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Contravariant form for reduction algebras and Pieri rule

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Abstract

We study properties and constructions of contravariant forms on reduction algebras. As an application we compute norms of highest weight vectors in the tensor product of an irreducible finite dimensional representation of the Lie algebra $\mathbf{gl}_n$ with a symmetric or wedge tensor power of its fundamental representation. Their zeroes describe Pieri rules.

1 Introduction

The contravariant (or Shapovalov) form on highest weight modules is a powerful tool in the representation theory of reductive Lie algebra. It is used for the construction of irreducible representations, description of singular vectors of Verma modules etc [D]. In this paper we define an analogue of the Shapovalov form for certain reduction algebras, compute it and apply to the the well-known problems of classical representation theory, calculating the norms of $\mathbf{n}_+$-invariant vectors in tensor products of irreducible finite-dimensional representation of the Lie algebra $\mathbf{gl}_n$ and symmetric or exterior powers of its fundamental representation. Here $\mathbf{n}_+$ is a Lie algebra of upper triangular matrices. Zeros of these norms describe Pieri rules. The norms themselves can be regarded as a generalization of particular Clebsch–Gordan coefficients.

To perform these calculations we pass to three particular reduction algebras: diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{gl}_n)$, and $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ together with its odd analogue $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$, see Section 2.3 for definitions. The first algebra may be regarded as a deformation of $\mathrm{U}(\mathbf{gl}_n)$ with coefficients in the localized universal enveloping algebra $\bar{\mathrm{U}}(\mathbf{h})$, and the latter two as analogous deformations of the algebras of polynomial differential operators in even or odd variables. These algebras possess their own $\bar{\mathrm{U}}(\mathbf{h})$-valued contravariant forms whose specializations to dominant weights is then used for the calculation of norms of $\mathbf{n}_+$-invariant vectors in tensor products of finite-dimensional representations of $\mathbf{gl}_n$.

The main ingredient of our applications of the theory of reduction algebras is the connection of the contravariant form on the reduction algebra to the Zhelobenko automorphism $\check{\xi}_{w_0}$, where w_0 is the longest element of the Weyl group of $\mathbf{gl}_n$. The origin of this connection goes back to Zhelobenko, see [Zh]. It was reformulated, proved and used

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in [KN] for the description of irreducible representations of Yangians. We reproduce here the arguments of [KN] and then compute the contravariant form on polynomial representations of the algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ in two ways: using $\check{\xi}_{w_0}$ and by direct computations in the latter reduction algebras.

The paper is organized as follows. In Sections 2.1-2.2 we recall the definition of Mickelsson algebras and their localizations called reduction algebras, introduce Zhelobenko automorphisms and describe in Section 2.3 our basic examples - reduction algebras $\bar{\mathcal{D}}(\mathbf{g})$, $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. In Section 3.3 we introduce a natural class of $(\bar{\mathrm{Z}}, \mathbf{h})$-modules over reduction algebras and a notion of $\bar{\mathrm{U}}(\mathbf{h})$-valued contravariant forms on them. We establish a connection of these forms with the contravariant forms on $\mathbf{n}_+$-invariants and $\mathbf{n}_-$-coinvariants of certain $\mathbf{g}$-modules. Here $\mathbf{g}$ is a reductive Lie algebra, $\mathbf{n}_\pm$ are their opposite nilpotent subalgebras. In Section 3.4 we describe analogues of the Harish-Chandra map for our basic examples of reduction algebras and define with their help contravariant forms on these algebras. Section 3.5 is devoted to the calculation of these forms on basic polynomial representations of the algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. Sections 4.1-4.2 are devoted to the justification of the evaluations of the computed contravariant forms and their use for the norms of $\mathbf{n}_+$-invariant vectors in tensor products of irreducible finite-dimensional representations of the Lie algebra $\mathbf{gl}_n$ and symmetric or exterior powers of its fundamental representation. In Section 4.3 we deduce the Pieri rules. Appendices contain an alternative derivation of norms of $\mathbf{n}_+$-invariant vectors.

2 Reduction algebras

2.1 Three types of reduction algebras

Let $\mathbf{g}$ be a finite-dimensional reductive Lie algebra with a fixed triangular decomposition $\mathbf{g} = \mathbf{n}_+ + \mathbf{h} + \mathbf{n}_-$, where $\mathbf{h}$ is Cartan subalgebra, $\mathbf{n}_+$ and $\mathbf{n}_-$ are two opposite nilpotent subalgebras. We denote by Δ the root system of $\mathbf{g}$ and by Δ_+ the set of positive roots. Let $\mathcal{A}$ be an associative algebra which contains the universal enveloping algebra $\mathrm{U}(\mathbf{g})$. In particular, $\mathcal{A}$ is a $\mathrm{U}(\mathbf{g})$-bimodule with respect to the left and right multiplications by elements of $\mathrm{U}(\mathbf{g})$. We assume that $\mathcal{A}$ is free as the left $\mathrm{U}(\mathbf{g})$-module and, moreover, that $\mathcal{A}$ contains a subspace V, invariant with respect to the adjoint action of $\mathrm{U}(\mathbf{g})$ such that $\mathcal{A}$ is isomorphic to $\mathrm{U}(\mathbf{g}) \otimes V$ as the left $\mathrm{U}(\mathbf{g})$ module. The action on $\mathrm{U}(\mathbf{g}) \otimes V$ is diagonal. The adjoint action of $\mathbf{g}$ on V is assumed to be reductive.

In this setting we have three natural reduction algebras. The Mickelsson [M] algebra $\mathrm{Z}_+ = \mathrm{Z}(\mathcal{A}, \mathbf{n}_+)$ is defined as the quotient of the normalizer of the left ideal $\mathrm{J}_+ = \mathcal{A}\mathbf{n}_+$ modulo J_+. The Mickelsson algebra $\mathrm{Z}_- = \mathrm{Z}(\mathcal{A}, \mathbf{n}_-)$ is defined as the quotient of the normalizer of the right ideal $\mathrm{J}_- = \mathbf{n}_-\mathcal{A}$ modulo J_-.

In the following we assume that $\mathcal{A}$ is equipped with an anti-involution ε whose restriction to $\mathrm{U}(\mathbf{g})$ coincides with the Cartan anti-involution:

$$\varepsilon(e_{\alpha_c}) = e_{-\alpha_c} \ , \qquad \varepsilon(h) = h \ \text{ for any } h \in \mathbf{h} \ , \tag{1}$$

where α_c, $c = 1, ..., r$, are simple roots in Δ_+ and $e_{\pm\alpha_c}$ and $h_{\alpha_c} = \check{\alpha}_c$ are Chevalley generators of $\mathbf{g}$, normalized by the conditions

$$[h_{\alpha_c}, e_{\pm\alpha_c}] = \pm 2 e_{\pm\alpha_c}, \qquad [e_{\alpha_c}, e_{-\alpha_c}] = h_{\alpha_c} \ .$$

Due to (1), $\varepsilon(\mathrm{J}_+) = \mathrm{J}_-$ and $\varepsilon(\mathrm{Z}_+) = \mathrm{Z}_-$ so that ε establishes an anti-isomorphism of the associative algebras Z_+ and Z_-

Denote by K the multiplicative set, which consists of finite products of elements

$$h_\alpha + k \ , \qquad k \in \mathbb{Z} \ . \tag{2}$$

Here $h_\alpha \in \mathbf{h}$ is the coroot corresponding to a root α of the root system Δ of the Lie algebra $\mathbf{g}$. For the construction of the third reduction algebra we localize with respect to K the enveloping algebras $\mathrm{U}(\mathbf{h})$, $\mathrm{U}(\mathbf{g})$ and the algebra $\mathcal{A}$, denoting by $\bar{\mathrm{U}}(\mathbf{h})$, $\bar{\mathrm{U}}(\mathbf{g})$ and $\bar{\mathcal{A}}$ the corresponding rings of fractions. Define Z and $\bar{\mathrm{Z}} = \bar{\mathrm{Z}}(\mathcal{A}, \mathbf{n}_\pm)$, $\mathrm{Z} \subset \bar{\mathrm{Z}}$, as the double coset spaces

$$\mathrm{Z} = \mathcal{A}/(\mathrm{J}_- + \mathrm{J}_+) \ , \qquad \bar{\mathrm{Z}} = \bar{\mathcal{A}}/(\bar{\mathrm{J}}_- + \bar{\mathrm{J}}_+) \ ,$$

where $\bar{\mathrm{J}}_+ = \bar{\mathcal{A}}\mathbf{n}_+$ and $\bar{\mathrm{J}}_- = \mathbf{n}_-\bar{\mathcal{A}}$. The localized double coset space $\bar{\mathrm{Z}}$ is an associative algebra with respect to the multiplication $\diamond$, see e.g. [KO1] for details. The multiplication $\diamond$ is described by the rule

$$x \diamond y = x \, \mathrm{P} \, y \qquad \mathrm{mod}\ \bar{\mathrm{J}}_+ + \bar{\mathrm{J}}_- \ , \tag{3}$$

where P is the extremal projector [AST] for $\mathbf{g}$, $\mathrm{P}^2 = \mathrm{P}$. The projector P belongs to a certain extension of $\bar{\mathrm{U}}(\mathbf{g})$ (see [Zh] for details), satisfies the properties

$$x \, \mathrm{P} = \mathrm{P} \, y = 0 \quad \text{for} \ \ x \in \mathbf{n}_+, \ y \in \mathbf{n}_- \ , \tag{4}$$

$$\mathrm{P} = 1 \quad \mathrm{mod}\ \mathbf{n}_-\bar{\mathrm{U}}(\mathbf{g}), \qquad \mathrm{P} = 1 \quad \mathrm{mod}\ \bar{\mathrm{U}}(\mathbf{g})\mathbf{n}_+ \ , \tag{5}$$

$$\varepsilon(\mathrm{P}) = \mathrm{P} \ , \tag{6}$$

and can be given [AST] by the explicit multiplicative formula (45). Alternatively, one can take representatives $\tilde{x} \in \bar{\mathcal{A}}$ and $\tilde{y} \in \bar{\mathcal{A}}$ of coset classes x and y such that either $\tilde{x}$ belongs to the normalizer of the left ideal $\bar{\mathcal{A}}\mathbf{n}_+$ or $\tilde{y}$ belongs to the normalizer of the right ideal $\mathbf{n}_-\bar{\mathcal{A}}$. Such representatives exist, see Lemma 2.1 (ii) below. Then $x \diamond y$ is the image in the coset space $\tilde{\mathcal{A}}$ of the product $\tilde{x} \cdot \tilde{y}$. The latter description shows that the maps $\iota_\pm : \mathrm{Z}_\pm \to \bar{\mathrm{Z}}$, defined as compositions of natural inclusions and projections

$$\begin{aligned} \iota_+ &: \mathrm{Z}_+ = \mathrm{Norm}(\mathrm{J}_+)/\,\mathrm{J}_+ \to \mathcal{A}/\,\mathrm{J}_+ \to \mathrm{Z} \subset \bar{\mathrm{Z}} \ , \\ \iota_- &: \mathrm{Z}_- = \mathrm{Norm}(\mathrm{J}_-)/\,\mathrm{J}_- \to \mathrm{J}_-/\mathcal{A} \to \mathrm{Z} \subset \bar{\mathrm{Z}} \ , \end{aligned} \tag{7}$$

are homomorphisms of algebras.

For each root α of the root system Δ of the Lie algebra $\mathbf{g}$ denote by $\mathring{h}_\alpha \in \bar{\mathrm{U}}(\mathbf{h})$ the element

$$\mathring{h}_\alpha = h_\alpha + (\rho, h_\alpha),$$

where $\rho \in \mathbf{h}^*$ is the half sum of positive roots. Denote by $\mathrm{K}_+ \subset \mathrm{K}$ the multiplicative set, which consists of finite products of elements $(\mathring{h}_\alpha + k)$ where k is a positive integer.

Lemma 2.1 *(i) The maps $\iota_\pm$ are injective.*

(ii) For each $z \in \mathrm{Z}$ there exist polynomials $d_+, d_- \in \mathrm{K}_+$ such that $d_+ \cdot z$ belongs to the image of i_+ and $z \cdot d_-$ belongs to the image of i_-.

(iii) The anti-involution ε induces an anti-automorphism of the double coset algebra $\bar{\mathrm{Z}}$, leaves invariant the subspace Z and maps the images of $\mathrm{Z}_\pm$ to the images of $\mathrm{Z}_\mp$.

Proof. (i) If $x \in \mathrm{Norm}(\mathrm{J}_+)$ then due to (5), $\mathrm{P}\, x \equiv x \mod \bar{\mathrm{J}}_+$ (in the above mentioned extension of $\bar{\mathcal{A}}$). If $\iota_+(x) = 0$ then $x \in \mathrm{J}_+ + \mathrm{J}_-$, but $\mathrm{P}\,\mathrm{J}_- = 0$ by the properties of the projector thus $x \in \mathrm{J}_+$.

(ii) For any $x \in \mathcal{A}$ the element $\mathrm{P}\, x$ (which is in the above extension of $\bar{\mathcal{A}}$) belongs to the normalizer of J_+ by the properties of the projectors. Present P as a series $\mathrm{P} = \sum_i d_i f_i e_i$, where d_i are elements of $\bar{\mathrm{U}}(\mathbf{h})$, $f_i \in \mathrm{U}(\mathbf{n}_-)$, $e_i \in \mathrm{U}(\mathbf{n}_+)$. Then

$$\mathrm{P}\, x \equiv \sum_i d_i f_i \hat{e}_i(x) \mod \bar{\mathrm{J}}_+ \ ,$$

where $\hat{e}_i(x)$ is the adjoint action of e_i on x. Since the adjoint action of $\mathbf{n}_+$ in $\bar{\mathcal{A}}$ is locally finite, the latter sum is finite and belongs to the normalizer of J_+ in $\bar{\mathcal{A}}$. Multiplying this sum by the common multiple of d_i we get the element of $\mathrm{Norm}(\mathrm{J}_+)$ in $\mathcal{A}$.

