under many other functions: quotients by an integer, roots, etc.

**Keywords.** Lattices, lattices of subsets of  $\mathbb{N}$ , regular subsets of  $\mathbb{N}$ , closure properties.

## 1 Introduction

### 1.1 Roadmap

We follow the terminology according to which a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is non decreasing if  $a \leq b \Rightarrow f(a) \leq f(b)$  for all  $a, b \in \mathbb{N}$ .

We prove in this paper the following result:

**Theorem 1.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a non decreasing function. The following conditions are equivalent:*

- (1) *Every lattice  $\mathcal{L}$  of regular subsets of  $\mathbb{N}$  which is closed under decrement (i.e.  $L \cap L'$ ,  $L \cup L'$  and  $L - 1$  are in  $\mathcal{L}$  whenever  $L, L' \in \mathcal{L}$ ) is also closed under  $f^{-1}$  (i.e.  $L \in \mathcal{L}$  implies  $f^{-1}(L) \in \mathcal{L}$ ).*
- (2) *The function  $f$  satisfies the following properties:*
  - (i)  $f(a) \geq a$  for all  $a \in \mathbb{N}$ ,
  - (ii)  $f(a) - f(b) \equiv 0 \pmod{a - b}$  for all  $a, b \in \mathbb{N}$ .

*Particular examples of such functions  $f$  are division by  $n$  and  $n$ -root for any  $n \geq 1$ .*

This problem, for finite sets and division by  $n$ , was submitted to us by Jean-Éric Pin & Zoltán Ésik, [2]. Jean-Éric Pin & Pedro Silva announce, in the framework of profinite topologies and uniformly continuous functions, a result related to our theorem 1.1 (see [4, 5]).

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<sup>†</sup>LACL, EA 4219, Université Paris-Est Créteil, France.

<sup>‡</sup>LIAFA, CNRS and Université Paris-Diderot, France.

<sup>§</sup>Emeritus at UPMC Univ Paris 6.

Any regular subset  $L$  of  $\mathbb{N}$  is ultimately periodic (cf. Lemma 1.9). For an arithmetic progression  $L$ , the fact that  $f^{-1}(L)$  is a union of decrements of  $L$  is an easy result (cf. Proposition 4.1). Difficulties arise with:

- (1) the finite set coming from the grouping of arithmetic progressions which constitutes the periodic part of  $L$ ,
- (2) the other finite set before periodicity (these two finite sets are the sets  $B$  and  $A$  of Proposition 1.8).

Prior to the general result (cf. Theorems 3.2 & 5.1), we prove particular instances, namely division by  $n$  and  $n$ th root, which give a clearer insight to the proof (cf. Theorems 2.1 and 2.2).

## 1.2 Lattices closed under decrementation

We recall some definitions and fix some notation.

**Definition 1.2.** A lattice  $\mathcal{L}$  over a set  $X$  is any non empty family of subsets of  $X$  such that  $L \cup M$  and  $L \cap M$  are in  $\mathcal{L}$  whenever  $L, M$  are in  $\mathcal{L}$ .

**Definition 1.3.** Let  $L$  be a subset of  $\mathbb{N}$ ,  $i \in \mathbb{N}$  and  $k \in \mathbb{N} \setminus \{0\}$ . The sets

$$L - i = \{x \in \mathbb{N} \mid x + i \in L\}, \quad L \div k = \{x \in \mathbb{N} \mid kx \in L\}, \quad \sqrt[k]{L} = \{x \in \mathbb{N} \mid x^k \in L\}$$

are respectively called the  $i$ -decrement,  $k$ -quotient and  $k$ -root of  $L$ . Observe that the  $i$ -decrement is defined as a subset of  $\mathbb{N}$ , excluding negative integers.

Let  $\mathcal{D}(L)$  denote the family  $\{L - i \mid i \in \mathbb{N}\}$  of decrements of  $L$ .

**Example 1.4.** 1) Let  $L = \{5, 6\} + 4\mathbb{N} = \{5, 6, 9, 10, 13, 14, \dots\}$ , then  $L \div 2 = 3 + 2\mathbb{N} = \{3, 5\} + 4\mathbb{N}$ . Moreover, for any integer  $x$ ,  $x^2 \equiv 0 \pmod{4}$  or  $x^2 \equiv 1 \pmod{4}$ , hence

$$\begin{aligned} x^2 \in \{5, 6\} + 4\mathbb{N} &\iff x^2 \geq 5 \wedge x^2 \equiv 1 \pmod{4} \\ &\iff x \geq 3 \wedge x \equiv 1, 3 \pmod{4} \\ &\iff x \in \{3, 5\} + 4\mathbb{N} \end{aligned}$$

Hence also  $\sqrt{L} = \{3, 5\} + 4\mathbb{N} = L \div 2$ .

2) Let  $L = \{1, 2\} + 4\mathbb{N}$ , then  $L \div 3 = \{2, 3\} + 4\mathbb{N}$  and  $\sqrt{L} = \{1, 3\} + 4\mathbb{N}$ .

