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# Norm saturating property of time optimal controls for wave-type equations \*

Jérôme Lohéac <sup>†</sup>      Enrique Zuazua <sup>‡</sup>

## Abstract

We consider a time optimal control problem with point target for a class of infinite dimensional systems governed by abstract wave operators. In order to ensure the existence of a time optimal control, we consider controls of energy bounded by a prescribed constant  $E > 0$ . Even when this control constraint is absent, in many situations, due to the hyperbolicity of the system under consideration, a target point cannot be reached in arbitrarily small time and there exists a minimal universal controllability time  $T_* > 0$ , so that for every points  $y_0$  and  $y_1$  and every time  $T > T_*$ , there exists a control steering  $y_0$  to  $y_1$  in time  $T$ . Simultaneously this may be impossible if  $T < T_*$  for some particular choices of  $y_0$  and  $y_1$ .

In this note we point out the impact of the strict positivity of the minimal time  $T_*$  on the structure of the norm of time optimal controls. In other words, the question we address is the following: If  $\tau$  is the minimal time, what is the  $L^2$ -norm of the associated time optimal control?

For different values of  $y_0, y_1$  and  $E$ , we can have  $\tau \leq T_*$  or  $\tau > T_*$ . If  $\tau > T_*$ , the time optimal control is unique, given by an adjoint problem and its  $L^2$ -norm is  $E$ , in the classical sense. In this case, the time optimal control is also a norm optimal control. But when  $\tau < T_*$ , we show, analyzing the string equation with Dirichlet boundary control, that, surprisingly, there exist time optimal controls which are not of maximal norm  $E$ .

**Keyword:** Wave equations, Optimal control, Open loop control systems, Point-to-point control, Reachable states, Norm-optimal controls, Minimal control time.

## 1 Introduction

Time optimal control problems have been intensively studied for finite dimensional systems showing that the optimal control satisfies a Pontryagin maximum

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principle, it is bang-bang and unique. For a survey of these results, we refer to the books [17] and [1] and to the original work by [2]. These results have been extended in [6] to infinite dimensional systems and reported in the books by [18] and [8].

Many new results have been obtained for parabolic type systems; see for instance [20], [23], [21] and [16]. However only few results exist for conservative systems and they only concern distributed controls; see for instance [7], [19] and [15, 14].

In all the above mentioned works, in order to ensure the existence of a time optimal control, the controls are assumed to be bounded in  $L^\infty$ . But for the wave equation, due to the finite velocity of propagation, the main difficulty arises from the fact that it is globally controllable only for a large enough control time.

In the present work, in order to analyse this delicate issue, we chose an Hilbertian approach and assume that the control is bounded in  $L^2$ . This simplification allows us to easily consider the case of boundary control operators. In section 4, we consider the example of the string equation with Dirichlet boundary control, where some computations are explicit.

According to [11] (Theorem 3.1), the string equation with Dirichlet boundary control cannot be controlled with classical bang-bang controls, i.e. controls taking their values in  $\{-1, 1\}$  for almost every time. In addition, for norm optimal control problems, which is a problem related to the one of finding time optimal controls as we will see later, [3] consider a string equation with Newman control at both ends and prove that for constant state targets (with constant initial data), the time optimal controls are of bang-off-bang type, i.e. controls taking values in  $\{-1, 0, 1\}$  for almost every time. Thus, even for constant data, time optimal controls are not, in general, of bang-bang form. A more general result on  $L^\infty$ -norm optimal controls, for the same system, can be found in [9] and its generalisation to any  $L^p$ -norm optimal controls in [10].

In order to give a precise statement of our result, let us first recall some classical definitions and notations from control theory, see for instance [22].

Throughout this paper,  $X$  and  $U$  are real Hilbert spaces identified with their duals. We denote by  $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$  a strongly continuous semigroup on  $X$  generated by an operator  $A : \mathcal{D}(A) \rightarrow X$ . In all this paper, we assume that  $A$  is skew-adjoint with nonempty resolvent  $\rho(A)$ .

The notation  $X_1$  stands for  $\mathcal{D}(A)$  equipped with the norm  $\|z\|_{X_1} = \|(\beta \text{Id} - A)z\|_X$ , where  $\beta \in \rho(A)$  is fixed, while  $X_{-1}$  is the completion of  $X$  with respect to the norm  $\|z\|_{X_{-1}} = \|(\beta \text{Id} - A)^{-1}z\|_X$ . Let us also denote by  $A$  and  $\mathbb{T}$  the extensions of  $A$  to  $X$  and  $\mathbb{T}$  to  $X_{-1}$ .

