Approximation of a compressible Navier-Stokes system by non-linear acoustical models

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Abstract

We analyse the existing derivation of the models of non-linear acoustics such as the Kuznetsov equation, the NPE equation and the KZK equation. The technique of introducing a corrector in the derivation *ansatz* allows to consider the solutions of these equations as approximations of the solution of the initial system (a compressible Navier-Stokes/Euler system). The validation of the approximation *ansatz* is given for the KZK equation case.

1 Introduction

There is a renewed interest in the study of wave propagation, in particular because of recent applications to ultrasound imaging (i.e. HIFU) or technical and medical applications such as lithotripsy or thermotherapy. Such new techniques rely heavily on the ability to model accurately the nonlinear propagation of a finite-amplitude sound pulse in thermo-viscous elastic media.

We analyse the derivation of different models of non-linear acoustics such as the Kuznetzov [1], the Nonlinear Progressive wave Equation (NPE) [2] and the Khokhlov-Zabolotskaya-Kuznetzov (KZK) [3] equations which are perturbative and paraxial approximations of small perturbations around a given state of a compressible nonlinear isentropic Navier-Stokes (for viscous media) and Euler (for the non-viscous case) systems. The direct derivation shows that the Kuznetzov equation is the first order approximation of the Navier-Stokes system, the KZK and NPE equations are the first order approximations of the Kuznetzov equation and the second order approximations of the Navier-Stokes system. In addition, the NPE equation can be considered as an approximation of the KZK equation.

To be able to validate the approximation of the exact solution of the Navier-Stokes/Euler systems by the solution of the Kuznetsov/KZK/NPE equation, we need to ensure that the derivation of our model, the Kuznetsov/KZK/NPE equation, allows us to reconstruct the

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solution of the initial Navier-Stokes system from the solution of the Kuznetsov/KZK/NPE equation. In this aim, following the ideas of Refs. [4, 6], we modify the initial physical derivation, given in Refs. [1, 3] for the KZK and the Kuznetsov equations and given in Ref. [2] for the NPE equation, introducing a corrector function in the derivation *ansatz*.

We also improve the validation of the KZK-approximation for the non-viscous and viscous cases obtained in Ref. [4], by the precision of the speed order of divergence between the solutions of the approximate and the exact systems.

Let us introduce some notations used throughout the paper. For a positive fixed small enough real number ϵ, we suppose that $\mathbb{R}_+$ consists of classes, which are characterized by the power of ϵ:

$$\dots, \epsilon^2, \dots, \epsilon, \dots, \sqrt{\epsilon}, \dots, \epsilon^0 = 1, \dots, \frac{1}{\epsilon}, \dots, \frac{1}{\epsilon^2}, \dots$$

$O(1)$ denotes the class of constants.

2 Approximation of the hydro-dynamic system by an isentropic Navier-Stokes system

We start from the Navier-Stokes system in $\mathbb{R}^n$:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

$$\rho[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}] = -\nabla p + \beta \nabla \operatorname{div} \mathbf{u}, \tag{2}$$

$$\rho T\,[\partial_t S + (\mathbf{u}\cdot\nabla) S] = \kappa \triangle T + \zeta (\operatorname{div} \mathbf{u})^2 + \frac{\eta}{2}\left(\partial_{x_k} u_i + \partial_{x_i} u_k - \frac{2}{3}\delta_{ik}\partial_{x_i} u_i\right)^2, \tag{3}$$

$$p = p(\rho, S), \tag{4}$$

where S is the entropy and the state law $p = p(\rho, S)$ is the pressure. The density ρ, the velocity $\mathbf{u}$, the temperature T and the entropy are unknown functions in the system (1)–(4). The coefficients β, κ and η are constant viscosity coefficients.