The following results are straightforward.

**Proposition 1.5** (Composing decrements).  $(L - i) - j = L - (i + j)$ .

**Proposition 1.6.** For  $L \subseteq \mathbb{N}$  let  $\mathcal{L}(L)$  be the family of sets of the form  $\bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)$  where  $J$  and the  $I_j$ 's are finite non empty subsets of  $\mathbb{N}$ . Then the family  $\mathcal{L}(L)$  is the smallest sublattice of  $\mathcal{P}(\mathbb{N})$  containing  $L$  and closed under decrement.

*Proof.* Observe that  $\left(\bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)\right) - k = \bigcup_{j \in J} \bigcap_{i \in I_j} (L - (i + k))$ .  $\square$

### 1.3 Regular sets of natural integers

**Definition 1.7.** 1. A set  $L \subseteq \mathbb{N}$  is periodic with period  $r$  if, for every  $x$ ,  $x \in L \implies x + r \in L$ .  
 2. A set  $L \subseteq \mathbb{N}$  is ultimately periodic with period  $r$  if there exists  $q \in \mathbb{N}$  such that  $L \cap \{x \mid x \geq q\}$  is periodic with period  $r$ , i.e. for every  $x \geq q$ ,  $x \in L \implies x + r \in L$ .

As we here we work with a semigroup and not a group, namely  $(\mathbb{N}, +)$ , the definition of periodicity is not given by an equivalence  $x \in L \iff x + r \in L$  but by an implication  $x \in L \implies x + r \in L$ .

Regular subsets of  $\mathbb{N}$  are subsets which are recognized by finite automata in unary notation (cf. [1], pages 100–103). Here, we will only use the following classical characterization of regular subsets of  $\mathbb{N}$  which goes back to Myhill, 1957 [3]. Recall that an arithmetic progression is a subset of  $\mathbb{N}$  of the form  $a + r\mathbb{N}$ .

**Proposition 1.8.** Let  $L \subseteq \mathbb{N}$ . The following conditions are equivalent:

- (i)  $L$  is regular,
- (ii)  $L$  is the union of a finite set with finitely many arithmetic progressions,
- (iii)  $L = A \cup (q + B + r\mathbb{N})$ , where  $q \in \mathbb{N}$ ,  $r \in \mathbb{N} \setminus \{0\}$ ,  $A \subseteq \{0, 1, \dots, \max(0, q-1)\}$  and  $B \subseteq \{0, 1, \dots, r-1\}$ .

Observe that in case  $B = \emptyset$ , the set  $A \cup (q + B + r\mathbb{N})$  reduces to the finite set  $A$ . The following lemmas will be useful.

**Lemma 1.9.** Any regular set is ultimately periodic and its family of decrements is finite.

More precisely, suppose  $L = A \cup (q + B + r\mathbb{N}) \subseteq \mathbb{N}$  where  $q \in \mathbb{N}$ ,  $r \in \mathbb{N} \setminus \{0\}$ ,  $A \subseteq \{0, 1, \dots, q-1\} \cap \mathbb{N}$ , and  $B \subseteq \{0, 1, \dots, r-1\}$ . Then

- (1)  $\forall x \geq q (x \in L \iff x + r \in L)$
- (2) The family  $\mathcal{D}(L)$  of decrements of  $L$  is equal to  $\{L - i \mid 0 \leq i < q + r\}$ .

*Proof.* (1) Let  $x \geq q$ , so that  $x = q + i + kr$  for some  $0 \leq i < r$ ,  $k \geq 0$ . Then  $x \in L = A \cup (q + B + r\mathbb{N}) \iff q + i + kr \in q + B + r\mathbb{N} \iff i \in B$ . Similarly,  $x + r \in L \iff q + i + (k+1)r \in q + B + r\mathbb{N} \iff i \in B$ . Thus,  $x \in L \iff x + r \in L$ .

(2) Let  $j \geq q$ . Then  $j = q + i + kr$  for some  $0 \leq i < r$ ,  $k \geq 0$ . For any  $x \in \mathbb{N}$ , we have  $x \in L - j \iff x + j \in L \iff x + q + i + kr \in L \iff x + q + i \in L \iff x \in L - (q + i)$ , the third in place of equivalence being obtained by applying  $k$  times point (1).  $\square$

**Example 1.10.** (Example 1.4 continued) 1) For  $L = \{5, 6\} + 4\mathbb{N}$ , the set  $\mathcal{D}(L)$  consists of 7 sets  $L, L - 1 = \{4, 5\} + 4\mathbb{N}, \dots, L - 5 = \{0, 1\} + 4\mathbb{N}, L - 6 = \{0, 3\} + 4\mathbb{N}, L - 7 = \{2, 3\} + 4\mathbb{N} = L - 3$ .

