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Discretized Jensen's inequality: an alternative vision of the reciprocally convex combination lemma

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Abstract

The analysis of time-varying delay systems has attracted many researchers over the last two decades. One of the major contribution within this field is the recent reciprocally convex combination lemma. The relevance of this lemma arises from the derivation of stability conditions using Jensen's inequality. The main interest of this lemma is the reduction of the number of decision variables while keeping the same level of conservatism. In this paper, we provide an alternative vision of this inequality through a new proof issued from the recent development on integral inequalities. The benefit of this proof relies on the derivation of an exact expression of the conservatism. A discussion is finally proposed at the end of the paper to point out the possible extensions of this approach.

Key words: Time-delay systems, integral inequalities, matrix inequality, reciprocally convex lemma.

1 Introduction

Time-delay systems have attracted many researchers over the past two decades. This interest arised notably from the perpectives of networked control systems where controllers and plants are connected through a network. This network eventually induces additional dynamics such as communications delays. Among the possible methods to tackle the problem of stability analysis, contributions in the time-domain framework using the Lyapunov-Krasovskii theorem have been successfully studied. Indeed this framework allows deriving stability conditions expressed in terms of linear matrix inequalities (LMI), that can be easily computed using semi-definite programming algorithms. Even if this direction of research is about 20 years-old, it still attracts many researchers, whose objectives are to reduce the conservatism and the number of decision variables of the stability conditions.

On the one hand, a possible way to reduce the conservatism relies on the consideration of more and more involved functionals including multiple integral quadratic terms at the price of an increase of the number of decision variables. On the other hand, more attention has been paid to the complexity of stability conditions, which depends on both the number of decision variables and the size of the LMI to resolve. Indeed [1,4] have provided discussions on the efficiency of introducing slack variables (or free weighting matrices). Following this discussion, the trend became the design of efficient inequalities able to reduce the conservatism and/or the computational complexity. One of the most popular inequalities is the reciprocally convex combination lemma, whose goal is to provide an efficient lower bound of reciprocally convex terms generally issued from the application of the Jensen's inequality. Its benefit is a reduction of the computational complexity with the same level of conservatism as the classical bounding methods. In this paper, we present an alternative proof of this inequality, which aims at delivering a different interpretation of this already well-known inequality. This new proof reveals that the reciprocally convex combination lemma can be seen as a discretized version of

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the celebrated Jensen's inequality, in the sense that an intermediate value is included. The benefits of the propose result are first to derive an exact expression of the resulting inequality and, second, potential directions for future researches.

Notations: Throughout the paper $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The notation $P \succ 0$, for $P \in \mathbb{R}^{n\times n}$, means that P is symmetric positive definite. For any matrix A, B, C of appropriate dimension, the matrix $\left[\begin{smallmatrix} A & B \\ * & C \end{smallmatrix}\right]$ stands for $\left[\begin{smallmatrix} A & B \\ B^T & C \end{smallmatrix}\right]$The matrix I represents the identity matrix of appropriate dimension. The notation $0_{n,m}$ stands for the matrix in $\mathbb{R}^{n\times m}$ whose entries are zero and, when no confusion is possible, the subscript will be omitted. For any $h > 0$ and any function $x : [-h, +\infty) \to \mathbb{R}^n$, the notation $x_t(\theta)$ stands for $x(t + \theta)$, for all $t \geq 0$ and all $\theta \in [-h, 0]$. Throughout the paper, $h_1 \leq h_2$ denotes two positive scalars and we use the notation $h_{21} = h_2 - h_1$.