(iii) Straightforward. □

2.2 Zhelobenko operators

It follows that the adjoint action of $\mathbf{g}$ on $\mathcal{A}$, $\hat{x}(a) := xa - ax$, $x \in \mathbf{g}$, $a \in \mathcal{A}$, is locally finite and semisimple. That is, $\mathcal{A}$ can be decomposed into a direct sum of finite-dimensional $\mathbf{g}$-modules with respect to the adjoint action of $\mathbf{g}$. We assume also that simple reflections σ_c, $c = 1, ..., r$, of $\mathbf{h}$, generating the Weyl group of $\mathbf{g}$ are extended to automorphisms of the algebra $\mathcal{A}$, preserving $\mathrm{U}(\mathbf{g})$. We denote them by the same symbols and assume that they still satisfy the corresponding braid group relations

$$\underbrace{\sigma_a \sigma_b \sigma_a \cdots}_{m_{ab}} = \underbrace{\sigma_b \sigma_a \sigma_a \cdots}_{m_{ab}} \ , \qquad a, b = 1, ..., r, \ a \neq b \ , \tag{8}$$

where $m_{ab} = 2$ if $c_{ab} = 0$, $m_{ab} = 3$ if $c_{ab}c_{ba} = 2$ and $m_{ab} = 6$ if $c_{ab}c_{ba} = 3$ with c_{ab} the Cartan matrix of $\mathbf{g}$.

Since the adjoint action of $\mathbf{g}$ in $\mathcal{A}$ is reductive, there is a common choice of such an extension[2], see e.g. [K],

$$\sigma_c(x) = e^{\mathrm{ed}_{e_{\alpha_c}}} \circ e^{-\mathrm{ad}_{e_{-\alpha_c}}} \circ e^{\mathrm{ad}_{e_{\alpha_c}}}(x) \ . \tag{9}$$

Denote by $\check{\mathrm{q}}_c$ the linear map $\check{\mathrm{q}}_c : \mathcal{A} \to \bar{\mathcal{A}}/\bar{\mathrm{J}}_+$ given by the relation

$$\check{\mathrm{q}}_c(x) := \sum_{k \geq 0} \frac{(-1)^k}{k!} \hat{e}^k_{\alpha_c}(\sigma_c(x)) e^k_{-\alpha_c} \prod_{j=1}^{k} (h_{\alpha_c} - j + 1)^{-1} \qquad \mathrm{mod}\ \bar{\mathrm{J}}_+ \ . \tag{10}$$

Properties of the operator $\check{\mathrm{q}}_c$ are listed in the following proposition [Zh], see also [KO1]. Here the shifted action of the Weyl group on $\mathbf{h}$ is used:

$$w \circ \mathring{h}_\alpha = \mathring{h}_{w(\alpha)}.$$

[2]Other extensions by automorphisms of $\mathcal{A}$ of the Weyl group action on $\mathbf{h}$ can be used here. First, one can use the inverse to (9) or switch the positive and negative roots in (9).

Proposition 2.2 *(i)* $\check{\mathrm{q}}_c(\mathrm{J}_+) = 0$;

(ii) $\check{\mathrm{q}}_c(\mathrm{J}_-) \subset \left(\bar{\mathrm{J}}_- + \bar{\mathrm{J}}_+\right)/\bar{\mathrm{J}}_+$;

(iii) $\check{\mathrm{q}}_c(hx) = (\sigma_c \circ h)\check{\mathrm{q}}_c(x)$ *for any* $x \in \mathcal{A}$ *and* $h \in \mathbf{h}$;

(iv) $\check{\mathrm{q}}_c(xh) = \check{\mathrm{q}}_c(x)(\sigma_c \circ h)$ *for any* $x \in \mathcal{A}$ *and* $h \in \mathbf{h}$.

The last two properties allow to extend the map $\check{\mathrm{q}}_c$ to the map of the localized algebras $\check{\mathrm{q}}_c : \bar{\mathcal{A}} \to \bar{\mathcal{A}}/\bar{\mathrm{J}}_+$. The properties (i) and (ii) show that the map $\check{\mathrm{q}}_c$ defines a linear map of the double coset algebra $\bar{\mathrm{Z}}$ to itself.

The Zhelobenko maps satisfy the braid group relations [Zh]:

$$\underbrace{\check{\mathrm{q}}_a\check{\mathrm{q}}_b\check{\mathrm{q}}_a\cdots}_{m_{ab}} = \underbrace{\check{\mathrm{q}}_b\check{\mathrm{q}}_a\check{\mathrm{q}}_a\cdots}_{m_{ab}}, \qquad a, b = 1, ..., r,\ a \neq b \tag{11}$$

and the inversion relation [KO1]:

$$\check{\mathrm{q}}_c^2(x) = (h_{\alpha_c} + 1)^{-1}\ \sigma_c^2(x)\ (h_{\alpha_c} + 1) \qquad \mathrm{mod}\ \bar{\mathrm{J}}_+\ . \tag{12}$$

In [KO1] we established the following homomorphism properties of the Zhelobenko maps $\check{\mathrm{q}}_c$.

Proposition 2.3 *Zhelobenko map* $\check{\mathrm{q}}_c$ *defines a homomorphism of the Mickelsson algebra* Z_+ *to the double coset algebra* $\bar{\mathrm{Z}}$ *and an automorphism of the double coset algebra* $\bar{\mathrm{Z}}$.

One can equally start from the right ideal J_- and define Zhelobenko operators $\check{\xi}_c = \varepsilon\check{\mathrm{q}}_c\varepsilon : \mathcal{A} \to \bar{\mathrm{J}}_-\backslash\bar{\mathcal{A}}$:

$$\check{\xi}_c(x) := \sum_{k\geq 0}\frac{1}{k!}\prod_{j=1}^{k}(h_{\alpha_c} - j + 1)^{-1}e^k_{\alpha_c}\hat{e}^k_{-\alpha_c}(\sigma_c(x)) \qquad \mathrm{mod}\ \bar{\mathrm{J}}_-\ . \tag{13}$$

As well as $\check{\mathrm{q}}_c$ the maps ξ_c determine the automorphisms $\xi_c :\ \bar{\mathrm{Z}} \to \bar{\mathrm{Z}}$ of the double coset algebra, satisfying the braid group relations (8).

Proposition 2.4 *The following relation between automorphisms* $\check{\mathrm{q}}_c$ *and* $\check{\xi}_c$ *of the double coset algebra* $\bar{\mathrm{Z}}$ *takes place*

$$\check{\xi}_c(x) = \check{\mathrm{q}}_c^{-1}\left((\sigma_c\varepsilon)^2(x)\right)\ , \tag{14}$$

where x *is a representative in* $\bar{\mathcal{A}}$ *of the double coset.*

Proof. It is sufficient to check (14) for the $\mathbf{sl}_2$ subalgebra $\mathbf{g}_c$ of $\mathbf{g}$ related to the simple root α_c. The operators $\check{\mathrm{q}}$ and $\check{\xi}$ are automorphisms of the algebra $\mathcal{P}_{\mathbf{h}}(2)$ (see precise definitions below) so it is sufficient to check (14) for the 2-dimensional representation since all other representations arise as the homogeneous components of $\mathcal{P}_{\mathbf{h}}(2)$. With the explicit formulas for $\check{\mathrm{q}}$, see [KO3], the calculation for the 2-dimensional representation is immediate. See also [KNS]. □

Note that the automorphism $(\sigma_c\varepsilon)^2$ is the involution which is -1 on even-dimensional irreducible representations of $\mathbf{g}_c$, and $+1$ on odd-dimensional irreducible representations of $\mathbf{g}_c$.

For $\mathbf{g} = \mathbf{gl}_n$, the symmetric group S_n acts on the universal enveloping algebra $\mathrm{U}(\mathbf{g})$ by permutation of indices. In the sequel we shall use this action to extend the automorphisms σ_c of the Weyl group action on $\mathbf{h}$. In this situation the automorphism $(\sigma_c\varepsilon)^2$ is identical.

2.3 Reduction algebras $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\bar{\mathcal{D}}(\mathbf{g})$

In the sequel we use the following notation for the Lie algebra $\mathbf{gl}_n$. The standard generators are denoted by e_{ij}, the Cartan elements e_{ii} by h_i. We set $h_{ij} = h_i - h_j$, $\mathring{h}_i = h_i - i$ and $\mathring{h}_{ij} = \mathring{h}_i - \mathring{h}_j$. The space $\mathbf{h}^*$ is spanned by the elements ϵ_i, $\epsilon_i(h_j) = \delta_i^j$.

Let $\mathrm{Diff}(n)$ be an associative ring of polynomial differential operators in n variables x^i, where $i = 1, ..., n$. It is generated by the elements x^i and ∂_i, $i = 1, ..., n$, subject to the defining relations

$$[x^i, x^j] = [\partial_i, \partial_j] = 0 \ , \qquad [\partial_i, x^j] = \delta_i^j \ . \tag{15}$$

Let $\psi : \mathrm{U}(\mathbf{gl}_n) \to \mathrm{Diff}(n)$ be the homomorphism of associative algebras, such that

$$\psi(e_{ij}) = x^i \partial_j \ . \tag{16}$$

Set

$$\mathcal{A} = \mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n) \ . \tag{17}$$

This algebra contains $\mathrm{U}(\mathbf{gl}_n)$ as a subalgebra generated by the elements

$$\psi(e_{ij}) \otimes 1 + 1 \otimes e_{ij}, \qquad i, j = 1, ..., n \ .$$

The corresponding double coset reduction algebra $\bar{\mathrm{Z}}$ is denoted further by $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and is called the algebra of $\mathbf{h}$-differential operators. The algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ is generated over $\bar{\mathrm{U}}(\mathbf{h})$ by the images of the elements $1 \otimes x^i$ and $1 \otimes \partial_i$, which we denote for simplicity by the same letters x^i and ∂_i. They satisfy quadratic relations which can be written in the R-matrix form, see [KO3, Proposition 3.3].

As an $\bar{\mathrm{U}}(\mathbf{h})$-module, $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ is freely generated by images in $\bar{\mathrm{Z}}$ of elements $1 \otimes d$, where $d \in \mathrm{Diff}(n)$. To distinguish elements in $\mathrm{Diff}(n)$ and in $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$, we use sometimes the notation $:d:$ for the image in $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ of a polynomial differential operator d. The anti-involution $\varepsilon : \mathrm{Diff}(n) \to \mathrm{Diff}(n)$ is given by the rule

$$\varepsilon(x^i) = \partial_i \ , \qquad \varepsilon(\partial_i) = x^i \ .$$

For the definition of the Zhelobenko operators we use the action of the symmetric group S_n, which permutes indices of the generators x^i and ∂_i.

The same construction applied to the ring $\mathcal{G}\mathrm{Diff}(n)$ of Grassmann differential operators, generated by the odd generators ζ^i and δ_i, $i = 1, ..., n$, with the defining relations

$$\zeta^i \zeta^j + \zeta^j \zeta^i = \delta_i \delta_j + \delta_j \delta_i = 0 \ , \qquad \zeta^i \delta_j + \delta_j \zeta^i = \delta_j^i \tag{18}$$

and the homomorphism $\varphi : \mathrm{U}(\mathbf{gl}_n) \to \mathcal{G}\mathrm{Diff}(n)$, such that

$$\varphi(e_{ij}) = \zeta^i \delta_j \tag{19}$$

gives rise to the reduction algebra $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$.

For any reductive Lie algebra $\mathbf{g}$ one can define the diagonal reduction algebra as follows. Set $\mathcal{A} = \mathrm{U}(\mathbf{g}) \otimes \mathrm{U}(\mathbf{g})$ and use the diagonally embedded $\mathrm{U}(\mathbf{g})$ as $\mathrm{U}(\mathbf{g})$-subalgebra of $\mathcal{A}$. This subalgebra is generated by the elements $x^{(1)} + x^{(2)}$, where, for $x \in \mathbf{g}$, $x^{(1)} := x \otimes 1$,

$x^{(2)} := 1 \otimes x$. The Chevalley anti-involution ε and the braid group action on $\mathrm{U}(\mathbf{g})$ is naturally extended to its tensor square. The corresponding reduction algebra is denoted by $\bar{\mathcal{D}}(\mathbf{g})$ and is called the diagonal reduction algebra.

There are two families of natural generators of $\bar{\mathcal{D}}(\mathbf{g})$. The first family is given by the images of the elements $x^{(1)}$, $x \in \mathbf{g}$. In particular, we denote the images of Cartan-Weyl generators $e_\alpha^{(1)}$ by $s_\alpha^{(1)}$, and the images of the elements $h_\alpha^{(1)}$, $h_\alpha \in \mathbf{h}$ by $t_\alpha^{(1)}$.

