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# Delay-dependent reciprocally convex combination lemma

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## Abstract

This paper deals with the stability analysis of linear systems subject to fast-varying delays. The main result of this paper is the derivation of a delay-dependent reciprocally convex lemma allowing a notable reduction of the conservatism of the resulting stability conditions at the reasonable price of additional decision variables. Several examples are studied to show the potential of the proposed method.

*Key words:* Time-delay systems, integral inequalities, matrix inequality, reciprocally convex lemma.

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## 1 Introduction

This paper aims at providing less conservatism and computationally efficient stability conditions for linear systems subject to fast-varying delays. This topic of research has attracted many researchers over the past decades (see for instance [2]). The main difficulties for the study of such a class of systems rely on two technical steps that are the derivation of efficient integral and matrix inequalities. Indeed, the differentiation of usual candidates for being Lyapunov-Krasovskii functionals leads to integral quadratic terms that cannot be included straightforwardly in a linear matrix inequality (LMI) setup. Including these terms requires the use of integral inequalities such as Jensen [3], Wirtinger-based [7], auxiliary-based [6] or Bessel inequalities [8]. Although these inequalities have shown a great interest for constant delay systems, their application to time- or fast-varying delays leads to additional difficulties related to the non convexity of the resulting terms. Then matrix inequalities are employed to derive convex conditions. The first method corresponds to the application of Young's inequality or Moon's inequality, which basically results from the positivity of a square

positive definite term. It can also be noted that the recent free-matrix inequality [12] can be interpreted as the merge of the Wirtinger-based inequality and Moon's inequality. Recently, the reciprocally convex lemma was proposed in [5]. This novelty of this method consists in merging the non convex terms into a single expression to derive an accurate convex inequality. It was notably shown that the conservatism of the reciprocally convex lemma [5] and the Moon's inequality are similar when considering Jensen-based stability criteria, with a lower computational burden.

In the present paper, the objective is to refine the reciprocally convex lemma by introducing delay dependent terms. The resulting lemma includes the initial reciprocally convex lemma as a particular case, and example show a clear reduction of conservatism with respect to the literature at a reasonable increase of the computational cost.

**Notations:** Throughout the paper  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space and  $\mathbb{R}^{n \times m}$  and  $\mathbb{S}^n$  are the set of  $n \times m$  real matrices and of  $n \times n$  real symmetric matrices, respectively. For any  $P \in \mathbb{S}^n$ ,  $P \succ 0$  means that  $P$  is symmetric positive definite. For any matrices  $A, B$  of appropriate dimension, the matrix  $\text{diag}(A, B)$  stands for  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ . The matrices  $I_n$  and  $0_{n,m}$  represent the identity and null matrices of appropriate dimension and, when no confusion is possible, the subscript will be omitted. For any  $h > 0$  and any function  $x : [-h, +\infty) \rightarrow \mathbb{R}^n$ , the notation  $x_t(\theta)$  stands for  $x(t+\theta)$ , for all  $t \geq 0$  and all  $\theta \in [-h, 0]$ . Finally, for given positive scalars  $h_1 \leq h_2$ , we use the notation  $h_{21} = h_2 - h_1$ .

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## 2 Problem formulation and preliminaries

Consider a linear time-delay system of the form:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-h(t)), & \forall t \geq 0, \\ x(t) = \phi(t), & \forall t \in [-h_2, 0], \end{cases} \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\phi$  is the initial condition and  $A, A_d \in \mathbb{R}^{n \times n}$  are constant matrices. There exist positive scalars  $h_1 \leq h_2$  such that

$$h(t) \in [h_1, h_2], \quad \forall t \geq 0. \quad (2)$$

No assumption on  $\dot{h}$  are included. When possible, the time argument of the delay function  $h$  will be omitted.

### 2.1 Objectives

Providing efficient stability conditions for time-varying or fast-varying delay systems relies on the accuracy of the matrix or integral inequalities, which are considered. On one hand, much attention has been paid recently to integral inequalities such as Jensen's [3], Wirtinger-based [7] or Bessel's inequalities [8], auxiliary functions [6] or on free-weighting matrix [12]. In this paper, we will only concentrate on the Wirtinger-based inequality stated in the next lemma, noting that the main result of this paper can be adapted to the other integral inequalities.