2) If  $L = \{1, 2\} + 4\mathbb{N}$ , then  $\mathcal{D}(L) = \{L, \{0, 1\} + 4\mathbb{N}, \{0, 3\} + 4\mathbb{N}, \{2, 3\} + 4\mathbb{N}\}$ .

In case of an arithmetic progression, Proposition 1.6 can be simplified.

**Lemma 1.11.** Let  $L = q + r\mathbb{N}$  be the range of an arithmetic sequence,  $r > 0$ .

1. The family  $\mathcal{D}(L)$  of decrements of  $L$  is equal to

$$\mathcal{D}(L) = \{s + r\mathbb{N} \mid 0 \leq s \leq \max(r-1, q)\} = \{L - j \mid 0 \leq j \leq \max(r-1, q)\}.$$

2. The smallest lattice  $\mathcal{L}(L)$  containing  $L$  and closed under decrement is equal to the family of sets

- (i)  $\{A + r\mathbb{N} \mid A \subseteq \{0, \dots, \max(r-1, q)\}\}$  if  $r \geq 2$ ,
- (ii)  $\{s + \mathbb{N} \mid 0 \leq s \leq q\}$  if  $r = 1$ .

In particular, every nonempty set of  $\mathcal{L}(L)$  is a finite union of decrements of  $L$ , and the empty set is in  $\mathcal{L}(L)$  just in case  $r \geq 2$  (obtained with  $A = \emptyset$ ).

*Proof.* 1. In case  $j \leq q$  then  $L - j = s + r\mathbb{N}$  with  $s = q - j \leq q$ . If  $j \geq q$ , i.e.  $j = q + i + kr$  with  $0 \leq i < r$  and  $k \in \mathbb{N}$ , then  $L - j = \{x \in \mathbb{N} \mid x + (q + i + kr) \in q + r\mathbb{N}\} = \{x \in \mathbb{N} \mid x + i \in r\mathbb{N}\} = r\mathbb{N} - i$ . If  $i = 0$  then  $L - j = r\mathbb{N} = L - q$ . If  $0 < i < r$  then  $L - j = r\mathbb{N} - i = (r - i) + r\mathbb{N} = L - (q - (r - i))$ .

2. Observe that the intersection of two sets in the family  $\mathcal{D}(L)$  is either empty (possible in case  $r \geq 2$  only) or equal to the smallest one. Then apply Proposition 1.6, noting that for  $r = 1$ ,  $A + r\mathbb{N} = \min(A) + \mathbb{N}$ .  $\square$

## 2 Closure under quotient and root

The following result was suggested for lattices of finite sets by Ésik & Pin [2].

**Theorem 2.1.** *Any lattice of regular subsets of  $\mathbb{N}$  which is closed under decrement is also closed under  $k$ -quotient, for  $k \in \mathbb{N} \setminus \{0\}$ .*

*Proof.* The case  $k = 1$  is trivial. We prove the theorem by induction on  $k \geq 1$ . For pedagogical reasons, we explicit the case  $k = 2$ .

*Case  $k = 2$ .* Consider some  $L \in \mathcal{L}$  and let  $J_a = (L - a) \cap \bigcap_{i \in L - a} (L - i)$  for any  $a \in \mathbb{N}$ . By Lemma 1.9, there are finitely many distinct sets  $(L - i)$ 's, so that  $J_a$  is a finite intersection of decrements of  $L$ . The assumed closure properties of  $\mathcal{L}$  insure that  $J_a \in \mathcal{L}$ .

In case  $a \in L \div 2$ , i.e.  $2a \in L$ , the following properties are true.

- (1)  $a \in J_a$ . In fact,  $a \in L - i$  for any  $i \in L - a$  and  $a \in L - a$  because  $2a \in L$ .
- (2)  $J_a \subseteq L \div 2$ . Indeed, if  $b \in J_a$  then  $b \in L - a$  and  $b$  is in all the  $(L - i)$ 's, for  $i \in L - a$ . Letting  $i = b$ , we get  $b \in L - b$ , i.e.  $2b \in L$  and  $b \in L \div 2$ .

Since there are finitely many  $L - a$ 's, there are finitely many  $J_a$ 's. Using closure under finite union, we see that  $K = \bigcup_{a \in L \div 2} J_a$  is in  $\mathcal{L}$ . Clearly,  $K = L \div 2$  because each element  $a \in L \div 2$  is in  $J_a$  and each  $J_a$  is included in  $L \div 2$ .

*Inductive case.* Assuming  $\mathcal{L}$  is closed under  $k$ -quotient, we prove that it is closed under  $(k + 1)$ -quotient. For  $L \in \mathcal{L}$ , set  $J_a = ((L - a) \div k) \cap \bigcap_{i \in (L - a) \div k} (L - ki)$  for any  $a \in \mathbb{N}$ . By Lemma 1.9, there are finitely many distinct  $(L - i)$ 's, so that  $J_a$  is a finite intersection of decrements of  $L$  and of a  $k$ -quotient of  $L$ . The assumed closure properties of  $\mathcal{L}$  and induction hypothesis insure that  $J_a \in \mathcal{L}$ .

