A Kleene theorem for Petri automata
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Abstract
While studying the equational theory of Kleene Allegories (KAI), we recently proposed two ways of defining sets of graphs [BP15]: from KAI expressions, that is, regular expressions with intersection and converse; and from a new automata model, Petri automata, based on safe Petri nets. To be able to compare the sets of graphs generated by KAI expressions, we explained how to construct Petri automata out of arbitrary KAI expressions.

In the present paper, we describe a reverse transformation: recovering an expression from an automaton. This has several consequences. First, it generalises Kleene theorem: the graph languages specified by Petri automata are precisely the languages denoted by KAI expressions. Second, this entails that decidability of the equational theory of Kleene Allegories is equivalent to that of language equivalence for Petri automata. Third, this transformation may be used to reason syntactically about the occurrence nets of a safe Petri net, provided they are parallel series.

Keywords regular expressions, graph language, Petri nets, finite automata, allegories, intersection, converse, Kleene theorem

1. Introduction
Petri Automata are Petri net-based automata, and are equipped with an operational semantics allowing them to recognise sets of graphs. They were introduced in a recent paper [2] to study Kleene Allegory expressions: terms built from a finite alphabet of variables, with the constants 0 and 1, the unary operators converse and Kleene star, and the binary operators union, composition and intersection. To any such expression, one can associate a set of labelled directed graphs with two distinguished vertices understood as input and output. A letter $a$ is associated to a singleton set containing the following graph:

![Graph Example](attachment://graph.png)

The input in this graph is the vertex with the unmarked incoming arrow, and the output is the other vertex, with the outgoing arrow. Similarly the constant 1 yields a singleton set containing the graph where the input is linked to the output by an edge labelled 1. We define the sequential composition of two such graphs by identifying the output of the first graph with the input of the second one. We also build the parallel composition (corresponding to the intersection operation) of two graphs by identifying both their inputs and their outputs. For instance $\mathcal{G}(a \cdot (b \cap 1))$ simply contains the graph:

![Graph Example](attachment://graph.png)

With these constructs we can define the set of graphs associated to an expression by structural induction: $\mathcal{G}(1)$ and $\mathcal{G}(a)$ are defined as before, $\mathcal{G}(e \cup f)$ is the union of $\mathcal{G}(e)$ and $\mathcal{G}(f)$; $\mathcal{G}(0)$ is the empty set; $\mathcal{G}(e \cdot f)$ (respectively $\mathcal{G}(e \cap f)$) is the set of graphs obtained by composing in sequence (resp. in parallel) a graph in $\mathcal{G}(e)$ with a graph in $\mathcal{G}(f)$; and finally the graphs in $\mathcal{G}(e^*)$ are either the graph of 1 or built by sequentially composing any finite number of graphs from $\mathcal{G}(e)$.

1.1 Expressions
For the bulk of the development, we restrict ourselves to a simpler setting: we consider expressions over the syntax of Kleene lattices, that is to say without the converse operation. To any such expression $e$, we associate a set $\mathcal{G}(e)$ of labelled directed graphs with two distinguished vertices understood as input and output. A letter $a$ is associated to a singleton set containing the following graph:

![Graph Example](attachment://graph.png)

In the remainder of this introduction, we explain informally what are the graphs associated with expressions and Petri automata and we give the outline of the computation of the expression corresponding to a given automaton.

1.2 Petri automata
A Petri automaton over the alphabet $\Sigma$ is defined by:

- a finite set of places (denoted in graphical representations by round vertices),
- a finite set $\mathcal{T}$ of transitions (denoted by rectangular vertices). Each transition is equipped with a set of input places (incoming arrows) and a set of outputs. These outputs are pairs of a label in $\Sigma \cup \{1\}$, and a place.
- an initial place and a finite set of final configurations, a configuration being a set of places.

Here is an example of Petri automaton:
The initial place is $A$, as denoted by the unmarked incoming arrow. There is a single final configuration, containing only the place $F$ (as indicated by the red dotted rectangle). In this paper, we will restrict ourselves to automata having a single final configuration, composed of one place.

Runs in such an automaton are plays of a token firing game: one starts by placing tokens on certain places, and if every place in the input of a transition holds a token, this transition is enabled and may be fired. Firing a transition will then remove one token from each of its inputs, and place one on each output. If the run starts with a single token on the initial place, we call it initial, and if it stops with one token on each place of a final configuration it is final. An accepting run is both initial and final. For instance, here is a graphical representation of an accepting run in the previous automaton:

We further impose that Petri automata should be safe, meaning that at any step of any initial run of the automaton at most one token is present on each place. The last step to get the language of an automaton is to generate a graph from any run of the automaton (called the trace of the run). We do so by considering a graph whose vertices are the transitions (plus an additional final vertex), and labelled edges are extrapolated from the outputs of the transitions. For instance the run from the previous example yields the graph:

We may compose these boxes by merging the outputs of the first box with the inputs of the second box, and reducing the graph by applying the operators $\cdot$ and $\cap$ on edges. For instance the first two boxes above compose to produce the box $\beta$, and the whole sequence can be composed as the box $\gamma$:

Types in this setting are trees with leaves labelled with places. We can type a box $\beta$ with a pair of trees $\sigma \rightarrow \tau$ if the tree $\sigma$ composed with the box $\beta$ yields the tree $\tau$. For instance the box $\beta$ has the type:

We build a finite state automata whose states are the types, and whose transitions between the states $\sigma$ and $\tau$ are labelled with the boxes built from transitions of the Petri automaton with type $\sigma \rightarrow \tau$. Using the standard Kleene Theorem, this allows us to produce a regular expression over boxes $e_{\sigma\tau}$. By changing the set of letters to finite sets of boxes labelled with expressions, we are then able to internalise all of the regular operations. This produces in the end a finite number of boxes with one input (corresponding to the initial place) and one output (corresponding to the final place), linked by an edge labelled with an expression. The union of these expressions is the expression $e^{(\sigma\tau)}$, that denotes the same set of graphs as $\sigma\tau$.

The technical difficulty here is the computation of the star of a set of boxes. For instance, consider the box:

Its star yields all boxes of the following shape, for $n \in \mathbb{N}$:

This set of boxes cannot be represented by a finite number of boxes. However, one must realise that because the two branches of the box $\delta$ are disconnected, they must correspond to concurrent processes in the Petri automaton, and thus can be iterated separately. Intuitively, the net cannot force two independent processes to perform exactly the same number of iterations. Because of this, the following box is the right choice to represent the star of the box $\delta$:

1.3 Main result

The main technical result of this paper is a construction allowing one to build from an automaton $\sigma\tau$ an expression $e^{(\sigma\tau)}$ such that:

Before properly describing the construction, we need a compositional way to represent a local and partial view of a graph. This will be achieved through the notions of boxes and types. In the same way a sequence of letters may be a factor of some word, boxes are meant to represent “slices” of graphs. Formally, if $A$ and $B$ are sets of places, a box going from $A$ to $B$ is a labelled directed graph $G$ with maps from $A$ to the vertices of $G$ and from $B$ to the vertices of $G$. For instance, in the run presented before as an example, we may associate a box to each transition. This yields the following sequence of boxes:
about them. Section 4 is dedicated to the main contribution of the paper, the construction extracting expressions from Petri automata. Finally, we list some consequences of this result, as well as some links with previous work, in Section 5.