Let us now introduce the control operator  $B \in \mathcal{L}(U, X_{-1})$ . Then the infinite dimensional system under consideration is:

$$y' = Ay + Bu \quad y(0) = y_0, \quad (1)$$

where  $y$  is the state of the system and  $u \in L^2(\mathbb{R}_+, U)$  is the control and  $y_0 \in X$  is the initial state position. The solution of (1) is:

$$y(t) = \mathbb{T}_t y_0 + \Phi_t u \quad (t \geq 0),$$

where  $\Phi_t \in \mathcal{L}(L^2([0, t], U), X_{-1})$  is the input to state map defined by:

$$\Phi_t u = \int_0^t \mathbb{T}_{t-s} B u(s) ds \quad (t \geq 0, u \in L^2([0, t], U)).$$

We will say that  $B$  is an *admissible control operator* for  $\mathbb{T}$  if there exists  $t > 0$  such that  $\text{Ran } \Phi_t \subset X$  and in the sequel we will assume that the pair  $(\mathbb{T}, B)$  satisfies this condition. Finally, we will say that the pair  $(A, B)$  is *exactly controllable in time  $T$*  ( $T > 0$ ) if  $\text{Ran } \Phi_T = X$ . In the sequel, we will assume that the pair  $(A, B)$  is exactly controllable in some time  $T > 0$  and we define the *universal controllability time*:

$$T_* = \inf\{T > 0, \text{Ran } \Phi_T = X\} \geq 0. \quad (2)$$

To be more precise, the time optimal control problem we address in this work is the following:

**Problem 1.** *Given  $E > 0$  and  $y_0, y_1 \in X$  with  $y_0 \neq y_1$ , find the minimal time  $T > 0$  such that there exists  $u \in L^2([0, T], U)$  satisfying:*

- $\|u\|_{L^2([0, T], U)} \leq E$ ;
- *the solution  $y$  of (1) with control  $u$  and initial condition  $y_0$  satisfies  $y(T) = y_1$ .*

In all this note,  $E$  defines a given nonnegative constant.

Our first result is as follows:

**Theorem 1.1.** *Let  $y_0, y_1 \in X$  with  $y_0 \neq y_1$ .*

*Assume that the pair  $(A, B)$  is exactly controllable and fix  $T > 0$ . Assume that a control  $u \in L^2([0, T], U)$  with  $\|u\|_{L^2([0, T], U)} \leq E$  steering  $y_0$  to  $y_1$  in time  $T$  exists.*

*Then there exists a minimal time  $\tau > 0$  such that  $y_0$  can be steered to  $y_1$  in time  $\tau = \tau(y_0, y_1; E)$  preserving this bound, i.e.*

$$\tau = \min \left\{ T > 0, \exists u \in L^2([0, T], U), \|u\|_{L^2([0, T], U)} \leq E \right. \\ \left. \text{and } \Phi_T u = y_1 - \mathbb{T}_T y_0 \right\}. \quad (3)$$

*Moreover, if  $\tau > T_*$  (with  $T_* \geq 0$  defined by (2)), there exists a unique control  $u \in L^2([0, \tau], U)$  with  $\|u\|_{L^2([0, \tau], U)} \leq E$  steering  $y_0$  to  $y_1$  in time  $\tau$ . In addition, we have*

$$\|u\|_{L^2([0, \tau], U)} = E \quad (4)$$

*and there exists  $\eta \in X \setminus \{0\}$  such that:*

$$u = \Phi_\tau^* \eta. \quad (5)$$

Let us remind that for every  $\eta \in X$ ,  $(\Phi_T^* \eta)(t) = B^* z(t)$  ( $t \in [0, T]$ ) where  $z$  is solution of:

$$z' = -A^* z, \quad z(T) = \eta.$$

The proof of the characterization of the optimal control Theorem 1.1 when  $\tau > T_*$  is similar to the one by [19, Theorem 1.4] and is not repeated here. We only give the key argument for the existence of  $\tau$  in section 2, see Proposition 2.1.

**Remark 1.1.** *If  $\tau > T_*$  the minimal time control is the minimal norm control in time  $\tau$  steering  $y_0$  to  $y_1$ . That is to say that, if  $\tau > T_*$ , the time optimal control is  $L^2([0, \tau], U)$ -norm optimal. This fact gives the same result as the one in [24] for the heat equation, where we have  $T_* = 0$ .*

Theorem 1.1 does not give any relevant information when  $\tau \leq T_*$ . In fact when  $\tau \leq T_*$  the situation is less clear. In section 3, we show in Proposition 3.1, under suitable assumptions on the reachable set, that for  $\tau < T_*$  there exists a time optimal control  $u \in L^2([0, \tau], U)$  with  $\|u\|_{L^2([0, \tau], U)} < E$ . That is to say that, when  $\tau < T_*$ , there exist time optimal controls which do not satisfy the norm saturating property (4). This situation appears at least when  $y_0 = 0$  and it is a consequence of the following two properties of the reachable sets: They are closed and strictly increasing as a function of  $\bar{t} < T_*$ .

