HAL
open science

Diagonal reduction algebra and the reflection equation

S. Khoroshkin, O. Ogievetsky

To cite this version:

S. Khoroshkin, O. Ogievetsky. Diagonal reduction algebra and the reflection equation . Israel Journal of Mathematics, 2017, 221 (1), pp.705-729. 10.1007/s11856-017-1571-2 . hal-01259066

HAL Id: hal-01259066

https://hal.science/hal-01259066v1

Submitted on 11 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Diagonal reduction algebra and reflection equation

S. Khoroshkin[◇⋆] **and O. Ogievetsky**[∘1]

◇ ITEP, B.Cheremushkinskaya 25, Moscow 117218, Russia
⋆ National Research University Higher School of Economics,
Myasnitskaya 20, Moscow 101000, Russia
∘ Aix Marseille Université, Université de Toulon,
CNRS, CPT UMR 7332, 13288, Marseille, France

arXiv:1510.05258v1 [math.RT] 18 Oct 2015

Abstract

We describe the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ of the Lie algebra $\mathbf{gl}_n$ in the R-matrix formalism. As a byproduct we present two families of central elements and the braided bialgebra structure of $\mathcal{D}(\mathbf{gl}_n)$.

1 Introduction

Reduction algebras were introduced in [M, AST2] for a study of representations of a Lie algebra with the help of the restriction to the space of highest weight vectors with respect to a reductive subalgebra $\mathbf{g}$. In an abstract setting, for an associative algebra $\mathcal{A}$ which contains $\mathrm{U}(\mathbf{g})$ as a subalgebra and satisfies certain finiteness conditions, the corresponding reduction algebra is the double coset (2) equipped with a nontrivial multiplication (3).

This associative multiplication, defined with the help of the extremal projector of Asherova-Smirnov-Tolstoy [AST], can be also described [K] by means of the so called universal dynamical twist J. This twist gives rise to a solution of the universal dynamical Yang–Baxter equation [ABRR].

The diagonal reduction algebra $\mathcal{D}(\mathbf{g})$ is a particular case, associated to the diagonal embedding of $\mathrm{U}(\mathbf{g})$ into $\mathrm{U}(\mathbf{g}) \otimes \mathrm{U}(\mathbf{g})$, of reduction algebras. The algebra $\mathcal{D}(\mathbf{g})$ acts in the space of highest weight vectors of the tensor product of two representations of $\mathbf{g}$, considered as the representation of $\mathbf{g}$. In [KO2, KO3] we presented a list of ordering defining relations for natural generators of the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ of the Lie algebra $\mathbf{gl}_n$.

The main goal, proposition 4.1 of this paper, is to relate the algebra $\mathcal{D}(\mathbf{gl}_n)$ to the R-matrix formalism. We exhibit a matrix L of certain generators of the algebra $\mathcal{D}(\mathbf{gl}_n)$ such that the defining relations can be collected into the operator equation usually called the reflection equation.

To this end we study and use the reduction algebras $\mathrm{Diff}_{\mathbf{h}}(n, N)$ of algebras of differential operators in nN variables, which we call the algebras of $\mathbf{h}$-deformed differential operators. These reductions were used in [KN] for the representation theory of Yangians. The algebras $\mathrm{Diff}_{\mathbf{h}}(n, N)$ are closely related, by means of the generalized Harish-Chandra isomorphism [KNV], to "relative Yangians" of A. Joseph [J] and "family algebras" of A. A. Kirillov [Kr].

In Section 3 we introduce a distinguished set of Heisenberg type generators of the algebras $\mathrm{Diff}_{\mathbf{h}}(n, N)$ and write down the defining relations for them. It turns out that

[1]On leave of absence from P. N. Lebedev Physical Institute, Leninsky Pr. 53, 117924 Moscow, Russia

the relations admit the $\hat{\mathrm{R}}$-matrix form, with the well-known solution $\hat{\mathrm{R}}$ of the dynamical Yang–Baxter equation. Next we exhibit homomorphisms from the diagonal reduction algebra $\mathcal{D}(\mathbf{gl}_n)$ to $\mathrm{Diff}_{\mathbf{h}}(n,N)$ ($\mathbf{h}$-analogues of the "oscillator representations"), which lead to the presentation of $\mathcal{D}(\mathbf{gl}_n)$ by means of the reflection equation.

The description of the algebra $\mathcal{D}(\mathbf{gl}_n)$ as the reflection equation algebra has many advantages. As an application we find two natural families of central elements of $\mathcal{D}(\mathbf{gl}_n)$ expressed as "quantum" traces of powers of L-operators, see Section 4.2. Also, the R-matrix formalism reveals the braided bialgebra structure of $\mathcal{D}(\mathbf{g})$, see Section 4.3.

The modules over the algebra $\mathcal{D}(\mathbf{g})$ and the braided tensor structure on certain categories of $\mathcal{D}(\mathbf{g})$-modules we study in a forthcoming publication.

2 Reduction algebras

In this section we recall the definition and some basic properties of reduction algebras. We restrict ourselves to reduction algebras related to general linear Lie algebra $\mathbf{gl}_n$ which we denote further by $\mathbf{g}$. Let e_{ij}, $i,j=1,\dots,n$, be the standard generators of the Lie algebra $\mathbf{g}$, with the commutation relations

$$[e_{ij},e_{kl}]=\delta_{jk}e_{il}-\delta_{il}e_{kj}\ . \tag{1}$$

We use the notation $\mathbf{h}$ and $\mathbf{n}_\pm$ for the Cartan and two opposite nilpotent subalgebras of $\mathbf{g}$; h_i denotes the element $e_{ii}\in\mathbf{h}$, and ε_i the elements in $\mathbf{h}^*$, so that $\varepsilon_i(h_j):=\delta_{i,j}$; $h_{ij}:=h_i-h_j\in\mathbf{h}$ and $\tilde{h}_i:=h_i-i$, $\tilde{h}_{ij}=\tilde{h}_i-\tilde{h}_j$ are the elements of $\mathrm{U}(\mathbf{h})$. We define $\bar{\mathrm{U}}(\mathbf{h})$ to be the ring of fractions of the commutative ring $\mathrm{U}(\mathbf{h})$ with respect to the multiplicative set of denominators, generated by the elements $(h_{ij}+k)^{-1}$, $k\in\mathbb{Z}$; and $\bar{\mathrm{U}}(\mathbf{g})$ to be the ring of fractions of the universal enveloping algebra $\mathrm{U}(\mathbf{g})$ with respect to the same set of denominators.

1. Let $\mathcal{A}$ be an associative algebra which contains $\mathrm{U}(\mathbf{g})$ as a subalgebra, and the adjoint action of $\mathbf{g}$ on $\mathcal{A}$ is locally finite. In particular $\mathcal{A}$ is a $\mathrm{U}(\mathbf{g})$-bimodule with respect to the multiplication by elements from $\mathrm{U}(\mathbf{g})$ on the left and on the right. Assume in addition that $\mathcal{A}$ is free as the left $\mathrm{U}(\mathbf{h})$-module and the adjoint action of $\mathrm{U}(\mathbf{h})$ is semisimple. Let $\bar{\mathcal{A}}$ be the localization $\bar{\mathcal{A}}=\mathcal{A}\otimes_{\mathrm{U}(\mathbf{h})}\bar{\mathrm{U}}(\mathbf{h})$. The double coset space

$$\tilde{\mathcal{A}}:=\mathbf{n}_-\bar{\mathcal{A}}\backslash\bar{\mathcal{A}}/\bar{\mathcal{A}}\mathbf{n}_+ \tag{2}$$

equipped with a natural associative multiplication $\diamond$, see e.g. [KO1] for details, is usually called the *reduction* (double coset) algebra. The multiplication $\diamond$ is described by the prescription

$$x\diamond y=x\,\mathrm{P}\,y\,, \tag{3}$$

where P is the extremal projector [AST]. The projector P belongs to a certain extension of $\bar{\mathrm{U}}(\mathbf{g})$, satisfies the properties

$$\begin{aligned} &x\,\mathrm{P}=\mathrm{P}\,y=0 && \text{for } x\in\mathbf{n}_+,\ y\in\mathbf{n}_-,\\ &\mathrm{P}=1 \mod \mathbf{n}_-\bar{\mathrm{U}}(\mathbf{g}), && \mathrm{P}=1 \mod \bar{\mathrm{U}}(\mathbf{g})\mathbf{n}_+, \qquad \mathrm{P}^2=\mathrm{P}\,. \end{aligned}$$

and can be given by the explicit multiplicative formula [AST]. Alternatively, one can find representatives $\tilde{x}\in\bar{\mathcal{A}}$ and $\tilde{y}\in\bar{\mathcal{A}}$ of coset classes x and y, such that $\tilde{x}$ belongs to the

normalizer of the left ideal $\bar{\mathcal{A}}\mathbf{n}_+$, or $\tilde{y}$ belongs to the normalizer of the right ideal $\mathbf{n}_-\bar{\mathcal{A}}$. Then $x\diamond y$ is the image in the coset space $\tilde{\mathcal{A}}$ of the product $\tilde{x}\cdot\tilde{y}$.

In the sequel the algebra $\mathcal{A}$ is free as the left $\mathrm{U}(\mathbf{g})$-module with respect to the multiplication by elements of $\mathrm{U}(\mathbf{g})$ on the left, and this $\mathrm{U}(\mathbf{g})$-module is generated by a linear space V, invariant with respect to the adjoint action (which is supposed to be locally finite) of the Lie algebra $\mathbf{g}$. In this case the reduction algebra is the free left $\bar{\mathrm{U}}(\mathbf{h})$-module, generated by V, see [Zh, KO1]. Note also that the construction of the double coset space $\tilde{\mathcal{A}}$ does not use the multiplication in $\mathcal{A}$ but the $\mathrm{U}(\mathbf{g})$-bimodule structure on $\mathcal{A}$ only. We denote sometimes by $:x:$ the image in $\tilde{\mathcal{A}}$ of the element $x\in\bar{\mathcal{A}}$. This notation is useful to distinguish between the multiplication $\cdot$ in the algebra $\mathcal{A}$ and the multiplication $\diamond$ in the reduction algebra $\tilde{\mathcal{A}}$.