The second family is given by the projections of the elements $x^{(2)}$, $x \in \mathbf{g}$, where we use analogous notations with the change of the upper index. Clearly,

$$s_\alpha^{(1)} + s_\alpha^{(2)} = 0 \ , \qquad \text{and} \qquad t_\alpha^{(1)} + t_\alpha^{(2)} = h_\alpha \ , \qquad \alpha \in \Delta \ .$$

We will be mainly interested in the diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{gl}_n)$. The algebraic structure of the $\bar{\mathcal{D}}(\mathbf{gl}_n)$ was studied in [KO2, KO3]. Note that the homomorphisms (16) and (19) define the homomorphisms of the reduction algebras

$$\psi : \bar{\mathcal{D}}(\mathbf{gl}_n) \to \overline{\mathrm{Diff}}_{\mathbf{h}}(n) \qquad \text{and} \qquad \varphi : \bar{\mathcal{D}}(\mathbf{gl}_n) \to \mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n) \ . \tag{20}$$

3 Contravariant forms

3.1 Extremal projector and $\mathbf{n}_\mp$-(co)invariants

Let M be an $\mathcal{A}$-module. Then the space $M^\circ = M^{\mathbf{n}_+}$ of $\mathbf{n}_+$-invariants (or singular vectors or highest weight vectors) is a Z_+-module, and the space $M_\circ = M_{\mathbf{n}_-} = M/\mathbf{n}_- M$ of $\mathbf{n}_-$-coinvariants is a Z_--module. Assume further that M is locally $\mathbf{n}_+$-finite, and the action of $\mathbf{h}$ is semisimple with *non-singular* (sometimes called dominant) weights, that is

$$M = \oplus_{\lambda \in \mathbf{h}^*} M_\lambda \ , \qquad hv = (h, \lambda) v \ \ v \in M_\lambda, \mathbf{h} \in \mathbf{h} \ ,$$

and

$$(h_\alpha, \lambda + \rho) \neq -1, -2, \dots \ , \ \alpha \in \Delta_+ \ . \tag{21}$$

Equivalently, the eigenvalues of all elements $\mathring{h}_\alpha$, $\alpha \in \Delta_+$, are not negative integers. In this case, the action of the extremal projector P on M is well defined, and the properties (4) of P imply that its image in $\mathrm{End}\, M$ establishes an isomorphism of $\mathbf{n}_+$- invariants M° and $\mathbf{n}_-$-coinvariants $M_\circ$:

$$\mathrm{P} : M_\circ \rightsquigarrow M^\circ, \qquad x \to Px \ . \tag{22}$$

If in addition the eigenvalues of all elements $\mathring{h}_\alpha$, $\alpha \in \Delta_+$, are generic, that is,

$$(h_\alpha, \lambda + \rho) \notin \mathbb{Z} \ , \ \alpha \in \Delta_+ \ , \tag{23}$$

then each of these isomorphic spaces comes equipped with a $\bar{\mathrm{Z}}$-module structure. The multiplication by elements $z \in \bar{\mathrm{Z}}$, which we sometimes denote by the symbol $\diamond$ of the multiplication in the double coset algebra, can be described in several ways. First, using Lemma 2.1, we can multiply z by a polynomial d_- from the right and get an element of Z_-, which we use for the action on coinvariants $M_\circ$; the multiplication by d_- on each weight space is then given as the multiplication by a nonzero number thus is an invertible operator on $M_\circ$, so this allows to define the action of z itself. Second, we can multiply

any representative of $M_\circ$ by $\mathrm{P}\, z\, \mathrm{P}$ (or $z\,\mathrm{P}$) and get another element of $M_\circ$. We can use analogous arguments for M° with the passage from z to an element of Z_+. Finally, we can multiply an element of M° directly by $\mathrm{P}\, z\, \mathrm{P}$ (or $\mathrm{P}\, z$) and get another element of M°.

There is another special case of a natural identification of $\mathbf{n}_+$-invariants with $\mathbf{n}_-$-coinvariants. Assume that the restriction of an $\mathcal{A}$-module M to $\mathbf{g}$ is decomposed into a direct sum of finite-dimensional $\mathbf{g}$-modules. In this case not all weights of M are non-singular, but the weights of M° and of $M_\circ$ are dominant, that is,

$$(h_\alpha, \lambda) = 0, 1, 2, 3, \dots \ , \ \alpha \in \Delta_+ \ ,$$

due to the structure of irreducible finite-dimensional $\mathbf{g}$-modules. Thus we have a well defined action of P on $M_\circ$ and M°, establishing an isomorphism of them. The action of Z_+ on M° can be extended to the action of elements from Z, and the action of Z_- on $M_\circ$ can be also extended to the action of elements from Z due to Lemma 2.1.

The functor, attaching to a $\mathcal{A}$-module M, whose restriction to $\mathbf{g}$ decomposes into a direct sum of finite-dimensional $\mathbf{g}$-modules, the Z_+-module M° is faithful and sends irreducible representations to irreducible representations. To show the latter property, we choose two highest weight vectors $v, u \in M^\circ$. If M is irreducible, then there exists $a \in \mathcal{A}$, such that $av = u$. Then $\mathrm{P}\, av = u$ as well. Repeating the arguments used in the proof of Lemma 2.1 we can replace $\mathrm{P}\, a$ by an element $d^{-1}a'$, where $a' \in \mathrm{Norm}(\mathrm{J}_+)$ and the denominator $d \in \mathrm{K}_+$ is such that $a'v = du$. Since any highest weight of the finite dimensional module is non-singular, d acts on u by multiplication by a nonzero scalar c, so the element $a'' = c^{-1}a' \in \mathrm{Norm}(\mathrm{J}_+)$ maps v to u, $a''v = u$. An analogous picture holds for the space $M_\circ$ of coinvariants and the algebra Z_-.

3.2 Contravariant forms

Let M be an $\mathcal{A}$-module. A symmetric bilinear form $(,) : M \otimes M \to \mathbb{C}$ is called contravariant[3] if

$$(ax, y) = (x, \varepsilon(a)y) \tag{24}$$

for any $x, y \in M$ and $a \in \mathcal{A}$. Let M be an $\mathcal{A}$-module equipped with a contravariant form $(,)$. Then this form induces a pairing

$$(\ ,\ ) : M_\circ \otimes M^\circ \to \mathbb{C}$$

which is contravariant for a pair of reduction algebras Z_- and Z_+, that is,

$$(gx, y) = (x, \varepsilon(g)y) \ , \qquad x \in M_\circ \ , \ y \in M^\circ \ , \ g \in \mathrm{Z}_- \ , \ \varepsilon(g) \in \mathrm{Z}_+ \ . \tag{25}$$

If M is locally $\mathbf{n}_+$-finite, and the action of $\mathbf{h}$ is semisimple with non-singular weights, see (21), then due to the isomorphism of the spaces $M_\circ$ and M° the contravariant form on M induces the contravariant form on the space $M_\circ$ of $\mathbf{n}_-$-coinvariants, so that its value $(u, u')_\circ$ on two elements u and u' of $M_\circ$ is equal to

$$(u, u')_\circ = (u, \mathrm{P}\, u') \ .$$

[3] The content of this section can be equally repeated for a sesquilinear contravariant form.

This form satisfies the following contravariant property:

$$(gu, u') = (u, \varepsilon(g)\,\mathrm{P}\,u') \tag{26}$$

for any $u, u' \in M_\circ$ and $g \in \mathrm{Z}_-$.

On the other hand, a contravariant form on M defines a symmetric bilinear form $(,)^\circ$ on M° by restriction. Under the above assumptions it satisfies the contravariant property

$$(gv, v')^\circ = (v, \mathrm{P}\,\varepsilon(g)v')^\circ \tag{27}$$

for any $u, v \in M^\circ$ and $g \in \mathrm{Z}_+$. The forms on $M_\circ$ and M° are related as follows. For any $u, v \in M_\circ$ vectors $\mathrm{P}\,u$ and $\mathrm{P}\,u'$ belong to M° and

$$(u, u') = (\mathrm{P}\,u, \mathrm{P}\,u').$$

If the eigenvalues of all elements $\mathring{h}_\alpha$, $\alpha \in \Delta_+$, in M are generic, then both forms on isomorphic spaces $M_\circ$ and M° are $\bar{\mathrm{Z}}$ -contravariant, $(gu, v) = (u, \varepsilon(g)v)$, for any $g \in \bar{\mathrm{Z}}$ and $u, v \in M_\circ$ (or $u, v \in M^\circ$).

3.3 $(\bar{\mathrm{Z}}, \mathbf{h})$-modules

Let now M be a left module over the reduction algebra $\bar{\mathrm{Z}}$. We call it a $(\bar{\mathrm{Z}}, \mathbf{h})$-module, or $\mathbf{h}$-module over the reduction algebra $\bar{\mathrm{Z}}$ if, in addition, M has a structure of a free right $\bar{\mathrm{U}}(\mathbf{h})$-module such that:

- $(z \diamond m) \cdot h = z \diamond (m \cdot h)$ for any $z \in \bar{\mathrm{Z}}$, $m \in M$ and $h \in \bar{\mathrm{U}}(\mathbf{h})$;
- the adjoint action of $\mathbf{h}$ on M is semisimple.

These conditions imply that M is also free as a left $\bar{\mathrm{U}}(\mathbf{h})$-module.

For example, the reduction algebra $\bar{\mathrm{Z}}$ itself is the $\mathbf{h}$-module over itself with respect to the left multiplication by elements of $\bar{\mathrm{Z}}$ and the right multiplication by elements of $\bar{\mathrm{U}}(\mathbf{h})$.

Assume that the weights of the adjoint action of $\mathbf{h}$ are generic, see (23). Then for any $\mu \in \mathbf{h}^*$ we can define the "evaluation" $\bar{\mathrm{Z}}$-module $M(\mu)$,

$$M(\mu) = M/MI_\mu \tag{28}$$

where I_μ is the (maximal) ideal in $\bar{\mathrm{U}}(\mathbf{h})$ generated by elements $h - (\mu, h)$ for all $h \in \mathbf{h}$.

We define a *contravariant form on an* $\mathbf{h}$*-module* M as a contravariant map $(\ ,\ ) : M \otimes M \to \bar{\mathrm{U}}(\mathbf{h})$, which is linear with respect to the right action of $\bar{\mathrm{U}}(\mathbf{h})$,

$$\begin{aligned} &(g \diamond u, v) = (u, \varepsilon(g) \diamond v)\ , \quad g \in \bar{\mathrm{Z}}\ , \\ &(uh, v) = (u, vh) = (u, v)h\ , \quad h \in \bar{\mathrm{U}}(\mathbf{h})\ , \end{aligned} \tag{29}$$

for any $u, v \in M$. For a generic $\mu \in \mathbf{h}^*$, the evaluation of a contravariant form on an $\mathbf{h}$-module M determines a $\mathbb{C}$-valued contravariant form on $M(\mu)$.

Here is the main example, which we use in this paper, of $\mathbf{h}$-modules over reduction algebras. Assume that we are given a pair $(\mathcal{B}, \gamma)$, which consists of an associative algebra $\mathcal{B}$ and an algebra homomorphism $\gamma : \mathrm{U}(\mathbf{g}) \to \mathcal{B}$. Let $\mathcal{A} = \mathcal{B} \otimes \mathrm{U}(\mathbf{g})$. Then $\mathcal{A}$ contains

the diagonally embedded subalgebra $\mathrm{U}(\mathbf{g})$, generated by the elements $\gamma(x)\otimes 1+1\otimes x$, $x\in\mathbf{g}$.

Let M be a $\mathcal{B}$-module, given as a quotient of $\mathcal{B}$ over its left ideal I, which contains all the elements $\gamma(x)$, $x\in\mathbf{n}_+$. Assume that the action of elements of $\gamma(h)$, $h\in\mathbf{h}$, is semisimple and all the weights ν of this action are integers, $\nu(h_\alpha)\in\mathbb{Z}$, for any $\alpha\in\Delta$. Consider the left $\mathcal{A}$ -module $N=M\otimes(\mathrm{U}(\mathbf{g})/\,\mathrm{U}(\mathbf{g})\mathbf{n}_+)$. Let $\bar N$ be the localization of N, which consists of the left fractions $d^{-1}n$, where $n\in N$ and d is an element of the multiplicative set K, generated by the elements $(b_\alpha+k)$, $k\in\mathbb{Z}$, where $b_\alpha=\gamma(h_\alpha)\otimes 1+1\otimes h_\alpha$ are elements of the diagonally embedded Cartan subalgebra. Define $M_{(\mathbf{h})}$ to be the space of $\mathbf{n}_-$-coinvariants of $\bar N$ with respect to the diagonally embedded $\mathbf{n}_-$,

$$M_{(\mathbf{h})}=\bar N/\mathbf{n}_-\bar N\ ,\ \text{where } N=M\otimes(\mathrm{U}(\mathbf{g})/\,\mathrm{U}(\mathbf{g})\mathbf{n}_+)\ .$$

By construction, $M_{(\mathbf{h})}$ is a quotient of $\bar{\mathcal{A}}$ by the sum of the right ideal $\bar{\mathrm{J}}_-$ and the left ideal containing $\bar{\mathrm{J}}_+$. Thus $M_{(\mathbf{h})}$ is a quotient of the double coset space $\bar{\mathrm{Z}}=\bar{\mathcal{A}}/(\bar{\mathrm{J}}_++\bar{\mathrm{J}}_-)$ by the image in $\bar{\mathrm{Z}}$ of some left ideal in $\bar{\mathcal{A}}$. Due to the structure of the multiplication in $\bar{\mathrm{Z}}$, $a\diamond b=a\,\mathrm{P}\,b$, this image is also a left ideal in $\bar{\mathrm{Z}}$, so $M_{(\mathbf{h})}$ is a left $\bar{\mathrm{Z}}$-module. For any $m\in M$ and $h\in\mathbf{h}$ we set

$$m\cdot h:=m(1\otimes h)\ . \tag{30}$$

Since elements $1\otimes h$ normalize all the ideals defining $M_{(\mathbf{h})}$, this is a well defined free right action of $\mathrm{U}(\mathbf{h})$, commuting with the $\bar{\mathrm{Z}}$-action on $M_{(\mathbf{h})}$. Moreover, due to the integer conditions on the weights of the initial module M, this action has a natural extension to the action of $\bar{\mathrm{U}}(\mathbf{h})$,

$$m\cdot(h_\alpha+k)^{-1}:=(h_\alpha+k-\nu(m)(h_\alpha))^{-1}m\ ,$$

where $\nu(m)\in\mathbf{h}^*$ is the weight of m. For a generic $\mu\in\mathbf{h}^*$ (that is, $\mu(h_\alpha)\notin\mathbb{Z}$ for any $\alpha\in\Delta$) the specialization $M_{(\mathbf{h})}(\mu)$ is isomorphic to the space of $\mathbf{n}_-$-coinvariants of the tensor product $M\otimes M_\mu$, where M_μ is the Verma module of $\mathbf{g}$ with the highest weight μ, $M_{(\mathbf{h})}(\mu)\simeq(M\otimes M_\mu)_\circ$, which is isomorphic, in its turn, to the space $(M\otimes M_\mu)^\circ$ of $\mathbf{n}_+$ -invariants, see (22). Denote by $\pi_\mu: M_{(\mathbf{h})}\to(M\otimes M_\mu)^\circ$ the composition of the evaluation map with the above isomorphisms. Then $\pi_\mu(xh)=\pi_\mu(x)\mu(h)$ for any $x\in M_{(\mathbf{h})}$, $h\in\mathbf{h}$, and

$$\pi_\mu((m\otimes 1)\cdot f)=\mathrm{P}\cdot(m\otimes 1_\mu)\cdot f(\mu) \tag{31}$$

for any $m\in M$ and $f\in\bar{\mathrm{U}}(\mathbf{h})$. Here $f(\gamma)$ is the evaluation of f, regarded as a rational function on $\mathbf{h}^*$ at the point $\gamma\in\mathbf{h}^*$. The space $(M\otimes M_\mu)^\circ$ has a natural structure of the module over the corresponding reduction algebra Z_+ which extends, due to conditions on μ, to the structure of $\bar{\mathrm{Z}}$ -module.