**Lemma 1** *Let  $R \succ 0$  be in  $\mathbb{S}^n$  and  $x$  be a continuously differentiable function from  $[-h_2, -h_1]$  to  $\mathbb{R}^n$ . The following equality holds*

$$h_{21} \int_{-h_2}^{-h_1} \dot{x}^T(s) R \dot{x}(s) ds = \omega_0^T R \omega_0^T + 3\omega_1^T R \omega_1^T,$$

where  $\omega_0 = x(-h_1) - x(-h_2)$ , and  $\omega_1 = x(-h_1) + x(-h_2) - 2/h_{21} \int_{-h_2}^{-h_1} x(s) ds$ .

On a second hand, when considering time varying delays, the problem often relies on finding lower bound  $\Theta_m$  of the parameter dependent matrix given by

$$\begin{bmatrix} \frac{1}{\alpha} R & 0 \\ 0 & \frac{1}{1-\alpha} R \end{bmatrix} \succeq \Theta_m.$$

There are two main methods to find lower bounds  $\Theta_m$ . The first one is based on the Moon's inequality (see for instance in the survey paper [11]). The second method is the so-called reciprocally convex combination lemma developed in [5]. The conservatism induced by these two inequalities are independent. While, in some cases, such as stability conditions resulting from the application of the Jensen inequality, the two methods lead to equivalent results on examples, the reciprocally convex combination

lemma is in general more conservative than Moon's inequality (see for instance [12]). In this paper, we present an extended version of the reciprocally convex combination lemma, which reduces notably the conservatism of the resulting stability conditions.

## 3 Extended reciprocally convex inequality

This section is devoted to the derivation of a new matrix inequality which refines the reciprocally convex combination lemma from [5]. It is presented in the next lemma.

**Lemma 2** *Let  $n$  be a positive integer, and  $R$  be a positive definite matrix in  $\mathbb{S}^n$ . If there exist  $X_1, X_2$  in  $\mathbb{S}^n$  and  $Y_1, Y_2$  in  $\mathbb{R}^{n \times n}$  such that*

$$\begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} - \alpha \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} - (1-\alpha) \begin{bmatrix} 0 & Y_2 \\ Y_2^T & X_2 \end{bmatrix} \succeq 0 \quad (3)$$

for all  $\alpha = 0, 1$ , then the following inequality

$$\begin{bmatrix} \frac{1}{\alpha} R & 0 \\ 0 & \frac{1}{1-\alpha} R \end{bmatrix} \succeq \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} + (1-\alpha) \begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix} \quad (4)$$

holds for all  $\alpha \in (0, 1)$ .

*Proof :* Following [5], the proof consists in noting that

$$\mathcal{R}(\alpha) = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} + \begin{bmatrix} \frac{1-\alpha}{\alpha} R & 0 \\ 0 & \frac{\alpha}{1-\alpha} R \end{bmatrix}. \quad (5)$$

The objective is to find a lower bound of the second term of the right-hand-side of (5). Using a convexity argument, if (3) holds for  $\alpha = 0, 1$ , it also holds for any  $\alpha$  in  $[0, 1]$ . Then, pre- and post-multiplying inequality (3) by  $\begin{bmatrix} \sqrt{\frac{1-\alpha}{\alpha}} I & 0 \\ 0 & \sqrt{\frac{\alpha}{1-\alpha}} I \end{bmatrix}$  yields, for all  $\alpha$  in  $(0, 1)$

$$\begin{aligned} \begin{bmatrix} \frac{1-\alpha}{\alpha} R & 0 \\ * & \frac{\alpha}{1-\alpha} R \end{bmatrix} &\succeq \alpha \begin{bmatrix} \frac{1-\alpha}{\alpha} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} + (1-\alpha) \begin{bmatrix} 0 & Y_2 \\ Y_2^T & \frac{\alpha}{1-\alpha} X_2 \end{bmatrix} \\ &= (1-\alpha) \begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix}. \end{aligned} \quad (6)$$

Re-injecting (6) into (5) concludes the proof.  $\diamond$

It is worth noting that (3) is affine with respect to  $\alpha$ , therefore it suffices to verify the inequality at the boundary of the interval  $[0, 1]$ . The second inequality of the previous lemma provides a lower bound of  $\mathcal{R}(\alpha)$  which is also affine, and consequently convex in  $\alpha$ . Note moreover that, selecting  $X_1 = X_2 = 0$ , and  $Y_1 = Y_2 = Y \in \mathbb{R}^{n \times n}$ ,

