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**UNIFORMLY ACCURATE TIME-SPLITTING METHODS FOR
THE SEMICLASSICAL SCHRÖDINGER EQUATION
PART 2 : NUMERICAL ANALYSIS OF THE LINEAR CASE**

PHILIPPE CHARTIER, LOÏC LE TREUST, AND FLORIAN MÉHATS

ABSTRACT. This article is second part of a twofold paper, devoted to the construction of numerical methods which remain insensitive to the smallness of the semiclassical parameter for the Schrödinger equation in the semiclassical limit. Here, we specifically analyse the convergence behavior of the first-order splitting introduced in Part I, for a linear equation with smooth potential. Our main result is a proof of uniform accuracy.

1. Introduction

This paper is the follow-up of a first part which introduces high-order uniformly accurate schemes in the non-linear case. We are concerned here with the numerical approximation of the solution $\Psi^\varepsilon : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{C}$, $d \geq 1$, of the *linear* Schrödinger equation in its semiclassical limit

$$i\varepsilon \partial_t \Psi^\varepsilon = -\frac{\varepsilon^2}{2} \Delta \Psi^\varepsilon + \mathcal{V} \Psi^\varepsilon \quad (1.1)$$

where \mathcal{V} is a smooth potential. The initial datum is assumed to be of the form

$$\Psi^\varepsilon(0, \cdot) = A_0(\cdot) e^{iS_0(\cdot)/\varepsilon} \quad \text{with} \quad \|A_0\|_{L^2(\mathbb{R}^d)} = 1. \quad (1.2)$$

As described in Part I, the problem is reformulated according to the strategy adopted in [2]. This is achieved by decomposing Ψ^ε as the product of a slowly varying amplitude and a fast oscillating factor

$$\Psi^\varepsilon(t, \cdot) = A^\varepsilon(t, \cdot) e^{iS^\varepsilon(t, \cdot)/\varepsilon}, \quad (1.3)$$

where $(S^\varepsilon, A^\varepsilon)$ satisfies

$$\partial_t S^\varepsilon + \frac{|\nabla S^\varepsilon|^2}{2} + \mathcal{V} = \varepsilon^2 \Delta S^\varepsilon, \quad (1.4a)$$

$$\partial_t A^\varepsilon + \nabla S^\varepsilon \cdot \nabla A^\varepsilon + \frac{A^\varepsilon}{2} \Delta S^\varepsilon = \frac{i\varepsilon \Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon \Delta S^\varepsilon \quad (1.4b)$$

with $S^\varepsilon(0, x) = S_0(x)$, $A^\varepsilon(0, x) = A_0(x)$ and $x \in \mathbb{R}^d$. Recall that system (1.4) is equivalent to the original equation (1.1) (see Part I). The existence and uniqueness of the solution of equation (1.1) is proved for instance in [4]. The corresponding result for equations 1.4 will be derived in Section 3.4.

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In this second part, we concentrate on the numerical analysis of the first-order splitting scheme introduced in the first part, which, we believe, is of interest for its own sake. For the sake of clarity, we now recall it in this specific case (linear). System (1.4) is split into four pieces as follows:

First flow: We denote φ_h^1 the approximate flow at time $h \in \mathbb{R}$ of the system

$$\partial_t S + \frac{|\nabla S|^2}{2} = 0, \quad (1.5a)$$

$$\partial_t A + \nabla S \cdot \nabla A + \frac{A}{2} \Delta S = \frac{i \Delta A}{2}. \quad (1.5b)$$

The eikonal equation (1.5a) is solved by means of the method of characteristics, while equation (1.5b) is dealt with by noticing that $w = A \exp(iS)$ satisfies the free Schrödinger equation $i \partial_t w = -\frac{1}{2} \Delta w$.

Second flow: We define φ_h^2 as the exact flow at time $h \in \mathbb{R}$ of the system

$$\partial_t S = 0, \quad (1.6a)$$

$$\partial_t A = \frac{i(\varepsilon - 1) \Delta A}{2}, \quad (1.6b)$$

which is solved in the Fourier space.

Third flow: The third flow φ_h^3 is defined as the exact flow at time $h \in \mathbb{R}$ of system

$$\partial_t S = -\mathcal{V}, \quad (1.7a)$$

$$\partial_t A = 0. \quad (1.7b)$$

Fourth flow: The fourth flow φ_h^4 is defined as the exact flow at time $h \in \mathbb{R}_+$ of

$$\partial_t S = \varepsilon^2 \Delta S, \quad (1.8a)$$

$$\partial_t A = -i \varepsilon A \Delta S. \quad (1.8b)$$

Equation (1.8a) is solved in the Fourier space and the solution of (1.8b) is simply obtained through the formula $A(h, \cdot) = \exp(-i \varepsilon^{-1} (S(h, \cdot) - S(0, \cdot))) A(0, \cdot)$. Notice that φ_h^4 can thus be viewed as a regularizing flow.

The first-order scheme that we consider for (1.4) is then the concatenation of all previous flows

$$\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4. \quad (1.9)$$

The main result of this part is the following theorem: it states that $\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4$ is uniformly accurate w.r.t. the semi-classical parameter ε . The proper statement of the result uses the norm $\|\cdot\|_s$ on the set $\Sigma_s = H^{s+2}(\mathbb{R}^d) \times H^s(\mathbb{R}^d)$ defined for $s \geq 0$ and $u = (S, A)$ by

$$\|u\|_s = \left(\|S\|_{H^{s+2}(\mathbb{R}^d)}^2 + \|A\|_{H^s(\mathbb{R}^d)}^2 \right)^{1/2}.$$

Theorem 1.1. *Let $s > d/2 + 1$, $\varepsilon_{max} > 0$, $u_0 \in \Sigma_{s+2}$ and $0 < T < T_{max}$ where*

$$T_{max} = \sup\{t > 0 : \tau \mapsto \phi_\tau^0(u_0) \in L^\infty([0, t]; \Sigma_{s+2})\}$$

and ϕ_τ^ε denotes the flow at time τ of (1.4). There exists $C > 0$ and $h_0 > 0$ such that the following error estimate holds true for any $\varepsilon \in (0, \varepsilon_{max}]$, any $h \in [0, h_0]$ and $n \in \mathbb{N}$ satisfying $nh \leq T$:

$$\|(\varphi_h^1 \circ \varphi_h^2 \circ \varphi_h^3 \circ \varphi_h^4)^n(u_0) - \phi_{nh}^\varepsilon(u_0)\|_s \leq Ch.$$

The constants C and h_0 do not depend on ε .

Remark 1.2. The constant T_{max} appearing in Theorem 1.1 is well-defined and positive (see Theorem 2.1).

Our proof is reminiscent of two previous results related to, on the one hand, splitting schemes for equations with Burgers nonlinearity [5] and on the other hand, splitting scheme for NLS in the semiclassical limit with [3]. Nonetheless, due to the finite-time existence of both exact and approximate flows, and to the peculiarity of the Lipschitz-type stability of the exact flows (see Lemma 2.3), our proof follows a different path. In particular, we lean the approximate solutions on the exact one to ensure that they do not blow up. Besides, the application of Lady Windermere's fan argument is somehow hidden in an induction procedure. Finally, let us mention that, in spite of the fact that we do not specifically address this case, it is our belief that this result can be extended to the Schrödinger equation with a nonlinearity of Hartree-type (see also [3, Remark 4.5]).

2. Numerical study of the scheme

2.1. Notations. Assume that $\varepsilon \in (0, \varepsilon_{max}]$ and $s > d/2 + 1$. For the sake of simplicity, we keep the notation of all the flows independent of ε . All the constants appearing in the proof depend on \mathcal{V} but not on $\varepsilon > 0$. We denote

$$\varphi_h^{ij} = \varphi_h^i \circ \varphi_h^j, \quad \varphi_h^{ijk} = \varphi_h^i \circ \varphi_h^{jk}, \quad \varphi_h^{1234} = \varphi_h^1 \circ \varphi_h^{234},$$

\mathcal{N}_i is the possibly nonlinear operator related to φ_h^i . The quantities $\partial_h \varphi_h(u)$ and $\partial_2 \varphi_h(u)$ are the Fréchet derivatives of φ with respect to h and u . The commutator of the nonlinear operators \mathcal{N}_i and \mathcal{N}_j is given by

$$[\mathcal{N}_i, \mathcal{N}_j](u) = D\mathcal{N}_i(u) \cdot \mathcal{N}_j(u) - D\mathcal{N}_j(u) \cdot \mathcal{N}_i(u).$$

2.2. Existence, uniqueness and uniform boundedness results. The following theorem study some properties of the solutions of equations (1.4).

Theorem 2.1. Let $\varepsilon_{max} > 0$, $s > d/2 + 1$ and $u_0 \in \Sigma_{s+2}$. The following two points are true.

(i) The quantity

$$T_{max} = \sup\{t > 0 : \phi^0(u_0) \in L^\infty([0, t]; \Sigma_{s+2})\} \quad (2.1)$$

is well-defined and positive.

(ii) Let $0 < T < T_{max}$. For all $\varepsilon \in [0, \varepsilon_{max}]$, there exists a unique solution

$$\phi^\varepsilon(u_0) \in C([0, T], \Sigma_{s+2})$$

of system of equations (1.4). Moreover, $\phi^\varepsilon(u_0)$ is bounded in

$$C([0, T], \Sigma_{s+2})$$

uniformly in $\varepsilon \in [0, \varepsilon_{max}]$.

The proof of Theorem 2.1 is given in Section 3.4.

2.3. The main lemmas. In this subsection, we present the main ingredients needed in the proof of Theorem 1.1. Their proof is postponed to Section 3.

Lemma 2.2. *Let $M > 0$ and $s > d/2 + 1$. There exist $h_1 = h_1(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_s$ satisfying*

$$\|u_0\|_s \leq M,$$

we have that the solution $\phi_t(u_0)$ of equation (1.4) is well-defined on $[0, h_1]$ and for all $t \in [0, h_1]$

$$\|\phi_t(u_0)\|_s \leq 2M.$$

Lemma 2.3. *Let $M > 0$ and $s > d/2 + 1$. There exist $C_2 = C_2(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$, any solutions $\phi_t(u_1) \in L^\infty([0, T], \Sigma_{s+1})$ and $\phi_t(u_2) \in L^\infty([0, T], \Sigma_s)$ of equation (1.4), satisfying for all $t \in [0, T]$*

$$\|\phi_t(u_1)\|_{s+1} + \|\phi_t(u_2)\|_s \leq M$$

we have

$$\|\phi_t(u_1) - \phi_t(u_2)\|_s \leq \|u_1 - u_2\|_s \exp(C_2 t).$$

Remark 2.4. *Let us insist on the fact that in Lemma 2.3, we have to control $\phi_t(u_1)$ in Σ_{s+1} and $\phi_t(u_2)$ in Σ_s to get Lipschitz-type stability in Σ_s .*

Lemma 2.5. *Let $M > 0$ and $s > d/2 + 1$. There exist $h_3 = h_3(M) > 0$ and $C_3 = C_3(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$, any $u_0 \in \Sigma_s$ satisfying $\|u_0\|_s \leq M$ and any $0 \leq t \leq h_3$, we have*

$$(a) \|\varphi_t^{1234}(u_0)\|_s \leq 8M.$$

(b) Furthermore, if $u_0 \in \Sigma_{s+2}$, then

$$\|\varphi_t^{1234}(u_0)\|_{s+2} \leq \exp(C_3 t) (\|u_0\|_{s+2} + t \|\mathcal{V}\|_{H^{s+4}}).$$

Lemma 2.6. *Let $M > 0$ and $s > d/2 + 1$. There exist $h_4 = h_4(M) > 0$ and $K_4 = K_4(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_{s+2}$ satisfying*

$$\|u_0\|_{s+2} \leq M,$$

we have for any $t \in [0, h_4]$ that

$$\|\phi_t(u_0) - \varphi_t^{1234}(u_0)\|_s \leq K_4 t^2.$$

2.4. Proof of Theorem 1.1. Let us denote

$$M_s^\varepsilon(T) := \sup\{\|\phi_t^\varepsilon(u_0)\|_s : 0 \leq t \leq T\}. \quad (2.2)$$

for $\varepsilon \geq 0$ and $T \geq 0$.