2. The Weyl group of the root system of $\mathbf{g}$ is the symmetric group S_n. Let $\sigma_1,\dots,\sigma_{n-1}$ be the generators of S_n; σ_i corresponds to the permutation $(i,i+1)$. The group S_n acts on vector spaces $\mathbf{h}^*$ and $\mathbf{h}$, so that $\sigma_i(h_j)=h_{\sigma_i(j)}$ and $\sigma_i(\varepsilon_j)=\varepsilon_{\sigma_i(j)}$. These actions are related by $\lambda(\sigma(h))=\sigma^{-1}(\lambda)(h)$, $\sigma\in\mathsf{S}_\mathsf{n}$, $h\in\mathbf{h}$ and $\lambda\in\mathbf{h}^*$. The action of S_n on $\mathbf{h}$ extends to the action of a cover $\tilde{\mathsf{S}}_n$ of the group S_n by automorphisms of the Lie algebra $\mathbf{g}$. We denote by the same symbols σ_i the following automorphisms of the algebra $\mathrm{U}(\mathbf{g})$:

$$\sigma_i(x):=\mathrm{Ad}_{\exp(e_{i,i+1})}\mathrm{Ad}_{\exp(-e_{i+1,i})}\mathrm{Ad}_{\exp(e_{i,i+1})}(x)\ ,$$

so that

$$\sigma_i(e_{kl})=(-1)^{\delta_{ik}+\delta_{il}}e_{\sigma_i(k)\sigma_i(l)}\ .$$

Let $\rho=-\sum_{k=1}^n k\varepsilon_k$. Then the *shifted action* $\circ$ of the group S_n on the vector space $\mathbf{h}^*$ is defined by setting

$$\sigma\circ\lambda:=\sigma(\lambda+\rho)-\rho\ . \tag{4}$$

With the help of (4) we induce the action $\circ$ of S_n on the commutative algebra $\mathrm{U}(\mathbf{h})$ by regarding the elements of this algebra as polynomial functions on $\mathbf{h}^*$. We have

$$\sigma\circ\tilde{h}_k=\tilde{h}_{\sigma(k)}\ ,\qquad \sigma\circ\tilde{h}_{ij}=\tilde{h}_{\sigma(i)\sigma(j)}\ .$$

We also use the following notation for shift automorphisms of the rings $\mathrm{U}(\mathbf{h})$ and $\bar{\mathrm{U}}(\mathbf{h})$. For any $\alpha\in\mathbf{h}^*$ and any element $f\in\bar{\mathrm{U}}(\mathbf{h})$ denote by $f[\alpha]$ the image of f under the shift automorphism $\bar{\mathrm{U}}(\mathbf{h})\to\bar{\mathrm{U}}(\mathbf{h})$, defined by the rule $h\to h+(\alpha,h)$. In particular, $h_i[\varepsilon_j]=h_i+\delta_{ij}$.

3. Assume now that the action of $\tilde{\mathsf{S}}_n$ on $\mathrm{U}(\mathbf{g})$ extends to the action by automorphisms of the algebra $\mathcal{A}$. For any $i=1,...,n-1$ define the map $\check{\mathrm{q}}_i:\mathcal{A}\to\tilde{\mathcal{A}}$ by

$$\check{\mathrm{q}}_i(x)=\sum_{k\geq 0}\frac{(-1)^k}{k!}\hat{e}^k_{i,i+1}\left(\sigma_i(x)\right)e^k_{i+1,i}\prod_{j=1}^k(\tilde{h}_{i,i+1}-j)^{-1} \tag{5}$$

Here $\hat{e}_{i,i+1}(x)=[e_{i,i+1},x]$ is the adjoint action of $e_{i,i+1}$ on x. The image of the sum $\mathbf{n}_-\mathcal{A}+\mathcal{A}\mathbf{n}_+$ of the ideals is a subspace of $\mathbf{n}_-\bar{\mathcal{A}}+\bar{\mathcal{A}}\mathbf{n}_+$, see [Zh], so the formula (5) defines the map, denoted by the same symbol $\check{\mathrm{q}}_i:\mathbf{n}_-\mathcal{A}\setminus\mathcal{A}/\mathcal{A}\mathbf{n}_+\to\tilde{\mathcal{A}}$. The map $\check{\mathrm{q}}_i$ satisfies the relations

$$\check{\mathrm{q}}_i(hx)=(\sigma_i\circ h)\check{\mathrm{q}}_i(x),\qquad \check{\mathrm{q}}_i(xh)=\check{\mathrm{q}}_i(x)(\sigma_i\circ h) \tag{6}$$

for any $h \in \mathrm{U}(\mathbf{h})$ and $x \in \mathcal{A}$. The use of (6) extends the map $\check{\mathrm{q}}_i : \mathbf{n}_-\mathcal{A} \setminus \mathcal{A}/\mathcal{A}\mathbf{n}_+ \to \tilde{\mathcal{A}}$ to the map $\check{\mathrm{q}}_i : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$. The maps $\check{\mathrm{q}}_i$ satisfy the braid group relations [Zh] and are automorphisms of the reduction algebra $\tilde{\mathcal{A}}$, see [KO1].

3 Algebra of h-deformed differential operators

3.1 Reduction of the algebras of differential operators

There is a homomorphism $\psi : \mathrm{U}(\mathbf{g}) \to \mathrm{Diff}(n)$ of the algebra $\mathrm{U}(\mathbf{g})$ to the algebra of polynomial differential operators in n variables $x_1, \dots, x_n$. The image of $e_{ij} \in \mathrm{U}(\mathbf{g})$ is

$$\psi(e_{ij}) = x^i \partial_j \ . \tag{7}$$

Denote by $\mathrm{Diff}_{\mathbf{h}}(n)$ the reduction algebra of $\mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{g})$ with respect to the diagonal embedding of $\mathrm{U}(\mathbf{g})$. We call $\mathrm{Diff}_{\mathbf{h}}(n)$ *algebra of* $\mathbf{h}$*-deformed differential operators* (the explicit definition, given below, makes it clear that this algebra admits a well-defined limit $\tilde{h}_{ij} \to \infty$ for $i < j$ and $\tilde{h}_1 > \tilde{h}_2 > \dots > \tilde{h}_n$ in which it becomes the usual algebra of differential operators with polynomial coefficients).

The algebra $\mathrm{Diff}_{\mathbf{h}}(n)$ is generated over $\bar{\mathrm{U}}(\mathbf{h})$ by the classes of x^i and ∂_j, which we denote by the same symbols. These elements are subject in $\mathrm{Diff}_{\mathbf{h}}(n)$ to quadratic-linear relations over $\bar{\mathrm{U}}(\mathbf{h})$, which we now describe.

These relations can be computed directly. However, almost all the required information can be found in the description of the reduction algebra of $\mathrm{U}(\mathbf{gl}_{n+1})$ with respect to $\mathrm{U}(\mathbf{gl}_n)$ (this description is a basic step in the derivation of the Gelfand–Tsetlin basis in [Zh]). Indeed, the generators $e_{i,n+1}$ and $e_{n+1,i}$, $i = 1, \dots, n$, of the corresponding double coset algebra form the bases of the fundamental representation ω and its dual ω^* with respect to the adjoint representation of $\mathbf{g} = \mathbf{gl}_n$. The elements x^i and ∂_i form the same bases of ω and ω^*, the only difference is that the commutators $[e_{i,n+1}, e_{n+1,j}]$ belong to the Cartan subalgebra of $\mathbf{gl}_{n+1}$ while $[x^i, \partial_j] = \delta_{ij}$. Thus, by [Zh, 4.5.3], we have

$$\begin{aligned} &x^i \diamond x^j = \alpha_{ij} x^j \diamond x^i, \qquad \partial_j \diamond \partial_i = \alpha_{ij} \partial_i \diamond \partial_j, \qquad i < j, \\ &x^i \diamond \partial_j = \partial_j \diamond x^i \qquad i \neq j, \\ &x^i \diamond \partial_i = \sum_j \beta_{ij} \partial_j \diamond x^j + \mu_i. \end{aligned} \tag{8}$$

Here

$$\alpha_{ij} = \frac{\tilde{h}_{ij} + 1}{\tilde{h}_{ij}}, \qquad \beta_{ij} = \frac{1}{1 - \tilde{h}_{ij}} \frac{\varphi_j[\varepsilon_j]}{\varphi_i} \quad \text{with} \quad \varphi_j = \prod_{k:k>j} \frac{\tilde{h}_{jk}}{\tilde{h}_{jk} - 1}, \tag{9}$$

and μ_i are the elements of $\bar{\mathrm{U}}(\mathbf{h})$ which are to be determined by another argument.

Lemma 3.1 *We have*

$$\mu_i = -\varphi_i^{-1} \ .$$

Proof. Note first that $\mu_n = -1$. Indeed, since ∂_n is a highest weight vector with respect to the adjoint action of $\mathrm{U}(\mathbf{g})$ on $\mathrm{Diff}(n)$,

$$x^n \diamond \partial_n =: x^n \partial_n :=: \partial_n x^n : -1.$$

On the other hand, the products $\partial_j \diamond x^j$ are equal to sums $\sum_m a_m : \partial_m x^m :$ with some $a_m \in \bar{\mathrm{U}}(\mathbf{h})$ and do not contain a constant term. Thus $\mu_n = -1 = -\varphi_n^{-1}$. For the derivation of other μ_j we use the Zhelobenko automorphisms $\check{\mathrm{q}}_i$.

It is not difficult to check that

$$\begin{aligned} &\check{\mathrm{q}}_i(x^i) = -x^{i+1}\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}, \qquad \check{\mathrm{q}}_i(x^{i+1}) = x^i, \qquad \check{\mathrm{q}}_i(x^j) = x^j, \quad j \neq i, i+1, \\ &\check{\mathrm{q}}_i(\partial_{i+1}) = \partial_i\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}, \qquad \check{\mathrm{q}}_i(\partial_i) = -\partial_{i+1}, \quad \check{\mathrm{q}}_i(\partial_j) = \partial_j, \quad j \neq i, i+1. \end{aligned} \tag{10}$$

Now we apply the automorphism $\check{\mathrm{q}}_{n-1}$ to the already known identity

$$x^n \diamond \partial_n = \sum_j \beta_{n,j}\partial_j \diamond x^j - 1.$$

We get the relation

$$\frac{\tilde{h}_{n-1,n}}{\tilde{h}_{n-1,n}-1}x^{n-1} \diamond \partial_{n-1} = \sum_j \check{\mathrm{q}}_{n-1}\left(\beta_{n,j}\partial_j \diamond x^j\right) - 1.$$

Taking into account the last line in (8), we rewrite this relation in the form

$$x^{n-1} \diamond \partial_{n-1} = \sum_j \beta_{n-1,j}\partial_j \diamond x^j - \frac{\tilde{h}_{n-1,n}-1}{\tilde{h}_{n-1,n}},$$

which implies that

$$\mu_{n-1} = -\frac{\tilde{h}_{n-1,n}-1}{\tilde{h}_{n-1,n}} = -\varphi_{n-1}^{-1}.$$

Next we apply $\check{\mathrm{q}}_{n-2}$ to the identity $x^{n-1} \diamond \partial_{n-1} = \sum_j \beta_{n-1,j}\partial_j \diamond x^j - \varphi_{n-1}^{-1}$ and find that $\mu_{n-2} = -\varphi_{n-1}^{-1}$ and then further $\mu_i = -\varphi_i^{-1}$ for all i. □

The last line in (8) can be now rewritten as

$$x^i \diamond \partial_i = \sum_j \frac{1}{1-\tilde{h}_{ij}}\,\partial_j\varphi_j \diamond x^j\varphi_i^{-1} - \varphi_i^{-1},$$

which suggests the change of variables

$$\bar{\partial}_j := \partial_j\varphi_j. \tag{11}$$

In the new variables the relations (8) and (10) look as follows:

$$\begin{aligned} &x^i \diamond x^j = \frac{\tilde{h}_{ij}+1}{\tilde{h}_{ij}}x^j \diamond x^i,\ i<j, \quad \bar{\partial}_i \diamond \bar{\partial}_j = \frac{\tilde{h}_{ij}-1}{\tilde{h}_{ij}}\,\bar{\partial}_j \diamond \bar{\partial}_i, \qquad i<j, \\ &x^i \diamond \bar{\partial}_j = \bar{\partial}_j \diamond x^i,\ i<j; \qquad\qquad x^i \diamond \bar{\partial}_j = \frac{\tilde{h}_{ij}(\tilde{h}_{ij}-2)}{(\tilde{h}_{ij}-1)^2}\,\bar{\partial}_j \diamond x^i,\ i>j, \\ &x^i \diamond \bar{\partial}_i = \sum_j \frac{1}{1-\tilde{h}_{ij}}\,\bar{\partial}_j \diamond x^j - 1, \end{aligned} \tag{12}$$

$$\check{\mathrm{q}}_i(x^i) = -x^{i+1}\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}, \qquad \check{\mathrm{q}}_i(x^{i+1}) = x^i, \quad \check{\mathrm{q}}_i(x^j) = x^j, \quad j \neq i, i+1,$$
$$\check{\mathrm{q}}_i(\bar{\partial}_i) = -\frac{\tilde{h}_{i,i+1}-1}{\tilde{h}_{i,i+1}}\bar{\partial}_{i+1}, \qquad \check{\mathrm{q}}_i(\bar{\partial}_{i+1}) = \bar{\partial}_i, \quad \check{\mathrm{q}}_i(\bar{\partial}_j) = \bar{\partial}_j, \quad j \neq i, i+1. \tag{13}$$