Note also that the homomorphism $\gamma:\mathrm{U}(\mathbf{g})\to\mathcal{B}$ induces the homomorphism of the reduction algebras

$$\bar\gamma:\bar{\mathcal{D}}(\mathbf{g})\to\bar{\mathrm{Z}} \tag{32}$$

so that $M_{(\mathbf{h})}$ carries as well the structure of a $(\bar{\mathcal{D}}(\mathbf{g}),\mathbf{h})$-module.

Assume that the module $M_{(\mathbf{h})}$ is equipped with a contravariant form $(,)$. For a generic μ this form induces a contravariant form on $(M\otimes M_\mu)^\circ$ by the rule

$$(\pi_\mu(x),\pi_\mu(y))=(x,y)(\mu)\ ,\ x,y\in M_{(\mathbf{h})}\ . \tag{33}$$

The $(\bar{\mathrm{Z}}, \mathbf{h})$-modules which we use in this paper arise from the rings $\mathcal{P}(n) = \mathbb{C}[x^1, ..., x^n]$ of polynomial functions in commuting variables and from the ring $\mathcal{G}(n) = \mathbb{C}[\zeta^1, ..., \zeta^n]$ of polynomial functions in anti-commuting variables. The ring $\mathcal{P}(n)$ is a module over the ring $\mathrm{Diff}(n)$ and over the Lie algebra $\mathbf{gl}_n$. Analogously, the ring $\mathcal{G}(n)$ is a module over the ring $\mathcal{G}\mathrm{Diff}(n)$ and over the Lie algebra $\mathbf{gl}_n$.

We define $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$-module $\mathcal{P}_{\mathbf{h}}(n)$ as a quotient of the reduction algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ over the left ideal I_∂, generated by all ∂_i, $i = 1, ..., n$. Since the Cartan subalgebra normalizes the ideal I_∂, $\mathrm{I}_\partial\, h \subset \mathrm{I}_\partial$ for any $h \in \mathbf{h}$, and the weight of any monomial is integer, we have the right action of $\bar{\mathrm{U}}(\mathbf{h})$ on $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ which supplies $\mathcal{P}_{\mathbf{h}}(n)$ with a structure of $\mathbf{h}$-module over the reduction algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. We define analogously the $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$-module $\mathcal{G}_{\mathbf{h}}(n)$.

In terms of the constructions above we set $\mathcal{B} = \mathrm{Diff}(n)$, $\gamma = \psi$, see (16), $M = \mathrm{Diff}(n)/\,\mathrm{Diff}(n)\{\partial_1, ..., \partial_n\}$ in the even case and $\mathcal{B} = \mathcal{G}\mathrm{Diff}(n)$, $\gamma = \varphi$, see (19), $M = \mathcal{G}\mathrm{Diff}(n)/\mathcal{G}\mathrm{Diff}(n)\{\delta_1, ..., \delta_n\}$ in the odd case.

Example. Let V be the two-dimensional tautological representation of $\mathbf{gl}_2$ with the basis $\{v^1, v^2\}$. The $(\bar{\mathcal{D}}\,(\mathbf{gl}_2), \mathbf{h})$-module $V_{(\mathbf{h})}$ is free as a one sided $\bar{\mathrm{U}}(\mathbf{h})$-module of rank 2. Its left $\bar{\mathcal{D}}\,(\mathbf{gl}_2)$-module structure is described by the following formulas:

$$s_{12}^{(1)} v^1 = 0 \ , \ s_{12}^{(1)} v^2 = v^1 \frac{h_{12}}{h_{12}+1} \ ,$$

$$s_{21}^{(1)} v^1 = v^2 \ , s_{21}^{(1)} v^2 = 0 \ ,$$

$$s_{11}^{(1)} v^1 = v^1 \ , \ s_{11}^{(1)} v^2 = v^2 \frac{1}{h_{12}+1} \ ,$$

$$s_{22}^{(1)} v^1 = 0 \ , \ s_{22}^{(1)} v^2 = v^2 \frac{h_{12}}{h_{12}+1} \ ,$$

$$\mathring{h}_i v^j = v^j (\mathring{h}_i + \delta_i^j) \ .$$

3.4 Harish-Chandra maps

Constructions of contravariant forms for reduction algebras refer to analogues of Harish-Chandra map for the universal enveloping algebras of reductive Lie algebras. We describe this map in our two basic examples.

Lemma 3.1 *(i) The left ideal* $\mathrm{I}_\partial = \overline{\mathrm{Diff}}_{\mathbf{h}}(n) \diamond \{\partial_1, \ldots, \partial_n\}$ *of* $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ *is generated over* $\bar{\mathrm{U}}(\mathbf{h})$ *by the classes of elements* $Y\partial_i$ *where* $Y \in \mathrm{Diff}(n)$, $i = 1, ..., n$.

(ii) The right ideal $\mathrm{I}_x = \{x^1, \ldots, x^n\} \diamond \overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ *of* $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ *is generated over* $\bar{\mathrm{U}}(\mathbf{h})$ *by the classes of elements* $x^i X$ *where* $X \in \mathrm{Diff}(n)$, $i = 1, ..., n$.

(iii) The natural inclusion $\bar{\mathrm{U}}(\mathbf{h}) \to \overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ *establishes the isomorphism of* $\bar{\mathrm{U}}(\mathbf{h})$*-modules* $\bar{\mathrm{U}}(\mathbf{h})$ *and* $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)/(\mathrm{I}_\partial + \mathrm{I}_x)$.

Proof. (i) This is a corollary of the property (4) of the extremal projector, together with the $\mathrm{ad}_{\mathbf{n}_+}$-invariance of the linear span of ∂_i.

(ii) Parallel to (i).

(iii) Follows from the Poincaré–Birkhoff–Witt property of the ring $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$: elements $:(x^1)^{a_1} \ldots (x^n)^{a_n} \partial_1^{b_1} \ldots \partial_n^{b_n}:$ form a basis of $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ over $\bar{\mathrm{U}}(\mathbf{h})$, see [KO4]. $\square$

The map $\overline{\mathrm{Diff}}_{\mathbf{h}}(n) \to \bar{\mathrm{U}}(\mathbf{h})$, which attaches to any element $x \in \overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ the unique element $x^{(0)} \in \bar{\mathrm{U}}(\mathbf{h})$ such that $x - x^{(0)} \in \mathrm{I}_\partial + \mathrm{I}_x$ is an analogue of the Harish-Chandra map $\mathrm{U}(\mathbf{g}) \to \mathrm{U}(\mathbf{h})$. With its use we define in a standard way the $\bar{\mathrm{U}}(\mathbf{h})$-valued bilinear form on $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and on its left module $\mathcal{P}_{\mathbf{h}}(n) = \overline{\mathrm{Diff}}_{\mathbf{h}}(n)/\,\mathrm{I}_\partial$:

$$(x, y) = (\varepsilon(x) \diamond y)^{(0)}. \tag{34}$$

Recall that $\varepsilon(x^i) = \partial_i$, $\varepsilon(\partial_i) = x^i$; in particular, $\varepsilon(\mathrm{I}_\partial) = \mathrm{I}_x$. It is not difficult to show that this form is contravariant, see (29), and symmetric

$$(x, y) = (y, x)$$

for any $x, y \in \overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ or $x, y \in \bar{\mathcal{P}}_n$. The same statements take place for $(\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n), \mathbf{h})$-module $\mathcal{G}_{\mathbf{h}}(n)$.

The diagonal reduction algebra $\bar{\mathcal{D}}(\mathbf{g})$ contains a family of commuting, see [Zh, KO2], elements $t_\alpha^{(1)}$ (in the notation of Section 2.3). Let $\mathbb{C}[\mathbf{t}]$ be the ring of polynomials in $t_\alpha^{(1)}$, $\alpha \in \Delta_+$.

Lemma 3.2 *(i) The left ideal* $\mathrm{I}_+ = \bar{\mathcal{D}}(\mathbf{g}) \diamond \{s_\alpha^{(1)}, \alpha \in \Delta_+\}$ *of* $\bar{\mathcal{D}}(\mathbf{g})$ *is generated over* $\bar{\mathrm{U}}(\mathbf{h})$ *by the classes of elements* $Y e_\alpha^{(1)}$*, where* $Y \in \mathrm{U}(\mathbf{g})^{(1)}$*,* $\alpha \in \Delta_+$*.*

(ii) The right ideal $\mathrm{I}_- = \{s_{-\alpha}^{(1)}, \alpha \in \Delta_+\} \diamond \bar{\mathcal{D}}(\mathbf{g})$ *of* $\bar{\mathcal{D}}(\mathbf{g})$ *is generated over* $\bar{\mathrm{U}}(\mathbf{h})$ *by the classes of elements* $e_{-\alpha}^{(1)} X$*, where* $X \in \mathrm{U}(\mathbf{g})^{(1)}$*,* $\alpha \in \Delta_+$*.*

(iii) The natural inclusion $\bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}[\mathbf{t}] \to \bar{\mathcal{D}}(\mathbf{g})$ *establishes the isomorphism of* $\bar{\mathrm{U}}(\mathbf{h})$*-modules* $\bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}[\mathbf{t}]$ *and* $\bar{\mathcal{D}}(\mathbf{g})/(\mathrm{I}_+ + \mathrm{I}_-)$*.*

Proof. (i) This is a corollary of the property (4) of the extremal projector, together with the $\mathrm{ad}_{\mathbf{n}_+}$-invariance of the linear space $\mathbf{n}_+ \otimes 1$.

(ii) Parallel to (i).

(iii) As in Lemma 3.1, this follows from Poincaré–Birkhoff–Witt property of $\bar{\mathcal{D}}(\mathbf{g})$, see [KO4]. □

The map $\bar{\mathcal{D}}(\mathbf{g}) \to \bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}[\mathbf{t}]$, which attaches to any element $u \in \bar{\mathcal{D}}(\mathbf{g})$ the unique element $u^{(0)} \in \bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}[\mathbf{t}]$ such that $u - u^{(0)} \in \mathrm{I}_+ + \mathrm{I}_-$ is an analogue of the Harish-Chandra map $\mathrm{U}(\mathbf{g}) \to \mathrm{U}(\mathbf{h})$. With its use we define in a standard way the $\bar{\mathrm{U}}(\mathbf{h}) \otimes \mathbb{C}[\mathbf{t}]$-valued bilinear form on $\bar{\mathcal{D}}(\mathbf{g})$:

$$(u, v) = (\varepsilon(u) \diamond v)^{(0)}. \tag{35}$$

The contravariant forms (34) and (35) are compatible. Namely, the bilinear form (34) on $\mathcal{P}_{\mathbf{h}}(n)$ is also contravariant with respect to $\bar{\mathcal{D}}(\mathbf{gl}_n)$ due to the homomorphism (16). As a $\bar{\mathcal{D}}(\mathbf{gl}_n)$-module, $\mathcal{P}_{\mathbf{h}}(n)$ decomposes into a direct sum of homogeneous components. The component $\mathcal{P}_{\mathbf{h}}(n; k)$ of degree k is generated by the element $: (x^1)^k :$ annihilated by the ideal I_+. We have

$$(: (x^1)^k :, : (x^1)^k :) = k!\ .$$

The restriction of the form (34) to $\mathcal{P}_{\mathbf{h}}(n; k)$ can be obtained by the evaluation, see (17), $t_{11} := e_{11}^{(1)} \mapsto k!$, $t_{jj} := e_{jj}^{(1)} \mapsto 0$, $j > 1$, of the form (35). For the anti-commuting variables, the homogeneous component of degree k is generated by the element $: \zeta^1 \zeta^2 \ldots \zeta^k :$ annihilated by the ideal I_+, for which

$$(: \zeta^1 \zeta^2 \ldots \zeta^k :, : \zeta^1 \zeta^2 \ldots \zeta^k :) = 1\ .$$

Now the evaluation is $t_{ii} \mapsto 1$, $i = 1, \dots, k$, and $t_{jj} := e^{(1)}_{jj} \mapsto 0$, $j > k$.

The bilinear form (34) is covariant with respect to the action of Zhelobenko operators in the following sense.

Lemma 3.3 *For any elements $x, y \in \mathcal{P}_{\mathbf{h}}(n)$ or $x, y \in \mathcal{G}_{\mathbf{h}}(n)$ we have*

$$\check{\mathrm{q}}_c(x, y) = (\check{\xi}_c(x), \check{\mathrm{q}}_c(y)) \ . \tag{36}$$

Proof consists in the following calculation:

$$\check{\mathrm{q}}_c(x, y) = \check{\mathrm{q}}_c((\varepsilon(x) \diamond y)^0) = (\check{\mathrm{q}}_c(\varepsilon(x) \diamond \check{\mathrm{q}}_c(y))^0 = (\varepsilon(\check{\xi}_c(x) \diamond \check{\mathrm{q}}_c(y))^0 = (\check{\xi}_c(x), \check{\mathrm{q}}_c(y)) \ . \qquad \square$$

3.5 Calculations of contravariant form on $\mathcal{P}_{\mathbf{h}}(n)$ and $\mathcal{G}_{\mathbf{h}}(n)$

We use the notation $x^{\uparrow a} = x(x+1)\dots(x+a-1)$ and $x^{\downarrow a} = x(x-1)\dots(x-a+1)$ for the Pochhammer symbols.