Let $s > d/2 + 1$, $\varepsilon \in (0, \varepsilon_{max}]$, $u_0 \in \Sigma_{s+2}$, $n \in \mathbb{N}$ and $h > 0$ be such that $nh \leq T < T_{max}$ (see (2.1)). By Theorem 2.1, there exist M_s , M_{s+1} and M_{s+2} independent of $\varepsilon \in (0, \varepsilon_{max}]$ such that for all $\varepsilon \in (0, \varepsilon_{max}]$,

$$M_s^\varepsilon \leq M_s, M_{s+1}^\varepsilon \leq M_{s+1} \text{ and } M_{s+2}^\varepsilon \leq M_{s+2}$$

(see (2.2)). We denote

$$\begin{aligned} a &= \sup \left\{ \frac{t}{e^t - 1}, t \geq 0 \right\}, \\ C &= C_3(2M_s), \\ c_0 &= \|u_0\|_{s+2} \exp(CT) + a\|\mathcal{V}\|_{H^{s+4}} e^{2TC}/C, \\ C' &= C_2(M_{s+1} + 4M_s), \\ \tilde{c} &= K_4(c_0)ae^{C'T}/C'. \end{aligned}$$

Assume that

$$0 \leq h \leq \min(h_3(c_0), M_s/\tilde{c}, h_1(2M_s), h_4(c_0)). \quad (2.3)$$

Here, h_1, C_2, h_3, h_4 and K_4 are defined in Lemmas 2.2, 2.3, 2.5 and 2.6.

We show by induction on $0 \leq k \leq n$ that

(i) $(\varphi_h^{1234})^k(u_0)$ is well-defined, belongs to Σ_{s+2} and

$$\|(\varphi_h^{1234})^k(u_0)\|_{s+2} \leq \|u_0\|_{s+2} \exp(Ch) + h\|\mathcal{V}\|_{H^{s+4}} \frac{e^{(k+1)hC} - e^{hC}}{e^{hC} - 1} \leq c_0,$$

(ii) $\|\phi_{kh}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \leq h^2 K_4(c_0) \frac{e^{C'hk} - 1}{e^{C'h} - 1} \leq \tilde{c}h$,

and Theorem 1.1 follows then from point (ii) with $k = n$.

The induction hypothesis are true for $k = 0$. Let us assume points (i) and (ii) true for $0 \leq k \leq n - 1$.

Lemma 2.5, point (i) and (2.3) ensure that

$$(\varphi_h^{1234})^{k+1}(u_0)$$

is well-defined and belongs to Σ_{s+2} . By Point (ii) and (2.3), we have

$$\|(\varphi_h^{1234})^k(u_0)\|_s \leq M_s + \|\phi_{kh}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \leq 2M_s.$$

By Lemma 2.5 and (2.3), we have

$$\|(\varphi_h^{1234})^{k+1}\|_{s+2} \leq \exp(Ch) \left(\|(\varphi_h^{1234})^k(u_0)\|_{s+2} + h\|\mathcal{V}\|_{H^{s+4}} \right)$$

and point (i) ensures that

$$\|(\varphi_h^{1234})^{k+1}\|_{s+2} \leq \|u_0\|_{s+2} \exp(C(k+1)h) + h\|\mathcal{V}\|_{H^{s+4}} \frac{e^{(k+2)hC} - e^{hC}}{e^{hC} - 1} \leq c_0.$$

By Lemma 2.2 and (2.3), $h' \mapsto \phi_{h'} \circ (\varphi_h^{1234})^k(u_0)$ is well-defined and satisfies for all $0 \leq h' \leq h$

$$\|\phi_{h'} \circ (\varphi_h^{1234})^k(u_0)\|_s \leq 4M_s.$$

By Lemma 2.3, we obtain that

$$\|\phi_{h(k+1)}(u_0) - \phi_h \circ (\varphi_h^{1234})^k(u_0)\|_s \leq \|\phi_{hk}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \exp(C'h).$$

By Lemma 2.6, point (i) and (2.3), we get

$$\|\phi_h \circ (\varphi_h^{1234})^k(u_0) - \varphi_h^{1234} \circ (\varphi_h^{1234})^k(u_0)\|_s \leq K_4(c_0)h^2,$$

so that

$$\|\phi_{h(k+1)}(u_0) - (\varphi_h^{1234})^{k+1}(u_0)\|_s \leq K_4(c_0)h^2 + \|\phi_{hk}(u_0) - (\varphi_h^{1234})^k(u_0)\|_s \exp(C'h).$$

By point (ii), we have then that

$$\|\phi_{h(k+1)}(u_0) - (\varphi_h^{1234})^{k+1}(u_0)\|_s \leq K_4(c_0)h^2 \left(\frac{e^{C'h(k+1)} - 1}{e^{C'h} - 1} \right).$$

Thus, points (i) and (ii) are true for $k + 1$.

3. Proof of the main lemmas

3.1. Auxiliary results. Let us denote by $\langle \cdot, \cdot \rangle$ the L^2 scalar product, for $s > 0$

$$\Lambda^s = (1 - \Delta)^{s/2},$$

$$\Pi_1 u = S, \quad \Pi_2 u = A, \quad \text{for } u = \begin{pmatrix} S \\ A \end{pmatrix} \quad (3.1)$$

and

$$\begin{aligned} \langle u_1, u_2 \rangle_s &= \langle \Pi_1 u_1, \Pi_1 u_2 \rangle \\ &+ \langle \Lambda^{s+1} \nabla \Pi_1 u_1, \Lambda^{s+1} \nabla \Pi_1 u_2 \rangle + \operatorname{Re} \langle \Lambda^s \Pi_2 u_1, \Lambda^s \Pi_2 u_2 \rangle. \end{aligned} \quad (3.2)$$

We recall two points that will be of constant use in the following: the Sobolev space $H^s \subset L^\infty$ is an algebra for $s > d/2$ and the Kato-Ponce [6] inequality holds true:

Proposition 3.1. *Let $s_0 > d/2 + 1$. There is $c > 0$ such that for all $f \in H^{s_0}(\mathbb{R}^d)$ and $g \in H^{s_0-1}(\mathbb{R}^d)$*

$$\|\Lambda^{s_0}(fg) - f\Lambda^{s_0}g\|_{L^2} \leq c(\|\nabla f\|_{L^\infty}\|g\|_{H^{s_0-1}} + \|f\|_{H^{s_0}}\|g\|_{L^\infty}).$$

The following lemmas will be used several times in our proof.

Lemma 3.2. *Let $s_0 > d/2 + 1$. There is $C > 0$ such that for all v_0, v_1 and $R \in L^\infty([0, h_0], H^{s_0}(\mathbb{R}^d)^d)$ satisfying*

$$\partial_t v_0 + (v_1 \cdot \nabla)v_0 = R,$$

we have

$$\begin{aligned} \partial_t \|v_0\|_{H^{s_0}}^2 &\leq C (\|v_0\|_{H^{s_0}}^2 \|\nabla v_1\|_{L^\infty} + \|v_0\|_{H^{s_0}} \|v_1\|_{H^{s_0}} \|\nabla v_0\|_{L^\infty}) + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle \\ &\leq C \|v_0\|_{H^{s_0}}^2 \|v_1\|_{H^{s_0}} + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle. \end{aligned}$$

Proof. We have by integration by parts that

$$\begin{aligned} \partial_t \frac{\|v_0\|_{H^{s_0}}^2}{2} &= \langle \Lambda^{s_0} v_0, \Lambda^{s_0} \partial_t v_0 \rangle = - \langle \Lambda^{s_0} v_0, \Lambda^{s_0} (v_1 \cdot \nabla)v_0 \rangle + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle \\ &\leq \frac{1}{2} \int_{\mathbb{R}^d} |\Lambda^{s_0} v_0|^2 \operatorname{div} v_1 + \|v_0\|_{H^{s_0}} \|[\Lambda^{s_0}, (v_1 \cdot \nabla)]v_0\|_{L^2} + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle. \end{aligned}$$

Proposition 3.1 ensures that

$$\partial_t \frac{\|v_0\|_{H^{s_0}}^2}{2} \leq c (\|v_0\|_{H^{s_0}}^2 \|\nabla v_1\|_{L^\infty} + \|v_0\|_{H^{s_0}} \|v_1\|_{H^{s_0}} \|\nabla v_0\|_{L^\infty}) + \langle \Lambda^{s_0} v_0, \Lambda^{s_0} R \rangle.$$

□

Lemma 3.3. *Let $s_0 > d/2 + 1$. There exists $C > 0$ such that for all $v_1 \in L^\infty([0, h_0], H^{s_0+1}(\mathbb{R}^d)^d)$ and $R \in L^\infty([0, h_0], H^{s_0}(\mathbb{R}^d))$ satisfying*

$$\partial_t A + v_1 \cdot \nabla A + A \frac{\operatorname{div} v_1}{2} = R,$$

we have,

$$\begin{aligned} \partial_t \|A\|_{H^{s_0}}^2 &\leq C \left(\|A\|_{H^{s_0}}^2 \|v_1\|_{W^{2,\infty}} + \|A\|_{H^{s_0}} \|v_1\|_{H^{s_0+1}} \|A\|_{W^{1,\infty}} \right) + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle \\ &\leq C \|A\|_{H^{s_0}}^2 \|v_1\|_{H^{s_0+1}} + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle. \end{aligned}$$

Proof. We have by integration by parts that

$$\begin{aligned} \partial_t \frac{\|A\|_{H^{s_0}}^2}{2} &= \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} \partial_t A \rangle = -\operatorname{Re} \left\langle \Lambda^{s_0} A, \left(v_1 \cdot \nabla + \frac{\operatorname{div} v_1}{2} \right) \Lambda^{s_0} A \right\rangle \\ &\quad - \operatorname{Re} \left\langle \Lambda^{s_0} A, \left[\Lambda^{s_0}, \left(v_1 \cdot \nabla + \frac{\operatorname{div} v_1}{2} \right) \right] A \right\rangle + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle \\ &\leq \|A\|_{H^{s_0}} \left\| \left[\Lambda^{s_0}, \left(v_1 \cdot \nabla + \frac{\operatorname{div} v_1}{2} \right) \right] A \right\|_{L^2} + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle. \end{aligned}$$