More precisely, the way we build a non saturation time optimal control is by choosing a target  $y_1 \in X$  so that  $y_1$  is accessible from 0 in a time  $\bar{t} > 0$  but not for times  $s < \bar{t}$ . In this case, it is clear that  $\tau(0, y_1; E) \geq \bar{t}$  whatever  $E > 0$  is. Choosing such a target  $y_1$  and choosing a constant  $E > 0$  large enough, we will obtain that  $\tau(0, y_1; E) = \bar{t}$  and the existence of a time optimal control whose norm is not  $E$ .

In section 4, we will show that the assumptions made in Proposition 3.1 are fulfilled for the string equation with Dirichlet boundary control.

## 2 Well posedness

In this paragraph, we will prove that  $\tau$  defined by (3) exists, i.e., the set

$$\{T > 0, \exists u \in L^2([0, T], U), \|u\|_{L^2([0, T], U)} \leq E \text{ and } y_1 - \mathbb{T}_T y_0 = \Phi_T u\}$$

admits a minimum.

Before going further, let us introduce some ad hoc notations and spaces. Let us define the set of points which can be reached from 0,

$$R_t^2 = \Phi_t(L^2([0, t], U)) \quad (t > 0), \quad (6)$$

with the convention  $R_0^2 = \{0\}$ .

Endowed with the norm:

$$\|y\|_{R_t^2} = \inf \{ \|u\|_{L^2([0, T], U)}, u \in L^2([0, T], U), y = \Phi_t u \},$$

it constitutes a Banach space.

We also define the closed ball of radius  $E$  in  $R_t^2$ :

$$B_t^2(E) = \{ y \in R_t^2, \|y\|_{R_t^2} \leq E \} \quad (t > 0).$$

Using these notations, we have:

$$\tau(y_0, y_1; E) = \inf \{T > 0, y_1 - \mathbb{T}_T y_0 \in B_T^2(E)\}. \quad (7)$$

By linearity of  $\Phi_t$ , for every  $E > 0$  and  $t > 0$ ,  $B_t^2(E)$  is a convex set of  $X$ .

Let us first give a basic property on the sets  $R_t^2$ .

**Lemma 2.1.** *Let  $0 < s \leq t$ , then we have*

$$R_t^2 = \mathbb{T}_s R_{t-s}^2 + R_s^2,$$

and the inclusions  $R_s^2 \subset R_t^2 \subset X$  are continuous.

In addition, the universal time  $T_*$  defined by (2) is given by

$$T_* = \inf \{T > 0, R_T^2 = X\}.$$

By a simple adaptation of [8] (Lemma 3.1.1 and Theorem 3.1.2), we obtain:

**Proposition 2.1.** *Let  $y_0, y_1 \in X$  with  $y_0 \neq y_1$  and assume that a time  $T > 0$  and a control  $u \in L^2([0, T], U)$  with  $\|u\|_{L^2([0, T], U)} \leq E$  steering  $y_0$  to  $y_1$  in time  $T$  exist.*

*Then the set  $\{T > 0, y_1 - \mathbb{T}_T y_0 \in B_T^2(E)\}$  admits a minimum  $\tau$ .*

This result ensures the existence of a time optimal control  $u \in L^2([0, \tau], U)$  whatever the minimal time  $\tau > 0$  is.

### 3 Lack of norm optimality when $\tau \leq T_*$

**Proposition 3.1.** *Assume that  $T_*$ , defined by (2), is positive.*

*Let  $0 < \bar{t} \leq T_*$  and assume there exists  $y_1 \in R_{\bar{t}}^2$  such that,*

$$y_1 \notin R_s^2 \quad (s \in (0, \bar{t})). \quad (8)$$

*Then for every  $\gamma \in \mathbb{R}^*$ , the minimal time  $\tau = \tau(0, \gamma y_1; E)$  (defined by (3)) needed to steer 0 to  $\gamma y_1$  is greater than  $\bar{t}$ .*

*In addition, for  $|\gamma| > 0$  small enough, we have  $\tau(0, \gamma y_1; E) = \bar{t}$  and there exists a time optimal control  $u \in L^2([0, \bar{t}], U)$  such that  $\|u\|_{L^2([0, \bar{t}], U)} < E$ .*

*Proof.* Since  $R_{\bar{t}}^2$  is a vector space, it is clear that for every  $\gamma \in \mathbb{R}$ ,  $\gamma y_1 \in R_{\bar{t}}^2$  and, similarly,  $\gamma y_1 \notin R_s^2$  for every  $s \in (0, \bar{t})$ . Consequently,  $\tau(0, \gamma y_1; E) \geq \bar{t}$ .