In case  $a \in L \div (k + 1)$ , i.e.  $(k + 1)a \in L$ , the following properties are true.

- (1)  $a \in J_a$ . In fact,  $a \in L - ki$  for any  $i \in (L - a) \div k$ . Also, since  $(k + 1)a \in L$ , we have  $ka \in L - a$  hence  $a \in (L - a) \div k$ .
- (2)  $J_a \subseteq L \div (k + 1)$ . If  $b \in J_a$  then  $b \in (L - a) \div k$  and  $b$  is in all the  $L - ki$ 's, for  $i \in (L - a) \div k$ . Letting  $i = b$ , we get  $b \in L - kb$ , i.e.  $(k + 1)b \in L$  and  $b \in L \div (k + 1)$ .

Since there are finitely many  $(L - a)$ 's, there are finitely many  $(L - a) \div k$ 's hence finitely many  $J_a$ 's. Using closure under finite union, we see that the set  $K = \bigcup_{a \in L \div (k+1)} J_a$  is in  $\mathcal{L}$ . Clearly,  $K = L \div (k + 1)$  because each element  $a \in L \div (k + 1)$  is in  $J_a$  and each  $J_a$  is included in  $L \div (k + 1)$ .  $\square$

**Theorem 2.2.** *Any lattice of regular subsets of  $\mathbb{N}$  which is closed under decrement is also closed under  $k$ -root, for  $k \in \mathbb{N} \setminus \{0\}$ .*

*Proof.* Adapt the above proof: substitute  $\times$  and division for  $+$  and subtraction, so that  $L - i$  becomes  $L \div i$ . In the argument, finiteness of the family  $\{L - i \mid i \in \mathbb{N}\}$  is replaced by that of  $\{L \div k \mid k \in \mathbb{N} \setminus \{0\}\}$  which holds since, by Lemma 1.9 and Proposition 1.6,  $\mathcal{L}(L)$  is always finite when  $L$  is regular.  $\square$

**Example 2.3.** (Examples 1.4 and 1.10 continued) If  $L = \{1, 2\} + 4\mathbb{N}$ , then  $L \div 3 = \{2 + 4\mathbb{N}\} \cup \{3 + 4\mathbb{N}\} = L - 3$  and  $\sqrt{L} = \{1, 3\} + 4\mathbb{N} = (L - 5) \cup (L - 3)$ .

For  $L = \{5, 6\} + 4\mathbb{N}$ , we have  $L \div 2 = \sqrt{L} = \{3, 5\} + 4\mathbb{N} = ((L - 2) \cap (L - 3)) \cup (L \cap (L - 1))$ .

### 3 More induced closures

We extend closure under quotient (cf. Theorem 2.1) and under  $n$ -root (cf. Theorem 2.2) to a more general class of functions  $f : \mathbb{N} \rightarrow \mathbb{N}$ . Given a regular set  $L \subseteq \mathbb{N}$  and  $n \in \mathbb{N}$ , the set  $L - n = \{x \in \mathbb{N} \mid x + n \in L\}$  is regular. Also, by Lemma 1.9, the family  $\{L - n \mid n \in \mathbb{N}\}$  is finite.

**Lemma 3.1.** *For any set  $L \subseteq \mathbb{N}$  and for any function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f(x) - f(y) \in (x - y)\mathbb{N}$  for every  $x, y \in \mathbb{N}$ , and such that  $f(x) \geq x$  for every  $x \in \mathbb{N}$ , we have:*

$$f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \left( \bigcap_{n \in L - a} L - n \right) \quad (1)$$

*Proof.* Let us first consider  $a \in f^{-1}(L)$ . Notice that for every  $n \in L - a$ , we have  $a + n \in L$  and thus  $a \in L - n$ . We deduce that  $a$  is in  $\bigcap_{n \in L - a} L - n$  and the inclusion  $\subseteq$  is proved.

For the other inclusion, let  $a \in f^{-1}(L)$  and  $b \in \bigcap_{n \in L - a} L - n$ . By the assumption on  $f$ , there exists  $k \in \mathbb{N}$  such that  $f(a) - f(b) = k(a - b)$ . Assume by contradiction that  $f(b) \notin L$ . Since  $f(a) \in L$  we get  $f(a) \neq f(b)$ , and in particular  $a \neq b$ .

Assume first that  $a < b$ . We consider the minimal natural number  $r \in \mathbb{N}$  such that  $f(a) + r(b - a) \notin L$ . Note that such a natural number exists since  $f(a) + k(b - a) = f(b) \notin L$ . Moreover, since  $f(a) \in L$  we get  $r \geq 1$ . By minimality of  $r$ , we get  $f(a) + (r - 1)(b - a) \in L$ . Thus,  $n + a \in L$  with  $n = f(a) + r(b - a) - b$ . Since  $f(a) \geq a$ , we get  $n \geq (r - 1)(b - a) \geq 0$ . Now  $n + a \in L$  implies  $n \in L - a$  and thus  $b \in L - n$ ; hence  $n + b = f(a) + r(b - a) \in L$ , contradicting the definition of  $r$ .