Proposition 3.1 ensures that

$$\begin{aligned} \partial_t \frac{\|A\|_{H^{s_0}}^2}{2} &\leq C \|A\|_{H^{s_0}} (\|\nabla v_1\|_{L^\infty} \|\nabla A\|_{H^{s_0-1}} + \|v_1\|_{H^{s_0}} \|\nabla A\|_{L^\infty}) \\ &\quad + C \|A\|_{H^{s_0}} (\|\nabla(\operatorname{div} v_1)\|_{L^\infty} \|A\|_{H^{s_0-1}} + \|\operatorname{div} v_1\|_{H^{s_0}} \|A\|_{L^\infty}) \\ &\quad + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle \\ &\leq C \left(\|A\|_{H^{s_0}}^2 \|v_1\|_{W^{2,\infty}} + \|A\|_{H^{s_0}} \|v_1\|_{H^{s_0+1}} \|A\|_{W^{1,\infty}} \right) + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle \\ &\leq C \|A\|_{H^{s_0}}^2 \|v_1\|_{H^{s_0+1}} + \operatorname{Re} \langle \Lambda^{s_0} A, \Lambda^{s_0} R \rangle. \end{aligned}$$

□

3.2. Study of the equation (1.4). Let us prove Lemma 2.2.

Proof. By the Cole-Hopf transform, we get that $w^\varepsilon = \exp\left(-\frac{S^\varepsilon}{2\varepsilon^2}\right) - 1$ is the solution of

$$\partial_t w^\varepsilon = \varepsilon^2 \Delta w^\varepsilon + \frac{\mathcal{V}}{2\varepsilon^2} (w^\varepsilon + 1), \quad w^\varepsilon(0) = \exp\left(-\frac{S_0}{2\varepsilon^2}\right) - 1,$$

Hence, global existence and uniqueness of the solution S^ε of (1.4a) for fixed $\varepsilon \in (0, \varepsilon_{max}]$, follows from standard semi-group theory. The function $v^\varepsilon = \nabla S^\varepsilon$ solves

$$\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla \mathcal{V} = \varepsilon^2 \Delta v^\varepsilon.$$

Since $s > d/2$, Lemma 3.2 and an integration by parts ensure that

$$\begin{aligned} \partial_t \|v^\varepsilon\|_{H^{s+1}}^2 &\leq c \|v^\varepsilon\|_{H^{s+1}}^3 + \langle \Lambda^{s+1} v^\varepsilon, \Lambda^{s+1} (-\nabla \mathcal{V} + \varepsilon^2 \Delta v^\varepsilon) \rangle \\ &\leq c \|v^\varepsilon\|_{H^{s+1}}^3 + \|v^\varepsilon\|_{H^{s+1}} \|\mathcal{V}\|_{H^{s+2}}. \end{aligned}$$

By (1.4a), we also have that

$$\partial_t \frac{\|S^\varepsilon\|_{L^2}^2}{2} \leq \|S^\varepsilon\|_{L^2} (\|\mathcal{V}\|_{L^2} + \|v^\varepsilon\|_{L^4}^2/2)$$

so that

$$\partial_t \|S^\varepsilon\|_{H^{s+2}}^2 \leq c \|\mathcal{V}\|_{H^{s+2}} \|S^\varepsilon\|_{H^{s+2}} + c \|S^\varepsilon\|_{H^{s+2}}^3.$$

The global existence and the uniqueness of a solution A^ε of equation (1.4b) follows from the fact that

$$\Psi^\varepsilon = A^\varepsilon \exp(iS^\varepsilon/\varepsilon)$$

satisfies equation (1.1). By Lemma 3.3, recalling that $s > d/2 + 1$, we also have

$$\partial_t \|A^\varepsilon\|_{H^s}^2 \leq c \|A^\varepsilon\|_{H^s}^2 \|S^\varepsilon\|_{H^{s+2}} + \operatorname{Re} \langle \Lambda^s A^\varepsilon, \Lambda^s R \rangle.$$

where $R = \frac{i\varepsilon \Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon \Delta S^\varepsilon$ so that an integration by parts gives us

$$\partial_t \|A^\varepsilon\|_{H^s}^2 \leq c \|A^\varepsilon\|_{H^s}^2 \|S^\varepsilon\|_{H^{s+2}}.$$

We obtain that

$$\partial_t \|\phi_t(u_0)\|_s^2 \leq c_1 \|\phi_t(u_0)\|_s \|\mathcal{V}\|_{H^{s+2}} + c_2 \|\phi_t(u_0)\|_s^3$$

and

$$\partial_t \|\phi_t(u_0)\|_s \leq c_1 \|\mathcal{V}\|_{H^{s+2}} + c_2 \|\phi_t(u_0)\|_s^2.$$

We get then that

$$\|\phi_t(u_0)\|_s \leq \sqrt{\frac{c_1 \|\mathcal{V}\|_{H^{s+2}}}{c_2}} \tan \left(t \sqrt{c_1 c_2 \|\mathcal{V}\|_{H^{s+2}}} + \arctan \left(M \sqrt{\frac{c_2}{c_1 \|\mathcal{V}\|_{H^{s+2}}}} \right) \right)$$

so that there is $h_1 = h_1(M) > 0$ such that for all $0 \leq t \leq h_1$

$$\|\phi_t(u_0)\|_s \leq 2M.$$

□

The following result will be used several times and in particular for the proof of the stability of equation (1.4) in Lemma 2.3.

Lemma 3.4. *Let $s_0 > d/2 + 1$. Let $u_1 = (S_1, A_1)$ be in $L^\infty([0, T], \Sigma_{s_0+1})$, $u_2 = (S_2, A_2)$, $(R_{1,S}, R_{1,A})$ and $(R_{2,S}, R_{2,A})$ be in $L^\infty([0, T], \Sigma_{s_0})$. Assume moreover that for $i = 1, 2$*

$$\begin{aligned} \partial_t S_i + \frac{|\nabla S_i|^2}{2} &= R_{i,S}, \\ \partial_t A_i + \nabla S_i \cdot \nabla A_i + A_i \frac{\Delta S_i}{2} &= R_{i,A}. \end{aligned}$$

Then, we have

$$\partial_t \|u_1 - u_2\|_{s_0}^2 \leq c \|u_1 - u_2\|_{s_0}^2 (\|u_1\|_{s_0+1} + \|u_2\|_{s_0}) + \langle u_1 - u_2, R_1 - R_2 \rangle_{s_0}$$

where $R_i = (R_{i,S}, R_{i,A})^T$.

Proof. Let $s_0 > d/2 + 1$. Let us define $v_1 = \nabla S_1$, $v_2 = \nabla S_2$, $w = v_1 - v_2$, $B = A_1 - A_2$ and $u = u_1 - u_2$.

We have that

$$\begin{aligned} \partial_t w &= -(v_1 \cdot \nabla)v_1 + (v_2 \cdot \nabla)v_2 + \nabla(R_{1,S} - R_{2,S}) \\ &= -(v_2 \cdot \nabla)w - (w \cdot \nabla)v_1 + \nabla(R_{1,S} - R_{2,S}) \end{aligned}$$

and Lemma 3.2 ensures that

$$\partial_t \|w\|_{H^{s_0+1}}^2 \leq c \|w\|_{H^{s_0+1}}^2 \|v_2\|_{H^{s_0+1}} + \langle \Lambda^{s_0+1} w, \Lambda^{s_0+1} R \rangle.$$

where $R = -(w \cdot \nabla)v_1 + \nabla(R_{1,S} - R_{2,S})$. We also have that

$$\|R\|_{H^{s_0+1}} \leq c \|w\|_{H^{s_0+1}} \|v_1\|_{H^{s_0+2}} + \langle \Lambda^{s_0+1} w, \Lambda^{s_0+1} \nabla(R_{1,S} - R_{2,S}) \rangle$$

and

$$\partial_t \|w\|_{H^{s_0+1}}^2 \leq c \|w\|_{H^{s_0+1}}^2 (\|S_1\|_{H^{s_0+3}} + \|S_2\|_{H^{s_0+2}}) + \langle \Lambda^{s_0+1} w, \Lambda^{s_0+1} \nabla (R_{1,S} - R_{2,S}) \rangle.$$

We also have

$$\partial_t (S_1 - S_2) = -\frac{1}{2} (v_1 + v_2) \cdot w + (R_{1,S} - R_{2,S})$$

so that

$$\partial_t \|S_1 - S_2\|_{L^2}^2 \leq c \|S_1 - S_2\|_{L^2} \|w\|_{L^2} (\|S_1\|_{W^{1,\infty}} + \|S_2\|_{W^{1,\infty}}) + \langle S_1 - S_2, R_{1,S} - R_{2,S} \rangle$$

and then

$$\begin{aligned} \partial_t \|S_1 - S_2\|_{H^{s_0+2}}^2 &\leq C \|S_1 - S_2\|_{H^{s_0+2}}^2 (\|S_1\|_{H^{s_0+3}} + \|S_2\|_{H^{s_0+2}}) \\ &\quad + \langle S_1 - S_2, R_{1,S} - R_{2,S} \rangle + \langle \Lambda^{s_0+1} \nabla (S_1 - S_2), \Lambda^{s_0+1} \nabla (R_{1,S} - R_{2,S}) \rangle \end{aligned}$$

Let us study B , we have

$$\partial_t B + \nabla S_2 \cdot \nabla B + \frac{\Delta S_2}{2} B = R$$

where

$$R = -w \cdot \nabla A_1 - \frac{\operatorname{div}(w)}{2} A_1 + (R_{1,A} - R_{2,A})$$

Hence, we obtain by Lemma 3.3

$$\begin{aligned} \partial_t \|B\|_{H^{s_0}}^2 &\leq c \|B\|_{H^{s_0}}^2 \|S_2\|_{H^{s_0+2}} + \operatorname{Re} \langle \Lambda^{s_0} B, \Lambda^{s_0} R \rangle \\ &\leq c \|B\|_{H^{s_0}}^2 \|S_2\|_{H^{s_0+2}} + c \|B\|_{H^{s_0}} \|w\|_{H^{s_0+1}} \|A_1\|_{H^{s_0+1}} \\ &\quad + \operatorname{Re} \langle \Lambda^{s_0} B, \Lambda^{s_0} (R_{1,A} - R_{2,A}) \rangle \end{aligned}$$

and

$$\partial_t \|u_1 - u_2\|_{s_0}^2 \leq c \|u_1 - u_2\|_{s_0}^2 (\|u_1\|_{s_0+1} + \|u_2\|_{s_0}) + \langle u_1 - u_2, R_1 - R_2 \rangle_{s_0}$$

The result follows. \square

Let us study now the stability of equation (1.4) and prove Lemma 2.3.