First, we assume that the temperature T and the entropy S have small increments $T = T_0 + \epsilon\tilde{T}$ and $S = S_0 + \epsilon^2\tilde{S}$. With the hypothesis of potential motion, we introduce constant states

$$\rho = \rho_0, \quad \mathbf{u} = \mathbf{u_0}.$$

Next, we assume that the density fluctuations (around the constant state ρ_0) and the velocity fluctuations (around $\mathbf{u_0}$, which can be taken equal to zero using a Galilean transformation), are of the same order of ϵ:

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon\,, \quad \mathbf{u}_\epsilon = \epsilon\tilde{\mathbf{u}}_\epsilon\,, \tag{5}$$

where ϵ is a dimensionless parameter which characterizes the smallness of the perturbation. For instance, in water with an initial power of the order of $0.3\,\mathrm{W/cm^2}$ ϵ is equal to

10^{-5}. We also suppose that all viscosity coefficients, for instance, β, ζ, η and κ, are small of the order ϵ:

$$\beta = \epsilon\tilde{\beta}.$$

Using the transport heat equation up to the terms of the order of ϵ^3

$$\epsilon^2 \rho_0 T_0 \partial_t \tilde{S} = \epsilon^2 \tilde{\kappa} \triangle \tilde{T} + O(\epsilon^3),$$

the approximate state equation

$$p = p_0 + c^2 \epsilon \tilde{\rho}_\epsilon + \frac{1}{2} \left(\partial_\rho^2 p\right)_S \epsilon^2 \tilde{\rho}_\epsilon^2 + \left(\partial_S p\right)_\rho \epsilon^2 \tilde{S} + O(\epsilon^3)$$

(where the notation $(\cdot)_S$ means that the expression in brackets is constant in S), can be replaced [3, 7, 8] by

$$p = p_0 + c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0} \epsilon^2 \tilde{\rho}_\epsilon^2 - \epsilon \tilde{\kappa} \left(\frac{1}{C_v} - \frac{1}{C_p} \right) \nabla . u_\epsilon + O(\epsilon^3). \tag{6}$$

Here $\gamma = C_p / C_v$ denotes the ratio of the heat capacities at constant pressure and at constant volume respectively. System (1)–(4) becomes an isentropic system

$$\partial_t \rho_\epsilon + \operatorname{div}(\rho_\epsilon \mathbf{u}_\epsilon) = 0\,, \tag{7}$$

$$\rho_\epsilon [\partial_t \mathbf{u}_\epsilon + (\mathbf{u}_\epsilon \cdot \nabla)\, \mathbf{u}_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta \mathbf{u}_\epsilon\,, \tag{8}$$

with the approximate state equation

$$p(\rho_\epsilon) = p_0 + c^2(\rho_\epsilon - \rho_0) + \frac{(\gamma - 1)c^2}{2\rho_0} (\rho_\epsilon - \rho_0)^2 \quad (O(\epsilon^3)) \tag{9}$$

and with a small enough and positive viscosity coefficient:

$$\epsilon \nu = \beta + \kappa \left(\frac{1}{C_v} - \frac{1}{C_p} \right).$$

3 Perturbative approach: Kuznetsov equation

First derived by Kuznetsov [1] from the isentropic Navier-Stokes system (7)–(9), the Kuznetsov equation

$$\partial_t^2 \tilde{\phi} - c^2 \triangle \tilde{\phi} = \partial_t \left((\nabla \tilde{\phi})^2 + \frac{\gamma - 1}{2c^2} (\partial_t \tilde{\phi})^2 + \frac{\epsilon \nu}{\rho_0} \Delta \tilde{\phi} \right), \tag{10}$$

written for the velocity potential

$$\mathbf{u}(\mathbf{x}, t) = -\nabla \tilde{\phi}(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}^+,$$

was latter derived by other methods and was discussed by a lot of authors (see for examples [8, 9]).

Here we focus on the introduction of the corrector $\epsilon^2\rho_2$ in the *ansatz* of Kuznetsov

$$\rho_\epsilon(\mathbf{x},t) = \rho_0 + \epsilon\rho_1(\mathbf{x},t) + \epsilon^2\rho_2(\mathbf{x},t) \tag{11}$$
$$\mathbf{u}_\epsilon(\mathbf{x},t) = -\epsilon\nabla\phi(\mathbf{x},t), \tag{12}$$

which allows to open the question about the approximation between the exact solution of the isentropic Navier-Stokes system (7)–(9) and its approximation given by the solution of the Kuznetsov equation, as it was done for the KZK equation in [4].