Proposition 3.4 *Images of monomials $: x^\nu :=: (x^1)^{\nu_1} \cdots (x^n)^{\nu_n} :$ in $\mathcal{P}_{\mathbf{h}}(n)$ have the following scalar products:*

$$\begin{aligned} (: x^\nu :, : x^{\nu'} :) &= \delta_{\nu,\nu'} \prod_{k=1}^n \nu_k! \cdot \prod_{i,j:i<j} \frac{(\mathring{h}_{ij} - \nu_j)^{\uparrow \nu_i + 1}}{\mathring{h}_{ij}^{\uparrow \nu_i + 1}} \\ &= \delta_{\nu,\nu'} \prod_{k=1}^n \nu_k! \cdot \prod_{i,j:i<j} \frac{\Gamma(\mathring{h}_{ij} - \nu_j + \nu_i + 1)\Gamma(\mathring{h}_{ij})}{\Gamma(\mathring{h}_{ij} - \nu_j)\Gamma(\mathring{h}_{ij} + \nu_i + 1)} \ . \end{aligned} \tag{37}$$

Proposition 3.5 *Images of monomials $: \zeta^\nu :=: (\zeta^1)^{\nu_1} \cdots (\zeta^n)^{\nu_n} :$ in $\mathcal{G}_{\mathbf{h}}(n)$ have the following scalar products:*

$$(: \zeta^\nu :, : \zeta^{\nu'} :) = \delta_{\nu,\nu'} \prod_{i,j:i<j} \frac{(\mathring{h}_{ij} - \nu_j)^{1-\nu_i}}{(\mathring{h}_{ij})^{1-\nu_i}} = \delta_{\nu,\nu'} \prod_{i,j:i<j} \frac{(\mathring{h}_{ij} - 1 + \nu_i)^{\nu_j}}{(\mathring{h}_{ij})^{\nu_j}} \tag{38}$$

We present two different proofs of Propositions 3.4 and 3.5.

The first proof is based on the description of the contravariant form for certain reduction algebras given in [KN]. We reproduce it in the particular case of the reduction algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. Let w_0 be the longest element of the symmetric group S_n, regarded as the Weyl group of Lie algebra $\mathbf{gl}_n$, $w_0 = (n, n-1, ..., 2, 1)$. Let $w_0 = s_{c_1} s_{c_2} \cdots s_{c_N}$, $N = \frac{n(n-1)}{2}$, be a reduced decomposition of w_0. Set

$$\check{\mathrm{q}}_{w_0} = \check{\mathrm{q}}_{c_1} \check{\mathrm{q}}_{c_2} \cdots \check{\mathrm{q}}_{c_N}, \qquad \check{\xi}_{w_0} = \check{\xi}_{c_1} \check{\xi}_{c_2} \cdots \check{\xi}_{c_N}.$$

Due to the braid relation (11), definitions of $\check{\mathrm{q}}_{w_0}$ and $\check{\xi}_{w_0}$ do not depend on a reduced decomposition of w_0 and

$$\check{\mathrm{q}}_{w_0} \check{\xi}_{w_0} = \check{\xi}_{w_0} \check{\mathrm{q}}_{w_0} = 1 \ .$$

For any two monomials $x^\nu = (x^1)^{\nu_1}(x^2)^{\nu_2}...(x^n)^{\nu_n}$ and $x^{\nu'} = (x^1)^{\nu'_1}(x^2)^{\nu'_2}...(x^n)^{\nu'_n}$ in commuting variables $x^1, ..., x^n$, and elements φ_1, $\varphi_2 \in \bar{\mathrm{U}}(\mathbf{h})$ set

$$<: x^\nu : \varphi_1, : x^{\nu'} : \varphi_2 >= \varphi_1\varphi_2\, \delta_{\nu,\nu'} \prod_{k=1}^n \nu_k! \ . \tag{39}$$

This defines a $\bar{\mathrm{U}}(\mathbf{h})$-valued pairing on $\mathcal{P}_{\mathbf{h}}(n)$, linear with respect to the right multiplication by elements of $\bar{\mathrm{U}}(\mathbf{h})$. Analogously, for any two monomials $\zeta^\nu = (\zeta^1)^{\nu_1}(\zeta^2)^{\nu_2}...(\zeta^n)^{\nu_n}$ and $\zeta^{\nu'} = (\zeta^1)^{\nu'_1}(\zeta^2)^{\nu'_2}...(\zeta^n)^{\nu'_n}$ in anti-commuting variables $\zeta_1, ..., \zeta_n$, and elements φ_1, $\varphi_2 \in \bar{\mathrm{U}}(\mathbf{h})$ set

$$<: \bar{\zeta}^\nu : \varphi_1, : \bar{\zeta}^{\nu'} : \varphi_2 >= \varphi_1\varphi_2 \prod_{k=1}^n \delta_{\nu_k,\nu'_k} \ . \tag{40}$$

This defines a $\bar{\mathrm{U}}(\mathbf{h})$-valued pairing on $\mathcal{G}_{\mathbf{h}}(n)$, linear with respect to the right multiplication by elements of $\bar{\mathrm{U}}(\mathbf{h})$. We have, see also eq. (3.18) in [KN, Section 3.3],

Proposition 3.6 *(i) For any two monomials x^ν, and $x^{\nu'}$ the contravariant pairing of their images in $\mathcal{P}_{\mathbf{h}}(n)$ is equal to*

$$(: x^\nu :, : x^{\nu'} :) = \check{\mathrm{q}}_{w_0}\Big(<: w_0(x^\nu) :, \check{\xi}_{w_0}(: x^{\nu'} :) > \Big) \ . \tag{41}$$

(ii) For any two monomials ζ^ν, and $\zeta^{\nu'}$ the contravariant pairing of their images in $\bar{\mathcal{G}}_n$ is equal to

$$(: \zeta^\nu :, : \zeta^{\nu'} :) = \check{\mathrm{q}}_{w_0}\Big(<: w_0(\xi^\nu) :, \check{\xi}_{w_0}(: \zeta^{\nu'} :) > \Big) \ . \tag{42}$$

Here in the right hand side of (41) and (42) we use the action of the symmetric group on monomials in $\mathcal{P}_n$ and $\mathcal{G}_n$ by permutations of indices. The outer action of $\check{\mathrm{q}}_{w_0}$ is simply the shifted Weyl group action on $\bar{\mathrm{U}}(\mathbf{h})$. Proposition 3.6 reduces the calculation of contravariant forms in $\mathcal{P}_{\mathbf{h}}(n)$ and $\mathcal{G}_{\mathbf{h}}(n)$ to the calculation of Zhelobenko operator $\check{\xi}_{w_0}$, which is a simple technical exercise; the result, e.g. for $\mathcal{P}_{\mathbf{h}}(n)$, is

$$\check{\xi}_{w_0}(: x^\nu :) =: x^{w_0\nu} : \ \check{\mathrm{q}}_{w_0}\left(\prod_{i,j:i<j} \frac{(\mathring{h}_{ij} - \nu_j)^{\uparrow \nu_i+1}}{\mathring{h}_{ij}^{\uparrow \nu_i+1}} \right) \ ,$$

cf (37). □

Remark. The following general statement holds. Let V be an irreducible finite-dimensional $\mathbf{gl}_n$-module and $<,>$ a contravariant form on V. Instead of (39) take its $\bar{\mathrm{U}}(\mathbf{h})$-linear extension to the $(\bar{\mathcal{D}}(\mathbf{gl}_n), \mathbf{h})$-module $V_{(\mathbf{h})}$. Then the formula (41) defines a contravariant form on $V_{(\mathbf{h})}$. This can be also deduced from [KN].

Proof of Proposition 3.6 (i). To find $(: x^\nu :, : x^{\nu'} :)$ we should calculate, see Lemma 3.1,

$$\Big((\varepsilon(x^\nu) \otimes 1)\Delta(\mathrm{P})(x^{\nu'} \otimes 1) \Big)^0 \tag{43}$$

in $\mathcal{A} = \mathrm{Diff}(n) \otimes \bar{\mathrm{U}}(\mathbf{gl}_n)$. Here $()^0$ means the projection of $\mathcal{A}$ to $\bar{\mathrm{U}}(\mathbf{h})$ parallel to the sum of the left ideal generated by all ∂_i and diagonally embedded e_{ij}, $i < j$ (equivalently,

by all ∂_i and $1\otimes e_{ij}$, $i<j$) and of the right ideal, generated by all x^i and diagonally embedded e_{ij}, $i>j$ (equivalently, by all x^i and $1\otimes e_{ij}$, $i>j$). The symbol Δ stands for the diagonal embedding of $\mathrm{U}(\mathbf{gl}_n)$. Present P in an ordered form

$$P=\sum_i d_i(h)x_iy_i\ ,\qquad \text{where}\quad d_i\in\bar{\mathrm{U}}(\mathbf{h}),\ x_i\in\mathrm{U}(\mathbf{n}_-),\ y_i\in\mathrm{U}(\mathbf{n}_+)\ .$$

Then $\Delta(P)=\sum_i d_i\left(h^{(1)}+h^{(2)}\right)\Delta(x_i)\Delta(y_i)$. Moving $\Delta(x_i)$ to the left and $\Delta(y_i)$ to the right, we conclude that their components in the second tensor factor do not affect the result so we can rewrite (43) as

$$\Big((\varepsilon(x^\nu\otimes 1))(\mathrm{P}[h^{(2)}]\otimes 1)(x^{\nu'}\otimes 1)\Big)^0 \tag{44}$$

where $\mathrm{P}[h^{(2)}]$ means the shift of $\bar{\mathrm{U}}(\mathbf{h})$-valued coefficients in P. The factorized expression for P, see [AST] for details, reads

$$\mathrm{P}=\prod_{\gamma\in\Delta_+}^{\rightarrow}\mathrm{P}_\gamma\quad\text{where}\quad \mathrm{P}_\gamma=\sum_{n\geq 0}\frac{(-1)^n}{n!(h_\gamma+\rho(h_\gamma)+1)^{\uparrow n}}e^n_{-\gamma}e^n_\gamma\ . \tag{45}$$

Then

$$\mathrm{P}[h^{(2)}]=\prod_{\gamma\in\Delta_+}^{\rightarrow}\mathrm{P}_\gamma[h^{(2)}]\quad\text{where}\quad \mathrm{P}_\gamma[h^{(2)}]=\sum_{n\geq 0}\frac{(-1)^n}{n!(h_\gamma+h^{(2)}_\gamma+\rho(h_\gamma)+1)^{\uparrow n}}e^n_{-\gamma}e^n_\gamma\ .$$

In (44) the elements $h_\gamma, e_{-\gamma}$ and e_γ should be understood as differential operators, see (16). Thus, the formula (44) defines a pairing in the $\bar{\mathrm{U}}(\mathbf{gl}_n)$-module $\mathcal{P}(n)$ with coefficients in $1\otimes\bar{\mathrm{U}}(\mathbf{h})$, so that

$$(:x^\nu:,:x^{\nu'}:)=<x^\nu,\psi\left(\mathrm{P}[h^{(2)}]\right)x^{\nu'}> \tag{46}$$

with the subsequent identification of elements $h^{(2)}=1\otimes h$ with elements $h\in\mathbf{h}$.

Next we compute $\check{\xi}_{w_0}(:x^{\nu'}:)$. Since the space $\mathcal{P}(n)\otimes 1$ is an $\mathrm{ad}_{\mathbf{gl}_n}$-invariant subspace of $\mathcal{A}=\mathrm{Diff}(n)\otimes\bar{\mathrm{U}}(\mathbf{gl}_n)$, the consecutive application of statements (iii) and (iv) of Proposition 2.2 leads to the following expression for $\check{\xi}_{w_0}(x^{\nu'})$, see [KO1, Section 8.1]:

$$\check{\xi}_{w_0}(:x^{\nu'}:)=\prod_{\gamma\in\Delta_+}^{\rightarrow}\sum_{n\geq 0}\frac{(-1)^n}{n!(h_{-\gamma}+\rho(h_{-\gamma})+1)^{\uparrow n}}\hat{e}^n_\gamma\hat{e}^n_{-\gamma}(w_0(x^{\nu'}\otimes 1))\ . \tag{47}$$

Here $\hat{g}$ means as before the operator of adjoint action of $g\in\mathbf{gl}_n$. The adjoint action of $g\in\mathbf{gl}_n$ on $\mathcal{P}(n)$ coincides with its action by the left multiplication by $\psi(g)$ on $\mathcal{P}(n)$, realized as the quotient of $\mathrm{Diff}(n)$ over the left ideal generated by ∂_i. Therefore the equality (47) can be understood as the relation in the $\mathbf{gl}_n$-module $\mathcal{P}(n)$ with coefficients in $1\otimes\bar{\mathrm{U}}(\mathbf{h})$:

$$\begin{aligned}\check{\xi}_{w_0}(:x^{\nu'}:)&=\prod_{\gamma\in\Delta_+}^{\rightarrow}\sum_{n\geq 0}\frac{(-1)^n}{n!(-h^{(1)}_\gamma-h^{(2)}_\gamma-\rho(h_\gamma)+1)^{\uparrow n}}e^n_\gamma e^n_{-\gamma}(w_0(x^{\nu'}))\otimes 1=\\&=(w_0\otimes 1)\prod_{\gamma\in\Delta_+}^{\rightarrow}\sum_{n\geq 0}\frac{(-1)^n}{n!(h^{(1)}_\gamma+h^{(2)}_{w_0(\gamma)}-\rho(h_\gamma)+1)^{\uparrow n}}e^n_{-\gamma}e^n_\gamma(x^{\nu'})\otimes 1\ .\end{aligned}$$

Here in the second line we changed the summation index from γ to $w_0(-\gamma)$ and used the property $\rho(h_\gamma) = -\rho(h_{w_0(\gamma)})$. We can rewrite the result using the shifted Weyl group action on the second tensor component:

$$\check{\xi}_{w_0}(:x^{\nu'}:) = (w_0 \otimes \check{\mathrm{q}}_{w_0}) \prod_{\gamma\in\Delta_+}^{\rightarrow} \sum_{n\geq 0} \frac{(-1)^n}{n!(h_\gamma^{(1)} + h_\gamma^{(2)} + \rho(h_\gamma) + 1)^{\uparrow n}} e_{-\gamma}^n e_\gamma^n (x^{\nu'}) \otimes 1$$

$$= (w_0 \otimes \check{\mathrm{q}}_{w_0}) \left(\mathrm{P}[h^{(2)}](x^{\nu'}) \otimes 1\right) . \tag{48}$$

The comparison of (44) and (48) gives the desired statement. The proof of the part (ii) is similar. □

Appendix B contains the second proof of Propositions 3.4 and 3.5 based on explicit calculations in the rings $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$.