2. Petri automata & expressions

Regular expressions and rational languages. We recall a few standard definitions and notations about regular expressions. A regular expression over the alphabet $\Sigma$ is a term built from variables taken in $\Sigma$, the constants $1$ and $0$, the binary operators $\cdot$ and $\cup$, and the unary "$^*$". The set is denoted by $\text{Reg}(\Sigma)$. The language denoted by $\varepsilon$, written $[\varepsilon]$, is a set of words over $\Sigma$ defined inductively by:

$$
\begin{align*}
[0] & : = \{ \varepsilon \}; \\
[1] & : = \{ 1 \}; \\
\varepsilon \cdot [e] & : = \{ \varepsilon \cup f \} := \{ \varepsilon \cup [f] \}; \\
\varepsilon \cdot f & : = \{ u \cdot v \mid u \in [e] \text{ and } v \in [f] \}; \\
\varepsilon^* & : = \{ u_1 \cdot \ldots \cdot u_n \mid n \in \mathbb{N}, \forall i, u_i \in [e] \}.
\end{align*}
$$

($\varepsilon$ being the empty word.)

If $[e] = [f]$, we say that regular laws prove the equality of $e$ and $f$, written $KA := e = f$.

2.1 Expressions

Expressions $e, f, \ldots$ are built on the signature $(1, \cdot, \cup, n^*)$\(^1\). We denote by $\text{Reg}_e$ the set of expressions over the finite alphabet $\Sigma$. A 2-pointed labelled directed graph $G$ over $\Sigma_e = \Sigma \cup \{ 1 \}$, simply called graph in the sequel, is given by a tuple $(V, E, \iota, \omega)$ with a finite set of vertices $V$, a set of labelled edges $E \subseteq V \times \Sigma \times V$, and $\iota, \omega : V$ two distinguished vertices, respectively called input and output. The sequential composition (written $\cdot$) of two graphs with disjoint sets of vertices can be performed by identifying the output of the first graph and the input of the second one. Their parallel composition (written $\cup$) consists in merging their inputs and merging their outputs. See Figure 1 for a graphical description of these constructions.

We assign to each expression a set of such graphs, called the graph language of the expression.

Definition 2 (Graph language of an expression: $\mathcal{G}(e)$)
The graph language is defined by structural induction as follows:

$$
\begin{align*}
\mathcal{G}(a) & : = \{ (V, E, \iota, \omega) \mid \iota = a \} \\
\mathcal{G}(0) & : = \emptyset \\
\mathcal{G}(e \cdot f) & : = \mathcal{G}(e) \cup \mathcal{G}(f) \\
\mathcal{G}(e \cup f) & : = \{ (V, E) \mid E \cup f \in \mathcal{G}(e) \cup \mathcal{G}(f) \} \\
\mathcal{G}(e^*) & : = \{ (E_1 \cdot \ldots \cdot E_n \mid n > 0, \forall i, E_i \in \mathcal{G}(e) \}
\end{align*}
$$

Those graphs were introduced independently by Freyd and Scedrov [3, page 208], and Andréka and Bredikhin [1] (except for the way $1$ is handled). Notice that the graphs produced by this construction are exactly Two Terminal Series Parallel graphs labelled with $\Sigma_e$ [8].

2.2 Petri automata

A Petri automaton is a Petri net whose transition’s outputs are labelled by the set $\Sigma_e$.

\(^1\) Notice that we don’t use the Kleene star, but rather its positive variant. It is more convenient here to have $e^*$ be defined as $1 \cup e^*$.

\[G \cap F \Rightarrow \]
\[G \cdot F \Rightarrow \]

Figure 1: Elementary graph constructions

\[\text{Figure 2: A Petri automaton. The initial place is } A, \text{ and the final configurations is } \{ J \}.\]

Definition 3 (Petri Automaton)

A Petri automaton $\mathcal{A}$ over the alphabet $\Sigma$ is a tuple $(P, \mathcal{T}, \iota, \omega)$ where:

- $P$ is a finite set of places,
- $\mathcal{T} \subseteq \mathcal{P}(P) \times \mathcal{P}(\Sigma)$ is a set of transitions,
- \(\iota \in P\) is the initial place of the automaton,
- \(\omega \in P\) is the final place of the automaton.

For each transition $t = (t_1, t_2) \in \mathcal{T}$, $t_1$ and $t_2$ are assumed to be non-empty; $t_1 \subseteq P$ is the input of $t$; and $t_2 \subseteq P \times P$ is the output of $t$.

This definition of Petri automata correspond to what we call simple Petri automata in [2]. However, this case is general enough, as discussed in Section 5.1. We use the graphical notation from the introduction to represent Petri automata; the Petri automaton from Figure 2 will be used as a running example.

From a configuration $\xi \subseteq P$, a transition $t = (t_1, t_2) \in \mathcal{T}$ is enabled if $\xi \subseteq \mathcal{T}$. If so, one may fire $t$, which produces a new configuration $\xi'$ defined by

$$
\xi' = \xi \setminus t_1 \cup \{ p \in P \mid \exists x \in \Sigma : (x, p) \in t_2 \}.
$$

We write $\xi \xrightarrow{t} \xi'$ in this case.

In the sequel, we assume all considered Petri automata to be safe. (i.e., in Petri nets terminology, such that any reachable marking has at most one token in each place [6]). Formally, with our definitions: a Petri automaton $(P, \mathcal{T}, \iota, \omega)$ is safe if for all configuration $\xi \subseteq P$ reachable from $\iota$ by firing any number of transitions, if $(t, \xi) \in \mathcal{T}$ is enabled from $\xi$, $p \in \xi$, and $(x, p) \in t_2$, then $p \in \xi'$. Now we recall how to use Petri automata to define languages of graphs. We first define the runs of an automaton.

Definition 4 (Run, accepting run)

A run $\xi$ in $\mathcal{A} = (P, \mathcal{T}, \iota, \omega)$ is a sequence $((\xi_k)_{0 \leq k < n}, (t_k)_{0 \leq k < n})$ of configurations and transitions, such that $\xi_k \subseteq P$, $t_k \in \mathcal{T}$ and $\forall k < n, \xi_k \xrightarrow{t_k} \xi_{k+1}$. When $\xi_0 = \{ \iota \}$ and $\xi_n = \{ \omega \}$, we call $\xi$ an accepting run. (Note that a run $\xi$ is uniquely determined by $\xi_0$ and the sequence $(t_k)$: all subsequent configurations can be computed deterministically.)