Puting expressions for the density and the velocity (11)–(12) into the isentropic Navier-Stokes system (7)–(9), we directely obtain

$$\begin{aligned}
&\partial_t\rho_\epsilon + \operatorname{div}(\rho_\epsilon\mathbf{u}_\epsilon) = \epsilon\frac{\rho_0}{c^2}\left[\partial_t^2\phi - c^2\triangle\phi - \right.\\
&\left.\epsilon\partial_t\left((\nabla\phi)^2 + \frac{\gamma-1}{2c^2}(\partial_t\phi)^2 + \frac{\nu}{\rho_0}\Delta\phi\right)\right] + O(\epsilon^3), \qquad (13)\\
&\rho_\epsilon[\partial_t\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\,\mathbf{u}_\epsilon] + \nabla p(\rho_\epsilon) - \epsilon\nu\Delta\mathbf{u}_\epsilon =\\
&\epsilon\nabla\left[\rho_1 - \frac{\rho_0}{c^2}\partial_t\phi\right] + \epsilon^2\nabla\left[c^2\rho_2 + \frac{\rho_0(\gamma-2)}{2c^2}(\partial_t\phi)^2\right.\\
&\left.+\frac{\rho_0}{2}(\nabla\phi)^2 + \nu\Delta\phi\right] + O(\epsilon^3). \qquad (14)
\end{aligned}$$

We see that the Kuznetsov equation

$$\partial_t^2\phi - c^2\triangle\phi = \epsilon\partial_t\left((\nabla\phi)^2 + \frac{\gamma-1}{2c^2}(\partial_t\phi)^2 + \frac{\nu}{\rho_0}\Delta\phi\right), \tag{15}$$

is the first order approximation, obtained from the equation of mass conservation up to the terms $O(\epsilon^3)$ with the relations for the density perturbations, found from the momentum conservation also up to the terms $O(\epsilon^3)$ with the help of the Sommerfeld radiation boundary condition at infinity:

$$\rho_1(\mathbf{x},t) = \frac{\rho_0}{c^2}\partial_t\phi(\mathbf{x},t), \tag{16}$$
$$\rho_2(\mathbf{x},t) = -\frac{\rho_0(\gamma+2)}{2c^4}(\partial_t\phi)^2 - \frac{\rho_0}{2c^2}(\nabla\phi)^2 - \frac{\nu}{c^2}\Delta\phi. \tag{17}$$

Since initially, we consider the state equation for the pressure p up to the terms of the order of ϵ^3, we conclude that the *ansatz* of the Kuznetsov equation gives the optimal approximation error of the same order.

Let us also notice, as it was originarily mentionned by Kuznetsov, that the Kuznetsov equation (15) contains terms of different orders, and hence, it is a wave equation with small size non-linear perturbations $\partial_t(\nabla\phi)^2$, $\partial_t(\partial_t\phi)^2$ and viscosity term $\partial_t\Delta\phi$. A way to obtain an approximate equation containning all terms of the same order without modification of the order of remainder terms is to perform a paraxial approximation, which we introduce in the next section. This time the approximation becomes the second order approximation and will be given by the KZK equation.

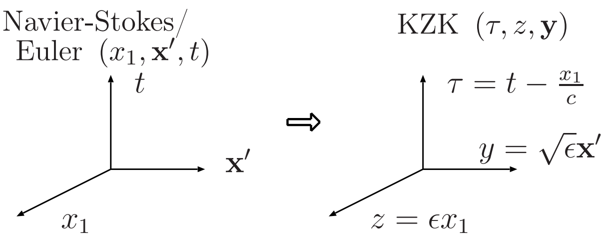

Figure 1: Paraxial change of variables for the profiles $U(t - x_1/c, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}')$.

4 Paraxial approximation

4.1 KZK equation

In the present Section we focus on the derivation of the KZK equation (19) in non-linear media using the following acoustical properties of beam's propagation

1. The beams are concentrated near the x_1-axis ;
2. The beams propagate along the x_1-direction;
3. The beams are generated either by an initial condition or by a forcing term on the boundary $x_1 = 0$.