Assume next that  $a > b$  and consider the minimal natural number  $r \in \mathbb{N}$  such that  $f(b) + r(a - b) \notin L$ . Again, such a natural number exists since  $f(b) + k(a - b) = f(a) \in L$ . Moreover, since  $f(b) \notin L$ , we get  $r \geq 1$ . Let  $n = f(b) - b + (r - 1)(a - b)$ . Since  $f(b) \geq b$  and  $a - b \geq 0$  we get  $n \geq 0$ . Moreover, as  $n + a = f(b) + r(a - b) \in L$ , we get  $n \in L - a$ . Thus,  $b \in L - n$  and we get  $n + b \in L$ . That means  $n + b = f(b) + (r - 1)(a - b) \in L$  which contradicts the minimality of  $r$ .

We have proved by contradiction that  $f(b) \in L$ . Thus,  $b \in f^{-1}(L)$  and we get the other inclusion.  $\square$

We can now prove the (2)  $\Rightarrow$  (1) implication of our main theorem 1.1.

**Theorem 3.2.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be non decreasing and such that (i)  $f(a) \geq a$  and (ii)  $f(a) - f(b) \equiv 0 \pmod{a - b}$  for all  $a, b \in \mathbb{N}$ . Every lattice of regular subsets of  $\mathbb{N}$  closed under decrement is also closed under  $f^{-1}$ .*

*Proof.* Let  $\mathcal{L}$  be a lattice of regular sets closed under decrement and let  $L \in \mathcal{L}$ . Consider the representation of  $f^{-1}(L)$  given by formula (1) of Lemma 3.1. In order to ensure that  $f^{-1}(L)$  belongs to the lattice  $\mathcal{L}$ , we have to show that both the intersection and the union are finite: since  $L$  is regular, the family  $\{L - n \mid n \in \mathbb{N}\}$  is finite by Lemma 1.9; this concludes the proof.  $\square$

**Remark 3.3.** Every non decreasing polynomial with integral coefficients mapping  $\mathbb{N}$  into  $\mathbb{N}$  satisfies the conditions of Theorem 3.2. Thus, Theorems 2.1 and 2.2 are consequences of Theorem 3.2; their proof gives a first idea and a better understanding to prove the more general Theorem 3.2).

## 4 About arithmetic progressions

For arithmetic progressions we sharpen Theorem 3.2 and give a simpler proof.

**Proposition 4.1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  non decreasing be such that for all  $a, b \in \mathbb{N}$  (i)  $f(a) \geq a$  and (ii)  $f(a) - f(b) \equiv 0 \pmod{a - b}$ . For every arithmetic progression  $L = q + r\mathbb{N}$ , with  $q, r \in \mathbb{N}$ ,  $r \geq 1$ , the following conditions hold:*

- (1)  $f^{-1}(L)$  is the union of at most  $r$  decrements of  $L$ ,
- (2) the smallest lattice  $\mathcal{L}(L)$  closed under decrement and such that  $L \in \mathcal{L}$  is closed under  $f^{-1}$ .

*Proof.* (1) If  $f(a) \in q + r\mathbb{N}$  then, using monotonicity of  $f$  and property (ii), for every  $k \in \mathbb{N}$  there exists  $\ell \in \mathbb{N}$  such that  $f(a + kr) = f(a) + \ell r$  hence  $f(a + kr) \in q + r\mathbb{N}$  and  $a + r\mathbb{N} \subseteq f^{-1}(L)$ . Thus,  $f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} a + r\mathbb{N}$ . Now, if  $a < b$  and  $a \equiv b \pmod{r}$  then  $b + r\mathbb{N} \subseteq a + r\mathbb{N}$ . Hence the last equality can be rewritten  $f^{-1}(L) = \bigcup_{a \in M} a + r\mathbb{N}$  where  $M$  picks the minimum element of  $f^{-1}(L) \cap (i + r\mathbb{N})$  for each  $i$  such that  $0 \leq i < r$  and  $f^{-1}(L) \cap (i + r\mathbb{N})$  is nonempty. In particular,  $M$  has at most  $r$  elements.

It remains to show that, for each  $a \in M$ , the set  $a + r\mathbb{N}$  is a decrement of  $L$ . Using Lemma 1.11, this amounts to show that  $a \leq \max(r - 1, q)$  for each  $a \in M$ . Let  $a \in M$ ,  $a = \min(f^{-1}(L) \cap (i + r\mathbb{N}))$  with  $0 \leq i < r$ . By way of contradiction, supposing  $a > \max(r - 1, q)$ , so that  $a - r \in \mathbb{N}$ , we show that  $f(a - r) \in L$ .