4 Specializations

4.1 Specialization to non-singular weights

The tensor product $\mathcal{P}_n \otimes M_\mu$, where M_μ is the $\mathbf{gl}_n$-Verma module with the highest weight μ, is a $\mathcal{A} = \mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n)$-module, generated by the vector $v_\mu = 1 \otimes 1_\mu$. Since this vector satisfies the conditions

$$(\partial_i \otimes 1)v_\mu = (1 \otimes x)v_\mu = 0 \ , \qquad i = 1, ..., n, \ x \in \mathbf{n}_+ \ ,$$

and $(1 \otimes h)v_\mu = \mu(h)v_\mu$ for any $h \in \mathbf{h}$, there is a unique $\mathcal{A}$-contravariant $\mathbb{C}$-valued form on $\mathcal{P}_n \otimes M_\mu$, normalized by the condition $(v_\mu, v_\mu) = 1$. This contravariant form can be constructed by means of the Harish-Chandra map $^{(0)} : \mathcal{A} \to \mathrm{U}(\mathbf{h})$, given by the prescription $x - x^{(0)} \in \mathrm{I} + \varepsilon(\mathrm{I})$, where I is the left ideal of $\mathcal{A}$, generated by $\partial_i \otimes 1$, $i = 1, ..., n$, and $1 \otimes x$, $x \in \mathbf{n}_+$. Then

$$(x \cdot v_\mu, y \cdot v_\mu) = (\varepsilon(x)y)^{(0)}(\mu) \qquad \text{for any } x, y \in \mathcal{A} \ . \tag{49}$$

Here $(\varepsilon(x)y)^{(0)}(\mu)$ in the right hand side of (49) means the evaluation of a polynomial on elements of the Cartan subalgebra at the point $\mu \in \mathbf{h}^*$. The restriction of this form to the space $(\mathcal{P}_n \otimes M_\mu)^\circ$ of $\mathbf{n}_+$-invariants defines on this space a bilinear form, satisfying the contravariance property (27). For a generic μ, the action of the Mickelsson algebra $\mathrm{Diff}(n)_+ = \mathrm{Norm}(\mathrm{J}_+)/\mathrm{J}_+$ on $(\mathcal{P}_n \otimes M_\mu)^\circ$ extends to the action of its localization, the reduction algebra $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. The actions of $\mathrm{Diff}(n)_+$ and $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ satisfy the contravariance property (24).

Due to Lemma 3.1, the Harish-Chandra maps, defining the contravariant forms for $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and for $\mathcal{A}$, are compatible, that is, they commute with the natural map from $\mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n)$ to its double coset $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$; thus the contravariant form on $(\mathcal{P}_n \otimes M_\mu)^\circ$ coincides with the evaluation at μ of the contravariant form on $\mathcal{P}_{\mathbf{h}}(n)$ under the isomorphism (31), see (33).

We conclude that for generic μ the square of the norm of the $\mathbf{n}_+$-invariant vector $\mathrm{P}(x^\nu \otimes 1_\mu)$ of $\mathcal{A}$-module $\mathcal{P}_n \otimes M_\mu$ is equal to, see Proposition 3.4,

$$(\mathrm{P}(x^\nu \otimes 1_\mu), \mathrm{P}(x^\nu \otimes 1_\mu)) = \prod_{k=1}^n \nu_k! \cdot \prod_{i,j:i<j} \frac{(\mathring{h}_{ij}(\mu) - \nu_j)^{\uparrow \nu_i + 1}}{(\mathring{h}_{ij}(\mu))^{\uparrow \nu_i + 1}} \ . \tag{50}$$

On the other hand, the space $\mathcal{P}_n$ decomposes into a direct sum of the spaces S^m of polynomials of degree m, each being an irreducible $\mathbf{gl}_n$-module,

$$\mathcal{P}_n = \oplus_{m\geq 0} S^m.$$

The $\mathbf{gl}_n$-module S^m gives rise to $(\bar{\mathcal{D}}(\mathbf{gl}_n), \mathbf{h})$-module $S^m_{(\mathbf{h})}$, see Section 3.3. It possesses a $\bar{\mathcal{D}}(\mathbf{gl}_n)$-contravariant form, which is the restriction of $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$-contravariant form on $\mathcal{P}_{\mathbf{h}}(n)$. The evaluation of this form at generic μ is a $\mathbb{C}$-valued $\bar{\mathcal{D}}(\mathbf{gl}_n)$-contravariant form on $S^m_{(\mathbf{h})}$. Up to a normalization, the map π_μ transforms it to the restriction to $(S^m \otimes M_\mu)^\circ$ of the unique $\mathrm{U}(\mathbf{gl}_n) \otimes \mathrm{U}(\mathbf{gl}_n)$-contravariant form on $S^m \otimes M_\mu$, see (33).

Thus the formula (50) describes norms of highest weight vectors in the tensor product of the m-th symmetric power of the fundamental representation normalized so that the square of the norm of the vector $(x^1)^m \otimes 1_\mu$ is equal to $m!$.

Denote $\lambda = \mu + \nu$. Since the denominators of the extremal projector P belong to the set K_+, defined in Section 2.1, the $\mathbf{n}_+$-invariant vector $\mathrm{P}(x^\nu \otimes 1_\mu)$ is well defined for any non-singular λ. The square of the norm of this vector is a rational function in λ. This function is equal to the right hand side of (50) for generic μ (that is, for generic λ). So for any non-singular λ the right hand side of (50) is finite and gives the square of the norm of $\mathrm{P}(x^\nu \otimes 1_\mu)$.

The similar considerations hold for the decomposition of the space $\mathcal{G}(n)$ into a direct sum of its homogeneous components, $\mathcal{G}(n) = \oplus_{m=0}^n \Lambda^m$ and the corresponding $(\bar{\mathcal{D}}(\mathbf{gl}_n), \mathbf{h})$-modules $\Lambda^m_{(\mathbf{h})}$.

4.2 Specialization to irreducible representations

Throughout this section the weight $\lambda \in \mathbf{h}^*$ is assumed to be non-singular.

Let M and N be two $\mathbf{g}$-modules from the category $\mathcal{O}$, that is, they are $\mathbf{n}_+$-locally finite, $\mathbf{h}$-semisimple and finitely generated $\mathbf{g}$-modules. Assume that M is generated by a highest weight vector 1_μ of the weight μ (that is, M is a quotient of the Verma module M_μ). Consider the $\mathbf{g}$-module $N \otimes M$.

Lemma 4.1 *(i) The space $(N \otimes M)_\circ$ is spanned by the images of vectors $v \otimes 1_\mu$, $v \in N$.*

(ii) For any non-singular $\lambda \in \mathbf{h}^*$ the space $(N \otimes M)^\circ_\lambda$ of $\mathbf{n}_+$-invariant vectors of the weight λ is spanned by the vectors $\mathrm{P}(v \otimes 1_\mu)$, where v has the weight $\nu = \lambda - \mu$.

Proof. Each element in M can be presented as $g \cdot 1_\mu$ with $g \in \mathrm{U}(\mathbf{n}_-)$. Using the relation

$$v \otimes xg\, 1_\mu \equiv -x\, v \otimes g\, 1_\mu \,(\mathrm{mod}\ \mathbf{n}_-(N \otimes M)) \quad \text{for any} \quad x \in \mathbf{n}_- \ , \ v \in N \ ,$$

we prove by induction on degree of $g \in \mathrm{U}(\mathbf{n}_-)$ that for any $v \in N$ we have an equality $v \otimes g\, v_\mu \equiv v' \otimes 1_\mu$ mod $\mathbf{n}_-(N \otimes M)$ for some $v' \in N$. This proves (i).

The statement (ii) follows from (i) since for a non-singular λ the extremal projector P establishes an isomorphism of $\mathbf{n}_-$-coinvariants and $\mathbf{n}_+$-invariants of the weight λ, see Section 3.1. □

Consider the tensor product $\mathcal{P}_n \otimes L_\mu$ of the $\mathrm{Diff}(n)$-module $\mathcal{P}_n$ and an irreducible $\mathrm{U}(\mathbf{gl}_n)$-module L_μ of the highest weight μ with the highest weight vector $\bar{1}_\mu$. The natural projection $1 \otimes \tau_\mu : \mathcal{P}_n \otimes M_\mu \to \mathcal{P}_n \otimes L_\mu$ defines maps

$$\tilde{\tau}_\mu : (\mathcal{P}_n \otimes M_\mu)^\circ \to (\mathcal{P}_n \otimes L_\mu)^\circ \qquad \text{and} \qquad \tilde{\tau}_{\lambda\mu} : (\mathcal{P}_n \otimes M_\mu)^\circ_\lambda \to (\mathcal{P}_n \otimes L_\mu)^\circ_\lambda \tag{51}$$

of the spaces of $\mathbf{n}_+$-invariant vectors and $\mathbf{n}_+$-invariant vectors of the weight λ.

Corollary 4.2 *(i) The map $\tilde{\tau}_{\lambda\mu}$ is an epimorphism.*

(ii) The square of the norm of each $\mathbf{n}_+$-invariant vector of the weight $\lambda = \mu + \nu$ of the $\mathrm{U}(\mathbf{gl}_n)$-module $\mathcal{P}_n \otimes L_\mu$ is given by the relation (50).

Proof. The statement (i) follows from Lemma 4.1(ii).

For the proof of the statement (ii) we note that the projection map $1 \otimes \tau_\mu$ transforms the contravariant form on $\mathcal{P}_n \otimes M_\mu$ to the contravariant form on $\mathcal{P}_n \otimes L_\mu$. In particular, $1 \otimes \tau_\mu$ transforms the restriction of the contravariant form to the space of $\mathbf{n}_+$-invariant vectors of weight λ in $\mathcal{P}_n \otimes M_\mu$ to the restriction of the contravariant form to the space of $\mathbf{n}_+$-invariant vectors of weight λ in $\mathcal{P}_n \otimes L_\mu$,

$$(u, v) = (\tilde{\tau}_{\lambda\mu}(u), \tilde{\tau}_{\lambda\mu}(v)) \ , \ u, v \in (\mathcal{P}_n \otimes L_\mu)^\circ_\lambda \ .$$

Thus for any non-singular λ the square of the norm of the vector $\mathrm{P}(\bar{x}^\nu \otimes \bar{1}_\mu) \in (\mathcal{P}_n \otimes L_\mu)^\circ_\lambda$ is given by the relation (50). $\square$

Assume now that both μ and λ are non-singular. For any $\nu \in \mathbf{h}^*$ denote by $\mathrm{Z}_\nu := \mathrm{Diff}_{\mathbf{h}}(n)_\nu$ the subspace of the reduction algebra $\bar{\mathrm{Z}} = \overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ generated by images in the double coset space $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ of elements in $\mathcal{A} = \mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n)$ of the weight ν,

$$\mathrm{Z}_\nu := \{x \bmod (\mathrm{J}_+ + \mathrm{J}_-) \mid x \in \mathcal{A} \ , \ [h, x] = \nu(h)x \ \text{ for any } h \in \mathbf{h}\} \ .$$

Since the $\mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n)$-module $\mathcal{P}_n \otimes L_\mu$ is irreducible, for any vector $v \in (\mathcal{P}_n \otimes L_\mu)^\circ_\lambda$ there exists $y \in (\mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n))_{\mu-\lambda}$ such that $y \cdot v = 1 \otimes \bar{1}_\mu$. Then

$$z \diamond v := \mathrm{P}\, y \cdot v = 1 \otimes \bar{1}_\mu \tag{52}$$

where $z \in \mathrm{Z}_{\mu-\lambda}$ is the image of y in $\bar{\mathrm{Z}}$. Due to Corollary 4.2, the map $\tilde{\tau}_{\lambda\mu} : (\mathcal{P}_n \otimes M_\mu)^\circ_\lambda \to (\mathcal{P}_n \otimes L_\mu)^\circ_\lambda$ is an epimorphism. We now describe its kernel for dominant λ and μ in two equivalent ways. Consider any element $u \in (\mathcal{P}_n \otimes M_\mu)^\circ_\lambda$.

Lemma 4.3 *(i) $u \in \mathrm{Ker}\, \tilde{\tau}_{\lambda\mu}$ iff $z \diamond u = 0$ for any $z \in \mathrm{Z}_{\mu-\lambda}$.*

(ii) $u \in \mathrm{Ker}\, \tilde{\tau}_{\lambda\mu}$ iff it is in the kernel of the contravariant form $(\ ,\ )$.