We have other sets of Heisenberg type generators in the algebra $\mathrm{Diff}_{\mathbf{h}}(n)$. Set

$$\varphi'_j = \prod_{k:k<j} \frac{\tilde{h}_{jk}}{\tilde{h}_{jk}-1}, \tag{14}$$

and put

$$\bar{\bar{\partial}}_j := \partial_j {\varphi'_j}^{-1}. \tag{15}$$

Arguments parallel to that of Lemma 3.1 show that

$$x^i \diamond x^j = \frac{\tilde{h}_{ij}+1}{\tilde{h}_{ij}} x^j \diamond x^i, i<j, \qquad \bar{\bar{\partial}}_i \diamond \bar{\bar{\partial}}_j = \frac{\tilde{h}_{ij}-1}{\tilde{h}_{ij}}\, \bar{\bar{\partial}}_j \diamond \bar{\bar{\partial}}_i, \qquad i<j,$$
$$\bar{\bar{\partial}}_j \diamond x^i = x^i \diamond \bar{\bar{\partial}}_j,\ i>j, \qquad \bar{\bar{\partial}}_j \diamond x^i = \frac{\tilde{h}_{ij}(\tilde{h}_{ij}-2)}{(\tilde{h}_{ij}-1)^2}\, x^i \diamond \bar{\bar{\partial}}_j,\ i<j, \tag{16}$$
$$\bar{\bar{\partial}}_i \diamond x^i = \sum_j \frac{1}{1+\tilde{h}_{ij}}\, x^j \diamond \bar{\bar{\partial}}_j + 1.$$

and the same as in (13) coefficients in the action of Zhelobenko automorphisms. Alternatively, we can leave the variables ∂_j unchanged and rescale the variables x^j. For the variables $\bar{x}^j := \varphi_j x^j$ and ∂_j we get the relations of the form (12) with the reversed inequalities between i and j in first two lines of the relations (12); for the variables $\bar{\bar{x}}^j := {\varphi'_j}^{-1} x^j$ and ∂_j we get the relations of the form (16) with the reversed inequalities between i and j in first two lines of relations (16).

3.2 Polarized form of relations

1. The first line of (12) can be written in the following polarized form:

$$x^i \diamond x^j = \frac{1}{\tilde{h}_{ij}} x^i \diamond x^j + \frac{\tilde{h}_{ij}^2-1}{\tilde{h}_{ij}^2} x^j \diamond x^i, \qquad i<j,$$
$$x^i \diamond x^j = \frac{1}{\tilde{h}_{ij}} x^i \diamond x^j + x^j \diamond x^i, \qquad i>j\,, \tag{17}$$

$$\bar{\partial}_i \diamond \bar{\partial}_j = -\frac{1}{\tilde{h}_{ij}} \bar{\partial}_i \diamond \bar{\partial}_j + \frac{\tilde{h}_{ij}^2-1}{\tilde{h}_{ij}^2} \bar{\partial}_j \diamond \bar{\partial}_i, \qquad i<j,$$
$$\bar{\partial}_i \diamond \bar{\partial}_j = -\frac{1}{\tilde{h}_{ij}} \bar{\partial}_i \diamond \bar{\partial}_j + \bar{\partial}_j \diamond \bar{\partial}_i, \qquad i>j\,. \tag{18}$$

Rewrite (17), (18) and the last two lines in (12) in an operator form[2]

$$x^i \diamond x^j = \hat{\mathrm{R}}^{ij}_{kl} x^k \diamond x^l, \qquad \bar{\partial}_i \diamond \bar{\partial}_j = \hat{\mathrm{R}}^{lk}_{ji} \bar{\partial}_k \diamond \bar{\partial}_l, \qquad x^i \diamond \bar{\partial}_j = \hat{\mathrm{T}}^{ik}_{jl} \bar{\partial}_k \diamond x^l - \delta^i_j, \tag{19}$$

[2]Unless the opposite is stated, we adopt the Einstein convention: if a tensor index appears in an expression twice, once as an upper index and once as a lower index, the summation over this index is assumed.

where $\hat{\mathrm{R}}^{ij}_{kl}$ and $\hat{\mathrm{T}}^{ik}_{jl}$ are matrix coefficients of operators $\hat{\mathrm{R}}, \hat{\mathrm{T}} : \mathbb{C}^n \otimes \mathbb{C}^n \to \bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. Their nonzero values are

$$\hat{\mathrm{R}}^{ij}_{ij} = \frac{1}{\tilde{h}_{ij}},\ i \neq j, \qquad \hat{\mathrm{R}}^{ij}_{ji} = \begin{cases} \dfrac{\tilde{h}^2_{ij} - 1}{\tilde{h}^2_{ij}}, & i < j, \\ 1, & i \geq j \end{cases} \tag{20}$$

$$\hat{\mathrm{T}}^{ij}_{ij} = -\frac{1}{\tilde{h}_{ij} - 1},\ i \neq j, \qquad \hat{\mathrm{T}}^{ji}_{ij} = \begin{cases} \dfrac{\tilde{h}_{ij}(\tilde{h}_{ij} + 2)}{(\tilde{h}_{ij} + 1)^2}, & i < j, \\ 1, & i \geq j \end{cases} \tag{21}$$

Note the following identity:

$$\hat{\mathrm{T}}^{kl}_{ij}[-\varepsilon_l] = \hat{\mathrm{R}}^{lk}_{ji}. \tag{22}$$

Similarly, the relations (16) can be presented in an operator form as

$$x^i \diamond x^j = \hat{\mathrm{R}}^{ij}_{kl} x^k \diamond x^l, \qquad \bar{\bar{\partial}}_i \diamond \bar{\bar{\partial}}_j = \hat{\mathrm{R}}^{lk}_{ji} \bar{\bar{\partial}}_k \diamond \bar{\bar{\partial}}_l, \qquad \bar{\bar{\partial}}_j \diamond x^i = \hat{\mathrm{S}}^{ik}_{jl} x^l \diamond \bar{\bar{\partial}}_k + \delta^i_j, \tag{23}$$

where the non-zero elements of $\hat{\mathrm{S}}$ are

$$\hat{\mathrm{S}}^{ij}_{ij} = \frac{1}{\tilde{h}_{ij} + 1}\ , \qquad \hat{\mathrm{S}}^{ij}_{ji} = \begin{cases} 1 & ,\ i > j \\ \dfrac{\tilde{h}_{ij}(\tilde{h}_{ij} - 2)}{(\tilde{h}_{ij} - 1)^2} & ,\ i < j \end{cases} \tag{24}$$

2. The relations in the last two lines of (16) can be obtained in another way, by inverting the last relation in (19). Let $\hat{\Psi}$ be the skew inverse to T (see e.g. [O], section 4.1.2 for details of the R-matrix technique needed here), that is,

$$\hat{\Psi}^{ik}_{jl} \hat{\mathrm{T}}^{lm}_{kn} = \delta^i_n \delta^m_j. \tag{25}$$

Multiplying the relation

$$x^l \diamond \bar{\partial}_k = \hat{\mathrm{T}}^{lm}_{kn} \bar{\partial}_m \diamond x^n - \delta^l_k$$

by $\hat{\Psi}^{ik}_{jl}$ from the left and contracting repeated indices, we get the relation

$$\bar{\partial}_j \diamond x^i = \hat{\Psi}^{ik}_{jl} x^l \diamond \bar{\partial}_k + \hat{\Psi}^{ik}_{jk}. \tag{26}$$

The relations (11), (15), and the last two lines of (16) imply that

$$\bar{\partial}_i = \bar{\bar{\partial}}_i (\mathrm{Q}^-_i)^{-1} = \mathrm{Q}^+_i\, \bar{\bar{\partial}}_i, \tag{27}$$

where

$$\mathrm{Q}^\pm_i = \prod_{k:k\neq i} \frac{\tilde{h}_{ik} \pm 1}{\tilde{h}_{ik}}, \qquad \mathrm{Q}^-_j[\varepsilon_j]\, \mathrm{Q}^+_j = 1\,, \tag{28}$$

and

$$\hat{\Psi}^{ij}_{ij} = \mathrm{Q}^+_i\, \mathrm{Q}^-_j\, \frac{1}{\tilde{h}_{ij} + 1}\ , \qquad \hat{\Psi}^{ij}_{ji} = \begin{cases} 1 & ,\ i < j \\ \dfrac{(\tilde{h}_{ij} - 1)^2}{\tilde{h}_{ij}(\tilde{h}_{ij} - 2)} & ,\ i > j \end{cases} \tag{29}$$

Comparing (26) and the last line in (16) we conclude that

$$\hat{\Psi}^{ik}_{jk} = \mathrm{Q}^{+}_{i}\,\delta^{i}_{j}. \tag{30}$$

Define operators $\mathrm{Q}^{\pm} : \mathbb{C}^n \to \mathbb{C}^n$ by

$$(\mathrm{Q}^{\pm})^{i}_{j} = \mathrm{Q}^{\pm}_{j}\,\delta^{i}_{j}.$$

Then the relation (30) can be rewritten as

$$\mathrm{Tr}_2\hat{\Psi}_{12} = \mathrm{Q}^{+}_{1}\,.$$

Here the lower index specifies the number of the copy of the space $\mathbb{C}^n$ in which the corresponding operator acts nontrivially. For example, Q^{+}_{1} stands for the operator Q^{+} acting in the first copy and Tr_2 means the contraction of indices in the second copy.

3.3 Reflection equation and copies

To lighten the notation, in the formulation of statements, the matrix multiplication of matrices with entries in a reduction algebra is written without the symbol $\diamond$ (which is always assumed).

Set $\tilde{\mathrm{L}}^{i}_{j} := x^i \diamond \bar{\partial}_j \in \mathrm{Diff}_{\mathbf{h}}(n)$.