Example 5

Consider the following sequence of transitions from the automaton
Let \( \xi = \{ (q_0, q_1, q_2, q_3, q_4, q_5) \} \), with
\[
\begin{align*}
q_0 & = \{ A \}, \\
q_1 & = \{ B, G \}, \\
q_2 & = \{ C, E, G \}, \\
q_3 & = \{ D, E, G \}, \\
q_4 & = \{ F, G \}, \\
q_5 & = \{ I \}.
\end{align*}
\]
We have:
\[
\begin{align*}
& \{ A \} \rightarrow \{ B, G \} \rightarrow \{ C, E, G \} \rightarrow \{ D, E, G \} \rightarrow \{ F, G \} \rightarrow \{ I \},
\end{align*}
\]
and since \( \{ A \} \) is the initial configuration and \( \{ I \} \) is the final one, this sequence is an accepting run. It can be represented graphically as in Figure 3.

In [2], the language of a Petri automaton is defined via an operational semantics, describing the way an automaton “reads” a graph. For the sake of clarity, we use here a different definition, more directly related to the set of accepting runs of the automaton. We associate to each accepting run \( \xi \) a 2-graph, written \( \mathcal{G}(\xi) \), called the trace of \( \xi \). The language of the automaton is then defined as the set of traces of its accepting runs.

The trace is constructed by creating a vertex \( k \) for each transition \( t_k = (t_{k-1}, t_k) \) of the run, plus a final vertex \( n \). We add an edge \((k, x, l)\) whenever there is some place \( q \) such that \( (x, q) \in t_{k-1} \), and \( t_k \) is the first transition after \( t_k \) in the run with \( q \) among its inputs, or \( l = n \) if there is no such transition in the run.

**Definition 6 (Trace of a run)**

Let \( \xi = \{ (t_{k0}, \ldots, t_{kn}) \in \mathcal{G} \} \) be run. For an index \( k \leq n \) and a place \( q \), let \( \nu(k, q) \) be either the smallest index \( l \) such that \( k \leq l \) and \( q \in \nu(k, q) \), or \( n \) if there is no such index.

The trace of \( \xi \) is the graph \( \mathcal{G}(\xi) = (\{0, \ldots, n\}, E_{\xi, 0, n}) \) with
\[
E_{\xi} = \{ (k, x, \nu(k + 1, q)) \mid (x, q) \in t_k \}. \tag{1}
\]

We write \( \mathcal{F}(\mathcal{A}) \) for the language of a Petri automaton \( \mathcal{A} \), defined as the set of traces of its accepting runs. The trace of the run presented in Figure 3 is depicted in Figure 4.

**Remark 7.** \( \mathcal{F}(\mathcal{A}) \) is different from \( \mathcal{L}(\mathcal{A}) \), as it is defined in [2]. See section Section 5.1 for a more detailed discussion.

**Theorem 8** ([2, Theorem 17]). For any expression \( e \in \mathcal{R}_{\mathcal{A}} \), there is a Petri automaton \( \mathcal{A}(e) \) such that
\[
\mathcal{F}(\mathcal{A}(e)) = \mathcal{F}(\mathcal{A}(\mathcal{A}(e))). \tag{2}
\]

## 2.3 SP-Petri automata

When trying to build expressions out of Petri automata, one immediately encounters a problem: traces of accepting runs are not necessarily series parallel graphs, which means they cannot be generated by any expression from \( \mathcal{R}_{\mathcal{A}} \). For instance the automaton given in Figure 6 has a single accepting run, whose trace is the graph drawn in Figure 5. To circumvent this problem, we consider only well-behaved automata: automata whose accepting runs have parallel-series graphs.

**Definition 9 (SP-automata)**

An automaton \( \mathcal{A} \) is said to be an SP-automaton if for any accepting run \( \xi \) in \( \mathcal{A} \), \( \mathcal{F}(\xi) \) is an SP-graph.

Despite its infinitary formulation, this property is decidable. In fact, the procedure we describe later on would fail (in finite time) if it is given a non-SP automaton: hence it may be used to decide this property.

### 3. Boxes

**Notations**

We call label space a structure \( (\mathcal{L}, \cdot, \cap) \) such that \( \cdot \) and \( \cap \) are two internal associative binary operations, and \( \cap \) is commutative. Examples of such spaces include:

\[\mathcal{W}_X : \text{the set of SP-graphs labelled by } X.\]
\[\mathcal{E}_X : \text{the set of expressions } \mathcal{R}_{\mathcal{A}}.\]

### 3.1 Graphs

We introduce a few definitions and results about graphs. Unless otherwise stated, all graphs are labelled with some fixed label space \( \mathcal{L} \).

\[\text{See Remark 18 for more details.}\]
DAGs and Series Parallel graphs Given a directed acyclic graph (DAG) \( G = (V, E) \), we may define \( \min G \) to be the set of vertices in \( V \) with no incoming edge, and \( \max G \) to be the set of vertices with no outgoing edge. A Two Terminal Parallel Series graph (SP-graph in the following) may be defined as a connected DAG that does not contain the graph from Figure 5 as a minor, and such that \( \min G \) and \( \max G \) are both singleton sets. A factor of \( (V, E) \) is a graph \( (V', E') \) such that:

- \( V' \subseteq V \) and \( E' \subseteq E \);
- \( e \in E' \) if and only if both its extremities are in \( V' \).

A prefix of \( (V, E) \) is a factor \( (V', E') \) such that if there is a path from \( x \in V \) to \( y \in V' \), then \( x \in V' \). When considering the graph as a partial order, a prefix is a downward closed set of vertices. Similarly, A suffix of \( (V, E) \) is an upward closed set, meaning a factor \( (V', E') \) such that if there is a path from \( x \in V' \) to \( y \in V \), then \( y \in V' \). Finally a slice of \( (V, E) \) is a convex set, i.e. a factor such that if there is a path from \( x \in V' \) to \( y \in V \) and from \( y \) to \( z \in V' \), then \( y \in V' \). Any slice may be obtained by removing from the graph a prefix and a suffix. A graph \( G \) is called an SP-slice if it is a slice of some SP-graph.

We define the SP-rewriting system, enriched with labels taken from a label space:

\[
\begin{align*}
\text{ } & \begin{array}{c}
\begin{array}{ccc}
a & \rightarrow & b
\end{array} \\
\begin{array}{ccc}
a \cap b & \rightarrow & a \cdot b
\end{array}
\end{array}
\end{align*}
\]

(This second rule can only be applied if the middle node on the left-hand side has no other adjacent edge.)

Given a DAG \( G \), we write \( \downarrow \) for the unique normal form of \( G \) with respect to this system. One may notice here that if \( G \) is a non-trivial SP-graph, then \( \downarrow \) has always the shape \( \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \circ \), and \( G \) is isomorphic to \( \triangle(e) \). This isomorphism means that we will often assimilate SP-graphs and terms built using the operators \( \cdot \) and \( \circ \).