2 Problem formulation

Providing efficient stability conditions for time-varying or fast-varying delay systems is related to the accuracy of the matrix or integral inequalities, which are considered. Among then let us recall one of the most recent and popular inequality dedicated to the analysis of time-varying or fast-varying delay systems, given by

$$V(\dot{x}_t) = h_{21} \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s \mathrm{d}\theta. \tag{1}$$

Differentiating this functional leads to

$$\dot{V}(\dot{x}_t) = h_{21}^2 \dot{x}^T(t) R \dot{x}(t) - h_{21} \int_{-h_2}^{-h_1} \dot{x}_t^T(s) R \dot{x}_t(s) \mathrm{d}s. \tag{2}$$

The notable aspects of such a class of functionals rely on the negative integral quadratic term. The objective is to transform this integral into a quadratic term so that stability conditions expressed in terms of LMI can be derived. In the context of time-varying or fast-varying delay systems, the objective is also to introduce an intermediate value $x(t - h)$, where h is any value within the interval $[-h_2, -h_1]$. One of most popular method is based on the application of the Jensen inequality [2] and the reciprocally convex combination lemma from [6]. In this note, we aim at giving an alternative interpretation of the combination of these two lemmas.

3 Integral and matrix inequalities

3.1 Jensen's inequality

In this paragraph, we recall the Jensen's inequality. In addition, a measure of the conservatism is considered based on the proof of the Bessel-Legendre inequality given in [10].

Lemma 1 *Let $R = R^T \in \mathbb{R}^{n\times n}$ and $x : [-h_2, -h_1] \to \mathbb{R}^n$ be a continuously differentiable function. The following equality holds*

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s = \omega_0^T R \omega_0^T + h_{21}\theta_0,$$

where $\omega_0 = x(-h_1) - x(-h_2)$, $\theta_0 = \int_{-h_2}^{-h_1} \epsilon_0^T(s) R \epsilon_0(s) \mathrm{d}s$ and $\epsilon_0(s) = \dot{x}(s) - \frac{1}{h_{21}}\omega_0$. Moreover, if $R \succ 0$, then $\theta_0 \geq 0$ and the Jensen inequality is retrieved.

Proof : The proof of this lemma is obtained by developing the expression of θ_0. ♢

In this lemma, θ_0 represents the conservatism of Jensen's inequality and depends on the vector ϵ_0. In light of [10], this vector can be seen as the approximation of the function $\dot{x}$ by a constant term $\omega_0 = \int_{-h_2}^{-h_1} \dot{x}(s) \mathrm{d}s$. The framework developed in [10] guarantees that ω_0 is the best constant vector function that approximates $\dot{x}$ in the sense of the inner product $\langle f, g\rangle := \int_{-h_2}^{-h_1} f^T(s) g(s) \mathrm{d}s$, where f and g are in $\mathcal{L}_2([-h_2, -h_1] \to \mathbb{R}^n)$.

3.2 Extension to deal with time-varying delays systems

In the situation of time-varying or fast-varying delay systems, the problem is often to derive a lower bound of (2), which depends on an intermediate value $x(-h)$ where h is in $[h_1, h_2]$. Therefore, the previous lemma cannot be applied directly. A preliminary step consists in splitting the integral into two parts and in applying the inequality to each integral to get:

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s \geq \Omega_0^T \begin{bmatrix} \frac{h_{21}}{h-h_1} R & 0 \\ * & \frac{h_{21}}{h_2-h} R \end{bmatrix} \Omega_0, \tag{3}$$

where $\Omega_0 = \left[\begin{smallmatrix} x(-h_1)-x(-h) \\ x(-h)-x(-h_2) \end{smallmatrix}\right]$. In the previous inequality, the lower bound of the integral, depends on h in a reciprocally convex manner as defined in [6]. In order to derive a delay-independant lower bound, an interesting result was provided in [6]. A corollary of the more general result presented in Theorem 1 of [6] is stated below.