Proposition 3.2 *The matrix* $\tilde{\mathrm{L}}$ *satisfies the reflection equation*

$$\hat{\mathrm{R}}_{12}\tilde{\mathrm{L}}_1\hat{\mathrm{R}}_{12}\tilde{\mathrm{L}}_1 - \tilde{\mathrm{L}}_1\hat{\mathrm{R}}_{12}\tilde{\mathrm{L}}_1\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\tilde{\mathrm{L}}_1 - \tilde{\mathrm{L}}_1\hat{\mathrm{R}}_{12}. \tag{31}$$

Proof. Consider the monomial $x^{i_1}\diamond x^{i_2}\diamond\bar{\partial}_{j_1}\diamond\bar{\partial}_{j_2}$. Reorder it in two ways. The first way:

$$x^{i_1}\diamond x^{i_2}\diamond\bar{\partial}_{j_1}\diamond\bar{\partial}_{j_2} = \hat{\mathrm{R}}^{i_1i_2}_{kl}x^k\diamond x^l\diamond\bar{\partial}_{j_1}\diamond\bar{\partial}_{j_2} = \hat{\mathrm{R}}^{i_1i_2}_{kl}x^k\diamond\hat{\mathrm{T}}^{ln}_{j_1m}\bar{\partial}_n\diamond x^m\diamond\bar{\partial}_{j_2} - \hat{\mathrm{R}}^{i_1i_2}_{kl}\delta^{l}_{j_1}x^k\diamond\bar{\partial}_{j_2}$$
$$= \sum_{k,l,m,n}\hat{\mathrm{R}}^{i_1i_2}_{kl}x^k\circ\bar{\partial}_n\diamond\hat{\mathrm{T}}^{ln}_{j_1m}[-\varepsilon_n]x^m\diamond\bar{\partial}_{j_2} - \hat{\mathrm{R}}^{i_1i_2}_{kj_1}x^k\diamond\bar{\partial}_{j_2} = \hat{\mathrm{R}}^{i_1i_2}_{kl}\tilde{\mathrm{L}}^{k}_{n}\diamond\hat{\mathrm{R}}^{nl}_{j_1m}\tilde{\mathrm{L}}^{m}_{j_2} - \hat{\mathrm{R}}^{i_1i_2}_{kj_1}\tilde{\mathrm{L}}^{k}_{j_2}\,.$$

The second way of reordering:

$$x^{i_1}\diamond x^{i_2}\diamond\bar{\partial}_{j_1}\diamond\bar{\partial}_{j_2} = x^{i_1}\diamond x^{i_2}\diamond\hat{\mathrm{R}}^{mn}_{j_2j_1}\bar{\partial}_n\diamond\bar{\partial}_m$$
$$=\hat{\mathrm{R}}^{mn}_{j_2j_1}[-\varepsilon_{i_1}-\varepsilon_{i_2}]x^{i_1}\diamond\hat{\mathrm{T}}^{i_2t}_{nk}\bar{\partial}_t\diamond x^k\diamond\bar{\partial}_m - \hat{\mathrm{R}}^{mn}_{j_2j_1}[-\varepsilon_{i_1}-\varepsilon_{i_2}]\delta^{i_2}_{n}x^{i_1}\diamond\bar{\partial}_m$$
$$=\hat{\mathrm{R}}^{mn}_{j_2j_1}[-\varepsilon_{i_1}-\varepsilon_{i_2}]\hat{\mathrm{T}}^{i_2t}_{nk}[-\varepsilon_{i_1}]x^{i_1}\diamond\bar{\partial}_t\diamond x^k\diamond\bar{\partial}_m - \hat{\mathrm{R}}^{mi_2}_{j_2j_1}[-\varepsilon_{i_1}-\varepsilon_{i_2}]x^{i_1}\diamond\bar{\partial}_m$$
$$= \sum_{t,k,m,n}x^{i_1}\diamond\bar{\partial}_t\diamond\hat{\mathrm{T}}^{i_2t}_{nk}[-\varepsilon_t]x^k\diamond\bar{\partial}_m\hat{\mathrm{R}}^{mn}_{j_2j_1}[-\varepsilon_t-\varepsilon_{i_2}+\varepsilon_k-\varepsilon_m] - \sum_{m}x^{i_1}\diamond\bar{\partial}_m\hat{\mathrm{R}}^{mi_2}_{j_2j_1}[-\varepsilon_m-\varepsilon_{i_2}]$$
$$= \sum_{t,k,m,n}\tilde{\mathrm{L}}^{i}_{t}\hat{\mathrm{T}}^{i_2t}_{nk}[-\varepsilon_t]\diamond\tilde{\mathrm{L}}^{k}_{m}\hat{\mathrm{R}}^{mn}_{j_2j_1}[-\varepsilon_t-\varepsilon_{i_2}+\varepsilon_k-\varepsilon_m] - \sum_{m}\tilde{\mathrm{L}}^{i_1}_{m}\hat{\mathrm{R}}^{mi_2}_{j_2j_1}[-\varepsilon_m-\varepsilon_{i_2}]\,.$$

Matrix elements $\hat{\mathrm{R}}^{ij}_{kl}$ and $\hat{\mathrm{T}}^{ij}_{kl}$ are nonzero only if $i=k$ and $j=l$, or $i=l$ and $j=k$. We can thus replace the shift by $\varepsilon_t+\varepsilon_{i_2}$ in the last displayed line with the shift by $\varepsilon_n+\varepsilon_k$.

Besides, the matrix coefficient $\hat{\mathrm{R}}^{mn}_{j_2j_1}$ depends only on the difference $h_n - h_m$ and is invariant with respect to the shift by $\varepsilon_m + \varepsilon_n$. Using the relation (22) we rewrite the result as

$$x^{i_1} \diamond x^{i_2} \diamond \bar{\partial}_{j_1} \diamond \bar{\partial}_{j_2} = \tilde{\mathrm{L}}^{i_1}_t \hat{\mathrm{R}}^{i_2t}_{kn} \diamond \tilde{\mathrm{L}}^k_m \hat{\mathrm{R}}^{mn}_{j_2j_1} - \tilde{\mathrm{L}}^{i_1}_m \hat{\mathrm{R}}^{mi_2}_{j_2j_1} \tag{32}$$

Comparing these two ways, we obtain the equality

$$\hat{\mathrm{R}}^{ij}_{kl} \tilde{\mathrm{L}}^k_n \diamond \hat{\mathrm{R}}^{nl}_{ms} \tilde{\mathrm{L}}^m_t - \hat{\mathrm{R}}^{ij}_{ks} \tilde{\mathrm{L}}^k_t = \tilde{\mathrm{L}}^i_t \hat{\mathrm{R}}^{tj}_{kn} \diamond \tilde{\mathrm{L}}^k_u \hat{\mathrm{R}}^{un}_{ts} - \tilde{\mathrm{L}}^i_u \hat{\mathrm{R}}^{un}_{ts},$$

that is the relation (31). □

Let now $\mathrm{Diff}(n, N)$ be the algebra of differential operators in nN variables $x^{i\alpha}$, $i = 1, \ldots, n$, $\alpha = 1, \ldots, N$. There is a homomorphism $\psi_N : \mathrm{U}(\mathbf{g}) \to \mathrm{Diff}(n, N)$. The image of $e_{ij} \in \mathrm{U}(\mathbf{g})$ is

$$\psi(e_{ij}) = \sum_\alpha x^{i\alpha} \partial_{j\alpha}. \tag{33}$$

Denote by $\mathrm{Diff}_{\mathbf{h}}(n, N)$ the reduction algebra of $\mathrm{Diff}(n, N) \otimes \mathrm{U}(\mathbf{g})$ with respect to the diagonal embedding of $\mathrm{U}(\mathbf{g})$. The algebra $\mathrm{Diff}_{\mathbf{h}}(n, N)$ is generated over $\bar{\mathrm{U}}(\mathbf{h})$ by the classes of all $x^{i\alpha}$ and $\partial_{j\beta}$, which we denote by the same symbols. Denote

$$\bar{\partial}_{j\beta} = \partial_{j\beta} \varphi_j \,. \tag{34}$$

The calculations from Sections 3.1 and 3.2 can be repeated for any N. The result is

Proposition 3.3 *The elements $x^{i\alpha}$ and $\bar{\partial}_{j\beta}$ satisfy the following relations*

$$\begin{gathered} x^{i\alpha} \diamond x^{j\beta} = \hat{\mathrm{R}}^{ij}_{kl} x^{k\beta} \diamond x^{l\alpha}, \qquad \bar{\partial}_{i\alpha} \diamond \bar{\partial}_{j\beta} = \hat{\mathrm{R}}^{lk}_{ji} \bar{\partial}_{k\beta} \diamond \bar{\partial}_{l\alpha}, \\ x^{i\alpha} \diamond \bar{\partial}_{j\beta} = \hat{\mathrm{R}}^{ki}_{lj}[\varepsilon_l] \bar{\partial}_{k\beta} \diamond x^{l\alpha} - \delta^\alpha_\beta \delta^i_j \,. \end{gathered} \tag{35}$$

The same proof as that of proposition 3.2 shows that the combinations

$$\tilde{\mathrm{L}}^i_j := \sum_\alpha x^{i\alpha} \diamond \bar{\partial}_{j\alpha} \tag{36}$$

satisfy the same reflection equation. We formulate this assertion in the separate proposition.

Proposition 3.4 *The matrix $\tilde{\mathrm{L}}$, defined in (36) satisfies the reflection equation*

$$\hat{\mathrm{R}}_{12} \tilde{\mathrm{L}}_1 \hat{\mathrm{R}}_{12} \tilde{\mathrm{L}}_1 - \tilde{\mathrm{L}}_1 \hat{\mathrm{R}}_{12} \tilde{\mathrm{L}}_1 \hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12} \tilde{\mathrm{L}}_1 - \tilde{\mathrm{L}}_1 \hat{\mathrm{R}}_{12}. \tag{37}$$

Remark. Equally well one can define the grassmanian version of $\mathbf{h}$-deformed differential operators, the reduction of the algebra $\mathrm{U}(\mathbf{g}) \otimes \mathbb{C}[\xi^{i\alpha}, d_{j\beta}]$ where $\xi^{i\alpha}$ are anti-commuting variables and $d_{j\beta}$ grassmanian derivatives in them. The defining relations differ by the following sign changes:

$$\begin{gathered} \xi^{i\alpha} \diamond \xi^{j\beta} = -\hat{\mathrm{R}}^{ij}_{kl} \xi^{k\beta} \diamond \xi^{l\alpha}, \qquad \bar{d}_{i\alpha} \diamond \bar{d}_{j\beta} = -\hat{\mathrm{R}}^{lk}_{ji} \bar{d}_{k\beta} \diamond \bar{d}_{l\alpha}, \\ \xi^{i\alpha} \diamond \bar{d}_{j\beta} = -\hat{\mathrm{R}}^{ki}_{lj}[\varepsilon_l] \bar{d}_{k\beta} \diamond x i^{l\alpha} - \delta^\alpha_\beta \delta^i_j \,. \end{gathered} \tag{38}$$

Here $\bar{d}_{j\beta} := d_{j\beta} \varphi_j$. Then the operator $\tilde{\mathrm{L}}$ with entries

$$\tilde{\mathrm{L}}^i_j := \sum_\alpha \xi^{i\alpha} \diamond \bar{d}_{j\alpha}$$

satisfies the same reflection equation (37).

3.4 R-matrix and its skew inverse

Using the first relation in (35) we can reorder a monomial $x^{i\alpha}\diamond x^{j\beta}\diamond x^{k\gamma}$, $\alpha\neq\beta\neq\gamma\neq\alpha$, as a combination of monomials of the form $x^{\bullet\gamma}\diamond x^{\bullet\beta}\diamond x^{\bullet\alpha}$ in two ways, as $(x^{i\alpha}\diamond x^{j\beta})\diamond x^{k\gamma}$ or as $x^{i\alpha}\diamond(x^{j\beta}\diamond x^{k\gamma})$. The ordered products form a basis in the reduction algebra (see [KO2], section 2) so the two ways of reordering lead to the same result. This implies certain compatibility conditions for the operator $\hat{\mathrm{R}}$ which are formulated in the following Proposition.