It is assumed that the variation of beam's propagation in the direction

$$\mathbf{x}' = (x_2, x_3, \ldots, x_n)$$

perpendicular to the x_1-axis is much larger than its variation along the x_1-axis, *i.e.* we suppose that the beam has the form $U(t - x_1/c, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}')$. The first argument $t - x_1/c$ describes the wave propagation in time along the x_1-axis with the sound speed c, two last arguments ϵx_1 and $\sqrt{\epsilon}\mathbf{x}'$ describe respectively the speed of the deformation of the wave along the x_1-axis and along the $\mathbf{x}'$-axis. We remark that $\epsilon \ll 1$ and consequently, $\epsilon \ll \sqrt{\epsilon}$.

We notice that if we perform the paraxial change of variables (see Fig. 1), the wave operator $\partial_t^2 - c^2\Delta$ becomes

$$\partial_t^2 - c^2\Delta = \epsilon\left[2c\partial^2_{\tau z} - c^2\Delta_{\mathbf{y}}\right] - \epsilon^2 c^2 \partial_z^2.$$

Therefore, if we suppose that the velocity potential $\phi(\mathbf{x}, t) = \Phi(t - x_1/c, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}')$, we directely obtain from the Kuznetsov equation (15) (see also [1]) that

$$\begin{aligned}
&\partial_t^2\phi - c^2\triangle\phi - \epsilon\partial_t\left((\nabla\phi)^2 + \frac{\gamma-1}{2c^2}(\partial_t\phi)^2 + \frac{\nu}{\rho_0}\Delta\phi\right)\\
&= \epsilon\left[2c\partial^2_{\tau z}\Phi - \frac{\gamma+1}{4c^2}c^2\partial_\tau(\partial_\tau\Phi)^2\right.\\
&\left.-\frac{\nu}{\rho_0 c^2}\partial_\tau^3\Phi - \Delta_{\mathbf{y}}\Phi\right] + O(\epsilon^2). \qquad (18)
\end{aligned}$$

Therefore, returning to the derivation of the Kuznetsov equation, after the paraxial approximation of ϕ, ρ_1 and ρ_2 with profiles Φ, I and J

$$\begin{aligned}
&\mathbf{u}_\epsilon(\mathbf{x},t) = -\epsilon\left(-\frac{1}{c}\partial_\tau\Phi + \epsilon\partial_z\Phi; \sqrt{\epsilon}\nabla_{\mathbf{y}}\Phi\right)(\tau,z,\mathbf{y})\\
&\rho_1(\mathbf{x},t) = I(\tau,z,\mathbf{y}) = \frac{\rho_0}{c^2}\partial_\tau\Phi(\tau,z,\mathbf{y}),\\
&\rho_2(\mathbf{x},t) = J(\tau,z,\mathbf{y}) =\\
&-\frac{(\gamma-1)\rho_0}{2c^4}(\partial_\tau\Phi)^2 - \frac{\nu}{c^4}\partial_\tau^2\Phi + O(\epsilon),
\end{aligned}$$

we find that the right-hand ϵ-order terms in Eq. (18) is exactly the KZK equation, originally written in Ref. [3] for the (first) perturbation I of the density ρ_ϵ:

$$c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\Delta_{\mathbf{y}} I = 0. \tag{19}$$

We notice that this model still contains terms describing the wave propagation $\partial_{\tau z}^2 I$, the non-linearity $\partial_\tau^2 I^2$ and the viscosity effects $\partial_\tau^3 I$ of the medium, as the Kuznetsov equation and adds a diffraction effects by the tranversal laplacian $\Delta_{\mathbf{y}} I$.

In addition, performing the paraxial approximation in the right-hand side of equations (13)–(14), we obtain that the KZK equation is the second order approximation of the isentropic Navier-Stokes system up to term of $O(\epsilon^3)$. In this sense, since the entropy and the pressure are approximated up to terms of the order of ϵ^3, the Kuznetsov-type *ansatz* (for the Kuznetsov or the KZK equations) is optimal, as the equations of the Navier-Stokes system also approximated up to $O(\epsilon^3)$-terms. For instance, the *ansatz* initially proposed by Khokhlov and Zabolotskaya [3] to derive the KZK equation, corrected with $\epsilon^2 v_1$ [4] for the velocity perturbation along the propagation axis,

$$\begin{aligned}
&\rho_\epsilon(x_1,\mathbf{x}',t) = \rho_0 + \epsilon I(t-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}'),\\
&\mathbf{u}_\epsilon(x_1,\mathbf{x}',t) = \epsilon(v+\epsilon v_1; \sqrt{\epsilon}\mathbf{w})(t-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}')
\end{aligned}$$

is not optimal since the equation of momentum in tranversal direction keeps the non-zero terms of the order of $\epsilon^{\frac{5}{2}}$ [4].