Proof. Let $s > d/2 + 1$ and $\varepsilon \in (0, \varepsilon_{max}]$. Let us define for $i = 1, 2$

$$\begin{aligned} R_{i,S} &= -\mathcal{V} + \varepsilon^2 \Delta S_i, \\ R_{i,A} &= i\varepsilon \frac{\Delta A_i}{2} - i\varepsilon A_i \Delta S_i. \end{aligned}$$

We apply Lemma 3.4 with $s_0 = s$. We have by integrations by parts that

$$\langle S_1 - S_2, R_{1,S} - R_{2,S} \rangle + \langle \Lambda^{s+1} \nabla (S_1 - S_2), \Lambda^{s+1} \nabla (R_{1,S} - R_{2,S}) \rangle \leq 0,$$

and

$$\begin{aligned} \operatorname{Re} \langle \Lambda^s (A_1 - A_2), \Lambda^s (R_{1,A} - R_{2,A}) \rangle &\leq c \|A_1 - A_2\|_{H^s}^2 \|S_1\|_{H^{s+2}} \\ &\quad + c \|A_1 - A_2\|_{H^s} \|S_1 - S_2\|_{H^{s+2}} \|A_2\|_{H^s}. \end{aligned}$$

so that

$$\partial_t \|\phi_t(u_1) - \phi_t(u_2)\|_s^2 \leq c \|u_1 - u_2\|_s^2 (\|\phi_t(u_1)\|_{s+1} + \|\phi_t(u_2)\|_s)$$

and the result follows. \square

3.3. Study of the numerical flow φ^{1234} . The following lemma is inspired by the work of Holden, Lubich and Risebro [5].

Lemma 3.5. *Let $s_0 > d/2 + 1$ and $M > 0$. There exists $h_5 = h_5(M) > 0$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ and any $0 \leq t \leq h_5$, the following two points are true.*

(i) *We have that $\|\varphi_t^1(u_0)\|_{s_0} \leq 2M$.*

(ii) *Let $s_1 \geq s_0$. There is $C_5 = C_5(M) > 0$ such that if $u_0 \in \Sigma_{s_1}$, then*

$$\|\varphi_t^1(u_0)\|_{s_1} \leq \exp(C_5 t) \|u_0\|_{s_1}.$$

Proof. The existence of the solution S of (1.5a) follows for instance from the method of characteristics. Lemma 3.2 ensures that for $s > d/2 + 1$

$$\partial_t \|\nabla S\|_{H^{s+1}}^2 \leq c \|\nabla S\|_{H^{s+1}}^2 \|\nabla(\nabla S)\|_{L^\infty} \leq C \|\nabla S\|_{H^{s+1}}^2 \|S(t)\|_{W^{2,\infty}}.$$

We also have

$$\partial_t \|S\|_{L^2}^2 \leq c \|S(t)\|_{L^2} \|\nabla S(t)\|_{L^4}^2$$

so that

$$\partial_t \|S\|_{H^{s+2}}^2 \leq C \|S(t)\|_{H^{s+2}}^2 \|S(t)\|_{W^{2,\infty}}.$$

The remaining of the proof follows exactly the same lines as the one of Lemma 2.2. By Lemma 3.3 and an integration by parts, we have

$$\begin{aligned} \partial_t \|A\|_{H^s}^2 &\leq C (\|A\|_{H^s}^2 \|S\|_{W^{3,\infty}} + \|A\|_{H^s} \|S\|_{H^{s+2}} \|A\|_{W^{1,\infty}}) \\ &\leq C \|\varphi_t^1(u_0)\|_s^2 \|\varphi_t^1(u_0)\|_{W^{3,\infty} \times W^{1,\infty}} \end{aligned}$$

and

$$\begin{aligned} \partial_t \|\varphi_t^1(u_0)\|_s^2 &\leq C \|\varphi_t^1(u_0)\|_s^2 \|\varphi_t^1(u_0)\|_{W^{3,\infty} \times W^{1,\infty}} \\ &\leq C \|\varphi_t^1(u_0)\|_s^3. \end{aligned}$$

Taking $s = s_0$, we get that

$$\|\varphi_t^1(u_0)\|_{s_0} \leq \frac{M}{1 - cMt}$$

and there is $h_5 = h_5(M) > 0$ such that for all $t \in [0, h_5]$

$$\|\varphi_t^1(u_0)\|_{s_0} \leq 2M.$$

We also obtain for $s = s_1 \geq s_0 > d/2 + 1$ and $t \in [0, h_5]$ that

$$\begin{aligned} \partial_t \|\varphi_t^1(u_0)\|_{s_1}^2 &\leq C \|\varphi_t^1(u_0)\|_{s_1}^2 \|\varphi_t^1(u_0)\|_{s_0} \\ &\leq 2CM \|\varphi_t^1(u_0)\|_{s_1}^2. \end{aligned}$$

and the result follows from Grönwall's Lemma. \square

We immediately get the following result for the second and the third flows.

Lemma 3.6. *Let $s_0 > 0$ and $M > 0$. There is $h_6 = h_6(M)$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ any $0 \leq t \leq h_6$, the following two points holds true.*

(i) *$\|\varphi_t^2(u_0)\|_{s_0} \leq M$ and $\|\varphi_t^3(u_0)\|_{s_0} \leq 2M$,*

(ii) *Let $s_1 \geq 0$. If moreover $u_0 \in \Sigma_{s_1}$, then, we have*

$$\|\varphi_t^2(u_0)\|_{s_1} \leq \|u_0\|_{s_1} \text{ and } \|\varphi_t^3(u_0)\|_{s_1} \leq \|u_0\|_{s_1} + t \|\mathcal{V}\|_{H^{s_1+2}}.$$

The following lemma study the fourth flow.

Lemma 3.7. *Let $s_0 > d/2 + 1$ and $M > 0$. There exists $h_7 = h_7(M) > 0$ such that for any $u_0 \in \Sigma_{s_0}$ satisfying $\|u_0\|_{s_0} \leq M$ and any $0 \leq t \leq h_7$, the following two points holds true.*

(i) $\|\varphi_t^4(u_0)\|_{s_0} \leq 2M,$

(ii) Let $s_1 \geq s_0$. There is $C_7 = C_7(M) > 0$ such that if $u_0 \in \Sigma_{s_1}$,

$$\|\varphi_t^4(u_0)\|_{s_1} \leq \exp(C_7 t) \|u_0\|_{s_1}.$$

Proof. Let $s > d/2 + 1$. By integration by parts, we have $\partial_t \|S(h)\|_{H^{s+2}}^2 \leq 0$ and

$$\begin{aligned} \partial_t \frac{\|A\|_{H^s}^2}{2} &= \operatorname{Re} \langle \Lambda^s A, \Lambda^s (-i\varepsilon A \Delta S) \rangle = \operatorname{Re} \langle \Lambda^s A, [\Lambda^s, -i\varepsilon \Delta S] A \rangle \\ &\leq c \|A\|_{H^s} (\|\nabla(\Delta S)\|_{L^\infty} \|A\|_{H^{s-1}} + \|\Delta S\|_{H^s} \|A\|_{L^\infty}) \\ &\leq c \|A\|_{H^s}^2 \|S\|_{W^{3,\infty}} + c \|A\|_{H^s} \|S\|_{H^{s+2}} \|A\|_{L^\infty}. \end{aligned}$$

We obtain for $s = s_0$ that

$$\partial_t \|\varphi_t^4(u_0)\|_{s_0}^2 \leq c \|\varphi_t^4(u_0)\|_{s_0}^3,$$

for $s = s_1$ that

$$\partial_t \|\varphi_t^4(u_0)\|_{s_1}^2 \leq c \|\varphi_t^4(u_0)\|_{s_1}^2 \|\varphi_t^4(u_0)\|_{s_0}$$

and the result follows from the arguments of the end of the proof of Lemma 3.5. \square

Taking $s_0 = s$ and $s_1 = s_0 + 2$, we immediately get Lemma 2.5 combining Lemmas 3.5, 3.6 and 3.7.

3.4. Proof of Theorem 2.1. Let $M > 0$. Lemma 2.2 ensures that there is $h_1 = h_1(M) > 0$ such that for any $\varepsilon \in (0, \varepsilon_{max}]$ and any $u_0 \in \Sigma_{s+2}$ satisfying $\|u_0\|_{s+2} \leq M$, the solutions $t \mapsto \phi_t^\varepsilon(u_0)$ of equation (1.4) are well-defined in $L^\infty([0, h_1], \Sigma_{s+2})$ and uniformly bounded with respect to ε .

Let $\varepsilon, \varepsilon' \in (0, \varepsilon_{max}]$, $u_0, u'_0 \in \Sigma_{s+2}$ such that $\|u_0\|_{s+2} \leq M$ and $\|u'_0\|_{s+2} \leq M$. We define $(S^\varepsilon, A^\varepsilon)^T = \phi^\varepsilon(u_0)$, $(S^{\varepsilon'}, A^{\varepsilon'})^T = \phi^{\varepsilon'}(u'_0)$ and

$$\begin{aligned} R_{1,S} &= -\mathcal{V} + \varepsilon^2 \Delta S^\varepsilon, & R_{2,S} &= -\mathcal{V} + \varepsilon'^2 \Delta S^{\varepsilon'}, \\ R_{1,A} &= i\varepsilon \frac{\Delta A^\varepsilon}{2} - i\varepsilon A^\varepsilon \Delta S^\varepsilon, & R_{2,A} &= i\varepsilon' \frac{\Delta A^{\varepsilon'}}{2} - i\varepsilon' A^{\varepsilon'} \Delta S^{\varepsilon'}. \end{aligned}$$

We apply Lemma 3.4 with $s_0 = s$, $u_1 = \phi^\varepsilon(u_0)$ and $u_2 = \phi^{\varepsilon'}(u_0)$. We have by integrations by parts that

$$\begin{aligned} &\left\langle S^\varepsilon - S^{\varepsilon'}, R_{1,S} - R_{2,S} \right\rangle + \left\langle \Lambda^{s+1} \nabla (S^\varepsilon - S^{\varepsilon'}), \Lambda^{s+1} \nabla (R_{1,S} - R_{2,S}) \right\rangle \\ &\leq c |\varepsilon - \varepsilon'| \|S^\varepsilon - S^{\varepsilon'}\|_{H^{s+2}} \|S^\varepsilon\|_{H^{s+4}}, \end{aligned}$$

and

$$\begin{aligned} &\operatorname{Re} \left\langle \Lambda^s (A^\varepsilon - A^{\varepsilon'}), \Lambda^s (R_{1,A} - R_{2,A}) \right\rangle \leq c |\varepsilon - \varepsilon'| \|A^\varepsilon - A^{\varepsilon'}\|_{H^s} \|A^\varepsilon\|_{H^{s+2}} \\ &+ c \|A^\varepsilon - A^{\varepsilon'}\|_{H^s}^2 \|S^{\varepsilon'}\|_{H^{s+2}} + c \|A^\varepsilon\|_{H^s} \|A^\varepsilon - A^{\varepsilon'}\|_{H^s} \|S^\varepsilon - S^{\varepsilon'}\|_{H^{s+2}} \\ &+ c |\varepsilon - \varepsilon'| \|A^\varepsilon - A^{\varepsilon'}\|_{H^s} \|A^\varepsilon\|_{H^s} \|S^\varepsilon\|_{H^{s+2}}. \end{aligned}$$

so that

$$\begin{aligned} \partial_t \|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u'_0)\|_s^2 &\leq c \|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u'_0)\|_s^2 \left(\|\phi_t^\varepsilon(u_0)\|_{s+1} + \|\phi_t^{\varepsilon'}(u'_0)\|_s \right) \\ &+ c |\varepsilon - \varepsilon'| \|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u'_0)\|_s (\|\phi_t^\varepsilon(u_0)\|_{s+2} + \|\phi_t^\varepsilon(u_0)\|_s^2). \end{aligned}$$

Grönwall's Lemma ensures that for all $t \in [0, h_1]$

$$\|\phi_t^\varepsilon(u_0) - \phi_t^{\varepsilon'}(u'_0)\|_s \leq C (\|u_0 - u'_0\|_s + |\varepsilon - \varepsilon'|) \quad (3.3)$$

where

$$C = C(\|\phi^\varepsilon(u_0)\|_{L^\infty([0, h_1], \Sigma_{s+2})}, \|\phi^{\varepsilon'}(u'_0)\|_{L^\infty([0, h_1], \Sigma_s)}) > 0.$$

Thus, $(\phi_t^\varepsilon(u_0))_{t \in [0, h_1]}$ is a Cauchy sequence of ε of $L^\infty([0, h_1], \Sigma_s)$. The limit $\phi^0(u_0)$ is solution of (1.4) with $\varepsilon = 0$. Uniqueness follows from (3.3). We get immediately that Lemma 2.2 is also true for $\varepsilon = 0$ and $\phi^0(u_0) \in L^\infty([0, h_1], \Sigma_{s+2})$.