Proof. Let $u \in \mathrm{Ker}\, \tilde{\tau}_{\lambda\mu}$. Then for each $z \in \mathrm{Z}_{\mu-\lambda}$ we have $z \diamond u = 0$. Indeed, the space $(\mathcal{P}_n \otimes M_\mu)^\circ_\mu$ is one-dimensional and is generated by the vector $1 \otimes 1_\mu$. The map $1 \otimes \tau_\mu$ is $\mathrm{Diff}(n) \otimes \mathrm{U}(\mathbf{gl}_n)$-equivariant thus the map $\tilde{\tau}_\mu$ commutes with action of elements of $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$. Moreover $\tilde{\tau}_\mu(1 \otimes 1_\mu) = 1 \otimes \bar{1}_\mu$. Then the vanishing of the left hand side of the equality

$$\tilde{\tau}_\mu(z \diamond u) = z \diamond \tilde{\tau}_\mu(u) \tag{53}$$

implies the relation $z \diamond u = 0$ since $z \diamond u$ is proportional to $1 \otimes 1_\mu$ by the weight reasons. On the other hand, if $z \diamond u = 0$ for any $z \in \mathrm{Z}_{\mu-\lambda}$ then (53) implies that $z \diamond \tilde{\tau}_\mu u = 0$ for any $z \in \mathrm{Z}_{\mu-\lambda}$. Then, by (52), we have $\tilde{\tau}_\mu u = 0$. This proves (i).

Next, Lemma 4.1 says that each vector $v \in (\mathcal{P}_n \otimes M_\mu)^\circ_\lambda$ can be presented as $x \diamond (1 \otimes 1_\mu)$ for some $x \in \mathrm{Z}_{\lambda-\mu}$. If $v = \mathrm{P}(x^\nu \otimes 1_\mu)$ then $x = x^\nu \otimes 1$. Here $\nu = \lambda - \mu$. Then for each $u \in \mathrm{Ker}\, \tilde{\tau}_{\lambda\mu}$,

$$(u, v) = (u, x \diamond (1 \otimes 1_\mu)) = (\varepsilon(x) \diamond u, 1 \otimes 1_\mu) = 0$$

since $\varepsilon(x) \in \mathrm{Z}_{\mu-\lambda}$. On the other hand, if $(u, \varepsilon(x)(1 \otimes 1_\mu)) = 0$ for any $x \in \mathrm{Z}_{\mu-\lambda}$ then $(x \diamond u, 1 \otimes 1_\mu) = 0$ and thus $x \diamond u = 0$. $\square$

Corollary 4.4 *For non-singular λ and μ the vector $\mathrm{P}(x^\nu \otimes \bar{1}_\mu)$ is a nonzero element of $(\mathcal{P}_n \otimes L_\mu)^\circ_\lambda$ iff its norm is nonzero.*

Let now μ be the highest weight of a finite-dimensional irreducible $\mathbf{gl}_n$-module L_μ. In particular, μ is dominant. Then the weights λ of all $\mathbf{n}_+$-invariant vectors of $\mathcal{P}_n \otimes L_\mu$ are highest weights of finite-dimensional irreducible $\mathbf{gl}_n$-modules and are dominant; so they are non-singular. Corollaries 4.2 and 4.4 describe all nonzero $\mathbf{n}_+$-invariant vectors of $\mathcal{P}_n \otimes L_\mu$ together with their norms.

The considerations are valid for Grassmann variables, where now the relation

$$(\mathrm{P}(\zeta^\nu \otimes \bar{1}_\mu), \mathrm{P}(\zeta^\nu \otimes \bar{1}_\mu)) = \prod_{i,j:i<j} \left(\frac{\mathring{h}_{ij}(\mu) - \nu_j}{\mathring{h}_{ij}(\mu)} \right)^{1-\nu_i} \tag{54}$$

describes the norms and nonvanishingness of $\mathbf{n}_+$-invariant vectors in the tensor product $\mathcal{G}_n \otimes L_\mu$.

We summarize the results in the following proposition.

Proposition 4.5 *Assume that the weight μ is dominant.*

(i) The square of the norm of an $\mathbf{n}_+$-invariant vector in $\mathcal{P}_n \otimes L_\mu$ is given by the relation (50). The square of the norm of an $\mathbf{n}_+$-invariant vector in $\mathcal{G}_n \otimes L_\mu$ is given by the relation (54).

(ii) Any $\mathbf{n}_+$-invariant vector in $\mathcal{P}_n \otimes L_\mu$ has a form $\mathrm{P}(x^\nu \otimes 1_\mu)$ (with dominant $\lambda = \mu + \nu$) and is nonzero iff its norm is nonzero.

4.3 Pieri rule

We recall some terminology concerning finite-dimensional representations of $\mathbf{gl}_n$. A finite-dimensional irreducible representation L_μ of highest weight $\mu = (\mu_1, \ldots, \mu_n)$ is visualized by the Young diagram with μ_j boxes in the j-th row. Let $|\mu| := \mu_1 + \ldots \mu_n$ and $\tilde{\mu}_i := \mu_i - i$.

The m-th symmetric power $L_{(m)}$ of the tautological representation of $\mathbf{gl}_n$ corresponds to the one-row diagram with m boxes. The m-th wedge power $L_{(1^m)}$ of the tautological representation of $\mathbf{gl}_n$ corresponds to the one-column diagram with m boxes.

For two diagrams μ and λ, $\mu \subset \lambda$, the set-theoretical difference $\lambda \setminus \mu$ is called a horizontal strip if it contains no more than one box in any column. The difference $\lambda \setminus \mu$ is called a vertical strip if it contains no more than one box in any row. The Pieri rule says that for any μ the product $L_{(m)} \otimes L_\mu$ is multiplicity free and is a direct sum of L_λ such that $\lambda \setminus \mu$ is a horizontal strip of cardinality m. The dual Pieri rule says that for any μ the product $L_{(1^m)} \otimes L_\mu$ is multiplicity free and is a direct sum of V_λ such that $\lambda \setminus \mu$ is a vertical strip of cardinality m. The multiplicity freeness follows since the $\mathbf{h}$-weights of the $(\bar{\mathcal{D}}(\mathbf{gl}_n), \mathbf{h})$-modules $S^m_{(\mathbf{h})}$ and $\Lambda^m_{(\mathbf{h})}$ are multiplicity free.

We keep the notation of Proposition 4.5. Rewrite the last product in the right hand side of the formula (50) in the form $\prod_{i,j:i<j} B^{(\nu)}_{i,j}(\mu)$ where

$$B^{(\nu)}_{i,j}(\mu) := \frac{(\mathring{h}_{ij}(\mu) - \nu_j)^{\uparrow \nu_i + 1}}{\mathring{h}_{ij}(\mu)^{\uparrow \nu_i + 1}} \ .$$

Let $\lambda = \mu + \nu$ be the weight of the $\mathbf{n}_+$-invariant vector $\mathrm{P}(x^\nu \otimes 1_\mu)$. The denominator of $B^{(\nu)}_{i,j}(\mu)$ is positive. The numerator of $B^{(\nu)}_{i,j}(\mu)$ is

$$(\tilde{\mu}_i - \tilde{\lambda}_j)(\tilde{\mu}_i - \tilde{\lambda}_j + 1) \dots (\tilde{\lambda}_i - \tilde{\lambda}_j) \ .$$

The last factor is a positive integer. So the product vanishes iff

$$\tilde{\mu}_i - \tilde{\lambda}_j \leq 0 \ \text{ for some } \ i, j \ , \ i < j \ . \tag{55}$$

It is sufficient to analyze only the neighboring indices, that is, to replace (55) by the condition

$$\tilde{\mu}_i - \tilde{\lambda}_{i+1} \leq 0 \ \text{ for some } \ i \ . \tag{56}$$

Indeed, if $\tilde{\mu}_i - \tilde{\lambda}_{i+1} > 0$ and $j > i$ then $\tilde{\mu}_i - \tilde{\lambda}_j > 0$ since $\tilde{\lambda}_{i+1} - \tilde{\lambda}_j \geq 0$. We conclude that the following lemma holds.

Lemma 4.6 *The tensor product $L_{(m)} \otimes L_\mu$ does not contain the representation L_λ, $\mu \subset \lambda$ and $|\lambda| = |\mu| + m$, iff the condition (55) holds.*

In other words, the tensor product $L_{(m)} \otimes L_\mu$ contains the representation L_λ, $\mu \subset \lambda$ and $|\lambda| = |\mu| + m$, if

$$\mu_i \geq \lambda_{i+1} \ \text{ for all } i \ . \tag{57}$$

The condition (57) says exactly that the difference $\lambda \backslash \mu$ is a horizontal strip so we obtain the Pieri rule.

The situation with the odd variables is different. The right hand side of (54) is $\prod_{i,j:i<j} C^{(\nu)}_{i,j}(\mu)$ where

$$C^{(\nu)}_{i,j}(\mu) := \left(\frac{\mathring{h}_{ij}(\mu) - \nu_j}{\mathring{h}_{ij}(\mu)} \right)^{1-\nu_i} \ .$$

The denominator of $C^{(\nu)}_{i,j}(\mu)$ is positive and the numerator is zero iff $\nu_i = 0$ and $\nu_j = 1$ for some i, j, $i < j$, and $\mathring{h}_{ij}(\mu) = 1$, or $\mu_i - \mu_j = i - j + 1$ which may occur only if $j = i + 1$ and $\mu_{i+1} = \mu_i$. But then $\lambda_i = \mu_i$ and $\lambda_{i+1} = \mu_{i+1} + 1 = \lambda_i + 1$ which cannot happen for a diagram λ. Thus all irreducible representations L_λ such that $\nu = \lambda - \mu$ is a weight of $\Lambda^m_{(\mathbf{h})}$ do appear in the tensor product $L_{(1^m)} \otimes L_\mu$ which is exactly the statement of the dual Pieri rule about the vertical strip.

A Rings $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$ and $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$

A.1 $\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$

The ring is generated by the elements x^i and ∂_i. We shall use, instead of the set of generators generators $\{x^i, \partial_i\}$ the set $\{x^i, \bar{\bar{\partial}}_i\}$ where, see [KO4],

$$\bar{\bar{\partial}}_j = \partial_j {\varphi'_j}^{-1} \ \text{ with } \ \varphi'_j = \prod_{k:k<j} \frac{\mathring{h}_{jk}}{\mathring{h}_{jk} - 1} \ . \tag{58}$$

The defining relations for the variables x^i are

$$x^i \diamond x^j = \frac{\mathring{h}_{ij}+1}{\mathring{h}_{ij}} x^j \diamond x^i \ , \ i<j \ . \tag{59}$$

The remaining defining relations read

$$\begin{aligned}
&\bar{\bar{\partial}}_i \diamond \bar{\bar{\partial}}_j = \frac{\mathring{h}_{ij}-1}{\mathring{h}_{ij}} \bar{\bar{\partial}}_j \diamond \bar{\bar{\partial}}_i \ , \qquad i<j \ , \\
&\bar{\bar{\partial}}_j \diamond x^i = x^i \diamond \bar{\bar{\partial}}_j, \ i>j \ , \qquad \bar{\bar{\partial}}_j \diamond x^i = \frac{\mathring{h}_{ij}(\mathring{h}_{ij}-2)}{(\mathring{h}_{ij}-1)^2} x^i \diamond \bar{\bar{\partial}}_j \ , \ i<j \ , \\
&\bar{\bar{\partial}}_i \diamond x^i = \sum_j \frac{1}{1+\mathring{h}_{ij}} x^j \diamond \bar{\bar{\partial}}_j + 1 \ .
\end{aligned} \tag{60}$$

We have, for $i<j$:

$$(x^i)^{\diamond a} \diamond (x^j)^{\diamond b} = (x^j)^{\diamond b} \diamond (x^i)^{\diamond a} \frac{(\mathring{h}_{ij}+1)^{\uparrow a}}{(\mathring{h}_{ij}-b+1)^{\uparrow a}} \ . \tag{61}$$

The proof is by induction, say, first on a and then on b.

We have

$$x^{j_1} \diamond x^{j_2} \diamond \ldots \diamond x^{j_k} =: x^{j_1} x^{j_2} \ldots x^{j_k} : \quad \text{if} \ \ j_1 \geq j_2 \geq \cdots \geq j_k \ . \tag{62}$$

The proof is by induction on k. Write the extremal projector in the form

$$\mathrm{P} = A_2 A_3 ... A_n \ \ \text{where} \ \ A_m = \mathrm{P}_{1,m} \, \mathrm{P}_{2,m} ... \, \mathrm{P}_{m-1,m} \ , \ m = 2, ..., n \ ,$$

with the notation $\mathrm{P}_{i,j} := \mathrm{P}_{\epsilon_i - \epsilon_j}$, see (45). By the induction hypothesis, $x^{j_2} \diamond \ldots \diamond x^{j_k} =: x^{j_2} \ldots x^{j_k} :$, so

$$x^{j_1} \diamond x^{j_2} \diamond \ldots \diamond x^{j_k} = x^{j_1} \diamond : x^{j_2} \ldots x^{j_k} : \equiv x^{j_1} \, \mathrm{P} : x^{j_2} \ldots x^{j_k} : \ .$$

The assertion (62) follows because $A_l : x^{j_2} \ldots x^{j_k} : \equiv : x^{j_2} \ldots x^{j_k} :$ for $l = j_2+1, \ldots, n$, and $x^{j_1} A_2 ... A_{j_1} \equiv x^{j_1}$.