Proposition 3.5 *The operator* $\hat{\mathrm{R}}$ *is a solution of the dynamical Yang–Baxter equation*

$$\sum_{a,b,u}\hat{\mathrm{R}}^{ij}_{ab}\hat{\mathrm{R}}^{bk}_{ur}[-\varepsilon_a]\hat{\mathrm{R}}^{au}_{mn}=\sum_{a,b,u}\hat{\mathrm{R}}^{jk}_{ab}[-\varepsilon_i]\hat{\mathrm{R}}^{ia}_{mu}\hat{\mathrm{R}}^{ub}_{nr}[-\varepsilon_m]\ . \tag{39}$$

This solution has already appeared several times in different contexts (see e.g. [I, ES] and references therein).

In addition, the operator $\hat{\mathrm{R}}$ satisfies the relations

$$\hat{\mathrm{R}}^{2}=\mathrm{Id}_{\mathbb{C}^n\otimes\mathbb{C}^n}\ , \tag{40}$$

$$\hat{\mathrm{R}}_{21}=\hat{\mathrm{R}}^{T}|_{\tilde{h}\mapsto-\tilde{h}}, \tag{41}$$

where $(\hat{\mathrm{R}}_{21})^{ik}_{jl}:=\hat{\mathrm{R}}^{ki}_{lj}$ and $(\hat{\mathrm{R}}^{T})^{ik}_{jl}:=\hat{\mathrm{R}}^{jl}_{ik}$ and

$$\mathrm{Q}^{+}_{i}[-\epsilon_i]\ \mathrm{Q}^{+}_{k}[-\epsilon_i-\epsilon_k]\,\hat{\mathrm{R}}^{ik}_{jl}=\hat{\mathrm{R}}^{ik}_{jl}\ \mathrm{Q}^{+}_{j}[-\epsilon_j]\ \mathrm{Q}^{+}_{l}[-\epsilon_j-\epsilon_l]\ ,$$

which is an immediate consequence of

$$\mathrm{Q}^{-}_{j}\ \mathrm{Q}^{-}_{i}[-\epsilon_j]=\mathrm{Q}^{-}_{i}\ \mathrm{Q}^{-}_{j}[-\epsilon_i]\ .$$

The operators $\hat{\Psi}$, $\hat{\mathrm{S}}$ and $\hat{\mathrm{R}}$ are related by

$$\hat{\Psi}^{ik}_{jl}=\mathrm{Q}^{+}_{i}[\epsilon_k-\epsilon_l]\,\hat{\mathrm{S}}^{ik}_{jl}\,(\mathrm{Q}^{+}_{l})^{-1}[-\epsilon_l]\ , \tag{42}$$

$$\hat{\mathrm{S}}^{ij}_{kl}=\hat{\mathrm{R}}^{ij}_{kl}[\epsilon_k]\ . \tag{43}$$

By (41), $\hat{\Psi}_{21}=\hat{\Psi}^{T}|_{\tilde{h}\mapsto-\tilde{h}}$ so

$$\hat{\Psi}^{am}_{an}=(\hat{\Psi}_{21})^{ma}_{na}=(\hat{\Psi}^{T}|_{\tilde{h}\mapsto-\tilde{h}})^{ma}_{na}=(\hat{\Psi}|_{\tilde{h}\mapsto-\tilde{h}})^{na}_{ma}=\mathrm{Q}^{+}_{n}\,|_{\tilde{h}\mapsto-\tilde{h}}\delta^{n}_{m}=\mathrm{Q}^{-}_{n}\,\delta^{m}_{n}\ , \tag{44}$$

or

$$\mathrm{Tr}_1\hat{\Psi}_{12}=\mathrm{Q}^{-}_{2}\ .$$

It follows from (25) together with (30) and (44) that

$$\sum_{a}\mathrm{Q}^{-}_{a}[-\epsilon_m]\hat{\mathrm{R}}^{ma}_{na}=\delta^{m}_{n}\ , \tag{45}$$

$$\sum_{a}\mathrm{Q}^{+}_{a}[\epsilon_m]\hat{\mathrm{R}}^{am}_{an}=\delta^{m}_{n}\ .$$

Remark. The reordering of the monomial

$$(\bar\partial_{i\alpha}\diamond\bar\partial_{j\beta})\diamond\bar\partial_{k\gamma}=\bar\partial_{i\alpha}\diamond(\bar\partial_{j\beta}\diamond\bar\partial_{k\gamma})$$

leads to a compatibility condition for the operator $\hat{\mathrm{R}}$ which is equivalent to the same dynamical Yang–Baxter equation for $\hat{\mathrm{R}}$.

The reordering of any of the monomials

$$(x^{i\alpha}\diamond x^{j\beta})\diamond\bar\partial_{k\gamma}=x^{i\alpha}\diamond(x^{j\beta}\diamond\bar\partial_{k\gamma}),$$
$$(x^{i\alpha}\diamond\bar\partial_{j\beta})\diamond\bar\partial_{k\gamma}=x^{i\alpha}\diamond(\bar\partial_{j\beta}\diamond\bar\partial_{k\gamma}),$$

$\alpha\neq\beta\neq\gamma\neq\alpha$, similarly leads to a compatibility condition for the operator $\hat{\mathrm{R}}$ which (in each case) now is equivalent to the dynamical Yang–Baxter equation for $\hat{\mathrm{R}}$ together with the equality (40). The verification of this statement uses the fact that the matrix element R^{ij}_{kl} can be nonzero only if $i=k$ and $j=l$, or $i=l$ and $j=k$.

4 Diagonal reduction algebra

4.1 R-matrix presentation

The diagonal reduction algebra $\mathcal{D}(\mathbf{g})$ is by definition the reduction algebra of $\mathcal{A}=\mathrm{U}(\mathbf{g})\otimes\mathrm{U}(\mathbf{g})$ with respect to the diagonal embedding of $\mathrm{U}(\mathbf{g})$. Let $e^{(1)}_{ij}$ and $e^{(2)}_{ij}$ be the standard generators e_{ij} of the Lie algebra $\mathbf{g}$ in the first and the second tensor components. Denote by s^i_j the generators of the diagonal reduction algebra defined as the images in $\mathcal{D}(\mathbf{g})$ of $e^{(1)}_{ij}$. In other words,

$$s^i_j=\mathrm{P}\ e^{(1)}_{ij}\,\mathrm{P}\,.$$

We will also need another set of generators

$$s'^i_{\ j}=\mathrm{P}\ e^{(2)}_{ij}\,\mathrm{P}\,.$$

The elements s^i_j and $s'^i_{\ j}$ are related by

$$s^i_j+s'^i_{\ j}=h_i\delta^i_j\,. \tag{46}$$

In addition to the elements φ_j, defined in (9), we need, for any $j=1,\dots,n$, and $m>j$, the following elements of $\bar{\mathrm{U}}(\mathbf{h})$:

$$\varphi_{jm}=\prod_{k:j<k<m}\frac{\tilde h_{jk}}{\tilde h_{jk}-1}.$$

The description of the algebra $\mathcal{D}(\mathbf{g})$ in terms of generators $s^i_j-s'^i_{\ j}$ was given in [KO2, KO3]. Here we suggest another presentation. Let L^i_j, $i,j=1,\dots,n$, be the following elements of $\mathcal{D}(\mathbf{g})$:

$$\mathrm{L}^i_j:=\begin{cases} s^i_j\varphi_j, & i\neq j\\ \left(s^i_j-\sum\limits_{m:m>i}s^m_m\dfrac{1}{\tilde h_{im}\varphi_{im}}\right)\varphi_j, & i=j\end{cases} \tag{47}$$

The elements L^i_j are linear combinations of s^k_l with the triangular transition matrix. Thus L^i_j generate $\mathcal{D}(\mathbf{g})$ as the algebra over $\bar{\mathrm{U}}(\mathbf{h})$.

Proposition 4.1 *Elements* $\mathrm{L}^i_j \in \mathcal{D}(\mathbf{g})$ *satisfy quadratic-linear relations collected in reflection equation*

$$\hat{\mathrm{R}}_{12}\mathrm{L}_1\hat{\mathrm{R}}_{12}\mathrm{L}_1 - \mathrm{L}_1\hat{\mathrm{R}}_{12}\mathrm{L}_1\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\mathrm{L}_1 - \mathrm{L}_1\hat{\mathrm{R}}_{12}. \tag{48}$$

The relations (48) form a complete list of defining relations over the field of fractions of $\bar{\mathrm{U}}(\mathbf{h})$.

It is plausible that the relations (48) form a complete list of defining relations over the ring $\bar{\mathrm{U}}(\mathbf{h})$ itself.

Proof of proposition 4.1 is based on the properties of homomorphisms of the diagonal reduction algebra $\mathcal{D}(\mathbf{g})$ to $\mathrm{Diff}_{\mathbf{h}}(n, N)$ and the proposition 3.4.

Using the map ψ, see (33), we define a homomorphism

$$\psi_1 := \psi \otimes 1 : \ \mathrm{U}(\mathbf{g}) \otimes \mathrm{U}(\mathbf{g}) \to \mathrm{Diff}(n, N) \otimes \mathrm{U}(\mathbf{g}). \tag{49}$$

The map ψ_1 sends the diagonal $\mathbf{g}$ to the diagonal $\mathbf{g}$ and thus defines the homomorphism

$$\tilde{\psi}_1 : \ \mathcal{D}(\mathbf{g}) \to \mathrm{Diff}_{\mathbf{h}}(n, N).$$

Lemma 4.2 *The map* $\tilde{\psi}_1$ *sends the generator* $\mathrm{L}^i_j \in \mathcal{D}(\mathbf{g})$ *to the element* $\tilde{\mathrm{L}}^i_j \in \mathit{Diff}_{\mathbf{h}}(n, N)$, *(see(36))*

$$\tilde{\psi}_1(\mathrm{L}^i_j) = \tilde{\mathrm{L}}^i_j.$$

Proof of proposition 4.1. The statement of proposition 4.1 follows from proposition 3.4, lemma 4.2 and the injectivity of the map ψ for $N \geq n$ (consider the map ψ as the tangent map for the group action $\mathrm{GL}_n \times \mathrm{Mat}_{n\times N} \to \mathrm{Mat}_{n\times N}$; for $N = n$ we obtain the classical isomorphism of $\mathrm{U}(\mathbf{g})$ with the ring of right invariant differential operators on $\mathrm{GL}_n \subset \mathrm{Mat}_{n\times n}$).

In particular, for $N \geq n$ the map $\tilde{\psi}_1$ is injective on the subspace of $\mathcal{D}(\mathbf{g})$, generated by polynomials in L^i_j of degree not bigger than two. This proves that the relations (48) are satisfied. In [KO2], we presented a complete list of relations for the generators $s^i_j - s'^i_j$ of the algebra $\mathcal{D}(\mathbf{g})$. These are ordering relations of degree at most two in these generators. This means, in particular, that the number of linearly independent over $\bar{\mathrm{U}}(\mathbf{h})$ quadratic monomials in these generators equals the number of ordered quadratic monomials in n^2 generators. Since the transition to the generators L^i_j has a triangular form, the relations are of degree at most two on generators L^i_j and the number of linearly independent quadratic monomials in L^i_j is as before. Therefore it is left to prove the completeness of relations (48) in degree at most two. To show this we write the relations (48) in the form

$$Z^A c^\mu_A(\tilde{h}) = \text{linear in } \mathrm{L}^i_j \text{ terms},$$

where $\{Z^A\}$ is the set of quadratic monomials in L^i_j.