Trees and bow ties A tree is a DAG \( T = (V, E) \) such that either \( \min T \) or \( \max T \) contains a single node called the root, and for any two nodes \( x, y \in V \) there exists at most one path from \( x \) to \( y \). If \( \min T \) is a singleton, then \( T \) is a top-down tree, otherwise it is a bottom-up tree. A tree is said to be proper if it does not contain any node with exactly one incoming edge and one outgoing edge. It is simple enough to check that there is only a finite number of proper trees with leaves chosen from a finite set.

A bow tie is a DAG \( G \) that does not contain the graph from Figure 5 as a minor and such that there is a node \( o \) such that every node in \( G \) either can reach \( o \) or is reachable from \( o \). By applying the SP-rewriting system to a bow tie, we get a bottom-up tree concatenated with a top-down tree, as illustrated in figure Figure 7 (hence the name). It follows easily that \( G \) is an SP-slice if and only if it is a disjoint union of bow ties.

3This is sometimes used as the definition of an SP-graph.

3.2 Categories of boxes

Definition 10 (Box)

A box from \( A \) to \( B \) labelled with a label space \( \mathbb{L} \) is a tuple \( \beta = (G, p, \bar{p}) \) where \( G \) is a DAG labelled with \( \mathbb{L} \), \( p : A \rightarrow \min G \) is a surjective map and \( \bar{p} : B \rightarrow \max G \) is a bijective map, respectively indicating the input ports and output ports. We write \( \beta : A \rightarrow B \) if \( \beta \) is a box from \( A \) to \( B \).

Examples of such boxes are given in Figures 10 and 11.

Definition 11 (Type)

A type over a set \( A \) is a structural \( \tau = (G_r, l_r) \) such that \( G_r \) is a proper top-down unlabelled tree and \( l_r \) is a bijection from \( A \) to \( \max G_r \). The set of types over subsets of a finite set \( P \) is written \( \mathbb{T}_P \).

Examples of such types are given in Figure 9.

Remark 12 In the sequel we reason about boxes and types modulo renaming of nodes.

A type \( \tau \) over \( A \) may be seen as a box from \( \{i\} \) to \( A \) labelled with \( \uplus \); we can build the box from \( \{i\} \) to \( A \) \( \downarrow \tau = (V_r, E_r, \{x \mapsto r_x\}, l_r), \) with \( E_r = \{(x, y) \mid (x, y) \in E_r\} \). Conversely, we may forget about the label information in a box: the erasing of a box \( \beta \) is the box \( \downarrow \beta \) where all labels are replaced by \( \uplus \). It is quite clear that for any type \( \tau, \downarrow \tau = \uplus \).

One of the most interesting features of boxes is their ability to compose. Let \( \beta = (G_1, p_1, \bar{p}_1) : A \rightarrow B \) and \( \gamma = (G_2, p_2, \bar{p}_2) : B \rightarrow C \). We define the relation \( \sim \) over \( G_1 \cup G_2 \) as the smallest equivalence relation containing all pairs \( x, y \) such that there is some \( b \in B \) such that \( p_1(b) = x \) and \( p_2(b) = y \). The composition of \( \beta \) and \( \gamma \), written \( \beta \circ \gamma \), is defined as \( (G_1 \circ \downarrow \beta \downarrow, \downarrow \gamma) : A \rightarrow C \), where:

- \( G \) is the quotient of the disjoint union of \( G_1 \) and \( G_2 \) by the relation \( \sim \);
- the function \( \bar{p} \) (respectively \( p \)) associates to \( x \in A \) (resp. \( B \)) the equivalence class of \( p_1(x) \) (resp. \( p_2(x) \)).

It can be checked that this composition operation is associative and that for any two boxes \( \beta : A \rightarrow B \) and \( \gamma : B \rightarrow C \), the following holds:

\[
[\beta \circ \gamma] = [\beta] \circ [\gamma].
\] (1)

This operation is illustrated in Figure 8.

Definition 13 (Typed boxes)

Let \( \beta : A \rightarrow B \) be a box over \( \mathbb{L} \), \( \sigma \) and \( \tau \) be types respectively over \( A \) and \( B \). \( \beta \) has the type \( \sigma \rightarrow \tau \) if the following holds:

\[
[\beta] \circ [\sigma] = [\tau].
\] (2)

We write \( \mathbb{B}_P^{\uplus, \mathbb{L}} \) for the set of boxes over \( \mathbb{L} \) of type \( \sigma \rightarrow \tau \), and \( \mathbb{B}_P^{\uplus} \) for the set of typed boxes over \( \mathbb{L} \) with types from \( \mathbb{T}_P \).

These boxes allow us to define the category \( \mathbb{B}^P \) of types and typed boxes over \( \mathbb{L} \) where:

- the objects are types;
3.3 Boxes over expressions

Consider boxes labelled with a label space $E_X$. The elements of $E_X$ are expressions taken from $\mathbb{Reg}^X$, and thus denote languages of SP-graphs, as defined earlier. This means that we can generate from such a box a set of boxes labelled with $\mathcal{H}_X$, by replacing each edge labelled with $e$ by a graph $G \in \mathcal{H}(e) \subseteq \mathcal{H}_X$.

For instance, the box in Figure 10 has type $\beta \cdot a \cdot b$, while the set of boxes generated by a box $B$ is displayed in Figure 11.

It is quite straightforward to check that for every accepting run $\beta \vdash_{\mathcal{E}_{\mathcal{A}}} o_{\mathcal{A}}$, the graph of a valid box has to be an SP-slice.

4. Automata and regular sets of boxes

4.1 A regular expression over boxes

It is then quite easy to generate from an SP-automaton $\mathcal{A}$ a regular expression over boxes $e \in \mathcal{E}_{\mathcal{A}}$ such that

$$\beta_1 \ldots \beta_n \in [e]_{\mathcal{A}} \iff \left\{ \begin{array}{l}
\beta_1 \otimes \cdots \otimes \beta_n = (G, \beta, \gamma) \quad \text{with} \quad G \in \mathcal{H}(\mathcal{A}) \\
\beta_1 \otimes \cdots \otimes \beta_n \text{ has type } (\mathcal{E}', \gamma) \quad \text{if} \quad (\mathcal{E}', \gamma) \in \mathcal{E}_{\mathcal{A}}.
\end{array} \right.$$
The property 2 allows us to check the other direction. The typing ensures that for any words \( u, v, w \) and \( u' \), if \( u, w \) compile to boxes with the same type then \( u \cdot w \cdot v \) is valid if and only if \( u \cdot u' \cdot v \) is valid as well, and they have the same type. Thus we know that \( e_{\text{af}}[f/g] \o \) only contain valid boxes of the same type as the ones from \( e_\text{af} \o \). As \( e_\text{af} \o \) is maximal for its type, it must contain \( e_{\text{af}}[f/g] \o \).