Let

$$T_{max} = \sup\{t > 0 : \phi^0(u_0) \in L^\infty([0, t]; \Sigma_{s+2})\} > 0,$$

then, for any $0 < T < T_{max}$, $\phi^0(u_0) \in L^\infty([0, T]; \Sigma_{s+2})$. Let us define $\tilde{T} = h_1(2M_s^0)$ (see Lemma 2.2 and (2.2)), $C = C(M_{s+2}^0, 2M_s^0)$ (see inequality (3.3)) and N the smallest $n \in \mathbb{N}$ such that

$$n\tilde{T} \geq T.$$

Let $\varepsilon_0 > 0$ be such that $\varepsilon_0 \sum_{j=1}^N C^j \leq M_s^0$ and $\varepsilon \in (0, \varepsilon_0]$. By inequality (3.3) and Lemma 2.2, we obtain by induction on $0 \leq k \leq N$ that

$$\|\phi_t^\varepsilon(u_0)\|_s \leq \|\phi_t^0(u_0)\|_s + \|\phi_t^0(u_0) - \phi_t^\varepsilon(u_0)\|_s \leq M_s^0 + \varepsilon \sum_{j=1}^k C^j \leq M_s^0 \leq 2M_s^0$$

for all $t \in [0, k\tilde{T}]$. Thus, $\phi^\varepsilon(u_0)$ is well-defined on $[0, T]$, belongs to $L^\infty([0, T]; \Sigma_s)$ and

$$\|\phi^\varepsilon(u_0)\|_{L^\infty([0, T]; \Sigma_s)} \leq 2M_s^0. \quad (3.4)$$

Following the arguments of the proofs of Lemmas 3.5, 3.6 and 3.7, we obtain that

$$\partial_t \|\phi^\varepsilon(u_0)\|_{s+2}^2 \leq c \|\phi^\varepsilon(u_0)\|_{s+2}^2 \|\phi^\varepsilon(u_0)\|_s + \|\phi^\varepsilon(u_0)\|_{s+2} \|\mathcal{V}\|_{H^{s+4}}.$$

Grönwall's lemma ensures that there is $\tilde{C} = \tilde{C}(M_s^0) > 0$ independent of ε such that

$$\|\phi_t^\varepsilon(u_0)\|_{s+2} \leq \exp(t\tilde{C}) (t\|\mathcal{V}\|_{H^{s+4}} + \|u_0\|_{s+2})$$

for all $t \in [0, T]$. Moreover, $\phi^\varepsilon(u_0)$ is well-defined in $L^\infty([0, T], \Sigma_{s+2})$ for any $\varepsilon \in (0, \varepsilon_{max}]$. Then, the same arguments ensure that $\varepsilon \in (0, \varepsilon_{max}] \mapsto \phi^\varepsilon(u_0)$ is continuous in $L^\infty([0, T], \Sigma_s)$ so that $(\phi^\varepsilon(u_0))_{\varepsilon \in [0, \varepsilon_{max}]}$ is uniformly bounded in $L^\infty([0, T], \Sigma_{s+2})$ and the result follows.

3.5. The local error estimates. The proof of Lemma 2.6 given in this section is inspired by [1] where the two flows case is treated. The local error of scheme (1.9) is defined by

$$\mathcal{R}(h, u) = \varphi_h^{1234}(u) - \phi_h(u).$$

3.5.1. Main lemmas. Let us give the main ingredients that will be used in the proof of Lemma 2.6. The balls in Σ_{s_0} are denoted by

$$B_{s_0}(M) = \{u \in \Sigma_{s_0} : \|u\|_{s_0} \leq M\} \quad (3.5)$$

for $s_0 \geq 0$ and $M > 0$. The strategy to get estimates on $\mathcal{R}(h, u)$ is to differentiate \mathcal{R} with respect to h . Hence, we will be in need of the following lemma whose proof is postponed to Appendix A.

Lemma 3.8. *Let $s > d/2 + 1$ and $M > 0$. There exists $h_8 = h_8(M) > 0$ such that the following two points hold true.*

(i) Let $s_1 \geq s$. The functions

$$\begin{aligned} (h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) &\mapsto \varphi_h^1(u) \in \Sigma_{s_1}, \\ (h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+2}) &\mapsto \varphi_h^2(u) \in \Sigma_{s_1}, \\ (h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1}) &\mapsto \varphi_h^3(u) \in \Sigma_{s_1}, \end{aligned}$$

are C^1 -applications.

(ii) Let $s_2 \geq s$ and $M_2 > 0$. There exists $C_8 = C_8(M, M_2) > 0$ such that for any $u \in B_s(M) \cap B_{s_2+1}(M_2)$, $h \in [0, h_8]$ and any $u_0 \in \Sigma_{s_2}$, we have

$$\begin{aligned} \|\partial_2 \varphi_h^1(u) \cdot u_0\|_{s_2} &\leq \exp(C_8 h) \|u_0\|_{s_2}, \\ \|\partial_2 \varphi_h^2(u) \cdot u_0\|_{s_2} &\leq \exp(C_8 h) \|u_0\|_{s_2}, \\ \|\partial_2 \varphi_h^3(u) \cdot u_0\|_{s_2} &\leq \exp(C_8 h) \|u_0\|_{s_2}, \end{aligned}$$

and

$$\begin{aligned} |\langle u_0, D\mathcal{N}_1(\varphi_h^1(u)) \cdot u_0 \rangle_{s_2}| &\leq C_8 \|u_0\|_{s_2}^2, \\ |\langle u_0, D\mathcal{N}_2(\varphi_h^2(u)) \cdot u_0 \rangle_{s_2}| &\leq C_8 \|u_0\|_{s_2}^2, \\ |\langle u_0, D\mathcal{N}_3(\varphi_h^3(u)) \cdot u_0 \rangle_{s_2}| &\leq C_8 \|u_0\|_{s_2}^2, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{s_0}$ is defined in (3.2) and $B_{s_0}(M)$ in (3.5).

The following lemma ensures that the object studied in the proof of Lemma 2.6 are well-defined.

Lemma 3.9. *Let $s > d/2 + 1$ and $M > 0$. There is $h_9 = h_9(M) > 0$ such that the following three points are true. Let $u \in \Sigma_{s+7}$ such that $\|u\|_{s+2} \leq M$.*

(i) We have for all $h \in [0, h_9]$,

$$\varphi_h^{1234}(u), \varphi_h^{234}(u), \varphi_h^{34}(u) \text{ and } \varphi_h^4(u)$$

are well-defined, belong to $L^\infty([0, h_9], \Sigma_{s+7})$ and satisfy

$$\max(\|\varphi_h^4(u)\|_{s+2}, \|\varphi_h^{34}(u)\|_{s+2}, \|\varphi_h^{234}(u)\|_{s+2}) \leq 4M.$$

(ii) The application $h \in [0, h_9] \mapsto \mathcal{R}(h, u) \in \Sigma_s$ is differentiable,

$$\partial_h \mathcal{R}(h, u) = \sum_{k=1}^4 \mathcal{N}_k(\varphi_h^{1234}(u)) - \mathcal{N}_k(\phi_h(u)) + \mathcal{S}(h, u),$$

$$\mathcal{R}(0, u) = 0.$$

where

$$\begin{aligned} \mathcal{S}(h, u) &= (\chi_{12} + \chi_{13} + \chi_{14})(h, \varphi_h^{234}(u)) \\ &\quad + \partial_2 \varphi^1(h, \varphi_h^{234}(u)) \cdot (\chi_{23} + \chi_{24})(h, \varphi_h^{34}(u)) \\ &\quad + \partial_2 \varphi^1(h, \varphi_h^{234}(u)) \cdot \partial_2 \varphi^2(h, \varphi_h^{34}(u)) \cdot \chi_{34}(h, \varphi_h^4(u)) \end{aligned}$$

and $\chi_{ij}(h, v) = \partial_2 \varphi_h^i(v) \cdot \mathcal{N}_j(v) - \mathcal{N}_j(\varphi_h^i(v))$ (see [1, Section 3]).

(iii) Let $v \in \Sigma_{s+7}$. We have,

$$\begin{aligned} \partial_h \chi_{ij}(h, v) &= D\mathcal{N}_i(\varphi_h^i(v)) \cdot \chi_{ij}(h, v) + [\mathcal{N}_i, \mathcal{N}_j](\varphi_h^i(v)) \\ \chi_{ij}(0, v) &= 0. \end{aligned}$$

The following lemma gives bounds on the commutators.

Lemma 3.10. *Let $s > d/2 + 1$. There is $C > 0$ such that for any $u \in \Sigma_{s+2}$ and any $1 \leq i < j \leq 4$, we have*

$$\|[\mathcal{N}_i, \mathcal{N}_j](u)\|_s \leq C \|u\|_{s+2}^2 (1 + \|u\|_{s+2}).$$

C does not depend on $\varepsilon \in (0, \varepsilon_{max}]$.