*Case  $q < r$ .* Then  $\max(r - 1, q) = r - 1 < a$ . Since  $f(a) \in L$  we have  $f(a) = q + kr$  for some  $k \in \mathbb{N}$ . Using property (ii), we get  $f(a) \equiv f(a - r) \equiv q \pmod{r}$ . Since  $q < r$ , this yields  $f(a - r) = q + \ell r$  for some  $\ell \in \mathbb{N}$  and thus  $f(a - r) \in L$ .

*Case  $q \geq r$ .* Then  $\max(r - 1, q) = q$  and  $a > q \geq r$ . Let  $q = i + kr$  with  $0 \leq i < r$  and  $k \geq 1$ . As above,  $f(a - r) \equiv f(a) \equiv i \pmod{r}$  hence  $f(a - r) = i + \ell r$  for some  $\ell \in \mathbb{N}$ . Now,  $f(a - r) \geq a - r$  by (i) hence  $i + \ell r \geq a - r > q - r = i + (k - 1)r$  so that  $\ell \geq k$ . Thus,  $f(a - r) = i + \ell r = q - kr + \ell r = q + (\ell - k)r \in q + r\mathbb{N} = L$ . In both cases, we have  $f(a - r) \in L$ , contradicting the minimality of  $a$  in the intersection of its congruence class modulo  $r$  with  $f^{-1}(L)$ .

(2) By Lemma 1.11 any set  $K$  in  $\mathcal{L}(L)$  is of the form  $K = A + r\mathbb{N}$  with  $A \subseteq \{0, \dots, \max(r - 1, q)\}$ , hence  $\forall a \in A \quad a \leq \max(r - 1, q)$ . Then  $f^{-1}(K) =$

$\cup_{a \in A} f^{-1}(a + r\mathbb{N})$ . By (1), each  $f^{-1}(a + r\mathbb{N})$  is of the form  $A_a + r\mathbb{N}$  with  $A_a \subseteq \{0, \dots, \max(r-1, a)\}$ . Hence  $f^{-1}(K) = \cup_{a \in A} (A_a + r\mathbb{N}) = (\cup_{a \in A} A_a) + r\mathbb{N}$ , and  $\cup_{a \in A} A_a$  is a subset of  $\{0, \dots, \max(r-1, q)\}$ . When  $r \geq 2$ , this concludes the proof that  $f^{-1}(K) \in \mathcal{L}(L)$  by Lemma 1.11 2(i). If  $r = 1$ , we must check also that  $f^{-1}(K) \neq \emptyset$ : indeed  $K = a + \mathbb{N}$  by Lemma 1.11 2(ii), as  $f(a) \geq a$  by hypothesis (i),  $f(a) \in a + \mathbb{N}$  and  $a \in f^{-1}(K)$  which is non empty; this concludes the proof that  $f^{-1}(K) \in \mathcal{L}(L)$  for the case  $r = 1$ .  $\square$

**Remark 4.2.** The statement of Proposition 4.1 is sharper than that of Theorem 3.2 applied to arithmetic progressions. In fact, the proof of Proposition 4.1 also shows that, for an arithmetic progression  $L$ , the lattice  $\mathcal{L}(L)$  is the smallest join-semilattice containing  $L$  and closed under decrement.

**Remark 4.3.** The proof of Proposition 4.1 cannot be extended to regular sets, not even to periodic sets. Let  $f: n \mapsto n^2$  and  $L$  the periodic set  $\{0, 4, 8\} + 3\mathbb{N} = \{0, 3, 4\} \cup (6 + \mathbb{N})$ . Then  $f^{-1}(L) = \sqrt{L} = \mathbb{N} \setminus \{1\} = L - 4$  is a decrement of  $L$ . However, this result cannot be obtained by the proof of Proposition 4.1 because this proof relies on the fact that, whenever  $a \in f^{-1}(L)$ , then  $a + r\mathbb{N}$  is a decrement of  $L$ ; here however,  $2 + 3\mathbb{N}$  is not a decrement of  $L$  and does not even belong to  $\mathcal{L}(L)$ . Indeed,  $\mathcal{D}(L)$  consists here of  $L, L-1, L-2, L-3, L-4, L-5, L-6 = \mathbb{N}$ , all of which are of the form  $D_i \cup (6 + \mathbb{N})$  with  $D_i$  a finite set. Thus,  $2 + 3\mathbb{N}$  cannot be obtained by finite unions, intersections and decrements of such sets, all of which contain *all* the integers larger than 6.

**Remark 4.4.** Proposition 4.1 does not hold for finite sets, nor general regular sets, nor periodic sets: unions of decrements are not sufficient to obtain  $f^{-1}(L)$ , intersections are needed.