\[ \square \]

4.2 Compiling an expression over boxes into \( \text{Reg}^\circ_{S_S} \)

What remains to do is to transform this regular expression over boxes into an expression from \( \text{Reg}^\circ_{S_S} \). We do so by moving from typed boxes labelled with \( \nu_{S_S} \) to finite sets of boxes sharing a common type and labelled with \( \Sigma_{S_S} \), thus internalising the operators. By mapping each box \( \beta \) to \( \{ \beta \} \), we may translate this way the expression \( e_{\text{af}} \) obtained in the previous section to a new expression \( \tau_{\text{af}} \). The operator \( \o \) may be lifted here in a type preserving way, as a pointwise application:

\[ S \o S' = \{ B \o B' \mid B \in S, B' \in S' \} \]

We also extend the notion of boxes over \( \nu_{S_S} \) generated by a box over \( S_{S_S} \) to a set \( S \) simply by stating that \( \Sigma(S) = \bigcup_{B \in S} \Sigma(B) \).

Definition 20 (Valid expression)

An expression \( e \in \text{Reg}(P_f(\text{Reg}^\circ_{S_S})) \) is valid if all \( S_1 \cdots S_n \in [e] \), the composition \( S_1 \cdots \circ S_n = S \) is defined and \( \Sigma(S) \) only contains valid boxes.

The definition of \( [\ ] \o \) we had for valid expressions with letters in \( \text{Reg}^\circ_{S_S} \) can thus be salvaged for expressions with letters in \( P_f(\text{Reg}^\circ_{S_S}) \). Lemma 14 can be adapted to this setting:

Lemma 21. Let \( S_1 \) and \( S_2 \) be finite sets of boxes over \( E_{S_S} \), such that

1. \( \forall \beta \in \Sigma(S), \beta \) is a valid box with type \( \sigma \rightarrow \tau \);
2. \( \forall \gamma \in \Sigma(S'), \gamma \) is a valid box with type \( \tau \rightarrow \upsilon \);

then \( \forall \beta \in \Sigma(S \o S') \), \( \beta \) is a valid box with type \( \sigma \rightarrow \upsilon \).

A regular expression \( e \) with letters in \( P_f(\text{Reg}^\circ_{S_S}) \) represent the set of boxes

\[ [e] = \bigcup_{S \in [e]_o} \Sigma(S) \]

We extend a definition from the previous section to this setting:

Definition 22 (Typed expression)

An expression valid \( e \in \text{Reg}(P_f(\text{Reg}^\circ_{S_S})) \) has type \( \sigma \rightarrow \tau \) if all \( \beta \in [e] \) have the type \( \sigma \rightarrow \tau \).

We can now equate two expressions if they denote the same set of boxes, introducing the equivalence relation \( \equiv_\beta \)

\[ e \equiv_\beta f = ([e] = [f]) \]

Notice that Lemma 19 can be restated for \( \tau_{\text{af}} \):

Lemma 23. Let \( f, g \in \text{Reg}(P_f(\text{Reg}^\circ_{S_S})) \) be valid expressions of type \( \sigma \rightarrow \tau \), such that \( [f] \o \subseteq [g] \o \). Then \( \tau_{\text{af}} \equiv_\beta \tau_{\text{af}}(g/f) \).

The operator \( \cup \) and \( \cdot \) are easy enough to define on finite sets of boxes labelled over \( E_{S_S} \):

\[ S \cup S' = \bigcup_{B \in S} \bigcup_{B' \in S'} B \cup B' \]

\[ S \cdot S' = \bigcup_{B \in S \cup S'} \{ B \mid B \in S \cup S' \} \]

The case of \( _{_{\cdot}} \) however is much more interesting. In fact, this is the main technical difficulty of the present paper, and where most of the machinery we have introduced up to this point will prove
useful. For typing reasons, notice that we only need to compute the star of a set of boxes of an identity type $\sigma \rightsquigarrow \sigma$. In the next section, we show how to compute from a set $S$ of boxes labelled over $\Sigma$, and having type $\sigma \rightsquigarrow \sigma$ a new finite set of boxes $S'$ such that:

1. $(\langle S' \rangle) \subseteq (\langle S \rangle)$
2. and $\forall \beta \in (\langle S' \rangle)$, $\beta$ is valid and has type $\sigma \rightsquigarrow \sigma$.

By Lemma 23 this entails the following equivalence:

$$\varepsilon_\sigma \equiv \varepsilon_\sigma [S'/\langle S' \rangle] \cup [\langle S' \rangle]$$

Using these transformation, we can now “compile” the expression $\varepsilon_\sigma$ into a finite set of boxes of type $\langle e \rangle \rightsquigarrow e \rightsquigarrow f$. Such a set may only have the shape:

$$\{ e \rightsquigarrow e_1 \rightsquigarrow f, \ldots, e \rightsquigarrow e_n \rightsquigarrow f \}$$

It is then straightforward to see that $\langle \varepsilon_\sigma \rangle$ is isomorphic to $\mathcal{G}(\varepsilon_\sigma)$. If we call $\varepsilon_\sigma$ the expression $e_1 \cdot \ldots \cdot e_n$, the following theorem holds:

**Theorem 24 (Main result).** For every SP-automaton $\mathcal{A}$ over $\Sigma$, there exists an expression $\varepsilon_\sigma(S') \in \text{Reg}_\Sigma$ such that

$$\mathcal{G}(\varepsilon_\sigma) = \mathcal{G}(\varepsilon_\sigma(S')).$$

### 4.3 Staring boxes

Instead of giving a formal description, which would be very involved, we describe the procedure on an example, outlining the important computation steps. Consider the following set of boxes, respectively named $\beta$ and $\gamma$.

$$\begin{align*}
\{ & A \rightsquigarrow A, A \rightsquigarrow B, B \rightsquigarrow D, D \rightsquigarrow E, \ldots, \} \\
\{ & A \rightsquigarrow A, A \rightsquigarrow B, B \rightsquigarrow D, D \rightsquigarrow E, \ldots, \}
\end{align*}$$

Both these boxes have type $\sigma \rightsquigarrow \sigma$, with the $\sigma$ as shown in Figure 9. First, we decompose each box, by separating disjoint non-trivial connected components. The idea behind this step is that the different connected component correspond to concurrent processes, and as such may be iterated independently. This yields the set of five so-called *elementary* boxes displayed in Figure 13, respectively named $\eta_1$ through $\eta_5$.