In the asymptotic regime $\tilde{h}_1 > \tilde{h}_2 > \cdots > \tilde{h}_n$ and $\tilde{h}_{ij} \to \infty$ for $i < j$ these relations are well defined and become

$$Z^A c^\mu_A(\infty) = \text{linear in } \mathrm{L}^i_j \text{ terms}.$$

Indeed, in this limit the matrix $\hat{\mathrm{R}}_{12}$ turns into the permutation matrix P_{12} so the relations (48) turn into the standard defining relations for the Lie algebra $\mathbf{g}$,

$$\Lambda_2\Lambda_1 - \Lambda_1\Lambda_2 = \mathrm{P}_{12}\,(\Lambda_1 - \Lambda_2)\,.$$

By general deformation arguments, $\mathrm{rk}\, c_A^\mu(\tilde{h}) \geq \mathrm{rk}\, c_A^\mu(\infty)$. Hence the number of linearly independent quadratic monomials for generic $\tilde{h}$ is not bigger than that in the limit. But in both cases, for generic $\tilde{h}$ and asymptotically, this number equals the number of ordered (for any linear order) quadratic monomials in L^i_j. □

Proof of lemma 4.2. Denote by $: x^{i\alpha}\partial_{j\alpha} :$ the image of the element $x^{i\alpha}\partial_{j\alpha}\otimes 1 \in \mathrm{Diff}(n,N)\otimes \mathrm{U}(\mathbf{g})$ in the reduction algebra $\mathrm{Diff}_{\mathbf{h}}(n,N)$. Due to the definitions of elements L^i_j, $\tilde{\mathrm{L}}^i_j$ and of the map ψ_1 it is sufficient to establish the equalities

$$x^{i\alpha}\diamond\bar{\partial}_{j\beta} = \begin{cases} : x^{i\alpha}\partial_{j\beta} : \varphi_j, & i \neq j \\ \left(: x^{i\alpha}\partial_{j\beta} : - \sum\limits_{m:m>i} : x^{m\alpha}\partial_{m\beta} : \dfrac{1}{\tilde{h}_{im}\varphi_{im}} \right)\varphi_j, & i = j \end{cases} \tag{50}$$

for fixed α and β and then sum them up over $\alpha = \beta = 1,\ldots,N$. First we note that

$$x^{n\alpha}\diamond\partial_{i\beta} =: x^{n\alpha}\partial_{i\beta} : \qquad \text{and} \qquad x^{i\alpha}\diamond\ \partial_{n\beta} =: x^{i\alpha}\partial_{n\beta} : \tag{51}$$

for any $i = 1,\ldots,n$. This is because $x^{n\alpha}$ and $\partial_{n\beta}$ are respectively lowest and highest weight vectors with respect to the adjoint action of the diagonal $\mathrm{U}(\mathbf{g})$:

$$[x^{n\alpha}, e_{ji}] = 0\ ,\ [e_{ij}, \partial_{n\beta}] = 0 \qquad \text{for any} \quad i < j,$$

so that $x^{n\alpha}\,\mathrm{P}\,\partial_{i\beta} =: x^{n\alpha}\partial_{i\beta} :$ and $x^{i\alpha}\,\mathrm{P}\,\partial_{n\beta} =: x^{i\alpha}\partial_{n\beta} :$ in the double coset space $\mathrm{Diff}_{\mathbf{h}}(n,N)$.

Next we apply Zhelobenko operator $\check{\mathrm{q}}_{n-1}$ to both sides of equality

$$x^{n\alpha}\diamond\partial_{j\beta} =: x^{n\alpha}\partial_{j\beta} :, \qquad j \neq n-1, n.$$

We have, using (10) and homomorphism property of Zhelobenko operators,

$$\check{\mathrm{q}}_{n-1}(x^{n\alpha}\diamond\partial_{j\beta}) = \check{\mathrm{q}}_{n-1}(x^{n\alpha)})\diamond\check{\mathrm{q}}_{n-1}(\partial_{j\beta}) = x^{n-1,\alpha}\diamond\partial_{j\beta}, \qquad j \neq n-1, n.$$

On the other hand, we can apply $\check{\mathrm{q}}_{n-2}$ to $: x^{n-1,\alpha}\partial_{n\beta} :$ as to elements of the adjoint representation of $\mathbf{g}$, see [KO2, eq. (4.5)]

$$\check{\mathrm{q}}_{n-1}(: x^{n\alpha}\partial_{j\beta}) =: x^{n-1,\alpha}\partial_{j,\beta} :\ .$$

This implies, due to (51), the equality

$$x^{n-1,\alpha}\diamond\partial_{j\beta} =: x^{n-1,\alpha}\partial_{j\beta} : \qquad \text{for} \quad j \neq n-1.$$

Proceeding further with application of other Zhelobenko automorphisms $\check{\mathrm{q}}_i$ we obtain similarly

$$x^{i\alpha}\diamond\partial_{j\beta} =: x^{i\alpha}\partial_{j\beta} : \qquad \text{for} \quad i \neq j\,. \tag{52}$$

Due to (34) this is equivalent to the first line of (50). For the derivation of the rest of the relations (50) we employ the action of the Zhelobenko operators on the elements

$: x^{i\alpha}\partial_{i\beta} :$ (no sum in i). Note that the sum $x^{i\alpha}\partial_{i\beta}+x^{i+1,\alpha}\partial_{i+1,\beta}$ is invariant with respect to the adjoint action of the $\mathbf{sl_2}$-subalgebra generated by e_{ii+1} and e_{i+1i}, while the difference $x^{i,\alpha}\partial_{i\beta}-x^{i+1,\alpha}\partial_{i+1,\beta}$ spans the zero weight subspace of the three-dimensional representation of this $\mathbf{sl_2}$-subalgebra, with the highest weight vector $x^{i\alpha}\partial_{i+1,\beta}$. This implies the relations

$$\begin{aligned}\check{\mathrm{q}}_i\left(: x^{i\alpha}\partial_{i\beta} :\right) &= - : x^{i\alpha}\partial_{i\beta} : \frac{1}{\tilde{h}_{i,i+1}-1} + : x^{i+1,\alpha}\partial_{i+1,\beta} : \frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1},\\ \check{\mathrm{q}}_i\left(: x^{i+1,\alpha}\partial_{i+1,\beta} :\right) &=: x^{i\alpha}\partial_{i\beta} : \frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1} - : x^{i+1,\alpha}\partial_{i+1,\beta} : \frac{1}{\tilde{h}_{i,i+1}-1}.\end{aligned} \tag{53}$$

Besides,

$$\check{\mathrm{q}}_i\left(: x^{j\alpha}\partial_{j\beta} :\right) =: x^{j\alpha}\partial_{j\beta} : \qquad \text{for} \quad j\neq i, i+1. \tag{54}$$

On the other hand, by (10) we have

$$\begin{aligned}\check{\mathrm{q}}_i\left(x^{i\alpha}\diamond\partial_{i\beta}\right) &= \check{\mathrm{q}}_i\left(x^{i\alpha}\right)\diamond\check{\mathrm{q}}_i\left(\partial_{i\beta}\right) = x^{i+1,\alpha}\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}\diamond\partial_{i+1,\beta} = x^{i+1,\alpha}\diamond\partial_{i+1,\beta}\frac{\tilde{h}_{i,i+1}+1}{\tilde{h}_{i,i+1}},\\ \check{\mathrm{q}}_i\left(x^{i+1,\alpha}\diamond\partial_{i+1,\beta}\right) &= \check{\mathrm{q}}_i\left(x^{i+1,\alpha}\right)\diamond\check{\mathrm{q}}_i\left(\partial_{i+1,\beta}\right) = x^{i}\diamond\partial_{i}\,\frac{\tilde{h}_{i,i+1}}{\tilde{h}_{i,i+1}-1}.\end{aligned} \tag{55}$$

We apply $\check{\mathrm{q}}_{n-1}$ to both sides of the equality $x^n\diamond\partial_n =: x^n\partial_n :$. Using (53) and (55) we get

$$x^{n-1,\alpha}\diamond\partial_{n-1,\beta}\frac{\tilde{h}_{n-1,n}}{\tilde{h}_{n-1,n}-1} =: x^{n-1,\alpha}\partial_{n-1,\beta} : \frac{\tilde{h}_{n-1,n}}{\tilde{h}_{n-1,n}-1} - : x^{n\alpha}\partial_{n\beta} : \frac{1}{\tilde{h}_{n-1,n}-1},$$

which implies the equality

$$x^{n-1,\alpha}\diamond\partial_{n-1,\beta} =: x^{n-1,\alpha}\partial_{n-1,\beta} : - : x^{n\alpha}\partial_{n\beta} : \frac{1}{\tilde{h}_{n-1,n}}. \tag{56}$$

Applying $\check{\mathrm{q}}_{n-2}$ to (56) we get

$$\begin{aligned}x^{n-2,\alpha}\diamond\partial_{n-2,\beta}\frac{\tilde{h}_{n-2,n-1}}{\tilde{h}_{n-2,n-1}-1} &=: x^{n-2,\alpha}\partial_{n-2,\beta} : \frac{\tilde{h}_{n-2,n-1}}{\tilde{h}_{n-2,n-1}-1}\\ &- : x^{n-1,\alpha}\partial_{n-1,\beta} : \frac{1}{\tilde{h}_{n-2,n-1}-1} - : x^{n\alpha}\partial_{n\beta} : \frac{1}{\tilde{h}_{n-2,n}},\end{aligned}$$

which gives

$$x^{n-2,\alpha}\diamond\partial_{n-2,\beta} =: x^{n-2,\alpha}\partial_{n-2,\beta} - x^{n-1,\alpha}\partial_{n-1,\beta}\frac{1}{\tilde{h}_{n-2,n-1}} - x^{n\alpha}\partial_{n\beta}\frac{\tilde{h}_{n-2,n-1}-1}{\tilde{h}_{n-2,n-1}\tilde{h}_{n-2,n}} : .$$

Proceeding further we obtain for any $i\leq n$ the relation

$$x^{i\alpha}\diamond\partial_{i\beta} =: x^{i\alpha}\partial_{i\beta} : - \sum_{m>i}\frac{1}{\tilde{h}_{im}\varphi_{im}} : x^{m\alpha}\partial_{m\beta} : \tag{57}$$

This is precisely the second line of (50). □

Note. Due to the realization $\mathrm{L}^i_j \mapsto x^{i\alpha}\diamond\bar{\partial}_{j\alpha}$, the action of the automorphisms $\check{\mathrm{q}}_i$ on the generators L^i_j can be directly read off the formulas (13). In particular, the action on the diagonal generators is standard, $\check{\mathrm{q}}_i(\mathrm{L}^j_j) = \mathrm{L}^{\sigma_i(j)}_{\sigma_i(j)}$.