4.2 NPE equation

The NPE equation (Nonlinear Progressive wave Equation), initially derived by McDonald and Kuperman [2], gives another example of a paraxial approximation in the aim to describe short-time pulses and a long-range propagation (see Fig. 2), for instance, in an ocean waveguide, where the refraction phenomena are important. To compare to the KZK-*ansatz*, the role of propagation distance and time was reversed [2]:

$$z_{\mathrm{NPE}} = -c\tau_{\mathrm{KZK}}, \quad \tau_{\mathrm{NPE}} = \epsilon\tau_{\mathrm{KZK}} + \frac{z_{\mathrm{KZK}}}{c}.$$

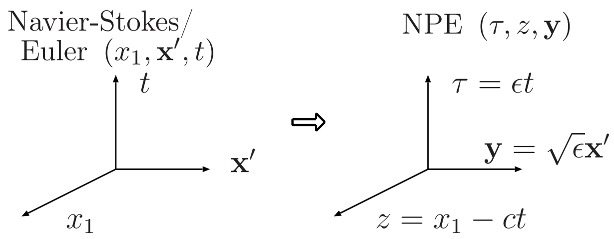

Figure 2: Paraxial change of variables for the profiles $U(\epsilon t, x_1 - ct, \sqrt{\epsilon}\mathbf{x}')$.

Consequently, from the KZK equation we directely have the NPE equation with the error $O(\epsilon)$:

$$
\begin{aligned}
&c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_{\mathbf{y}} I = \\
&-c\partial^2_{\tau_{\mathrm{NPE}} z_{\mathrm{NPE}}} I - \frac{c^2(\gamma+1)}{4\rho_0}\partial^2_{z_{\mathrm{NPE}}} I^2 + \frac{c\nu}{2\rho_0}\partial^3_{z_{\mathrm{NPE}}} I \\
&-\frac{c^2}{2}\Delta_{\mathbf{y}_{\mathrm{NPE}}} I + O(\epsilon).
\end{aligned}
$$

The fact that the NPE equation is an approximation of the KZK equation does not allow to keep, by the analogy to the derivation of the KZK, the Kuznetsov-*ansatz* of perturbations (11)–(12) just by introducing the new paraxial profiles Ψ for ϕ , P_1 for ρ_1 and P_2 for ρ_2 . Indeed, if we do this, the Kuznetsov equation, appeared in the conservation of mass, gives the NPE equation for the potential profile Ψ [compare with Eq. (18)]

$$
\begin{aligned}
&\partial^2_t\phi - c^2\triangle\phi - \epsilon\partial_t\left((\nabla\phi)^2 + \frac{\gamma-1}{2c^2}(\partial_t\phi)^2 + \frac{\nu}{\rho_0}\Delta\phi\right) \\
&= \epsilon\left[-2c\partial^2_{\tau z}\Psi + \frac{\gamma+1}{2}c\partial_z(\partial_z\Psi)^2\right. \\
&\left.+\frac{\nu c^2}{\rho_0}\partial^3_z\Psi - c^2\Delta_{\mathbf{y}}\Psi\right] + O(\epsilon^2),
\end{aligned}
\tag{20}
$$

but in the conservation of momentum, we obtain that the corrector P_1 has a term of the order of ϵ :

$$
\rho_1(\mathbf{x}, t) = P_1(\tau, z, \mathbf{y}) = -\frac{\rho_0}{c}\partial_z\Psi + \epsilon\frac{\rho_0}{c^2}\partial_\tau\Psi,
$$

what will not allow to keep equal to zero just the terms of the same order without any arrangement between the first and the second order terms. Thus we need to suppose that