Consider  $f: n \mapsto n^2$ . Let  $L = \{5, 6\} + 4\mathbb{N}$  (periodic); then (cf. Example 2.3)  $L \div 2 = \sqrt{L} = \{3, 5\} + 4\mathbb{N} = ((L-2) \cap (L-3)) \cup (L \cap (L-1))$  cannot be obtained as a union of decrements of  $L$ : in order to obtain 5, we must include either  $L, L-1, L-4$  or  $L-5$ , but each of these decrements contains numbers not in  $\sqrt{L}$  (respectively 6, 4, 2 and 0) which must be excluded by a suitable intersection.

Let  $L = \{1, 2\}$  then  $f^{-1}(L) = \{1\}$ ; the decrements of  $L$  are the sets  $\{1, 2\}, \{0, 1\}, \{0\}, \emptyset$ , no union of which is  $f^{-1}(L)$ , intersections are required to get  $f^{-1}(L)$ .

This is why the proof in both the general and the finite case does exclude the elements which are not in  $f^{-1}(L)$  by using carefully chosen intersections.

## 5 Characterizing induced closures

We characterize the functions  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that closure under decrement yields closure under  $f^{-1}$ .

**Theorem 5.1.** *Let  $f: \mathbb{N} \rightarrow \mathbb{N}$ . The following conditions are equivalent.*

- (i) *Every lattice of regular subsets of  $\mathbb{N}$  closed under decrement is closed under  $f^{-1}$ .*
- (ii) *For every finite subset  $L$  of  $\mathbb{N}$ , the lattice  $\mathcal{L}(L)$  is closed under  $f^{-1}$ .*
- (iii) *For every arithmetic progression  $L = q + r\mathbb{N}$ ,  $r > 0$ , the lattice  $\mathcal{L}(L)$  is closed under  $f^{-1}$ .*



(iv) *The map  $f$  is non decreasing and satisfies  $f(a) \geq a$  and  $f(a) - f(b) \equiv 0 \pmod{(a-b)}$  for all  $a, b \in \mathbb{N}$ .*

*Proof.* (iv)  $\Rightarrow$  (i). This is Theorem 3.2.

(i)  $\Rightarrow$  (ii). Finite sets are regular sets.

(i)  $\Rightarrow$  (iii). Arithmetic progressions are regular sets.

(ii)  $\Rightarrow$  (iv). We first prove that  $f(a) \geq a$ , for all  $a \in \mathbb{N}$ . Let  $a \in \mathbb{N}$  and  $L = \{f(a)\}$ . Observe that the smallest lattice containing the set  $\{f(a)\}$  and closed under decrementation is the family of subsets of  $\{0, 1, \dots, f(a)-1, f(a)\}$ . As a consequence, all elements of  $f^{-1}(L)$  must be less than  $f(a)$ . In particular  $a \leq f(a)$ , since  $a \in f^{-1}(L)$ .

We prove now that  $f(a) - f(b) \in (a-b)\mathbb{N}$  for all  $a, b \in \mathbb{N}$  such that  $a > b$ . In particular,  $f$  is monotone non decreasing and  $f(a) - f(b) \equiv 0 \pmod{(a-b)}$ . We argue by contradiction. Suppose that  $f(a) \notin f(b) + (a-b)\mathbb{N}$ . Let

$$\ell = \left\lfloor \frac{f(a) - a}{a - b} \right\rfloor, \quad k = \left\lfloor \frac{f(a)}{a - b} \right\rfloor, \quad L = \{f(a) - j(a - b) \mid 0 \leq j \leq k\}.$$

Since  $f(a) \geq a$ , we have  $\ell \geq 0$ ; moreover,

$$k = \left\lfloor \frac{f(a)}{a - b} \right\rfloor = \left\lfloor \frac{f(a) - a}{a - b} + 1 + \frac{b}{a - b} \right\rfloor$$

hence  $k \geq \ell + 1$ .

For  $j \in \{0, \dots, k\}$ ,  $f(a) \neq f(b) + j(a - b)$  hence  $f(b) \neq f(a) - j(a - b)$ . Thus,  $f(b) \notin L$  and  $b \notin f^{-1}(L)$ . Of course,  $f(a) \in L$  and  $a \in f^{-1}(L)$ . To get a contradiction, we show that  $f^{-1}(L)$  is not in  $\mathcal{L}(L)$ . Since  $f^{-1}(L)$  contains  $a$  but not  $b$ , it suffices to show that every set  $X \in \mathcal{L}(L)$  which contains  $a$  also contains  $b$ . Since  $\mathcal{L}(L)$  is generated by the  $L - i$ 's, we reduce to show that, for all  $i$ , if  $a$  is in  $L - i$  then so is  $b$ . Now, using the definition of  $\ell$ , for all  $i \in \mathbb{N}$

$$\begin{aligned} a \in L - i &\iff \exists \alpha \in \{0, \dots, k\} \quad a = f(a) - \alpha(a - b) - i \\ &\iff \exists \alpha \in \{0, \dots, k\} \quad i = f(a) - a - \alpha(a - b) \\ &\iff \exists \alpha \in \{0, \dots, \ell\} \quad i = f(a) - a - \alpha(a - b) \end{aligned}$$

and, for  $i$  associated to such an  $\alpha \in \{0, \dots, \ell\}$ ,

$$L - i = \mathbb{N} \cap \{a + (\alpha - j)(a - b) \mid j \in \{0, \dots, k\}\}$$

letting  $j = \alpha$  and  $j = \alpha + 1$  (which is  $\leq k$  since  $\alpha \leq \ell < k$ ), we see that

$$L - i \supseteq \{a, b\}.$$

This gives the required contradiction.