Corollary 1 *Let $R = R^T \in \mathbb{R}^{n\times n}$ be positive definite matrix. If there exists a matrix $X \in \mathbb{R}^{n\times n}$ such that $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$, then the following inequality holds*

$$\min_{\alpha\in(0,\ 1)} \begin{bmatrix} \frac{1}{\alpha} R & 0 \\ * & \frac{1}{1-\alpha} R \end{bmatrix} \succeq \begin{bmatrix} R & X \\ * & R \end{bmatrix}$$

Proof : Following [6], the proof consists in noting that

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ * & \frac{1}{1-\alpha}R \end{bmatrix} = \begin{bmatrix} R & X \\ * & R \end{bmatrix} + \begin{bmatrix} \frac{1-\alpha}{\alpha}R & -X \\ * & \frac{\alpha}{1-\alpha}R \end{bmatrix} \tag{4}$$

To conclude, we need to show $\begin{bmatrix} \frac{1-\alpha}{\alpha}R & -X \\ * & \frac{\alpha}{1-\alpha}R \end{bmatrix} \succeq 0$. There are two methods to prove this inequality.

i) The first one presented in [6] consists in introducing $\beta = \sqrt{(1-\alpha)/\alpha}I$ and noting that, if $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$, then

$$\begin{bmatrix} \frac{1-\alpha}{\alpha}R & -X \\ * & \frac{\alpha}{1-\alpha}R \end{bmatrix} = \begin{bmatrix} \beta & 0 \\ 0 & -\beta^{-1} \end{bmatrix} \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & -\beta^{-1} \end{bmatrix} \succeq 0.$$

The last term of the previous equation is positive and the result holds.

ii) The positivity of the last matrix of (4) is equivalent, by Schur Complement, to $\frac{1-\alpha}{\alpha}(R - X^T R^{-1} X) \succeq 0$, whose positivity is ensured by the constraints $\alpha \in (0,1)$ and $R - X^T R^{-1} X$ which is equivalent, again by the Schur Complement to $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$. $\diamondsuit$

Thanks to Corollary 1 and defining $\alpha = (h - h_1)/h_{21}$, provided that there exists a matrix $X \in \mathbb{R}^{n\times n}$ such that $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$, the following inequality, so-called Jensen-Park's inequality, is derived from (3):

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s \geq \Omega_0^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \Omega_0. \tag{5}$$

4 Discretized Jensen's inequality

4.1 Alternative proof of Jensen-Park's inequality (5)

In this section, we aim at providing an extension of Lemma 1, which includes an intermediate value, namely $x(-h)$, where $h \in [h_1, h_2]$. The main idea relies on a slight modification of the reminder θ_0 of Jensen's inequality and more particularly on the approximation error vector ϵ_0. We propose to replace $\dot{x}$ by the following augmented and discontinuous vector $\dot{\bar{x}}$ defined by

$$\dot{\bar{x}}(s) = \begin{cases} \begin{bmatrix} \dot{x}(s) \\ 0 \end{bmatrix} & if \quad s \in [-h, -h_1], \\ \begin{bmatrix} 0 \\ \dot{x}(s) \end{bmatrix} & if \quad s \in [-h_2, -h). \end{cases} \tag{6}$$

This leads to the following lemma.

Lemma 2 *Let $R = R^T, X \in \mathbb{R}^{n\times n}$ and $x : [-h_2, -h_1] \to \mathbb{R}^n$ be a piecewise differentiable function. The equality*

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s = \Omega_0^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \Omega_0 + h_{21}\Theta_0, \tag{7}$$

holds, where Ω_0 is defined in (3) *and*

$$\begin{aligned} \Theta_0 &= \int_{-h_2}^{-h_1} \eta_0^T(s) \begin{bmatrix} R & X \\ * & R \end{bmatrix} \eta_0(s) \mathrm{d}s, \\ \eta_0(s) &= \dot{\bar{x}}(s) - \tfrac{1}{h_{21}}\Omega_0, \end{aligned} \tag{8}$$

and where h is any scalar in $[h_1, h_2]$ and $\dot{\bar{x}}$ is given in (6)*. Moreover, if $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$, then inequality* (5) *holds for any $h \in [h_1, h_2]$.*