$$
\begin{aligned}
&\mathbf{u}_\epsilon(\mathbf{x}, t) = -\epsilon\nabla\phi(\mathbf{x}, t) = -\epsilon\left(\partial_z\Psi; \sqrt{\epsilon}\nabla_{\mathbf{y}}\Psi\right)(\tau, z, \mathbf{y}), \\
&\rho_\epsilon(\mathbf{x}, t) = \rho_0 + \epsilon P_1(\tau, z, \mathbf{y}) + \epsilon^2 P_2(\tau, z, \mathbf{y}),
\end{aligned}
$$

where

$$P_1(\tau, z, \mathbf{y}) = \frac{\rho_0}{c} \partial_z \Psi(\tau, z, \mathbf{y}),$$
$$P_2(\tau, z, \mathbf{y}) = \frac{\rho_0}{c^4} \partial_\tau \Psi - \frac{\rho_0(\gamma+3)}{2c^2}(\partial_z \Psi)^2 - \frac{\nu}{c^2}\partial_z^2 \Psi,$$

to obtain the NPE equation for the profile of the potential

$$\partial^2_{\tau z}\Psi - \frac{\gamma+1}{4}\partial_z(\partial_z\Psi)^2 - \frac{\nu}{2\rho_0}\partial_z^3\Psi + \frac{c}{2}\Delta_{\mathbf{y}}\Psi = 0 \tag{21}$$

as the second order approximation of the isentropic Navier-Stokes system up to the terms of the order of $O(\epsilon^3)$.

5 Approximation results

We precise the approximation results for the KZK equation, given in Ref. [4], by the evaluation of the size of the difference between the exact and the approximate solutions. As it was explained in Ref. [4], the isentropic Euler system for $\widetilde{\mathbf{U}}_\epsilon = (\rho_\epsilon, \rho_\epsilon \mathbf{u}_\epsilon)$ and $\mathbf{F}(\widetilde{\mathbf{U}}_\epsilon) = (\rho_\epsilon \mathbf{u}_\epsilon, \rho_\epsilon \mathbf{u}_\epsilon^{\mathbf{2}} + p(\rho_\epsilon))^T$ can be written as a system of conservation laws

$$\partial_t \widetilde{\mathbf{U}}_\epsilon + \nabla.\mathbf{F}(\widetilde{\mathbf{U}}_\epsilon) = 0. \tag{22}$$

The KZK-ansatz allows to find from the solution I of the KZK equation (19) the correctors v, $\mathbf{w}$, v_1 and to obtain for

$$\overline{\mathbf{U}}_\epsilon = (\overline{\rho}_\epsilon, \overline{\rho}_\epsilon \overline{\mathbf{u}}_\epsilon), \tag{23}$$

with

$$\overline{\rho}_\epsilon = \rho_0 + \epsilon I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}'),$$
$$\overline{\mathbf{u}}_\epsilon = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\mathbf{w}))(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}'),$$

the approximate system

$$\partial_t \overline{\mathbf{U}}_\epsilon + \nabla.\mathbf{F}(\overline{\mathbf{U}}_\epsilon) = \epsilon^{\frac{5}{2}}\mathbf{R}. \tag{24}$$

More precisely, for the non-viscous case, we have the following theorem:

Theorem 1 *Let $I_0(\tau, 0, \mathbf{y}) \in H^{s'}(\mathbb{R}^n)$, $s' > [\frac{n}{2}] + 5$ be the initial data for the KZK equation (19) L-periodic and with mean value zero with respect to τ. Then there exists a unique solution I of the KZK equation such that*

- *$I(\tau, z, \mathbf{y})$ is L-periodic and with mean value zero with respect to τ and defined for $|z| \leq K$ (K is a positive constant depending only on s', L and $\|I_0\|_{H^{s'}}$) and $y \in \mathbb{R}^{n-1}$,*
- *for $\Omega = \mathbb{R}/L\mathbb{Z} \times \mathbb{R}^{n-1}_y$ $z \mapsto I(\tau, z, \mathbf{y}) \in C(]-K, K[; H^{s'}(\Omega)) \cap C^1(]-K, K[; H^{s'-2}(\Omega))$.*