A.2 $\mathcal{G}\overline{\mathrm{Diff}}_{\mathbf{h}}(n)$

Now, the defining relations for the variables ζ^i are

$$\zeta^i \diamond \zeta^j = -\frac{\mathring{h}_{ij}-1}{\mathring{h}_{ij}} \zeta^j \diamond \zeta^i \ , \ i<j \ . \tag{63}$$

Let $\bar{\bar{\delta}}_j = \delta_j {\varphi'_j}^{-1}$. The remaining defining relations read

$$\begin{aligned}
&\bar{\bar{\delta}}_i \diamond \bar{\bar{\delta}}_j = -\frac{\mathring{h}_{ij}+1}{\mathring{h}_{ij}} \bar{\bar{\delta}}_j \diamond \bar{\bar{\delta}}_i \ , \qquad i<j \ , \\
&\bar{\bar{\delta}}_j \diamond \zeta^i = -\zeta^i \diamond \bar{\bar{\delta}}_j \ , \ i>j \ , \qquad \bar{\bar{\delta}}_j \diamond \zeta^i = -\frac{\mathring{h}_{ij}(\mathring{h}_{ij}-2)}{(\mathring{h}_{ij}-1)^2} \zeta^i \diamond \bar{\bar{\delta}}_j \ , \ i<j \ , \\
&\bar{\bar{\delta}}_i \diamond \zeta^i = -\sum_j \frac{1}{1+\mathring{h}_{ij}} \zeta^j \diamond \bar{\bar{\delta}}_j + 1 \ .
\end{aligned} \tag{64}$$

Similarly to (62), for the **h**-Grassmann variables,

$$\zeta^{j_1}{}_\diamond\zeta^{j_2}{}_\diamond\ldots{}_\diamond\zeta^{j_k} =: \zeta^{j_1}\zeta^{j_2}\ldots\zeta^{j_k}: \quad \text{if} \;\; j_1 > j_2 > \cdots > j_k \;. \tag{65}$$

A.3 Zhelobenko automorphisms

Recall that we use the action of the symmetric group as the extension by the automorphisms of the Weyl group action. The action of Zhelobenko automorphisms on generators is

$$\begin{aligned}
&\check{\mathrm{q}}_i(x^i) = x^{i+1}\frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1}-1}\;, \qquad \check{\mathrm{q}}_i(x^{i+1}) = x^i, \quad \check{\mathrm{q}}_i(x^j) = x^j\;, \quad j\neq i,i+1\;,\\
&\check{\mathrm{q}}_i(\bar{\bar{\partial}}_i) = \frac{\mathring{h}_{i,i+1}-1}{\mathring{h}_{i,i+1}}\bar{\bar{\partial}}_{i+1}\;, \qquad \check{\mathrm{q}}_i(\bar{\bar{\partial}}_{i+1}) = \bar{\bar{\partial}}_i\;, \quad \check{\mathrm{q}}_i(\bar{\bar{\partial}}_j) = \bar{\bar{\partial}}_j\;, \quad j\neq i,i+1\;,\\
&\check{\mathrm{q}}_i(\mathring{h}_j) = \mathring{h}_{s_i(j)}\;.
\end{aligned} \tag{66}$$

The Zhelobenko automorphisms act on ζ^i with the same coefficients as on x^i .

Let $\check{\mathrm{q}}$ be the Zhelobenko automorphism for $\mathbf{gl}_2$. Then

$$\check{\mathrm{q}}\big((x^2)^{\diamond a}{}_\diamond(x^1)^{\diamond b}\big) = (x^2)^{\diamond b}{}_\diamond(x^1)^{\diamond a}\,\frac{(\mathring{h})^{\uparrow a+1}}{(\mathring{h}-b)^{\uparrow a+1}} \tag{67}$$

and

$$\check{\mathrm{q}}^{-1}\big((x^2)^{\diamond a}{}_\diamond(x^1)^{\diamond b}\big) = (x^2)^{\diamond b}{}_\diamond(x^1)^{\diamond a}\,\frac{(\mathring{h})^{\uparrow a}}{(\mathring{h}-b)^{\uparrow a}}\;. \tag{68}$$

B Calculation of norms

Even variables. Let $\nu = (\nu_1,\ldots,\nu_n)$ be a milti-index, $\nu! = \nu_1!\ldots\nu_n!$ and $x^\nu = (x^n)^{\diamond\nu_n}{}_\diamond\ldots{}_\diamond(x^1)^{\diamond\nu_1}$. The monomials x^ν for $\nu\in\mathbb{Z}^n_{\geq 0}$ form a basis of $\mathcal{P}_{\mathbf{h}}(n)$. Define a bilinear form on $\mathcal{P}_{\mathbf{h}}(n)$ by

$$(x^\nu, x^{\nu'}) = \delta_{\nu,\nu'}\nu!\gamma_\nu \;\; \text{where} \;\; \gamma_\nu := \prod_{i,j:i<j} B^{(\nu)}_{i,j} \;\; \text{and} \;\; B^{(\nu)}_{i,j} := \frac{(\mathring{h}_{ij}-\nu_j)^{\uparrow\nu_i+1}}{\mathring{h}_{ij}^{\uparrow\nu_i+1}}\;. \tag{69}$$

We denote by $s_i\nu$ the multi-index $(\nu_1,\ldots,\nu_{i-1},\nu_{i+1},\nu_i,\nu_{i+2},\ldots,\nu_n)$.

Proposition B.1 *The form (69) coincides with the contravariant form on $\mathcal{P}_{\mathbf{h}}(n)$.*

By (62), this proposition is equivalent to Proposition 3.4.

Proof of Proposition B.1. Since the subspaces of different **h**-weight are orthogonal with respect to a contravariant form, it is sufficient to analyze the products (x^ν, x^ν).

1. We first check the covariance (36) of the form (69). Collecting pairs $B^{(\nu)}_{m,i}$ and $B^{(\nu)}_{m,i+1}$ for $m<i$ and pairs $B^{(\nu)}_{i,m}$ and $B^{(\nu)}_{i+1,m}$ for $m>i+1$ in the product for γ_ν, we find

$$\check{\mathrm{q}}_i(\gamma_\nu) = \gamma_{s_i\nu}\frac{\check{\mathrm{q}}_i(B^{(\nu)}_{i,i+1})}{B^{(s_i\nu)}_{i,i+1}} = \gamma_{s_i\nu}\frac{(\mathring{h}_{i,i+1}+\nu_{i+1})^{\downarrow\nu_i+1}}{\mathring{h}_{i,i+1}^{\downarrow\nu_i+1}}\;\frac{\mathring{h}_{i,i+1}^{\uparrow\nu_{i+1}+1}}{(\mathring{h}_{i,i+1}-\nu_i)^{\uparrow\nu_{i+1}+1}}\;. \tag{70}$$

We have

$$\check{\mathrm{q}}_i(x^\nu) = x^{s_i\nu}\,\frac{\mathring{h}_{i,i+1}^{\uparrow\nu_{i+1}+1}}{(\mathring{h}_{i,i+1}-\nu_i)^{\uparrow\nu_{i+1}+1}} \quad\text{and}\quad \check{\mathrm{q}}_i^{-1}(x^\nu) = x^{s_i\nu}\,\frac{(\mathring{h}_{i,i+1}+\nu_{i+1})^{\downarrow\nu_i+1}}{\mathring{h}_{i,i+1}^{\downarrow\nu_i+1}}\ . \tag{71}$$

Therefore, the transformation laws $(x^\nu,x^\nu)\mapsto\big(\check{\mathrm{q}}_i^{-1}(x^\nu),\check{\mathrm{q}}_i(x^\nu)\big)$ and $(x^\nu,x^\nu)\mapsto\check{\mathrm{q}}_i(x^\nu,x^\nu)$ are the same so it is sufficient to prove (69) for an arbitrary permutation of $(\nu_1,\dots,\nu_n)$.

2. We prove the assertion by induction on degree $|\nu|=\nu_1+\dots+\nu_n$, the induction base is $(1,1)=1$. Assume that $|\nu|>0$. By part **1**, it is sufficient to verify the statement for ν such that $\nu_1>0$. By induction hypothesis, $(x^\upsilon,x^\upsilon)=\upsilon!\gamma_\upsilon$ where $\upsilon=(\nu_n,\dots,\nu_2,\nu_1-1)$. We have

$$x^\nu = x^\upsilon\diamond x^1 \quad\text{and}\quad \gamma_\nu=\gamma_\upsilon\prod_{j>1}\frac{\mathring{h}_{1j}-\nu_j+\nu_1}{\mathring{h}_{1j}+\nu_1}\ .$$

It follows from (61) that

$$x^\upsilon\diamond x^1 = x^1\diamond x^\upsilon\prod_{j>1}\frac{\mathring{h}_{1j}-\nu_j+\nu_1}{\mathring{h}_{1j}+\nu_1}\ .$$

Therefore,

$$(x^\upsilon\diamond x^1,x^\nu)=(x^1\diamond x^\upsilon,x^\nu)\prod_{j>1}\frac{\mathring{h}_{1j}-\nu_j+\nu_1}{\mathring{h}_{1j}+\nu_1}=(x^\upsilon,\partial_1\diamond x^\nu)\prod_{j>1}\frac{\mathring{h}_{1j}-\nu_j+\nu_1}{\mathring{h}_{1j}+\nu_1}\ .$$

We used the contravariance in the last equality. By (58), $\partial_1=\bar{\bar{\partial}}_1$. Now, $\bar{\bar{\partial}}_1\diamond x^\nu=(x^n)^{\diamond\nu_n}\diamond\dots\diamond(x^2)^{\diamond\nu_2}\diamond\bar{\bar{\partial}}_1\diamond(x^1)^{\diamond\nu_1}$ by (60). We have

$$\bar{\bar{\partial}}_1\diamond x^1 = 1+x^1\diamond\bar{\bar{\partial}}_1+\text{linear combination of } x^j\diamond\bar{\bar{\partial}}_j \text{ with } j>1\ .$$

The **h**-derivative $\bar{\bar{\partial}}_j$, $j>1$, then moves to the right through remaining x^1 without a constant term, so $\bar{\bar{\partial}}_1\diamond(x^1)^{\diamond\nu_1}=\nu_1+$ (a linear combination of terms $F_i\diamond\bar{\bar{\partial}}_i$) which does not contribute to the scalar product. □

Odd variables. Let now $\nu=(\nu_1,\dots,\nu_n)$ where $\nu_j\in\{0,1\}$, $j=1,\dots,n$, and $\zeta^\nu=(\zeta^n)^{\nu_n}\diamond\dots\diamond(\zeta^1)^{\nu_1}$. The monomials ζ^ν form a basis of $\mathcal{G}_{\mathbf{h}}(n)$. Define a bilinear form on $\mathcal{G}_{\mathbf{h}}(n)$ by

$$(\zeta^\nu,\zeta^{\nu'})=\delta_{\nu,\nu'}\kappa_\nu \quad\text{where}\quad \kappa_\nu:=\prod_{i,j:i<j}C_{i,j}^{(\nu)} \tag{72}$$

and

$$C_{i,j}^{(\nu)}:=\left(\frac{\mathring{h}_{ij}-\nu_j}{\mathring{h}_{ij}}\right)^{1-\nu_i}=\left(\frac{\mathring{h}_{ij}-1}{\mathring{h}_{ij}}\right)^{\nu_j(1-\nu_i)}\ .$$

We denote by $s_i\nu$ the string $(\nu_1,\dots,\nu_{i-1},\nu_{i+1},\nu_i,\nu_{i+2},\dots,\nu_n)$.

Proposition B.2 *The form (72) coincides with the contravariant form on $\mathcal{G}_{\mathbf{h}}(n)$.*

By (65), this proposition is equivalent to Proposition 3.5. The proof is along the same lines as for Proposition B.1.

Proof. **1.** Analogues of formulas (70) and (71) are

$$\check{\mathrm{q}}_i(\kappa_\nu) = \kappa_{s_i\nu} \left(\frac{\mathring{h}_{i,i+1} + \nu_{i+1}}{\mathring{h}_{i,i+1}} \right)^{1-\nu_i} \left(\frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1} - \nu_i} \right)^{1-\nu_{i+1}} ,$$

$$\check{\mathrm{q}}_i(\zeta^\nu) = (-1)^{\nu_i\nu_{i+1}} \zeta^{s_i\boldsymbol{\nu}} \left(\frac{\mathring{h}_{i,i+1}}{\mathring{h}_{i,i+1} - \nu_i} \right)^{1-\nu_{i+1}} , \ \check{\mathrm{q}}_i^{-1}(\zeta^\nu) = (-1)^{\nu_i\nu_{i+1}} \zeta^{s_i\boldsymbol{\nu}} \left(\frac{\mathring{h}_{i,i+1} + \nu_{i+1}}{\mathring{h}_{i,i+1}} \right)^{1-\nu_i} .$$

Again, the transformation laws $(\zeta^\nu, \zeta^\nu) \mapsto \big(\check{\mathrm{q}}_i^{-1}(\zeta^\nu), \check{\mathrm{q}}_i(\zeta^\nu)\big)$ and $(\zeta^\nu, \zeta^\nu) \mapsto \check{\mathrm{q}}_i(\zeta^\nu, \zeta^\nu)$ are the same so it is sufficient to prove (72) for an arbitrary permutation of $(\nu_1, \dots, \nu_n)$.

2. Induction now is on degree $|\nu| = \nu_1 + \dots + \nu_n$. Assume that $|\nu| > 0$. By part **1**, it is sufficient to verify the statement for ν such that $\nu_1 = 1$. By induction hypothesis, $(\zeta^\upsilon, \zeta^\upsilon) = \kappa_\upsilon$ where $\upsilon = (\nu_n, \dots, \nu_2, 0)$. We have

$$\zeta^\nu = \zeta^\upsilon \diamond \zeta^1 \ \text{ and } \ \kappa_\nu = \kappa_\upsilon \prod_{j>1} \frac{\mathring{h}_{1j}}{\mathring{h}_{1j} - \nu_j} .$$

On the other hand,

$$\zeta^\upsilon \diamond \zeta^1 = (-1)^{|\upsilon|} \zeta^1 \diamond \zeta^\upsilon \prod_{j>1} \left(\frac{\mathring{h}_{1j}}{\mathring{h}_{1j} - 1} \right)^{\nu_j} = (-1)^{|\upsilon|} \zeta^1 \diamond \zeta^\upsilon \prod_{j>1} \frac{\mathring{h}_{1j}}{\mathring{h}_{1j} - \nu_j} .$$

The rest of the proof follows, as for the even variables, from the covariance and the fact that $\bar{\bar{\delta}}_1 = \delta_1$. □

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