Since  $y_1 \in R_{\bar{t}}^2$ , for every  $\varepsilon > 0$ , there exists  $\bar{u}_\varepsilon \in L^2([0, \bar{t}], U)$  such that  $y_1 = \Phi_{\bar{t}} \bar{u}_\varepsilon$  and  $\|\bar{u}_\varepsilon\|_{L^2([0, \bar{t}], U)} \leq \|y_1\|_{R_{\bar{t}}^2} + \varepsilon$ . For every  $\gamma \in \mathbb{R}$ , it is obvious that  $\gamma y_1 = \Phi_{\bar{t}}(\gamma \bar{u}_\varepsilon)$  and  $\|\gamma \bar{u}_\varepsilon\|_{L^2([0, \bar{t}], U)} \leq \|\gamma y_1\|_{R_{\bar{t}}^2} + |\gamma| \varepsilon$ . Consequently, for

$$0 < |\gamma| < \frac{E}{\|y_1\|_{R_{\bar{t}}^2} + \varepsilon},$$

we found a control  $u = \gamma \bar{u}_\varepsilon \in L^2([0, \bar{t}], U)$  steering 0 to  $\gamma y_1$  satisfying  $\|u\|_{L^2([0, \bar{t}], U)} < E$ .

This ends the proof, since  $\tau(0, \gamma y_1; E) \geq \bar{t}$ .  $\square$

**Remark 3.1.** *With the assumptions of Proposition 3.1 and if, in addition,  $\text{Ker } \Phi_{\bar{t}} \neq \{0\}$ , then, for  $|\gamma| > 0$  small enough, there exists an infinite number of time optimal controls steering 0 to  $\gamma y_1$ .*

Let us also notice that with some closure property on  $R_t^2$ , there exists a target  $y_1$  such that the assumption (8) holds.

**Lemma 3.1.** *Assume that for every  $t > 0$ ,  $R_t^2$  is a closed set in  $X$  and  $T_*$  defined by (2) is nonnegative.*

*Then, there exists  $y_1 \in X \setminus \{0\}$ , such that the set  $\{t > 0, y_1 \in R_t^2\}$  admits a minimum.*

*Proof.* For every  $y \in X \setminus \{0\}$  and every  $t > 0$ , let us define  $f_y(t) = \sup_{\varphi \in R_t^2} \frac{\langle y_1, \varphi \rangle}{\|y_1\|_X \|\varphi\|_X}$ .

We have  $f_y(t) \in [0, 1]$  and for every  $t > T_*$ ,  $f_y(t) = 1$ . Thus,  $t_y = \inf \{t > 0, f_y(t) = 1\}$  exists.

By contradiction, assume that  $t_y = 0$ , that is to say that for every  $t > 0$  and every  $y \in X \setminus \{0\}$ ,

$$\sup_{\varphi \in R_t^2} \frac{\langle y_1, \varphi \rangle}{\|y_1\|_X \|\varphi\|_X} = 1$$

and hence,  $R_t^2$  is dense in  $X$ . Finally, using the closure property of  $R_t^2$ , we obtain  $R_t^2 = X$  for every  $t > 0$  and this is in contradiction with  $T_* > 0$ .

Consequently, there exists  $y_1 \in X \setminus \{0\}$  such that  $t_{y_1} > 0$ . In particular, there exists  $\varepsilon_0 > 0$  such that  $f_{y_1}(\varepsilon_0) < 1$  and, consequently,  $y_1 \notin R_{\varepsilon_0}^2$  and since  $R_{\varepsilon_0}^2$  is a closed set, we can choose  $y_1 \in X \setminus \{0\}$  such that  $y_1 \in (R_{\varepsilon_0}^2)^\perp$ . In addition, for every  $\varepsilon < \varepsilon_0$ , using  $R_\varepsilon^2 \subset R_{\varepsilon_0}^2$ , we have  $y_1 \in (R_\varepsilon^2)^\perp$ .

Due to the closure property, to end the proof, we only need to prove that  $f_{y_1}(t_{y_1}) = 1$ .