4.2 Central elements

1. Let Λ be a $n \times n$ matrix with noncommutative entries belonging to some $\bar{\mathrm{U}}(\mathbf{h})$-bimodule, such that the weight of Λ^i_j equals $\varepsilon_i - \varepsilon_j$, that is, $\tilde{h}_k \Lambda^i_j = \Lambda^i_j(\tilde{h}_k + \delta^i_k - \delta^j_k)$. Assume that Λ verifies the reflection equation

$$\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12}\Lambda_1 - \Lambda_1\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\Lambda_1 - \Lambda_1\hat{\mathrm{R}}_{12} \ .$$

Proposition 4.3 *For any nonnegative integer N the elements $tr(\Lambda^N \mathrm{Q}^-)$ commute with Λ^i_j for all i and j.*

This immediately implies

Corollary 4.4 *For any nonnegative integer N the elements $tr(\mathrm{L}^N \mathrm{Q}^-)$ are central in the algebra $\mathcal{D}(\mathbf{g})$.*

Proof of proposition 4.3. The defining relation

$$(\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12} - \hat{\mathrm{R}}_{12})\Lambda_1 = \Lambda_1(\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12} - \hat{\mathrm{R}}_{12})$$

implies that

$$(\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12} - \hat{\mathrm{R}}_{12})\Lambda_1^N = \Lambda_1^N(\hat{\mathrm{R}}_{12}\Lambda_1\hat{\mathrm{R}}_{12} - \hat{\mathrm{R}}_{12})$$

for any non-negative integer N, or, using (40),

$$\Lambda_1\hat{\mathrm{R}}_{12}\Lambda_1^N\hat{\mathrm{R}}_{12} - \Lambda_1^N\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\Lambda_1^N\hat{\mathrm{R}}_{12}\Lambda_1 - \hat{\mathrm{R}}_{12}\Lambda_1^N \ . \tag{58}$$

Let $\mathrm{Tr}_{(\mathbf{h})2}$ be the linear map from the space of tensors $\Xi^{i_1 i_2}_{j_1 j_2}$ to the space of tensors $\Upsilon^{i_1}_{j_1}$ such that $\big(\mathrm{Tr}_{(\mathbf{h})2}(\Xi)\big)^{i_1}_{j_1} = (\mathrm{Q}^-)^u_v[-\epsilon_{i_1}]\Xi^{i_1 v}_{j_1 u}$. This is the $\mathbf{h}$-analogue of the R-matrix trace in the second space. We calculate $\mathrm{Tr}_{(\mathbf{h})2}$ of each term of (58).

• The image of the expression $\Lambda_1\hat{\mathrm{R}}_{12}\Lambda_1^N\hat{\mathrm{R}}_{12}$ under the map $\mathrm{Tr}_{(\mathbf{h})2}$ is

$$\sum_{u,a,b,c,d} \mathrm{Q}^-_u[-\epsilon_{i_1}]\,\Lambda^{i_1}_c\,(\Lambda^N)^a_b\,\hat{\mathrm{R}}^{cu}_{ad}[\epsilon_a - \epsilon_b]\,\hat{\mathrm{R}}^{bd}_{j_1 u}$$

$$= \sum_{u,a,b,c,d} \Lambda^{i_1}_c\,(\Lambda^N)^a_b\,\hat{\mathrm{R}}^{cu}_{ad}[\epsilon_a - \epsilon_b]\,\hat{\mathrm{R}}^{bd}_{j_1 u}\,\mathrm{Q}^-_u[\epsilon_a - \epsilon_b - \epsilon_c]$$

which we rewrite, using (25) and (42), as

$$= \sum_{a,b,c} \Lambda^{i_1}_c\,(\Lambda^N)^a_b\,\mathrm{Q}^-_a[\epsilon_a - \epsilon_b]\,\delta^b_a\,\delta^c_{j_1} = \Lambda^{i_1}_{j_1}\,\mathrm{Tr}\,(\Lambda^N\,\mathrm{Q}^-) \ .$$

• The image of $\hat{\mathrm{R}}_{12}\Lambda_1^N\hat{\mathrm{R}}_{12}\Lambda_1$ is

$$\sum_{u,a,b,d,f} (\Lambda^N)^a_b\,\mathrm{Q}^-_u[-\epsilon_{i_1} + \epsilon_a - \epsilon_b]\,\hat{\mathrm{R}}^{i_1 u}_{ad}[\epsilon_a - \epsilon_b]\,\hat{\mathrm{R}}^{bd}_{fu}\,\Lambda^f_{j_1} \ ,$$

which, using again (25) and (42), equals

$$= \sum_{a,b,f} (\Lambda^N)^a_b\,Q^-_a\,\delta^b_a\,\delta^{i_1}_f\,\Lambda^f_{j_1} = \mathrm{Tr}\,(\Lambda^N\,\mathrm{Q}^-)\,\Lambda^{i_1}_{j_1} \ .$$

• The image of $\hat{\mathrm{R}}_{12}\Lambda_1^N$ is

$$\sum_{u,a} \mathrm{Q}_u^-[-\epsilon_{i_1}]\hat{\mathrm{R}}_{au}^{i_1 u}(\Lambda^N)_{j_1}^a \ .$$

Thus, by (45), $\mathrm{Tr}_{(\mathbf{h})2}(\hat{\mathrm{R}}_{12}\Lambda_1^N) = \Lambda_1^N$.
• The image of $\Lambda_1^N\hat{\mathrm{R}}_{12}$ is

$$\sum_{u,a} \mathrm{Q}_u^-[-\epsilon_{i_1}]\,(\Lambda^N)_a^{i_1}\,\hat{\mathrm{R}}_{j_1 u}^{au} = (\Lambda^N)_a^{i_1}\ \mathrm{Q}_u^-[-\epsilon_a]\,\hat{\mathrm{R}}_{j_1 u}^{au}$$

and we again obtain Λ_1^N.
Combining these calculations we find that the application of the map $\mathrm{Tr}_{(\mathbf{h})2}$ to the relation (58) gives $\Lambda\mathrm{Tr}\,(\Lambda^N\,\mathrm{Q}^-) = \mathrm{Tr}\,(\Lambda^N\,\mathrm{Q}^-)\Lambda$ as stated. □

Notes. 1. $\mathrm{Tr}\,(\mathrm{L}^N\,\mathrm{Q}^-) = \mathrm{Tr}\,(\mathrm{Q}^-\,\mathrm{L}^N)$ since the diagonal elements of L^N have weight 0.

2. The reflection equation (48) admits shifts $\mathrm{L}_j^i \to \mathrm{L}_j^i + \mathrm{const}\cdot\delta_j^i$.

3. $\mathrm{Tr}\ \mathrm{Q}^+ = \mathrm{Tr}\ \mathrm{Q}^- = n$. Indeed, with the explicit form (29) of the tensor $\hat{\Psi}$, the relation (30) is

$$\sum_j \frac{1}{1+\tilde{h}_{ij}}\,\mathrm{Q}_j^- = 1 \text{ for any } i\ .$$

Write this relation for $\mathbf{gl}_{n+1}$, with indices in the range $\{0,1,\ldots,n\}$, for $i=0$:

$$\prod_{l=1}^n \frac{\tilde{h}_{0l}-1}{\tilde{h}_{0l}} + \sum_{j=1}^n \frac{1}{\tilde{h}_{0l}}\,\mathrm{Q}_j^- = 1$$

(here Q_j^- corresponds to $\mathbf{gl}_n$). Decomposing into a power series in $\tilde{h}_0^{-1}$ and comparing coefficients at $\frac{1}{\tilde{h}_0}$ we find $\mathrm{Tr}\ \mathrm{Q}^- = n$. Since $Q_a^-|_{\tilde{h}\mapsto-\tilde{h}} = Q_a^+$, we have $\mathrm{Tr}\ \mathrm{Q}^+ = n$ as well.

2. The images, in the reduction algebra, of $e_{ij}^{(2)}$ satisfy the same relations as the images of $e_{ij}^{(1)}$. We have therefore another set of generators of $\mathcal{D}(\mathbf{g})$

$$\mathrm{L}'^i_j = \begin{cases} s'^i_j\varphi_j, & i\neq j \\ \left(s'^i_j - \sum\limits_{m:m>i} s'^m_m \dfrac{1}{\tilde{h}_{im}\varphi_{im}}\right)\varphi_j, & i=j \end{cases} \tag{59}$$

which satisfy the same algebra

$$\hat{\mathrm{R}}_{12}\mathrm{L}'_1\hat{\mathrm{R}}_{12}\mathrm{L}'_1 - \mathrm{L}'_1\hat{\mathrm{R}}_{12}\mathrm{L}'_1\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\mathrm{L}'_1 - \mathrm{L}'_1\hat{\mathrm{R}}_{12}\ .$$

Using (46), one can check that the matrices L and L′ are related by

$$\mathrm{L}' + \mathrm{L} = \mathrm{H}\ .$$

Here H is the operator $\mathbb{C}^n \to \bar{\mathrm{U}}(\mathbf{h})\otimes\mathbb{C}^n$ with matrix coefficients

$$\mathrm{H}_j^i := (\tilde{h}_j + n)\delta_j^i\ .$$

By proposition 4.3, for any nonnegative integer N the elements $\mathrm{Tr}\,(\mathrm{L}'^N\,\mathrm{Q}^-)$ are central in the algebra $\mathcal{D}(\mathbf{g})$.

Substituting into the reflection equation the expression for L' in terms of L we find

$$\hat{\mathrm{R}}_{12}\,\mathrm{H}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{H}_1 - \mathrm{H}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{H}_1\,\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\,\mathrm{H}_1 - \mathrm{H}_1\,\hat{\mathrm{R}}_{12}\ , \tag{60}$$

$$\hat{\mathrm{R}}_{12}\,\mathrm{L}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{H}_1 + \hat{\mathrm{R}}_{12}\,\mathrm{H}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{L}_1 - \mathrm{L}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{H}_1\,\hat{\mathrm{R}}_{12} - \mathrm{H}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{L}_1\,\hat{\mathrm{R}}_{12} = 2\hat{\mathrm{R}}_{12}\,\mathrm{L}_1 - 2\,\mathrm{L}_1\,\hat{\mathrm{R}}_{12}\ .$$

Note that the latter equality (one can check it directly) holds for any matrix L whose matrix element L^i_j has the weight $\varepsilon_i - \varepsilon_j$.

Presumably the center is generated by the elements $\mathrm{Tr}\,(\mathrm{L}^N\,\mathrm{Q}^-)$ and $\mathrm{Tr}\,(\mathrm{L}'^N\,\mathrm{Q}^-)$.

Notes. 1. The center of the algebra of **h**-deformed differential operators is non-trivial. It is described in [HO].

2. The relation (60) shows that the assignment $\mathrm{L} \to \mathrm{H}$ is a realization of the reflection equation algebra (48).

4.3 Braided bialgebra structure

Consider the $\bar{\mathrm{U}}(\mathbf{h})$-bimodule $\mathcal{D}(\mathbf{g}) \otimes_{\bar{\mathrm{U}}(\mathbf{h})} \mathcal{D}(\mathbf{g})$. The elements M^i_j generate the first copy of $\mathcal{D}(\mathbf{g})$ and $\tilde{\mathrm{M}}^i_j$ generate the second copy. These elements satisfy the relations

$$\begin{aligned} &\hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12}\mathrm{M}_1 - \mathrm{M}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\,\mathrm{M}_1 - \mathrm{M}_1\,\mathrm{R}_{12}, \\ &\hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1 - \tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1 - \tilde{\mathrm{M}}_1\,\mathrm{R}_{12}\,. \end{aligned} \tag{61}$$

We impose the commutation relation

$$\hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1 = \tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12}\ . \tag{62}$$

By virtue of the dynamical Yang–Baxter equation, with this setting, the $\bar{\mathrm{U}}(\mathbf{h})$-bimodule $\mathcal{D}(\mathbf{g}) \otimes_{\bar{\mathrm{U}}(\mathbf{h})} \mathcal{D}(\mathbf{g})$ becomes the associative algebra, which we denote by $\mathcal{D}(\mathbf{g}) \circledcirc \mathcal{D}(\mathbf{g})$.