Proof : Noting that $\int_{-h_2}^{-h_1} \dot{\bar{x}}(s) = \Omega_0$, Lemma 1 yields

$$\Theta_0 = \int_{-h_2}^{-h_1} \dot{\bar{x}}^T(s) \begin{bmatrix} R & X \\ * & R \end{bmatrix} \dot{\bar{x}}(s) \mathrm{d}s - \frac{1}{h_{21}} \Omega_0^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \Omega_0.$$

Spliting the first integral over the two intervals $[-h, -h_1]$ and $[-h_2, -h]$ yields

$$\begin{aligned} \Theta_0 = & \int_{-h}^{-h_1} \begin{bmatrix} \dot{x}(s) \\ 0 \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ 0 \end{bmatrix} \mathrm{d}s \\ & + \int_{-h_2}^{-h} \begin{bmatrix} 0 \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \begin{bmatrix} 0 \\ \dot{x}(s) \end{bmatrix} \mathrm{d}s \\ & - \tfrac{1}{h_{21}} \Omega_0^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \Omega_0 \end{aligned}$$

or, equivalently

$$\Theta_0 = \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) \mathrm{d}s - \frac{1}{h_{21}} \Omega_0^T \begin{bmatrix} R & X \\ * & R \end{bmatrix} \Omega_0,$$

which concludes the proof on the first statement of the lemma. Moreover, the inequalty $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$ implies that Θ_0 is positive and the second statement holds. $\diamondsuit$

The inequality derived through Lemma 2 is not a new inequality since it is exactly the same inequality as in (5). The novelty of Lemma 2 relies on an exact expression of its conservatism, Θ_0.

4.2 Analysis of the conservatism

In this section, we want to take advantages of the expression of the conservatism of the Jensen-Park's inequality,

Θ_0 given in (8) to potentially derive a more accurate inequality. A first attempt would be to apply Jensen's inequality over the whole interval to Θ_0, i.e.

$$h_{21}\Theta_0 \geq \left(\int_{-h_2}^{-h_1} \eta_0^T(s)\mathrm{d}s\right) \begin{bmatrix} R & X \\ * & R \end{bmatrix} \left(\int_{-h_2}^{-h_1} \eta_0(s)\mathrm{d}s\right).$$

However the computation of the integral terms leads to

$$\int_{-h_2}^{-h_1} \eta_0(s)\mathrm{d}s = \int_{-h_2}^{-h_1} \left(\dot{\bar{x}}(s) - \frac{\Omega_0}{h_{21}}\right)\mathrm{d}s = 0,$$

which implies $\Theta_0 \geq 0$, which is already captured by the matrix inequality $\left[\begin{smallmatrix} R & X \\ * & R \end{smallmatrix}\right] \succeq 0$. Then this first attempt does not bring valuable information. A second attempt consists in splitting the integral into two parts, corresponding to the intervals $[-h_2, -h]$ and $[-h, -h_1]$, before applying Jensen's inequality. Following this procedure the following lower bound of Θ_0 is obtained.

$$\begin{aligned} h_{21}\Theta_0 \geq\ & \tfrac{h_{21}}{h-h_1}\left(\int_{-h}^{-h_1} \eta_0^T(s)\mathrm{d}s\right) \begin{bmatrix} R & X \\ * & R \end{bmatrix} \left(\int_{-h}^{-h_1} \eta_0(s)\mathrm{d}s\right) \\ & + \tfrac{h_{21}}{h_2-h}\left(\int_{-h_2}^{-h} \eta_0^T(s)\mathrm{d}s\right) \begin{bmatrix} R & X \\ * & R \end{bmatrix} \left(\int_{-h_2}^{-h} \eta_0(s)\mathrm{d}s\right). \end{aligned}$$