One can show that the resulting boxes are always valid and have the type $\sigma \rightsquigarrow \sigma$ (a formal proof of the properties of this decomposition is given in Appendix A). Because $\beta = \eta_1 \otimes \eta_2$ and $\gamma = \eta_3 \otimes \eta_4 \otimes \eta_5$, we get that:

$$[\langle \beta \cup \gamma \rangle]_0 \subseteq [\langle \eta_1 \cup \eta_2 \cup \eta_3 \cup \eta_4 \cup \eta_5 \rangle]_0$$

On the other hand, because of Lemma 21 we can establish that the set $[\langle \eta_1 \cup \eta_2 \cup \eta_3 \cup \eta_4 \cup \eta_5 \rangle]$ only contains valid boxes of type $\sigma \rightsquigarrow \sigma$.

By Lemma 23, we know that we may replace in the expression $\varepsilon_\sigma$ the sub expression $(\beta \cup \gamma)$ by $(\eta_1 \cup \eta_2 \cup \eta_3 \cup \eta_4 \cup \eta_5)$, without changing the set of produced graphs. This means that we may continue the computation with the set of elementary boxes $H = \{ \eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \}$. We define the *support* of a box with a single non-trivial connected component as the set of ports that are mapped inside this component. Figure 14 lists the supports of the boxes $\eta_i$. Notice that for every two boxes $\eta_i, \eta_j$ in $H$, their supports are either included one in the other, or of empty intersection. This can be shown formally by realising that for a box with a single connected component of type $\sigma \rightsquigarrow \sigma$, the support is the set of leaves of a sub-tree of $\sigma$ (i.e. the set of all leaves reachable from some node of $\sigma$). This means we can endow the set $\{ \eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \}$ with a preorder relation $\leq$ defined by $\eta \leq \eta'$ if $\text{support}(\eta) \subseteq \text{support}(\eta')$.

The graph of the order on our example set is given in Figure 15.

The order has an interesting property: if $\eta$ and $\eta'$ are incomparable, then $\eta \otimes \eta' \equiv \eta' \otimes \eta$. Thus $[\langle \eta \cup \eta' \rangle] \equiv [\langle \eta' \cup \eta \rangle]$. With that in mind we will compute the expression:

$$\langle \eta_1 \cup \eta_2 \rangle \cdot \langle \{ \eta_1, \eta_5 \} \cup \{ \eta_3, \eta_4 \} \rangle.$$

The first step in this computation is the computation of the star of a single box. Notice that we may define easily the star of a “linear” elementary box, i.e. of the shape:

$$\begin{align*}
\{ & A \rightsquigarrow e, A \rightsquigarrow B_1, B_1 \rightsquigarrow B_2, B_2 \rightsquigarrow B_3, \ldots, \} \\
\{ & A \rightsquigarrow e^*, A \rightsquigarrow B_1, B_1 \rightsquigarrow B_2, B_2 \rightsquigarrow B_3, \ldots, \}
\end{align*}$$

This construction gives us $\eta_1^*$ and $\eta_2^*$. Now for general elementary boxes, notice that the non-trivial connected component of an elementary box is always a bow tie. Hence we can always decompose such a box $\eta$ of type $\sigma \rightsquigarrow \sigma$ as $\delta_1 \otimes \delta_2$ such that the non-trivial connected component of $\delta_1$ is a bottom-up tree, and that of $\delta_2$ is a top-down tree, with the middle node of $\eta$ of the bow tie being the root of both trees. See Figure 16 for examples. This means that $\delta_2 \otimes \delta_1$ is a linear elementary box. Furthermore, the definition of $\delta_1, \delta_2$ ensures $[\langle \eta \rangle] \equiv [\langle \delta_1 \cdot \delta_2 \rangle]$. The regular laws entail that for every expressions $e$ and $f$ we have $[\langle e \cdot f \rangle] = [\langle e \rangle \cdot [\langle f \rangle \cdot f \rangle]$. Thus we can use the previous construction to get a set of boxes representing $\{ \langle \delta_1 \cdot \delta_2 \rangle \} \equiv [\langle \delta_2 \otimes \delta_1 \rangle]^*$, and then compute:

$$[\{ \eta \}]^* \equiv [\langle \text{id}_\sigma \rangle \cup \{ \delta_1 \} \cdot [\langle \delta_2 \otimes \delta_1 \rangle]^* \cdot \{ \delta_2 \}].$$

The last thing to do is to compute stars of the shape $(S^* \cup \eta_1)^*$ where:

- $S^*$ is a finite set of boxes computed in an earlier step,
- $\text{support}(\eta_1)$ contains the “support” of $S^*$: the set of inputs mapped to a non-trivial connected component in some $B \in S^*$. 

\[\text{Figure 13: Elementary decomposition of } \{\beta, \gamma\}.\]

\[\text{Figure 14: Supports of the } \eta_i.\]

\[\text{Figure 15: Graph of } \leq.\]
Again, we decompose $\eta$ into $\delta_1$ and $\delta_2$ as in the previous construction, and notice that $\delta_2 \circ S^* \circ \delta_1$ is set of a linear boxes, along the same line. We may thus bundle them together in a single box with a line labelled with the union of their labels. For example:

$$
\begin{align*}
\eta_1 &= \begin{array}{c}
A \\
B \\
C \\
D \\
E \\
\end{array} \\
\eta_2 &= \begin{array}{c}
A \\
B \\
C \\
\end{array} \\
\eta_3 &= \begin{array}{c}
A \\
B \\
C \\
\end{array} \\
\eta_4 &= \begin{array}{c}
A \\
B \\
C \\
\end{array}
\end{align*}
$$

Figure 16: Decomposition as a product of trees of $\eta_1$, $\eta_2$ and $\eta_3$.

This allow us to produce a set of (two) boxes representing the star of the box ($\delta_2 \circ S^* \circ \delta_1$). The regular laws allow us to conclude, by stating that:

$$
(S^* \cup \delta_1 \cup \delta_2)^* = S^* \cup \delta_1 \cup (\delta_2 \circ S^* \circ \delta_1)^* \cup \delta_2
$$

We can now give the bulk of the computation, in Figure 17. The result is then obtained by performing the composition

$$
(\eta_3^* \cup \eta_2^*)^* \cup (\eta_4^* \cup \eta_1^*)^*,
$$

thus yielding 32 boxes: 8 for $((\eta_3^* \cup \eta_2^*) \cup \eta_1^*)^*$ times 4 for $((\eta_4^* \cup \eta_1^*) \cup \eta_1^*)^*$.

5. Consequences

5.1 Kleene Allegories

The main motivation for this work was the study of the equational theory of Kleene Allegories (KAl). In this line of work we consider binary relations and the operations of union ($\cup$), intersection ($\cap$), composition ($\circ$), converse ($\circ^*$), transitive closure ($\circ^+$), and the constants identity ($1$) and empty relation ($0$). Terms made from those operations and some variables $a, b, \ldots \in \Sigma$ constitute the set $\text{Reg}_e$. A pair $e, f \in \text{Reg}_e$ is then relationally equivalent, denoted $\text{Rel} \equiv e \equiv f$, if the corresponding equality holds universally when the expressions are interpreted as binary relations.