For every  $0 < \varepsilon \leq \varepsilon_0$ , we have:

$$1 = f_{y_1}(t_{y_1} + \varepsilon) = \sup_{\varphi \in R_{t_{y_1} + \varepsilon}^2} \frac{\langle \varphi, y_1 \rangle}{\|\varphi\|_X \|y_1\|_X}.$$

But, according to Lemma 2.1,

$$\begin{aligned} 1 = f_{y_1}(t_{y_1} + \varepsilon) &= \sup_{\substack{\varphi_1 \in R_{t_{y_1}}^2 \\ \varphi_0 \in R_\varepsilon^2}} \frac{\langle \mathbb{T}_\varepsilon \varphi_1 + \varphi_0, y_1 \rangle}{\|\mathbb{T}_\varepsilon \varphi_1 + \varphi_0\|_X \|y_1\|_X} \\ &= \sup_{\substack{\varphi_1 \in R_{t_{y_1}}^2 \\ \varphi_0 \in R_\varepsilon^2}} \frac{\langle \mathbb{T}_\varepsilon \varphi_1, y_1 \rangle}{\|\mathbb{T}_\varepsilon \varphi_1 + \varphi_0\|_X \|y_1\|_X} \\ &\leq \sup_{\varphi_1 \in R_{t_{y_1}}^2} \frac{\langle \mathbb{T}_\varepsilon \varphi_1, y_1 \rangle}{\|\mathbb{T}_\varepsilon \varphi_1\|_X \|y_1\|_X} \\ &\leq \sup_{\varphi_1 \in R_{t_{y_1}}^2} \frac{\langle \varphi_1, \mathbb{T}_\varepsilon^* y_1 \rangle}{\|\varphi_1\|_X \|y_1\|_X} = f_{\mathbb{T}_\varepsilon^* y_1}(t_{y_1}). \end{aligned}$$

Using that  $\lim_{\varepsilon \rightarrow 0} \mathbb{T}_\varepsilon^* y_1 = y_1$ , it is easy to see that  $\lim_{\varepsilon \rightarrow 0} f_{\mathbb{T}_\varepsilon^* y_1}(t_{y_1}) = f_{y_1}(t_{y_1})$ . Consequently, we have proved that  $f_{y_1}(t_{y_1}) = 1$  and this ends the proof.  $\square$

In the next section, we show that the assumptions of Proposition 3.1 are fulfilled for the string equation with Dirichlet boundary control.

## 4 Time optimal control of the string equation

In this section, we focus on the string equation with Dirichlet boundary control:

$$\ddot{w}(t, x) = \partial_x^2 w(t, x) \quad (t > 0, x \in (0, 1)), \quad (9a)$$

$$w(t, 0) = u(t) \quad (t > 0), \quad (9b)$$

$$w(t, 1) = 0 \quad (t > 0). \quad (9c)$$

This system is supplemented with initial conditions:

$$w(0, x) = w_0(x) \quad \text{and} \quad \dot{w}(0, x) = w_1(x) \quad (x \in (0, 1)). \quad (9d)$$

In this paragraph, we chose a Dirichlet boundary control at one end, but similar results could have been obtained with Newman or Dirichlet boundary control at one or both ends.

Using the formal representation introduced in section 1, see also § 10.3 by [22], we have  $\begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix} = \mathbb{T}_t \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} + \Phi_t u$  with the operator  $A$  defined by:

$$\begin{aligned} A : \mathcal{D}(A) &\rightarrow X := L^2(0, 1) \times H^{-1}(0, 1) \\ \begin{bmatrix} f \\ g \end{bmatrix} &\mapsto \begin{bmatrix} 0 & \text{Id} \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{with } \mathcal{D}(A) = H_0^1(0, 1) \times L^2(0, 1), \end{aligned}$$

where  $A_0$  is the operator with values in  $L^2(0, 1)$  defined by:

$$\mathcal{D}(A_0) = H^2(0, 1) \cap H_0^1(0, 1) \quad \text{and} \quad A_0 f = -\frac{d^2}{dx^2} f.$$

According to [22], see § 11.6.2,  $A$  can be seen has a skew-adjoint operator on  $X$ .

The control operator  $B$  is given by its adjoint:

$$\begin{aligned} B^* : \mathcal{D}(A) &\rightarrow \mathbb{R} \\ \begin{bmatrix} \varphi \\ \psi \end{bmatrix} &\mapsto \frac{d}{dx} (A_0^{-1} \psi) \Big|_{x=0}. \end{aligned}$$

For  $T > 0$ , let us define:

$$(\Psi_T z)(t) = B^* \mathbb{T}_t z \quad (t \in [0, T], z \in \mathcal{D}(A)).$$

We have [22, Proposition 4.4.1],

$$\Psi_T^* = \Phi_T \mathcal{R}_T, \quad (10)$$



with  $\mathfrak{H}_T \in \mathcal{L}(L^2(0, T))$  defined by  $\mathfrak{H}_T u(t) = u(T - t)$ .