Simple computations show that

$$\begin{aligned} \int_{-h}^{-h_1} \eta_0(s)\mathrm{d}s &= \int_{-h}^{-h_1} \left(\begin{bmatrix} \dot{x}(s) \\ 0 \end{bmatrix} - \frac{\Omega_0}{h_{21}}\right)\mathrm{d}s \\ &= \begin{bmatrix} \frac{h_2-h}{h_{21}} I & 0 \\ 0 & -\frac{h-h_1}{h_{21}} I \end{bmatrix} \Omega_0 \end{aligned}$$

and, similarly, we have

$$\int_{-h_2}^{-h} \eta_0(s)\mathrm{d}s = -\begin{bmatrix} \frac{h_2-h}{h_{21}} I & 0 \\ 0 & -\frac{h-h_1}{h_{21}} I \end{bmatrix} \Omega_0.$$

Reinjecting these expressions into the lower bound of Θ_0 leads to

$$h_{21}\Theta_0 \geq \Omega_0^T \begin{bmatrix} \frac{(h_2-h)^2}{(h_2-h)(h-h_1)} R & -\frac{(h_2-h)(h-h_1)}{(h_2-h)(h-h_1)} X \\ * & \frac{(h-h_1)^2}{(h_2-h)(h-h_1)} R \end{bmatrix} \Omega_0,$$

or, equivalently

$$h_{21}\Theta_0 \geq \Omega_0^T \begin{bmatrix} \frac{h_2-h}{h-h_1} R & -X \\ * & \frac{h-h_1}{h_2-h} R \end{bmatrix} \Omega_0. \tag{9}$$

The resulting expression of the lower bound of Θ_0 coincides with the one obtained in the original proof of Park's inequality in (4). Indeed it represents the conservatism of the application of Jensen's inequality and the reciprocally convex lemma [6]. This lemma 2 provides another intepretation Jensen-Park's inequality, corresponding to a discretized Jensen's inequality.

4.3 A particular inequality

Based on inequalities (5) or (7), a particular case can be derived by selecting $X = -R$. This leads to the following corollary

Corollary 2 *Let $R = R^T$ be a positive definite matrix in $\mathbb{R}^{n\times n}$ and $x : [-h_2, -h_1] \to \mathbb{R}^n$ be a piecewise differentiable function. The inequality*

$$h_{21}\int_{-h_2}^{-h_1} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s \geq \Omega_1^T R \Omega_1 \tag{10}$$

holds for h in $[h_1, h_2]$ where $\Omega_1 = x(-h_1) - 2x(-h) + x(-h_2)$.

Proof : Selecting $X = -R$ in Lemma 2 implies $\left[\begin{smallmatrix} R & -R \\ * & R \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \\ -I \end{smallmatrix}\right] R \left[\begin{smallmatrix} I \\ -I \end{smallmatrix}\right]^T \succeq 0$. The proof is concluded by noting that $\Omega_0^T \left[\begin{smallmatrix} R & -R \\ * & R \end{smallmatrix}\right] \Omega_0 = \Omega_1^T R \Omega_1$. $\diamondsuit$

The benefits of this particular inequality are first that the number of decision variables is reduced by n^2 since no matrix X is required and, second, the size of the LMI is also reduced by $2n\times 2n$ since the contraint $\left[\begin{smallmatrix} R & -R \\ * & R \end{smallmatrix}\right] \succeq 0$ is already captured by the condition $R \succ 0$. Moreover it was shown in [8], that the same level of conservatism as in [6] was obtained for fast-varying delay systems.

5 Conclusion and perspectives

In this paper, we have provided an alternative proof of an integral inequality that is usually derived from the application of the Jensen's inequality and the reciprocally convex combination lemma. This proof shows that this integral inequality can be seen as a discretized Jensen's inequality. The benefits of this approach are the derivation of an exact expression of the conservatism of this inequality and, potentially, a method to extend this class of integral inequalities. Therefore, it seems relevant to look for discretized version of these recent inequalities to obtain less conservative inequalities.

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