Notice that we may restrict expressions so that the operator $\circ^*$ is only applied to variables, by using the following rewriting system:

- $(a \cup b)^* = a^* \cup b^* \quad 0^* = 0 \quad (a^*)^* = (a^*)^*$
- $(a \cdot b)^* = b^* \cdot a^* \quad 1^* = 1 \quad a^* = a^* \quad a^* = a^*$
- $(a \cap b)^* = a^* \cap b^* \quad b^* \cdot a^* = b^* \cdot a^*$

The result of this transformation on an expression $e$ is an expression $e'$ always satisfying $\text{Rel} \equiv e \equiv e'$. We may associate to these terms graph languages, as in Definition 2.

Definition 25 (Graph language of a KAl expression: $G'(e)$)

The graph language is defined by structural induction as follows:

- $G'(a) = \{a\}
- G'(0) = \emptyset
- G'(1) = \{\emptyset\}
- G'(e \cup f) = G'(e) \cup G'(f)
- G'(e \cdot f) = \{E \cdot F \mid E \in G'(e) \text{ and } F \in G'(f)\}
- G'(e \cap f) = \{E \cap F \mid E \in G'(e) \text{ and } F \in G'(f)\}
- G'(e^*) = \{E_1 \cdot \ldots \cdot E_n \mid n > 0, \forall i, E_i \in G'(e)\}

* In [2], we showed the following result:

\[ \text{Rel} \equiv e \equiv f \iff G'(e) = G'(f). \]

We need not concern ourselves with the definition of $^*$ here, as it plays no role in the present result.
Notice that the only two differences between \( \mathcal{G}(\_\_\_) \) and \( \mathcal{G}^\prime(\_\_\_) \) are the construction for 1 and the case of \( _-_- \). We can mend that gap, by computing \( \mathcal{G}^\prime(\_\_\_) \) from \( \mathcal{G}(\_\_\_) \). The first step is to consider the sub-expressions \( a^* \) as letters, i.e. to see \( \mathcal{G}^\prime(\_\_\_) \) as an expression in \( \mathcal{G}(\_\_\_) \). Let \( \equiv_G \) be the smallest equivalence relation on \( V \) containing all pairs \((i,j)\) such that \((i,j)\in E\). Then \( \Phi(G) \) is the graph defined by

\[
\Phi(G) := \{(i)_{[G]} | i \in V \}, E', \alpha_{[G],\alpha_G} \in \{ k \in V \mid i \equiv_G k \}
\]

\[
E': \left\{ \left( [i]_{[G],\alpha_{[G]}}, [j]_{[G],\alpha_{[G]}} \right) \bigg| x \in \Sigma \text{ and } \exists k \in [i]_{[G],\alpha_{[G]}}, l \in [j]_{[G],\alpha_{[G]}} : (k,x,l) \in E \text{ or } (l,x',k) \in E \right\}
\]

It is then a simple matter to check that

\[
\forall e \in \text{Reg}_{\Sigma}^G, \mathcal{G}^\prime(\_\_\_) = \{ \Phi(G) \mid G \in \mathcal{G}(\_\_\_) \}.
\]

In the same paper we also introduced Petri automata, but again we used a different definition of the language \( \mathcal{L}(\_\_\_) \) of an automaton \( \_\_\_ \). However, this set can be expressed with our notations as \( \mathcal{L}(\_\_\_) = \Phi(\mathcal{G}(\_\_\_)) \). A function \( \mathcal{G}(\_\_\_) \) was provided, mapping an expression \( e \in \text{Reg}_{\Sigma}^G \) to a Petri automaton \( \mathcal{G}(\_\_\_) \) such that \( \mathcal{G}(\_\_\_) = \mathcal{G}(\_\_\_) \).\(^7\) Thus:

\[
\mathcal{L}(\_\_\_) = \Phi(\mathcal{G}(\_\_\_)).
\]

Theorem 24 allow us to relate the language of any SP-automaton \( \_\_\_ \) to the graph language of \( \mathcal{G}(\_\_\_) \):

\[
\mathcal{L}(\_\_\_) = \Phi(\mathcal{G}(\_\_\_)).
\]

This proves that from a decidability stand point, language equivalence of Petri automata and relational equivalence of KAI expressions. Formally:

\[
\forall e, f, \text{ Rel } e = f \iff \mathcal{L}(\_\_\_) = \mathcal{L}(\_\_\_)
\]

\[
\forall \_\_\_, \_\_\_, \_\_\_, \_\_\_ \in \text{Reg}_{\Sigma}^G, \_\_\_ = \_\_\_ \iff \_\_\_ = \_\_\_.
\]

One could also read this result from a graph theoretical viewpoint. What we show here is that the sets of SP-graphs one may express trough regular expressions enhanced with a parallel operator \( \cap \) are exactly those generated by SP-automata.

5.2 Petri nets

The construction presented here also can provide a finite regular expression with intersection to represent all finite occurrence nets of a given Petri net \( \_\_\_ \), provided this net is safe and Parallel Series. In terms of concurrent processes, these two conditions roughly amount to requiring that there is a bounded number of processes running at any given time, and that communication between processes is synchronous.

More precisely, suppose we are given a net \( \_\_\_ \), with a finite set \( P \) of places and a finite set \( T \subseteq \mathcal{P}(P) \times \mathcal{P}(\Sigma \times P) \) of transitions.\(^8\) If \( I, F \subseteq \mathcal{P}(P) \) are respectively a source set and a target set, we can produce a Petri automaton \( \_\_\_ \) such that the set of occurrence nets of process going from a source configuration in \( I \) to a target configuration in \( F \) is isomorphic to \( \mathcal{G}(\_\_\_ \_\_) \). To achieve this, we add to the net:

- two new places \( t \) and \( f \) and a fresh label \( \# \);  
- for each source configuration \( \{ p_1, \ldots, p_n \} \in I \) a transition \( \{(t), \{\#, p_1, \ldots, \#, p_n\}\} \);  
- for each target configuration \( \{ q_1, \ldots, q_n \} \in F \) a transition \( \{\{q_1, \ldots, q_n\}, \{\#, \#\}\} \).

It is then a simple matter of unfolding definitions to see that \( \mathcal{G}(\_\_\_) \) indeed corresponds to the set of occurrence nets we wanted. Of course, this construction is ill-defined if the net \( \_\_\_ \) is not safe for one of the source configurations in \( I \).

If furthermore the produced automaton is SP, which is again a property of the net \( \_\_\_ \), then the construction of this paper yields an expression \( \mathcal{E}(\_\_\_) \) representing faithfully the set of occurrence nets of processes going from any configuration in \( I \) to any configuration in \( F \).

References


A. Omitted proofs

Definition 27 (Connected component)
A set $C$ of nodes of a graph $G = (V, E)$ is a connected component if $\forall x \in C, \forall y \in V, y$ is in $C$ if and only if there is a non-oriented path from $x$ to $y$ in $G$.