For every  $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathcal{D}(A) = H_0^1(0, 1) \times L^2(0, 1)$ , we have:

$$(\Psi_{Tz})(t) = \left. \frac{d}{dx} \zeta(t) \right|_{x=0}, \quad (11)$$

where  $\zeta$  is solution of:

$$\ddot{\zeta}(t, x) = \partial_x^2 \zeta(t, x) \quad ((t, x) \in (0, T) \times (0, 1)), \quad (12a)$$

$$\zeta(t, 0) = \zeta(t, 1) = 0 \quad (t \in [0, T]), \quad (12b)$$

$$\zeta(0, \cdot) = A_0^{-1} z_2 \in H_0^1(0, 1) \quad \text{and} \quad \dot{\zeta}(0, \cdot) = -z_1 \in L^2(0, 1). \quad (13)$$

Decomposing any solution of (12) in Fourier series and using Ingham inequalities (see [12] and, for result related to control theory, see for instance [13] and § 8.1 of [22]), we obtain that the pair  $(A^*, B^*)$  is exactly observable in any time  $T > T_* = 2$ , i.e., there exists  $c(T) > 0$  such that:

$$c(T) \|\Psi_{Tz}\|_{L^2(0, T)}^2 \geq \|z\|_X^2 \quad (z \in X).$$

In addition, the observability constant  $c(T)$  is of order  $\frac{1}{T}$  when  $T$  goes to infinity. Thus for every  $T > 2$ , using a HUM control, there exists a control with  $L^2(0, T)$ -norm of order  $\frac{1}{\sqrt{T}}$  when  $T$  goes to infinity [25, Remark 2.3].

We sum up the above remarks in the following lemma:

**Lemma 4.1.** *For every  $y_0, y_1 \in X$  and  $E > 0$ , there exists  $T > 0$  large enough such that  $y_1 - \mathbb{T}_T y_0 \in B_T^2(E)$ .*

The above lemma combined with Proposition 2.1 ensures that a time optimal control steering  $y_0$  to  $y_1$  exists.

For this particular control system, with the help of D'Alembert formula, it is possible to compute the reachable space  $R_T^2$  for all times  $T$  (including  $T \leq T_* = 2$ ). To this end, let us first compute the kernel of  $\Psi_T$ .

**Lemma 4.2.**

- If  $T \in (0, 1)$ ,

$$\text{Ker } \Psi_T = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in L^2(0, 1) \times H^{-1}(0, 1), \right. \\ \left. \mathbf{1}_{[0, T]} \left( z_1(x) - \frac{d}{dx} (A_0^{-1} z_2) \right) = 0 \right\};$$

- If  $T \in [1, 2)$ ,

$$\text{Ker } \Psi_T = \left\{ \begin{bmatrix} \frac{d}{dx} (A_0^{-1} z_2) \\ z_2 \end{bmatrix}, z_2 \in H^{-1}(0, 1), z_2 \mathbf{1}_{[2-T, 1]} = 0 \right\};$$

- If  $T \in [2, \infty)$ ,  $\text{Ker } \Psi_T = \{0\}$ .

*Proof.*  $\text{Ker } \Psi_T$  is the set of points  $(-\zeta_2, A_0\zeta_1) \in L^2(0, 1) \times H^{-1}(0, 1)$ , where  $(\zeta_1, \zeta_2) \in H_0^1(0, 1) \times L^2(0, 1)$  are such that the solution  $\zeta$  of (12), with initial conditions

$$\zeta(0, \cdot) = \zeta_1 \quad \text{and} \quad \dot{\zeta}(0, \cdot) = \zeta_2,$$

satisfies

$$\partial_x \zeta(t, \cdot)|_{x=0} = 0 \quad (t \in [0, T] \text{ a.e.}).$$

It is natural to extend  $\zeta_1$  and  $\zeta_2$  as odd functions on  $[-1, 1]$  and then as 2-periodic functions on  $\mathbb{R}$ . Consequently, using D'Alembert formula, it follows that  $\zeta(t, x)$  is given by:

$$\zeta(t, x) = \frac{1}{2} \left( \zeta_1(x+t) + \zeta_1(x-t) + \int_{x-t}^{x+t} \zeta_2(s) ds \right).$$

From the above formula, we obtain:

$$\partial_x \zeta(t, \cdot)|_{x=0} = \zeta_1'(t) + \zeta_2(t). \quad (14)$$

Let us now compute the set  $\mathcal{K}_T$  of points  $(\zeta_1, \zeta_2)$  such that  $\partial_x \zeta(t, \cdot)|_{x=0} = 0$  for almost every  $t \in [0, T]$  for different values of  $T$ .

- If  $T \in (0, 1)$ ,

$$\mathcal{K}_T = \{(\zeta_1, \zeta_2) \in H_0^1(0, 1) \times L^2(0, 1), \forall x \in [0, T], \zeta_2(x) = -\zeta_1'(x)\};$$

- If  $T \in [1, 2)$ , set  $T = 1 + \theta$ , then  $(\zeta_1, \zeta_2) \in \mathcal{K}$  if and only if their extensions satisfy  $\zeta_2(s) = -\zeta_1'(s)$  for almost every  $s \in (0, 1 + \theta)$ . But, for almost every  $s \in [0, \theta]$ ,

$$\zeta_i(1+s) = -\zeta_i(1-s) \quad (i \in \{1, 2\}).$$

This implies that:

$$\mathcal{K}_T = \{(\zeta_1, -\zeta_1'), \zeta_1 \in H_0^1(0, 1), \forall x \in [2-T, 1], \zeta_1(x) = 0\};$$

- If  $T \geq 2$ ,  $\mathcal{K}_T = \{0\}$ .