(iii)  $\Rightarrow$  (iv). Note first that if (iii) holds,  $f$  cannot be constant: indeed, for any constant function  $f(x) = a$ , there exists an arithmetic progression, namely  $L = a + 1 + \mathbb{N}$ , such that the lattice  $\mathcal{L}(L)$  is not closed under  $f^{-1}$ . In fact,  $f^{-1}(L) = \emptyset \notin \mathcal{L}(L)$  because all sets of  $\mathcal{D}(L)$  are of the form  $\{\ell + \mathbb{N} \mid 0 \leq \ell \leq a + 1\}$ , hence all their finite intersections contain  $a + 1 + \mathbb{N}$  and so are not empty.

By Lemma 1.11, if  $L = q + r\mathbb{N}$ , with  $q, r \in \mathbb{N}$ ,  $r \geq 1$ , then  $\mathcal{L}(L)$  is the family of sets of the form  $B + r\mathbb{N}$  with  $B \subseteq \{0, \dots, \max(q, r - 1)\}$ , and  $B \neq \emptyset$  if  $r = 1$ .

First, we check that  $f$  is non decreasing. Let  $a < b$  and let  $L = f(a) + \mathbb{N}$ . Note that, since  $r = 1$ , a set  $B + \mathbb{N}$  is equal to  $\min(B) + \mathbb{N}$  and  $\emptyset \notin \mathcal{L}(L)$ , hence  $f^{-1}(L) = s + \mathbb{N}$ , with  $s \leq f(a)$ ; as  $a \in f^{-1}(L)$ , then  $a \in s + \mathbb{N}$ , i.e.  $a \geq s$ , and  $f(s + \mathbb{N}) \subseteq L$ . Since  $b > a \geq s$  we get  $b \in s + \mathbb{N}$  and  $f(b) \in L$  hence  $f(b) \geq f(a)$ .

Second, we show that  $f(b) - f(a) \in (b-a)\mathbb{N}$  whenever  $a < b$ . Let  $L = f(a) + (b-a)\mathbb{N}$ . Then, in view of Lemma 1.11, we may write  $f^{-1}(L) = A + (b-a)\mathbb{N}$ . Since  $a \in f^{-1}(L)$ , we have  $a + (b-a)\mathbb{N} \subseteq f^{-1}(L)$  hence  $b = a + (b-a) \in f^{-1}(L)$ , i.e.  $f(b) \in L$ , whence for some  $k$ ,  $f(b) = f(a) + k(b-a)$ .

Finally, we show that  $f(a) \geq a$  for all  $a \in \mathbb{N}$ . Suppose that, for some  $a$ ,  $f(a) < a$ . Since  $a$  divides  $f(a) - f(0) \leq f(a) < a$ , we have  $f(a) = f(0)$  hence  $f$  is constant on  $\{0, \dots, a\}$  with value  $< a$ .

*Case 1.* There are infinitely many  $a$ 's such that  $f(a) < a$ . Then  $f$  is constant, contradicting what was proved above.

*Case 2.* There is a largest  $a$  such that  $f(a) < a$ . Then  $f(x) = f(0) < a$  for  $x \leq a$  and  $f(x) \geq x > a$  for  $x > a$ . Thus,  $\emptyset \neq f^{-1}(a + \mathbb{N}) \subseteq (a+1) + \mathbb{N}$  is not in  $\mathcal{L}(a + \mathbb{N})$ , contradicting (iii).  $\square$

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## References

- [1] S. Eilenberg, Automata, languages and machines, vol. A, Academic Press, New York, 1974.
- [2] Z. Ésik, and J.-É. Pin. *Personal communication*. July, 2011.
- [3] J. Myhill, Finite automata and the representation of events, Wright Air Development Command Tech. Rep. 5764, 1957, pp. 112-137.
- [4] J.-É. Pin and P.V. Silva, On profinite uniform structures defined by varieties of finite monoids, International Journal of Algebra and Computation, 21, 2011, pp 295-314.
- [5] J.-É. Pin and P.V. Silva, On uniformly continuous functions for some profinite topologies, in preparation.
- [6] R. Stansifer, Presburgers Article on Integer Arithmetic: Remarks and Translation, Cornell tech reports, <http://ecommons.library.cornell.edu/bitstream/1813/6478/1/84-639.pdf>.