In other words, $\mathcal{D}(\mathbf{g}) \circledcirc \mathcal{D}(\mathbf{g})$ is the associative algebra over $\bar{\mathrm{U}}(\mathbf{h})$, generated by elements M^i_j and $\tilde{\mathrm{M}}^i_j$ of weight $\varepsilon_i - \varepsilon_j$ subject to the defining relations (61)–(62). It is isomorphic to $\mathcal{D}(\mathbf{g}) \otimes_{\bar{\mathrm{U}}(\mathbf{h})} \mathcal{D}(\mathbf{g})$ as a $\bar{\mathrm{U}}(\mathbf{h})$-bimodule.

Lemma 4.5 *The matrix* $\mathrm{M} + \tilde{\mathrm{M}}$ *satisfies the reflection equation (48).*

Proof. Straightforward. □

Corollary 4.6 *The map*

$$\mathrm{L} \mapsto \mathrm{M} + \tilde{\mathrm{M}} \tag{63}$$

is a homomorphism $\mathcal{D}(\mathbf{g}) \to \mathcal{D}(\mathbf{g}) \circledcirc \mathcal{D}(\mathbf{g})$ *of algebras.*

In a similar fashion we define the product $\circledcirc$ of three and more copies of $\mathcal{D}(\mathbf{g})$. For instance, $\mathcal{D}(\mathbf{g}) \circledcirc \mathcal{D}(\mathbf{g}) \circledcirc \mathcal{D}(\mathbf{g})$ is generated by M^i_j, $\tilde{\mathrm{M}}^i_j$ and $\tilde{\tilde{\mathrm{M}}}^i_j$, with the defining relations (61)–(62) and, in addition,

$$\begin{gathered} \hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1\,\hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1 - \tilde{\tilde{\mathrm{M}}}_1\,\hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1\,\hat{\mathrm{R}}_{12} = \hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1 - \tilde{\tilde{\mathrm{M}}}_1\,\mathrm{R}_{12}, \\ \hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1 = \tilde{\tilde{\mathrm{M}}}_1\,\hat{\mathrm{R}}_{12}\,\mathrm{M}_1\,\hat{\mathrm{R}}_{12}\ \ ,\ \ \hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12}\,\tilde{\tilde{\mathrm{M}}}_1 = \tilde{\tilde{\mathrm{M}}}_1\,\hat{\mathrm{R}}_{12}\tilde{\mathrm{M}}_1\hat{\mathrm{R}}_{12}. \end{gathered}$$

The coproduct (63) is clearly coassociative. We have therefore defined the braided bialgebra structure on $\mathcal{D}(\mathbf{g})$.

The coproduct (63) has a natural interpretation in terms of differential operators. Divide the interval $\{1,2,\dots,N\}$ into two subintervals, $\{1,2,\dots,\nu\}$ and $\{\nu+1,2,\dots,N\}$ for an arbitrary ν, $1\le\nu<N$. Then $M^i_j\mapsto\sum_{\alpha=1}^{\nu}x^{i\alpha}\diamond\bar\partial_{j\alpha}$ and $\tilde M^i_j\mapsto\sum_{\alpha=\nu+1}^{N}x^{i\alpha}\diamond\bar\partial_{j\alpha}$ is a realization of the algebra $\mathcal{D}(\mathbf{g})\circledcirc\mathcal{D}(\mathbf{g})$ while $L^i_j\mapsto\sum_{\alpha=1}^{N}x^{i\alpha}\diamond\bar\partial_{j\alpha}$ is a realization of the algebra $\mathcal{D}(\mathbf{g})$.

Appendix. Basic relations for $n=2$

Denote $\tilde h=\tilde h_{12}=h_1-h_2+1$. The defining relations between different copies of generators in the algebra $\mathrm{Diff}_{\mathbf{h}}(2,2)$, see (35), look as follows:

$$
\begin{aligned}
&x^1\diamond x'^2=\frac{1}{\tilde h}x'^1\diamond x^2+\frac{\tilde h^2-1}{\tilde h^2}x'^2\diamond x^1, && \bar\partial_1\diamond\bar\partial'_2=-\frac{1}{\tilde h}\bar\partial'_1\diamond\bar\partial_2+\frac{\tilde h^2-1}{\tilde h^2}\bar\partial'_2\diamond\bar\partial_1,\\
&x^2\diamond x'^1=x'^1\diamond x^2-\frac{1}{\tilde h}x'^2\diamond x^1, && \bar\partial_2\diamond\bar\partial'_1=\bar\partial'_1\diamond\bar\partial_2+\frac{1}{\tilde h}\bar\partial'_2\diamond\bar\partial_1,\\
&x^i\diamond x'^i=x'^i\diamond x^i, && \bar\partial_i\diamond\bar\partial'_i=\bar\partial'_i\diamond\bar\partial_i,\qquad i=1,2,
\end{aligned}
$$

$$
\begin{aligned}
&x^1\diamond\bar\partial'_2=\bar\partial'_2\diamond x^1, && x^2\diamond\bar\partial'_1=\frac{\tilde h(\tilde h+2)}{(\tilde h+1)^2}\,\bar\partial'_1\diamond x^2,\\
&x^1\diamond\bar\partial'_1=\bar\partial'_1\diamond x^1+\frac{1}{1-\tilde h}\,\bar\partial'_2\diamond x^2-1, && x^2\diamond\bar\partial'_2=\frac{1}{1+\tilde h}\,\bar\partial'_1\diamond x^1+\bar\partial'_2\diamond x^2-1.
\end{aligned}
$$

Here the elements $\{x^1,x^2,\bar\partial_1,\bar\partial_2\}$ belong to the first copy and the elements $\{x'^1,x'^2,\bar\partial'_1,\bar\partial'_2\}$ belong to the second copy.

The ordering form of the relations (48) is

$$
\begin{aligned}
&\mathrm{L}^1_1\,\mathrm{L}^1_2=\frac{\tilde h-3}{\tilde h-2}\,\mathrm{L}^1_2\,\mathrm{L}^1_1+\frac{1}{\tilde h-2}\,\mathrm{L}^1_2\,\mathrm{L}^2_2+\mathrm{L}^1_2\ ,\\
&\mathrm{L}^2_2\,\mathrm{L}^1_2=\frac{\tilde h-3}{(\tilde h-2)(\tilde h+1)}\,\mathrm{L}^1_2\,\mathrm{L}^1_1+\frac{(\tilde h-1)^2}{(\tilde h-2)(\tilde h+1)}\,\mathrm{L}^1_2\,\mathrm{L}^2_2-\frac{\tilde h-1}{\tilde h+1}\,\mathrm{L}^1_2\ ,\\
&\mathrm{L}^1_1\,\mathrm{L}^2_1=\frac{(\tilde h+1)^2}{(\tilde h-1)(\tilde h+2)}\,\mathrm{L}^2_1\,\mathrm{L}^1_1-\frac{\tilde h+3}{(\tilde h-1)(\tilde h+2)}\,\mathrm{L}^2_1\,\mathrm{L}^2_2-\frac{\tilde h+1}{\tilde h-1}\,\mathrm{L}^2_1\ ,\\
&\mathrm{L}^2_2\,\mathrm{L}^2_1=-\frac{1}{\tilde h+2}\,\mathrm{L}^2_1\,\mathrm{L}^1_1+\frac{\tilde h+3}{\tilde h+2}\,\mathrm{L}^2_1\,\mathrm{L}^2_2+\mathrm{L}^2_1\ ,\\
&\mathrm{L}^1_1\,\mathrm{L}^2_2=\mathrm{L}^2_2\,\mathrm{L}^1_1\ ,\\
&\mathrm{L}^1_2\,\mathrm{L}^2_1=\mathrm{L}^2_1\,\mathrm{L}^1_2-\frac{1}{\tilde h}(\mathrm{L}^1_1-\mathrm{L}^2_2)^2+\mathrm{L}^1_1-L^2_2\ .
\end{aligned}
$$

The central elements of Corollary 4.4 have the form

$$
\frac{\tilde h-1}{\tilde h}(\mathrm{L}^N)^1_1+\frac{\tilde h+1}{\tilde h}(\mathrm{L}^N)^2_2\ .
$$

Acknowledgments. The work of S. K. was supported within the framework of a subsidy granted to the HSE by the Government of the Russian Federation for the implementation of the Global Competitiveness Program, by the RFBR grant 14-01-00547 and joint RFBR-Ukrainian Academy of Sciences grant 14-01-90405-Ukr.

References

[ABRR] D. Arnaudon, E. Buffenoir, E. Ragoucy and Ph. Roche, *Universal Solutions of quantum dynamical Yang-Baxter equations*; Lett. Math. Phys. **44** (1998) 201–214.

[AST] R. M. Asherova, Yu. F. Smirnov and V. N. Tolstoy, *Description of a certain class of projection operators for complex semi-simple Lie algebras* (Russian); Matem. Zametki **26** (1979) 15–25.

[AST2] R. M. Asherova, Yu. F. Smirnov and V. N. Tolstoy, *Projection operators for simple Lie groups. II. General scheme for construction of lowering operators. The groups SU(n)* (Russian); Teoret. Mat. Fiz. **15** (1973) 107–119.

[ES] P. Etingof P. and O. Schiffmann, *Lectures on the dynamical Yang-Baxter equations*; London Math. Soc. Lecture Note Ser. **290** (2001) 89–129.

[HO] B. Herlemont and O. Ogievetsky, *Rings of* $\mathbf{h}$*-deformed differential operators*, to appear.

[I] A. P. Isaev, *Twisted Yang-Baxter equations for linear quantum (super) groups*; Journal of Physics A: Mathematical and General **29** (1996) 6903–6910.

[J] A. Joseph, *Modules for Relative Yangians (Family Algebras) and Kazhdan-Lusztig Polynomials*; Transformation Groups **19**.1 (2014) 105–129.

[Kr] A. A. Kirillov, *Introduction to family algebras*; Moscow Math. J. **1**.1 (2001) 49–63.

[KN] S. Khoroshkin and M. Nazarov, *Mickelsson algebras and representations of Yangians*; Trans. Amer. Math. Soc. **364** (2012), 1293–1367.

[KNV] S. Khoroshkin, M. Nazarov, and E. Vinberg. *A generalized Harish-Chandra isomorphism*; Advances in Mathematics **226**.2 (2011) 1168–1180.

[K] S. Khoroshkin, *Extremal projector and dynamical twist*; Theoret. Math. Phys. **139** (2004) 582–597.

[KO1] S. Khoroshkin and O. Ogievetsky, *Mickelsson algebras and Zhelobenko operators*; J. Algebra **319** (2008) 2113–2165;

[KO2] S. Khoroshkin and O. Ogievetsky, *Diagonal reduction algebras of* $\mathbf{gl}$ *type*; Funktsional. Anal. i Prilozhen. **44** no.3 (2010) 27–49

[KO3] S. Khoroshkin and O. Ogievetsky, *Structure constants of diagonal reduction algebras of gl type*; SIGMA **7** (2011), 064, 34 pp. doi:10.3842/SIGMA.2011.06.

[M] J. Mickelsson, *Step algebras of semi-simple subalgebras of Lie algebras*; Rep. Math. Phys. **4**:4 (1973) 303–318.

[O] O. Ogievetsky, *Uses of quantum spaces*; Contemp. Math. **294** (2002) 161–232.

[Zh] D. Zhelobenko, *Representations of reductive Lie algebras*; Nauka, Moscow (1994).