From  $\mathcal{K}_T$ , we easily obtain  $\text{Ker } \Psi_T$ . This ends the proof.  $\square$

By duality, the reachable state  $R_T^2$  is:

**Proposition 4.1.**

- If  $T \in (0, 1]$ ,

$$R_T^2 = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in L^2(0, 1) \times H^{-1}(0, 1), \mathbf{1}_{[T, 1]} f = \mathbf{1}_{[T, 1]} \frac{d}{dx} (A_0^{-1} g) = 0, \right. \\ \left. \mathbf{1}_{[0, T]} \left( f - \frac{d}{dx} (A_0^{-1} g) \right) = 0 \right\}.$$

- If  $T \in (1, 2]$ ,

$$R_T^2 = \left\{ \begin{bmatrix} f \\ g \end{bmatrix} \in L^2(0, 1) \times H^{-1}(0, 1), \mathbf{1}_{[0, 2-T]} \left( f - \frac{d}{dx} (A_0^{-1} g) \right) = 0 \right\}.$$

- If  $T > 2$ ,  $R_T^2 = X = L^2(0, 1) \times H^{-1}(0, 1)$ .

*Proof.* According to (10),  $R_t^2 = \text{Ran } \Phi_t$  is equal to  $\text{Ran } \Psi_t^*$  (since  $\mathfrak{A}_t$  is a unitary operator). If  $\Psi_T$  is coercive on  $X/\text{Ker } \Psi_T$  (with the quotient norm) then,

$$\text{Ran } \Psi_T^* = (\text{Ker } \Psi_T)^\perp.$$

Let us then show that  $\Psi_T$  is coercive on  $X/\text{Ker } \Psi_T$ .

To this end, define the two semi-norms on  $X = L^2(0, 1) \times H^{-1}(0, 1)$ :

$$\begin{aligned} N_1(z_1, z_2) &= \left\| z_1 + \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(0, 1)} \\ \text{and } N_2(z_1, z_2) &= \left\| z_1 - \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(0, 1)} \quad ((z_1, z_2) \in X). \end{aligned}$$

Clearly, for every  $(z_1, z_2) \in X$ ,

$$N_1(z_1, z_2)^2 + N_2(z_1, z_2)^2 = 2 \left( \|z_1\|_{L^2(0, 1)}^2 + \|z_2\|_{H^{-1}(0, 1)}^2 \right) = 2\|(z_1, z_2)\|_X^2.$$

For every  $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in X$ , using (11) and (14), we have:

$$\|\Psi_T z\|_{L^2(0, T)}^2 = \int_0^T |\zeta_1'(x) + \zeta_2(x)|^2 dx,$$

with  $\zeta_1$  (resp.  $\zeta_2$ ) defined by  $\zeta_1 = A_0^{-1} z_2$  (resp.  $\zeta_2 = -z_1$ ) on  $(0, 1)$  and extend as an odd function on  $[-1, 1]$  and then extended by 2-periodicity on  $\mathbb{R}$ .

Thus,

- if  $T \in (0, 1]$ , we have,

$$\|\Psi_T z\|_{L^2(0, T)}^2 = \left\| \frac{d}{dx} (A_0^{-1} z_2) - z_1 \right\|_{L^2(0, T)}^2$$

and

$$\begin{aligned} 2\|z\|_{X/\text{Ker } \Phi_T}^2 &= 2 \inf_{\tilde{z} \in \text{Ker } \Phi_T} \|z + \tilde{z}\|_X^2 \\ &= \inf_{\tilde{z} \in \text{Ker } \Phi_T} \left( \left\| z_1 + \tilde{z}_1 + \frac{d}{dx} (A_0^{-1} (z_2 + \tilde{z}_2)) \right\|_{L^2(0, 1)}^2 \right. \\ &\quad \left. + \left\| z_1 + \tilde{z}_1 - \frac{d}{dx} (A_0^{-1} (z_2 + \tilde{z}_2)) \right\|_{L^2(0, 1)}^2 \right) \\ &= \inf_{\tilde{z}_1 \in L^2(0, T)} \left\| 2\tilde{z}_1 + z_1 + \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(0, T)}^2 \\ &\quad + \left\| z_1 - \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(0, T)}^2 \\ &= \|\Psi_T z\|_{L^2(0, T)}^2. \end{aligned}$$