Let $\overline{U}_\epsilon$ *be the approximate solution of the isentropic Euler system deduced from a solution of the KZK equation with the help of the correctors* v *,* w *,* v_1 *, found by I following the formulae of the derivation KZK-ansatz, ensuring the remainder term of the order of* $\epsilon^{\frac{5}{2}}$ *. Then the function* $\overline{U}_\epsilon(x_1,\mathbf{x}',t)=\overline{U}_\epsilon(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}\mathbf{x}')$ *given by formula (23) is defined in*

$$\mathbb{R}_t\times(\Omega_\epsilon=\{|x_1|<\frac{K}{\epsilon}-ct\}\times\mathbb{R}^{n-1}_{\mathbf{x}'})$$

and is smooth enough according to the above procedure and the remainder term $\mathbf{R}$ *in Eq. (24) is in* $[L_\infty((-K,K);L_2)]^2$.

Let us now consider the solution of the Euler system (22) in a cone (see Fig. 3)

$$C(t)=\cup_{0<s<t}\{s\}\times Q_\epsilon(s)=$$
$$\{x=(x_1,\mathbf{x}'):\,|x_1|\leq\frac{K}{\epsilon}-Ms,\ M\geq c,\ \mathbf{x}'\in\mathbb{R}^{n-1}\}$$

with the initial data

$$(\bar{\rho}_\epsilon-\rho_\epsilon)|_{t=0}=0,\quad(\bar{\mathbf{u}}_\epsilon-\mathbf{u}_\epsilon)|_{t=0}=0.\tag{25}$$

Consequently, there exists T_0 *such that for the time interval* $0\leq t\leq\frac{T_0}{\epsilon}$ *there exists the*

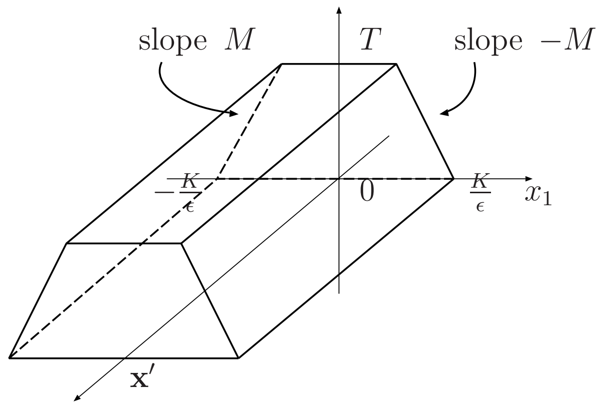

Figure 3: The cone $C(T)$.

classical solution $\mathbf{U}_\epsilon=(\rho_\epsilon,\mathbf{u}_\epsilon)$ *of the Euler system (22) in a cone*

$$C(T)=\{0<t<T|T<\frac{T_0}{\epsilon}\}\times Q_\epsilon(t)\tag{26}$$

with

$$\|\nabla.\mathbf{U}_\epsilon\|_{L_\infty([0,\frac{T_0}{\epsilon}[;H^{s'-5})}<\epsilon C\ \textit{ for }\ s'>[\frac{n}{2}]+5.$$

Moreover, there exist positive constants C_1 *and* C_2 *such that for any* ϵ *small enough, the solutions* $\widetilde{\mathbf{U}}_\epsilon\overset{note}{=}(\rho_\epsilon,\rho_\epsilon\mathbf{u}_\epsilon)$ *and* $\bar{\mathbf{U}}_\epsilon\overset{note}{=}(\overline{\rho}_\epsilon,\overline{\rho}_\epsilon\overline{\mathbf{u}}_\epsilon)$ *, which were determined as above in cone (26) with the same initial data (25), satisfy the estimate*

$$C_1\epsilon^{\frac{7}{2}}t\leq\|\bar{\mathbf{U}}_\epsilon-\widetilde{\mathbf{U}}_\epsilon\|^2_{L_2(Q_\epsilon(t))}\leq\epsilon^5e^{C_2\epsilon t}.\tag{27}$$

Let now consider the viscous case.