3.5.2. *Proof of Lemma 2.6.* Let $s > d/2 + 1$ and $M > 0$. Let us define $h_4 = h_4(M) = h_9(M)$. Assume for the moment that $u \in \Sigma_{s+7}$ and $\|u\|_{s+2} \leq M$. By Lemmas 3.8, 3.9, 3.10 and Grönwall's Lemma, there is $C = C(M) > 0$ such that for any $h \in [0, h_4]$

$$\begin{aligned} \|\chi_{12}(h, \varphi_h^{234}(u))\|_s + \|\chi_{13}(h, \varphi_h^{234}(u))\|_s + \|\chi_{14}(h, \varphi_h^{234}(u))\|_s &\leq Ch, \\ \|\chi_{23}(h, \varphi_h^{34}(u))\|_s + \|\chi_{24}(h, \varphi_h^{34}(u))\|_s &\leq Ch \\ \|\chi_{34}(h, \varphi_h^4(u))\|_s &\leq Ch. \end{aligned}$$

Using again Lemmas 3.8 and 3.9, we obtain that

$$\|\mathcal{S}(h, u)\|_s \leq Ch.$$

Let us define

$$\begin{aligned} R_{1,S} &= -\mathcal{V} + \varepsilon^2 \Delta \Pi_1 \phi_h(u), \\ R_{2,S} &= -\mathcal{V} + \varepsilon^2 \Delta \Pi_1 \varphi_h^{1234}(u) + \Pi_1 \mathcal{S}(h, u), \\ R_{1,A} &= \frac{i\varepsilon \Delta \Pi_2 \phi_h(u)}{2} - i\varepsilon \frac{\Delta \Pi_1 \phi_h(u)}{2} \Pi_2 \phi_h(u), \\ R_{2,A} &= \frac{i\varepsilon \Delta \Pi_2 \varphi_h^{1234}(u)}{2} - i\varepsilon \frac{\Delta \Pi_1 \varphi_h^{1234}(u)}{2} \Pi_2 \varphi_h^{1234}(u) + \Pi_2 \mathcal{S}(h, u) \end{aligned}$$

where Π_1 and Π_2 are defined in (3.1). Then, Lemma 3.4 ensures that

$$\partial_t \|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq C \|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 + C \|\varphi_h^{1234}(u) - \phi_h(u)\|_s \|\mathcal{S}(h, u)\|_s.$$

Grönwall's lemma ensures that there is $K_4 = K_4(M)$ such that

$$\|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq K_4 h^2.$$

Let us insist on the fact that K_4 and h_4 only depend on M . Hence, using the fact that for all $h \in [0, h_4]$, the applications

$$u \in \Sigma_{s+2} \mapsto \phi_h(u) \in \Sigma_s$$

and

$$u \in \Sigma_{s+2} \mapsto \varphi_h^{1234}(u) \in \Sigma_s$$

are continuous (see Lemma 2.3 and the proof of Lemma 3.8), we get that

$$\|\varphi_h^{1234}(u) - \phi_h(u)\|_s^2 \leq K_4 h^2.$$

holds true for any $u \in \Sigma_{s+2}$ such that $\|u\|_{s+2} \leq M/2$ and the result follows.

3.5.3. *Proof of Lemma 3.9.* Let $u \in \Sigma_{s+7}$ such that $\|u\|_{s+2} \leq M$. Let us define

$$0 < h_9 = h_9(M) := \min(h_5(4M), h_6(2M), h_7(M), h_8(4M)) \quad (3.6)$$

where h_5, h_6, h_7 and h_8 are defined by Lemmas 3.5, 3.6, 3.7 and 3.8. Using these lemmas, we get that for all $h \in [0, h_9]$,

$$\varphi_h^{1234}(u), \varphi_h^{234}(u), \varphi_h^{34}(u) \text{ and } \varphi_h^4(u)$$

are well-defined, belong to $L^\infty([0, h_9], \Sigma_{s+7})$ and satisfy

$$\max(\|\varphi_h^4(u)\|_{s+2}, \|\varphi_h^{34}(u)\|_{s+2}, \|\varphi_h^{234}(u)\|_{s+2}) \leq 4M.$$

Let define for $i = 1, 2, 3, 4$, $h \geq 0$ and $u_0 \in \Sigma_{s+2}$, the applications

$$\vartheta^i(h, u_0) = (h, \varphi_h^i(u_0))^T \text{ and } \Xi(h, u_0) = u_0.$$

By Lemma 3.8, we obtain that

$$h \in [0, h_9] \mapsto \varphi_h^{1234}(u) \in B_s(8M)$$

is a C^1 -application since $\varphi_h^{1234}(u) = \Xi \circ \vartheta^1 \circ \vartheta^2 \circ \vartheta^3 \circ \vartheta^4(h, u)$. We have that

$$\begin{aligned} \partial_h \varphi_h^{1234}(u) &= \mathcal{N}_1 \varphi_h^{1234}(u) + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \mathcal{N}_2 \varphi_h^{234}(u) \\ &\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \mathcal{N}_3 \varphi_h^{34}(u) \\ &\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \partial_2 \varphi_h^3(\varphi_h^4(u)) \cdot \mathcal{N}_4 \varphi_h^4(u) \end{aligned}$$

so that

$$\begin{aligned} \partial_h \varphi_h^{1234}(u) &= \mathcal{N}_1 \varphi_h^{1234}(u) + \mathcal{N}_2 \varphi_h^{1234}(u) + \mathcal{N}_3 \varphi_h^{1234}(u) + \mathcal{N}_4 \varphi_h^{1234}(u) \\ &\quad + \chi_{12}(h, \varphi_h^{234}(u)) + \chi_{13}(h, \varphi_h^{234}(u)) + \chi_{14}(h, \varphi_h^{234}(u)) \\ &\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot (\chi_{23}(h, \varphi_h^{34}(u)) + \chi_{24}(h, \varphi_h^{34}(u))) \\ &\quad + \partial_2 \varphi_h^1(\varphi_h^{234}(u)) \cdot \partial_2 \varphi_h^2(\varphi_h^{34}(u)) \cdot \chi_{34}(h, \varphi_h^4(u)) \end{aligned}$$

Let us show the last point. We have for $u_0 \in \Sigma_{s+7}$ that,

$$\partial_h (\partial_2 \varphi_h^i(v) \cdot u_0) = D\mathcal{N}_i(\varphi_h^i(v)) \cdot (\partial_2 \varphi_h^i(v) \cdot u_0)$$

so that

$$\begin{aligned} \partial_h \chi_{ij}(h, v) &= D\mathcal{N}_i(\varphi_h^i(v)) \cdot \partial_2 \varphi_h^i(v) \cdot \mathcal{N}_j(v) - D\mathcal{N}_j(\varphi_h^j(v)) \cdot \partial_h \varphi_h^i(v) \\ &= D\mathcal{N}_i(\varphi_h^i(v)) \cdot \chi_{ij}(h, v) + [\mathcal{N}_i, \mathcal{N}_j](\varphi_h^i(v)). \end{aligned}$$

3.5.4. *Proof of Lemma 3.10.* Let us consider

$$u = \begin{pmatrix} S \\ A \end{pmatrix} \text{ and } u_0 = \begin{pmatrix} S_0 \\ A_0 \end{pmatrix}.$$

We have

$$\begin{aligned} D\mathcal{N}_1(u) \cdot u_0 &= \begin{pmatrix} -\nabla S \cdot \nabla S_0 \\ -\nabla S \cdot \nabla A_0 - A_0 \frac{\Delta S}{2} - \nabla S_0 \cdot \nabla A - A \frac{\Delta S_0}{2} + i \frac{\Delta A_0}{2} \end{pmatrix}, \\ D\mathcal{N}_2(u) \cdot u_0 &= \mathcal{N}_2 u_0 = \begin{pmatrix} 0 \\ i(\varepsilon - 1) \frac{\Delta A_0}{2} \end{pmatrix}, \\ D\mathcal{N}_3(u) \cdot u_0 &= 0, \\ D\mathcal{N}_4(u) \cdot u_0 &= \begin{pmatrix} \varepsilon^2 \Delta S_0 \\ -i\varepsilon (A_0 \Delta S + A \Delta S_0) \end{pmatrix}, \end{aligned}$$

so that, $[\mathcal{N}_1, \mathcal{N}_3](u) = 0$, $[\mathcal{N}_2, \mathcal{N}_3](u) = 0$, $[\mathcal{N}_3, \mathcal{N}_4](u) = 0$ and

$$\begin{aligned} & [\mathcal{N}_1, \mathcal{N}_2](u) = D\mathcal{N}_1(u) \cdot \mathcal{N}_2(u) - D\mathcal{N}_2(u) \cdot \mathcal{N}_1(u) \\ &= \frac{i(\varepsilon - 1)}{2} \left(\begin{array}{c} 0 \\ \nabla \Delta S \cdot \nabla A + A \frac{\Delta^2 S}{2} + 2 \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k A + \partial_k A \frac{\Delta \partial_k S}{2} \end{array} \right). \end{aligned}$$

We obtain

$$\|[\mathcal{N}_1, \mathcal{N}_2](u)\|_s \leq C \|u\|_{s+2}^2.$$

We also have

$$\begin{aligned} & [\mathcal{N}_1, \mathcal{N}_4](u) = D\mathcal{N}_1(u) \cdot \mathcal{N}_4(u) - D\mathcal{N}_4(u) \cdot \mathcal{N}_1(u) \\ &= \left(\begin{array}{c} \varepsilon^2 \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k S \\ (\varepsilon - \varepsilon^2) \left(\nabla \Delta S \cdot \nabla A + A \frac{\Delta^2 S}{2} \right) - i\varepsilon A \sum_{k=1}^d \nabla \partial_k S \cdot \nabla \partial_k S \end{array} \right), \end{aligned}$$

and

$$\|[\mathcal{N}_1, \mathcal{N}_4](u)\|_s \leq \varepsilon C \|u\|_{s+2}^2 (1 + \|u\|_{s+2}).$$

We also get

$$\begin{aligned} & [\mathcal{N}_2, \mathcal{N}_4](u) = D\mathcal{N}_2(u) \cdot \mathcal{N}_4(u) - D\mathcal{N}_4(u) \cdot \mathcal{N}_2(u) \\ &= \frac{\varepsilon(\varepsilon - 1)}{2} \left(\begin{array}{c} 0 \\ A \Delta^2 S + 2 \nabla A \cdot \nabla \Delta S \end{array} \right), \end{aligned}$$

so that

$$\|[\mathcal{N}_2, \mathcal{N}_4](u)\|_s \leq \varepsilon C \|u\|_{s+2}^2$$

and the result follows.

Appendix A. Proof of Lemma 3.8

A.1. Study of the differentiability of φ^1 . The proof of this lemma is divided in several steps. Let us fix $s > d/2 + 1$ and $M > 0$.

A.1.1. Notations. For any Banach space E and F , we denote $\mathcal{L}(E, F)$ the set of continuous linear maps between E and F endowed with the norm

$$\|l\|_{\mathcal{L}(E, F)} = \sup\{\|l(x)\|_F, x \in E, \|u\|_E \leq 1\}$$

where $\|\cdot\|_E$ and $\|\cdot\|_F$ are the norm of E and F .

Let us define for $u_0 = (S_0, A_0)$

$$\Theta_h^1 \cdot u_0 = \begin{pmatrix} \tilde{S}_h^1 \\ \tilde{A}_h^1 \end{pmatrix}$$

the solution of

$$\begin{aligned} \partial_h \Theta_h^1 &= D\mathcal{N}_1(\varphi_h^1(u)) \cdot \Theta_h^1 \\ \Theta_0^1 \cdot u_0 &= u_0. \end{aligned}$$

We denote $\Gamma_h^1 = \varphi_h^1(u + u_0) - \varphi_h^1(u) - \Theta_h^1 \cdot u_0$,

$$\varphi_h^1(u) = \begin{pmatrix} S_h^1 \\ A_h^1 \end{pmatrix}, \quad \varphi_h^1(u + u_0) = \begin{pmatrix} \underline{S}_h^1 \\ \underline{A}_h^1 \end{pmatrix},$$

$v_h^1 = \nabla S_h^1$, $\underline{v}_h^1 = \nabla \underline{S}_h^1$, $\tilde{v}_h^1 = \nabla \tilde{S}_h^1$, $\omega_h^1 = \underline{S}_h^1 - S_h^1 - \tilde{S}_h^1$ and $B_h^1 = \underline{A}_h^1 - A_h^1 - \tilde{A}_h^1$.