- if  $T \in (1, 2]$ , we have,

$$\begin{aligned}\|\Psi_{TZ}\|_{L^2(0,T)}^2 &= \int_0^1 |\zeta_1'(x) + \zeta_2(x)|^2 dx + \int_1^T |\zeta_1'(x) + \zeta_2(x)|^2 dx \\ &= \int_0^1 |\zeta_1'(x) + \zeta_2(x)|^2 dx + \int_{2-T}^1 |\zeta_1'(x) - \zeta_2(x)|^2 dx,\end{aligned}$$

that is to say,

$$\|\Psi_{TZ}\|_{L^2(0,T)}^2 = \left\| z_1 - \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(0,1)^2}^2 + \left\| z_1 + \frac{d}{dx} (A_0^{-1} z_2) \right\|_{L^2(2-T,1)}^2$$

and, as for the previous item, we obtain:

$$2\|z\|_{X/\text{Ker } \Psi_T}^2 = \|\Psi_{TZ}\|_{L^2(0,T)}^2.$$

- if  $T > 2$ , we have  $\text{Ker } \Psi_T = \{0\}$  and

$$\|\Psi_{TZ}\|_{L^2(0,T)}^2 \geq \|\Psi_{TZ}\|_{L^2(0,2)}^2 = 2\|z\|_X^2.$$

All in all, we have proved that, for every  $T > 0$ ,

$$\|\Psi_{TZ}\|_{L^2(0,T)} \geq 2\|z\|_{X/\text{Ker } \Psi_T} \quad (z \in X).$$

Consequently,  $R_T^2 = (\text{Ker } \Psi_T)^\perp$  and a straight forward computation leads to the result.  $\square$

Consequently, by considering for instance the target

$$y_1 = \left[ \begin{array}{c} (-2x + 1)\mathbf{1}_{[0,\bar{t}]}(x) \\ 2 \end{array} \right],$$

with  $\bar{t} \in (0, 1)$ , we have  $y_1 \in R_{2-\bar{t}}^2$  and  $y_1 \notin R_s^2$  for every  $s < 2 - \bar{t}$ .

Thus, using Proposition 3.1, there exists  $\gamma \neq 0$ , such that the minimal time  $\tau(0, \gamma y_1; E)$  needed to steer 0 to  $\gamma y_1$  is  $2 - \bar{t}$  and in addition, there exists a control  $\bar{u} \in L^2(0, 2 - \bar{t})$  steering 0 to  $\gamma y_1$  in time  $\bar{t}$  together with  $\|\bar{u}\|_{L^2(0, 2 - \bar{t})} < E$ .

## 5 Concluding remarks

In this note we proved, on the example of the string equation with Dirichlet boundary control, that there exist time optimal controls with  $L^2$ -norm strictly lower than  $E$ .

More generally, if,  $\tau > T_*$ , time optimal controls are  $L^2([0, \tau], U)$ -norm optimal controls, but when  $\tau < T_*$ , time optimal controls and  $L^2([0, \tau], U)$ -norm optimal controls can differ. In particular, in the situation of Remark 3.1, there is an infinite number of time optimal controls which are not norm optimal ones.

This result is due to the fact that the reachable space is not the full space in arbitrary small time. This is a great difference between Schrödinger and wave

systems from the point of view of time optimal controllability. For the first one exact controllability holds in all time and, therefore, the reachable space is the full space for all time and, in particular, time independent, something that does not occur for the wave equation.

One important question which is not addressed here, and that constitutes still and interesting open problem, is the effective numerical computation of the minimal control time and of time optimal controls. In fact, this issue seems to be highly complex.

For instance for the string equation with boundary control, in order to compute time optimal controls, we should design a numerical method able to build controls even for times  $t < T_*$  and, as far as we know, this has not been done so far.

Indeed, there has been extensive work done on the development of efficient numerical methods for approximating wave control problems. But this has been always done beyond the universal control time  $T_*$ , i.e. for time intervals  $[0, T]$  with  $T > T_*$ . In fact, the existing methods try to mimic the behavior of the continuous wave equation and, in particular, its finite velocity of propagation property and, therefore, do not allow yielding any effective approximation method to build controls, specific to given data, when the control time is strictly smaller than  $T_*$ . A survey of these methods can be found in [5]. Another approach, based on Russell's principle, "stabilization implies controllability", can be found in [4]. But this method also needs the control time to be larger than the universal controllability time  $T_*$ .

In addition, with the result of Proposition 4.1, we have a lower bound on the minimal time needed to steer 0 to a given final target. But we do not have a lower bound on the minimal time needed to steer a non null starting point to a given target. Further developments in this direction could be derived out of Theorem 2.1 of [11].

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