This means any two nodes in $C$ are linked by non-oriented path, and that the set $C$ is maximal for this property (with the containment order).

Definition 28 (Elementary boxes)
A box $\eta$ is said to be elementary if its graph contains a single non-trivial connected component.

Lemma 29. If $\eta$ is a valid elementary box, then its graph is necessarily a bow tie.

Proof. Validity means the graph has to be an SP-slice, and reduced connected SP-slices are always bow ties. \qed

Definition 30 (Support and target of an elementary box)
Let $\eta$ be an elementary box. The support (respectively target) of $\eta$ is the set of input (resp. output) ports of $\eta$ which are mapped inside its unique non-trivial connected component.

Lemma 31. Any valid box $\beta$ can be decomposed into a set of valid elementary boxes $\eta_1, \ldots, \eta_p$, such that:

1. $\beta = \eta_1 \cap \cdots \cap \eta_p$;
2. $\forall i \neq j$, $\text{support}(\eta_i) \cap \text{support}(\eta_j) = \emptyset$;
3. if $\beta$ has type $\sigma \rightarrow \sigma$, then each $\eta_i$ has the same type and $\text{support}(\eta_i) = \text{target}(\eta_i)$.

Proof. We number the connected components of $\beta$ from $C_1$ to $C_p$.

Suppose $\beta$ is valid, then there is a run $\pi : \xi_0 \xrightarrow{t_1} \cdots \xrightarrow{t_n} \xi_n$ such that $\beta = \beta(\pi) = \beta_{t_1, \xi_0} \cap \cdots \cap \beta_{t_n, \xi_n-1}$. Notice that for any index $i$, the box $\beta_{t_{i-1}, \xi_{i-1}}$ is elementary (see the definition of $\beta_{t,c}$). Thus it can be associated with a single non-trivial connected component in $\beta$. We introduce the function $c : \{1, \ldots, n\} \rightarrow \{1, \ldots, p\}$, such that the transition $i$ is associated with the component $C_{c(i)}$.

The key argument is the following. For any index $i$, if the set of output places of transition $t_i$ contains a place that appears in the input of the transition $t_{i+1}$, then the box $\beta_{t_{i-1}, \xi_{i-1}} \cap \beta_{t_{i+1}, \xi_i}$ is elementary. This means that $c(i) = c(i+1) + 1$. By contraposition, we may then deduce that if $c(i) = c(i+1) + 1$, then the intersection of the outputs of $t_i$ and the inputs of $t_{i+1}$ is empty. This can in turn be used to build a configuration $\xi_i$ such that:

- $\xi_{i-1} \xrightarrow{t_{i+1}} \xi_i \xrightarrow{t_i} \xi_{i+1}$;
- and $\beta_{t_{i-1}, \xi_{i-1}} \cap \beta_{t_i, \xi_i} = \beta_{t_{i+1}, \xi_{i+1}} \cap \beta_{t_{i+2}, \xi_{i}}$;

$(\xi' = \xi_{i+1} \cap t_{i+1} \cup \{p \mid \beta_{a, \eta}(a, p) \in \xi_{i+1}\})$

This means we may group the transitions that contribute to the same connected component. Formally we apply a permutation $\pi$ to the run $\xi$, such that:

- $\pi(\xi) = \xi_0 \xrightarrow{t_{\pi(1)}} \cdots \xrightarrow{t_{\pi(|\xi|)}} \xi_n$
- $\beta(\pi(\xi)) = \beta(\xi)$;
- $\pi(\xi)$ can be decomposed as a sequence of runs:

$\pi(\xi) = \xi_0 \xrightarrow{\chi_1} \xi_1 \cdots \xrightarrow{\chi_{n-1}} \xi_n$

such that $t_i \in \chi_j \iff c(i) = j$.

We now define $\eta_j := \beta(\chi_j)$. By construction $\eta_j$ is valid, it contains only the connected component $C_j$ so it is elementary, and $\beta = \beta(\pi(\xi)) = \eta_1 \cap \cdots \cap \eta_p$. Furthermore, for each $p \in \text{support}(\eta_j)$ if and only if $p(p) \in C_j$. As the connected components are pairwise disjoint, so are the supports of the $\eta_j$. Further analysis of the run $\pi(\xi)$ also shows that $\forall i < j$, $\text{target}(\eta_i) \cap \text{support}(\eta_j) = \emptyset$.

If $\beta$ has type $\sigma \rightarrow \sigma$, that means in particular $\xi_0 = \xi_n$. This entails the following equality:

$$\bigcup_j \text{support}(\eta_j) = \bigcup_j \text{target}(\eta_j).$$

(The remaining places are those untouched by the execution.) Hence:

$$\text{target}(\eta_1) \subseteq \bigcup_j \text{target}(\eta_j) = \bigcup_j \text{support}(\eta_j)$$

$$\Rightarrow \text{support}(\eta_1) \subseteq \bigcup_j \text{support}(\eta_j) = \bigcup_j \text{target}(\eta_j)$$

We know that $\forall 1 < j$, $\text{target}(\eta_j) \cap \text{support}(\eta_1) = \emptyset$, hence $\text{target}(\eta_1) \subseteq \text{support}(\eta_1)$. To get the other direction, one has to notice that the run $\chi_2 \cdots \chi_p$ may be executed starting from $\xi_0$ (rather than $\xi_1$). This is again because of the possibility to exchange “independent” transitions. Safety of the automaton ensures that $\text{support}(\eta_1) \cap \bigcup_j \text{target}(\eta_j) = \emptyset$.

Thus we obtain $\text{support}(\eta_1) \subseteq \text{target}(\eta_1)$, finally proving $\text{support}(\eta_1) = \text{target}(\eta_1)$. This in turn ensures that $\xi_1 = \xi_0$, and enable us to repeat the proof for $\eta_2, \ldots, \eta_p$.

The last thing to do is to “type check” these elementary boxes. Let’s detail the computation of the type $\beta$ outputs when given a tree $o$. Each connected component $C_j$ is a bow tie of centre $o_j$, with inputs $S_j$. For each of those, there must be a sub-tree $T_j$ in $o$ with leaves $S_j$ (otherwise there would be no way of reducing $\sigma \cap \beta$ to a tree). We then remove that sub-tree, and replace it by the top-down rooted tree in $o_j$. Because all $S_j$ are pairwise disjoint, the trees $T_j$ are also pairwise disjoint (otherwise they would share some leaves). With this description of the type-checking, one realises that 1) each of the $\eta_j$ contribute to the final type on disjoint sub-trees 2) and each of them leaves the rest of the tree alone. Because $\forall j, S_j \subseteq \text{support}(\eta_j) = \text{target}(\eta_j)$ we can conclude that $\eta_j$ replaces the tree with leaves $S_j$ with the tree with leaves $S_j$, and thus preserves the type $\sigma \rightarrow \sigma$. \qed