A.1.2. *Definition of h_8 .* Lemma 3.5 ensures that for any $u \in B_s(2M)$, we have for $h \in [0, h_5(2M)]$ that

$$\|\varphi_h^1(u)\|_s \leq 4M. \quad (\text{A.1})$$

We denote $h_8(M) = h_5(2M)$.

Let $s' \geq s$. If moreover, $u \in \Sigma_{s'}$, then we have

$$\|\varphi_h^1(u)\|_{s'} \leq \exp(C_5(2M)h) \|u\|_{s'}. \quad (\text{A.2})$$

A.1.3. *Continuity of φ^1 .* Let $s' \geq s$, $M' > 0$ and $u_1, u_2 \in B_s(M) \cap B_{s'+1}(M')$.

By (A.1) and (A.2), we obtain that $\varphi_h^1(u_1)$ and $\varphi_h^1(u_2)$ are well-defined on $[0, h_8]$ and satisfy

$$\|\varphi_h^1(u_1)\|_{s'+1} + \|\varphi_h^1(u_2)\|_{s'+1} \leq 2 \exp(C_5(2M)h_8) M'$$

for all $h \in [0, h_8]$. By Lemma 3.4 and an integration by parts, we get that there exists $C = C(M, M') > 0$ such that for all $h \in [0, h_8]$

$$\|\varphi_h^1(u_1) - \varphi_h^1(u_2)\|_{s'} \leq C \|u_1 - u_2\|_{s'+1}. \quad (\text{A.3})$$

Moreover, for fixed $u \in B_s(M) \cap B_{s'+1}(M')$, Lemma 3.5 ensures that $h \in [0, h_8] \mapsto \varphi_h^1(u) \in \Sigma_{s'}$ is continuous so that

$$(h, u) \in [0, h_8] \times \Sigma_{s'+1} \mapsto \varphi_h^1(u) \in \Sigma_{s'} \quad (\text{A.4})$$

is also continuous.

A.1.4. *Well-posedness, continuity and estimates on the norm for Θ_h^1 .* Let $s_2 \geq s$, $M_2 > 0$, $u \in B_s(M) \cap B_{s_2+1}(M_2)$ and $u_0 \in \Sigma_{s_2}$. We recall that the function $\Theta_h^1 \cdot u_0 =: (\tilde{S}_h^1, \tilde{A}_h^1)^T$ satisfies

$$\partial_h \tilde{S}_h^1 + \nabla S_h^1 \cdot \nabla \tilde{S}_h^1 = 0$$

$$\partial_h \tilde{A}_h^1 + \nabla S_h^1 \cdot \nabla \tilde{A}_h^1 + \frac{\tilde{A}_h^1}{2} \Delta S_h^1 = -\nabla \tilde{S}_h^1 \cdot \nabla A_h^1 - \frac{A_h^1}{2} \Delta \tilde{S}_h^1 + \frac{i}{2} \Delta \tilde{A}_h^1$$

and $\Theta_0^1 \cdot u_0 = u_0$. The existence and uniqueness of \tilde{S}_h^1 follows for instance from the method of characteristics. We have

$$\partial_h \tilde{v}_h^1 + (\tilde{v}_h^1 \cdot \nabla) \tilde{v}_h^1 = -(\tilde{v}_h^1 \cdot \nabla) v_h^1$$

and Lemma 3.2 with $R = -(\tilde{v}_h^1 \cdot \nabla) v_h^1$ gives us that

$$\partial_h \|\tilde{v}_h^1\|_{H^{s_2+1}}^2 \leq C \|\tilde{v}_h^1\|_{H^{s_2+1}}^2 \|S_h^1\|_{H^{s_2+3}} \leq C \|\tilde{v}_h^1\|_{H^{s_2+1}}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

We also have

$$\partial_h \|\tilde{S}_h^1\|_{L^2}^2 \leq C \|\tilde{S}_h^1\|_{L^2} \|\tilde{S}_h^1\|_{H^1} \|S_h^1\|_{W^{1,\infty}}$$

so that

$$\partial_h \|\tilde{S}_h^1\|_{H^{s_2+2}}^2 \leq C \|\tilde{S}_h^1\|_{H^{s_2+2}}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

The existence and uniqueness of \tilde{A}_h^1 follows from the fact that $\tilde{w}_h^1 = \tilde{A}_h^1 \exp(iS_h^1)$ satisfies

$$i\partial_h \tilde{w}_h^1 = -\frac{\Delta}{2} \tilde{w}_h^1 - \left(\nabla \tilde{S}_h^1 \cdot \nabla A_h^1 + \frac{A_h^1}{2} \Delta \tilde{S}_h^1 \right) \exp(iS_h^1).$$

Lemma 3.3 with $R = -\nabla \tilde{S}_h^1 \cdot \nabla A_h^1 - \frac{A_h^1}{2} \Delta \tilde{S}_h^1 + \frac{i}{2} \Delta \tilde{A}_h^1$ ensures that

$$\partial_t \|\tilde{A}_h^1\|_{H^{s_2}}^2 \leq C \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \|\varphi_h^1(u)\|_{s_2+1}$$

so that

$$\partial_t \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \leq C \|\Theta_h^1 \cdot u_0\|_{s_2}^2 \|\varphi_h^1(u)\|_{s_2+1}.$$

By (A.2) and Grönwall's Lemma, there is $C_8 = C_8(M, M_2) > 0$ such that for any $h \in [0, h_8]$,

$$\|\Theta_h^1 \cdot u_0\|_{s_2} \leq \exp(C_8 h) \|u_0\|_{s_2}. \quad (\text{A.5})$$

Using directly the integrations by parts of the proof of Lemmas 3.2 and 3.3, we obtain actually that

$$|\langle u_0, D\mathcal{N}_1(\varphi_h^1(u)) \cdot u_0 \rangle_{s_2}| \leq C_8 \|u_0\|_{s_2}^2,$$

for all $u_0 \in \Sigma_{s_2}$.

A.1.5. *Differentiability of φ^1 .* By Lemma 3.5 and equations (1.5), the application

$$h \in [0, h_8] \mapsto \varphi_h^1(u) \in \Sigma_{s_1}$$

is differentiable in h for any $u \in B_s(M) \cap \Sigma_{s_1+2}$.

Let us prove that φ_h^1 is differentiable in u and that Θ_h^1 is its derivative.

Let $M_1 > 0$ and $u, u_0 \in B_s(M) \cap B_{s_1+2}(M_1)$. We have that $u, u + u_0 \in B_s(2M) \cap B_{s_1+2}(2M_1)$. By (A.1) and (A.2), we obtain that for all $h \in [0, h_8]$,

$$\|\varphi_h^1(u)\|_{s_1+2} + \|\varphi_h^1(u + u_0)\|_{s_1+2} \leq 4 \exp(C_5(2M)h) M_1.$$

We have

$$\partial_h \nabla \omega_h^1 = -(v_h^1 + \tilde{v}_h^1) \cdot \nabla (\nabla \omega_h^1) - (\nabla \omega_h^1 \cdot \nabla) \underline{v}_h^1 - (\tilde{v}_h^1 \cdot \nabla) \tilde{v}_h^1.$$

By Lemma 3.2, we obtain taking $v_1 = v_h^1 + \tilde{v}_h^1$ and

$$R = -(\nabla \omega_h^1 \cdot \nabla) \underline{v}_h^1 - (\tilde{v}_h^1 \cdot \nabla) \tilde{v}_h^1$$

that

$$\begin{aligned} \partial_t \|\nabla \omega_h^1\|_{H^{s_1+1}}^2 &\leq C \|\nabla \omega_h^1\|_{H^{s_1+1}}^2 (\|v_h^1\|_{H^{s_1+1}} + \|\tilde{v}_h^1\|_{H^{s_1+1}} + \|\underline{v}_h^1\|_{H^{s_1+2}}) \\ &\quad + C \|\nabla \omega_h^1\|_{H^{s_1+1}} \|\tilde{v}_h^1\|_{H^{s_1+2}}^2. \end{aligned}$$

Moreover, we have

$$\partial_h \omega_h^1 = -\frac{1}{2} (\nabla \omega_h^1 \cdot (\tilde{v}_h^1 + v_h^1) + \underline{v}_h^1 \cdot \nabla \omega_h^1 + |\tilde{v}_h^1|^2)$$

so that

$$\begin{aligned} \partial_h \|\omega_h^1\|_{H^{s_1+2}}^2 &\leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ &\quad + C \|\omega_h^1\|_{H^{s_1+2}}^2 (1 + \|\varphi_h^1(u)\|_{s_1} + \|\varphi_h^1(u + u_0)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1}) \end{aligned}$$

We also have

$$\begin{aligned} \partial_h B_h^1 &= -\nabla \underline{S}_h^1 \cdot \nabla B_h^1 - B_h^1 \frac{\Delta \underline{S}_h^1}{2} - \nabla \omega_h^1 \cdot \nabla (A_h^1 + \tilde{A}_h^1) - (A_h^1 + \tilde{A}_h^1) \frac{\Delta \omega_h^1}{2} \\ &\quad + i \frac{\Delta B_h^1}{2} - \nabla \tilde{S}_h^1 \cdot \nabla \tilde{A}_h^1 - \tilde{A}_h^1 \frac{\Delta \tilde{S}_h^1}{2} \end{aligned}$$

and Lemma 3.3 ensures taking

$$R = -\nabla \omega_h^1 \cdot \nabla (A_h^1 + \tilde{A}_h^1) - (A_h^1 + \tilde{A}_h^1) \frac{\Delta \omega_h^1}{2} + i \frac{\Delta B_h^1}{2} - \nabla \tilde{S}_h^1 \cdot \nabla \tilde{A}_h^1 - \tilde{A}_h^1 \frac{\Delta \tilde{S}_h^1}{2}$$

that,

$$\begin{aligned} \partial_t \|B_h^1\|_{H^{s_1}}^2 &\leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ &+ C \|\Gamma_h^1\|_{s_1}^2 (1 + \|\varphi_h^1(u + u_0)\|_{s_1} + \|\varphi_h^1(u)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1+1}) \end{aligned}$$

and

$$\begin{aligned} \partial_h \|\Gamma_h^1\|_{s_1}^2 &\leq \|\Theta_h^1 \cdot u_0\|_{s_1+1}^4 \\ &+ C \|\Gamma_h^1\|_{s_1}^2 (1 + \|\varphi_h^1(u + u_0)\|_{s_1+1} + \|\varphi_h^1(u)\|_{s_1+1} + \|\Theta_h^1 \cdot u_0\|_{s_1+1}). \end{aligned}$$

By (A.5) with $s_2 = s_1 + 1$ and Grönwall's Lemma, we get that there exists $C = C(M, M_1) > 0$ such that for all $h \in [h, h_8]$,

$$\|\Gamma_h^1\|_{s_1} \leq C \|u_0\|_{s_1+1}^2 \leq C \|u_0\|_{s_1+2}^2$$

We proved that for any $h \in [0, h_8]$

$$\varphi_h^1 : B_s(M) \cap \Sigma_{s_1+2} \rightarrow \Sigma_{s_1}$$

is differentiable in $B_s(M) \cap \Sigma_{s_1+2}$.