For the viscous case we have

$$\partial_t \widetilde{\mathbf{U}}_\epsilon + \nabla.\mathbf{F}(\widetilde{\mathbf{U}}_\epsilon) - \epsilon\nu \left[\begin{array}{c} 0 \\ \triangle \mathbf{u}_\epsilon \end{array} \right] = 0 \tag{28}$$

for the exact system, and

$$\partial_t \overline{\mathbf{U}}_\epsilon + \nabla.\mathbf{F}(\overline{\mathbf{U}}_\epsilon) - \epsilon\nu \left[\begin{array}{c} 0 \\ \triangle \bar{\mathbf{u}}_\epsilon \end{array} \right] = \epsilon^{\frac{5}{3}} \mathbf{R} \tag{29}$$

for the approximate system.

Theorem 2 *Suppose that the initial data of the KZK Cauchy problem* $I_0(t,\mathbf{y}) = I_0(t, \sqrt{\epsilon}\mathbf{x}')$ *is such that*

1. I_0 *is* L*-periodic in* t *and with mean value zero,*
2. *for fixed* t, I_0 *has the same sign for all* $\mathbf{y} \in \mathbb{R}^{n-1}$, *and for* $t \in]0, L[$ *the sign changes, i.e.* $I_0 = 0$, *only for a finite number of times,*
3. $I_0(t,\mathbf{y}) \in H^{s'}(\{t \geq 0\} \times \mathbb{R}^{n-1})$ *for* $s' > \max\{6, [\frac{n}{2}] + 1\}$,
4. I_0 *is sufficiently small such that*

$$\|I_0\|_{H^{s'}} < \frac{\nu}{2c^2\rho_0} \frac{C_1(L)}{C_2(s')} \quad \textit{(see [5, p.20])},$$

and $I_0 = \epsilon^\alpha \tilde{I}_0$, $\alpha \geq 0$.

Then there exists a unique global solution in time $\overline{\mathbf{U}}_\epsilon = (\bar{\rho}_\epsilon, \bar{\mathbf{u}}_\epsilon)$ *of the approximate system (29) deduced from a solution of the KZK equation with the help of correctors* v, $\mathbf{w}$, v_1, *found by* I *following the formulae of the derivation KZK-ansatz, ensuring the remainder term of the order of* $\epsilon^{\frac{5}{2}}$. *The function* $\overline{\mathbf{U}}_\epsilon(x_1, \mathbf{x}', t) = \overline{\mathbf{U}}_\epsilon(x_1 - ct, \epsilon x_1, \sqrt{\epsilon}\mathbf{x}')$, *given by formula (23), is defined in the half space (see [4] for its regularity)*

$$\{x_1 > 0, \quad t > 0, \quad \mathbf{x}' \in \mathbb{R}^{n-1}\}. \tag{30}$$

The Navier-Stokes system (28) in the half space with initial data (25) and following boundary conditions

$$(\bar{\mathbf{u}}_\epsilon - \mathbf{u}_\epsilon)|_{x_1=0} = 0,$$

with positive first component of the velocity $u_{\epsilon,1}|_{x_1=0} > 0$ *(i.e. at points where the fluid enters the domain) has the additional boundary condition*

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $u_{\epsilon,1}|_{x_1=0} \leq 0$ *there is no any boundary condition for* ρ_ϵ.

Then there exists a constant $T_0 > 0$ *such that for all* $t < \frac{T_0}{\epsilon^{2+\alpha}}$ *there exists a unique solution* $\mathbf{U}_\epsilon = (\rho_\epsilon, \mathbf{u}_\epsilon)$ *of the Navier-Stokes system (28) with the same smoothness as* $\overline{\mathbf{U}}_\epsilon$.

In addition, there exist positive constants $C_1 > 0$ and $C_2 > 0$ such that for all small enough ϵ

$$C_1\epsilon^{\frac{5}{2}}\sqrt{t} \leq \|\rho_\epsilon - \overline{\rho}_\epsilon\|_{L_2} + \|\rho_\epsilon \mathbf{u}_\epsilon - \overline{\rho}_\epsilon\overline{\mathbf{u}}_\epsilon\|_{L_2} \leq \epsilon^{\frac{5}{2}} e^{C_2\epsilon t}. \tag{31}$$

Estimate (31) ensures that its left-hand side remains smaller than the order of ϵ for any finite time

$$0 < t < \frac{T}{\epsilon}\ln\frac{1}{\epsilon},$$

where T is a positive constant and $T = O(1)$.

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