A.1.6. *Proof of point (i).* Let us prove that the application

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+4}) \mapsto \varphi_h^1(u) \in \Sigma_{s_1}$$

is a C^1 -function.

Using equations (1.5) and (A.4), we get that

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto \varphi_h^1(u) \in \Sigma_{s_1+2}$$

is continuous so that the partial derivative

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto \partial_h \varphi_h^1(u) = \mathcal{N}_1 \varphi_h^1(u) \in \Sigma_{s_1}$$

is also continuous. Let us study the continuity of

$$(h, u) \mapsto \partial_2 \varphi_h^1(u).$$

Let $u_1, u_2 \in B_s(M) \cap B_{s_1+2}(M_1)$. We denote $\varphi_h^1(u_i) = (S_h^{1,i}, A_h^{1,i})$ and $\partial_2 \varphi_h^1(u_i) \cdot u_0 = (\tilde{S}_h^{1,i}, \tilde{A}_h^{1,i})$ for $i = 1, 2$. We have

$$\partial_h \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) + \nabla S_h^{1,1} \cdot \nabla \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) = -\nabla \left(S_h^{1,1} - S_h^{1,2} \right) \cdot \nabla \tilde{S}_h^{1,2}$$

so that

$$\begin{aligned} &\partial_h \left(\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2} \right) + \left(\nabla S_h^{1,1} \cdot \nabla \right) \left(\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2} \right) \\ &= - \left(\left(\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2} \right) \cdot \nabla \right) \left(\nabla S_h^{1,1} + \tilde{S}_h^{1,2} \right) \\ &\quad - \left(\nabla \tilde{S}_h^{1,2} \cdot \nabla \right) \left(\nabla S_h^{1,1} - \nabla S_h^{1,2} \right). \end{aligned}$$

By Lemma 3.2 with $v_1 = \nabla S_h^{1,1}$ and

$$\begin{aligned} R &= - \left(\left(\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2} \right) \cdot \nabla \right) \left(\nabla S_h^{1,1} + \tilde{S}_h^{1,2} \right) \\ &\quad - \left(\nabla \tilde{S}_h^{1,2} \cdot \nabla \right) \left(\nabla S_h^{1,1} - \nabla S_h^{1,2} \right), \end{aligned}$$

which satisfies

$$\begin{aligned} \|R\|_{H^{s_1+1}} &\leq C \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^{s_1+2}} \left(\|S_h^{1,1}\|_{H^{s_1+3}} + \|\tilde{S}_h^{1,2}\|_{H^{s_1+3}} \right) \\ &\quad + C \|S_h^{1,1} - S_h^{1,2}\|_{H^{s_1+3}} \|\tilde{S}_h^{1,2}\|_{H^{s_1+2}} \end{aligned}$$

we obtain that

$$\begin{aligned} \partial_t \|\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2}\|_{H^{s_1+1}}^2 &\leq C \|\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2}\|_{H^{s_1+1}}^2 \left(\|S_h^{1,1}\|_{H^{s_1+3}} + \|\tilde{S}_h^{1,2}\|_{H^{s_1+3}} \right) \\ &\quad + C \|\nabla \tilde{S}_h^{1,1} - \nabla \tilde{S}_h^{1,2}\|_{H^{s_1+1}} \|S_h^{1,1} - S_h^{1,2}\|_{H^{s_1+3}} \|\tilde{S}_h^{1,2}\|_{H^{s_1+2}}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \partial_t \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{L^2}^2 &\leq C \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{L^2} \left(\|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^1} \|S_h^{1,1}\|_{W^{1,\infty}} + \|S_h^{1,1} - S_h^{1,2}\|_{H^1} \|\tilde{S}_h^{1,2}\|_{W^{1,\infty}} \right) \end{aligned}$$

so that

$$\begin{aligned} \partial_t \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^{s_1+2}}^2 &\leq C \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^{s_1+2}}^2 \left(\|S_h^{1,1}\|_{H^{s_1+3}} + \|\tilde{S}_h^{1,2}\|_{H^{s_1+3}} \right) \\ &\quad + C \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^{s_1+2}} \|S_h^{1,1} - S_h^{1,2}\|_{H^{s_1+3}} \|\tilde{S}_h^{1,2}\|_{H^{s_1+2}}. \end{aligned}$$

We also have

$$\begin{aligned} \partial_h \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right) + \nabla S_h^{1,1} \cdot \nabla \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right) + \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right) \frac{\Delta S_h^{1,1}}{2} \\ = -\nabla \left(S_h^{1,1} - S_h^{1,2} \right) \cdot \nabla \tilde{A}_h^{1,2} - \frac{\tilde{A}_h^{1,2}}{2} \Delta \left(S_h^{1,1} - S_h^{1,2} \right) \\ - \nabla \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) \cdot \nabla A_h^{1,1} - \frac{A_h^{1,1}}{2} \Delta \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) \\ - \nabla \tilde{S}_h^{1,2} \cdot \nabla \left(A_h^{1,1} - A_h^{1,2} \right) - \frac{\left(A_h^{1,1} - A_h^{1,2} \right)}{2} \Delta \tilde{S}_h^{1,2} + \frac{i}{2} \Delta \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right). \end{aligned}$$

Using Lemma 3.3 with $v_1 = \nabla S_h^{1,1}$ and

$$\begin{aligned} R &= -\nabla \left(S_h^{1,1} - S_h^{1,2} \right) \cdot \nabla \tilde{A}_h^{1,2} - \frac{\tilde{A}_h^{1,2}}{2} \Delta \left(S_h^{1,1} - S_h^{1,2} \right) \\ &\quad - \nabla \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) \cdot \nabla A_h^{1,1} - \frac{A_h^{1,1}}{2} \Delta \left(\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2} \right) \\ &\quad - \nabla \tilde{S}_h^{1,2} \cdot \nabla \left(A_h^{1,1} - A_h^{1,2} \right) - \frac{\left(A_h^{1,1} - A_h^{1,2} \right)}{2} \Delta \tilde{S}_h^{1,2} + \frac{i}{2} \Delta \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right). \end{aligned}$$

which satisfies

$$\begin{aligned} \operatorname{Re} \left\langle \Lambda^{s_1} \left(\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2} \right), \Lambda^{s_1} R \right\rangle &\leq C \|\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2}\|_{H^{s_1}} \|S_h^{1,1} - S_h^{1,2}\|_{H^{s_1+2}} \|\tilde{A}_h^{1,2}\|_{H^{s_1+1}} \\ &\quad + C \|\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2}\|_{H^{s_1}} \|\tilde{S}_h^{1,1} - \tilde{S}_h^{1,2}\|_{H^{s_1+2}} \|A_h^{1,1}\|_{H^{s_1+1}} \\ &\quad + C \|\tilde{A}_h^{1,1} - \tilde{A}_h^{1,2}\|_{H^{s_1}} \|A_h^{1,1} - A_h^{1,2}\|_{H^{s_1+1}} \|\tilde{S}_h^{1,2}\|_{H^{s_1+2}} \end{aligned}$$

we obtain that

$$\begin{aligned} \partial_h \left\| \left(\partial_2 \varphi_h^1(u_1) - \partial_2 \varphi_h^1(u_2) \right) \cdot u_0 \right\|_{s_1}^2 &\leq C \left\| \left(\partial_2 \varphi_h^1(u_1) - \partial_2 \varphi_h^1(u_2) \right) \cdot u_0 \right\|_{s_1}^2 \left(\|\varphi_h^1(u_1)\|_{s_1+1} + \|\partial_2 \varphi_h^1(u_2) \cdot u_0\|_{s_1+1} \right) \\ &\quad + C \left\| \left(\partial_2 \varphi_h^1(u_1) - \partial_2 \varphi_h^1(u_2) \right) \cdot u_0 \right\|_{s_1} \|\varphi_h^1(u_1) - \varphi_h^1(u_2)\|_{s_1+1} \|\partial_2 \varphi_h^1(u_2) \cdot u_0\|_{s_1+1}. \end{aligned}$$

Let us recall that $u_1, u_2 \in B_s(M) \cap B_{s_1+2}(M_1)$. By (A.2), (A.3) and (A.5) with $s_2 = s_1$ and Grönwall's Lemma, we get that for all $h \in [0, h_8]$,

$$u \in B_s(M) \cap \Sigma_{s_1+2} \mapsto \partial_2 \varphi_h^1(u) \in \mathcal{L}(\Sigma_{s_1+2}, \Sigma_{s_1})$$

is continuous. Hence, we obtain that

$$(h, u) \in [0, h_8] \times (B_s(M) \cap \Sigma_{s_1+3}) \mapsto (\partial_2 \varphi_h^1(u), \partial_h \varphi_h^1(u)) \in \mathcal{L}(\Sigma_{s_1+3}, \Sigma_{s_1}) \times \Sigma_{s_1}$$

is continuous and the result follows.

A.2. Study of the differentiability of φ^2 and φ^3 . Let $u, u_0 \in \Sigma_s$. Since \mathcal{N}_2 is linear, we have that

$$\Theta_h^2 \cdot u_0 = \varphi_h^2(u_0),$$

φ_h^2 is differentiable on Σ_s and for any $h \geq 0$,

$$\begin{aligned} \|\partial_2 \varphi_h^2(u) \cdot u_0\|_s &= \|\varphi_h^2(u_0)\|_s = \|u_0\|_s, \\ |\langle u_0, D\mathcal{N}_2(\varphi_h^2(u)) \cdot u_0 \rangle_s| &\leq C \|u_0\|_s^2. \end{aligned}$$

and the result follows. We easily prove that φ_i^3 is differentiable, that for any $h \geq 0$, $\Theta_h^3 \cdot u_0 = u_0$, that

$$\|\Theta_h^3 \cdot u_0\|_s = \|\Theta_0^3 \cdot u_0\|_s = \|u_0\|_s.$$

and

$$|\langle \chi, (D\mathcal{N}_3(\varphi_h^3(u)) \cdot \chi) \rangle_s| \leq C \|\chi\|_s^2,$$

for all $\chi \in \Sigma_s$.

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E-mail address: Philippe.Chartier@inria.fr

INRIA RENNES, IRMAR AND ENS RENNES, IPSO PROJECT TEAM, CAMPUS DE BEAULIEU, F-35042 RENNES

E-mail address: loic.letreust@univ-rennes1.fr

IRMAR, UNIVERSITÉ DE RENNES 1 AND INRIA, IPSO PROJECT

E-mail address: florian.mehats@univ-rennes1.fr

IRMAR, UNIVERSITÉ DE RENNES 1 AND INRIA, IPSO PROJECT