$$\begin{aligned}
g_0 &= \begin{bmatrix} A & 0 & A_d & 0 & 0 & 0 & 0 \end{bmatrix}, \quad G_0 = \begin{bmatrix} A & 0 & A_d & 0 & 0 & 0 & 0 \\ I & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & -I & 0 & 0 & 0 \end{bmatrix}, \quad G_1(\theta) = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\theta - h_1)I & (h_2 - \theta)I \end{bmatrix}, \\
G_2 &= \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & -2I & 0 & 0 \end{bmatrix}, \quad G_3 = \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 & -2I & 0 \end{bmatrix}, \quad G_4 = \begin{bmatrix} 0 & 0 & I & -I & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & -2I \end{bmatrix}, \quad \Gamma = \begin{bmatrix} G_2 \\ G_3 \\ G_4 \end{bmatrix}.
\end{aligned} \tag{7}$$

inequalities (3) and (4) recover the reciprocally convex combination lemma from [5]. Therefore, Lemma 2 brings additional degree of freedom and is potentially less conservative than the lemma from [5].

The conditions of Lemma 2 can be straightforwardly extended to find a lower bound of the matrix

$$\mathcal{R}_{12}(\alpha) = \begin{bmatrix} \frac{1}{\alpha} R_1 & 0 \\ 0 & \frac{1}{1-\alpha} R_2 \end{bmatrix},$$

where  $R_1$  and  $R_2$  are in  $\mathbb{S}^n$  and  $\mathbb{S}^m$ , respectively, where  $n$  and  $m$  in  $\mathbb{N}$  are not necessarily equal.

**Remark 1** Lemma 2 involves four matrix variables, namely,  $X_1, X_2, Y_1$  and  $Y_2$ . One may reduce the number of decision variables by including the constraints  $X_1 = X_2$  and/or  $Y_1 = Y_2$ .

#### 4 Main result

Based on the previous developments, the following stability theorem is provided.

**Theorem 1** Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R_1, R_2$  in  $\mathbb{S}_+^n$ ,  $X_1, X_2$  in  $\mathbb{S}_+^{2n}$ , and two matrices  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$ , such that the conditions

$$\begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} - \begin{bmatrix} X_1 & Y_1 \\ Y_1^T & 0 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} - \begin{bmatrix} 0 & Y_2 \\ Y_2^T & X_2 \end{bmatrix} \succeq 0, \tag{8}$$

$$\begin{aligned} \Phi(h_1) &= \Phi_0(h_1) - \Gamma^T \Psi(h_1) \Gamma \prec 0, \\ \Phi(h_2) &= \Phi_0(h_2) - \Gamma^T \Psi(h_2) \Gamma \prec 0, \end{aligned} \tag{9}$$

are satisfied, where

$$\begin{aligned} \Phi_0(\theta) &= G_1^T(\theta) P G_1(\theta) + \hat{S} \\ &\quad + g_0^T (h_1^2 R_1 + h_2^2 R_2) g_0, \\ \hat{S} &= \text{diag}(S_1, -S_1 + S_2, 0_n, -S_2, 0_{3n}), \\ \tilde{R}_i &= \text{diag}(R_i, 3R_i), \quad \forall i = 1, 2, \end{aligned} \tag{10}$$

$$\begin{aligned} \Psi(h_1) &= \text{diag} \left( \tilde{R}_1, \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} + \begin{bmatrix} X_1 & Y_2 \\ Y_2^T & 0 \end{bmatrix} \right), \\ \Psi(h_2) &= \text{diag} \left( \tilde{R}_1, \begin{bmatrix} \tilde{R}_2 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} + \begin{bmatrix} 0 & Y_1 \\ Y_1^T & X_2 \end{bmatrix} \right), \end{aligned} \tag{11}$$

and where the matrices  $g_0, \Gamma$  and  $G_i$ , for  $i = 0, 1, \dots, 4$  are given in (7). Then system (1) is asymptotically stable for all time-varying delay  $h$  satisfying (2).

*Proof :* Consider the same Lyapunov-Krasovskii functional as in [9], given by

$$\begin{aligned} V(x_t, \dot{x}_t) &= \begin{bmatrix} x(t) \\ \int_{t-h_1}^t x(s) ds \\ \int_{t-h_2}^{t-h_1} x(s) ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \int_{t-h_1}^t x(s) ds \\ \int_{t-h_2}^{t-h_1} x(s) ds \end{bmatrix} \\ &\quad + \int_{t-h_1}^t x^T(s) S_1 x(s) ds + \int_{t-h_2}^{t-h_1} x^T(s) S_2 x(s) ds, \\ &\quad + h_1 \int_{-h_1}^0 \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds \\ &\quad + h_{12} \int_{-h_2}^{-h_1} \int_{t+\theta}^t \dot{x}^T(s) R \dot{x}(s) ds, \end{aligned} \tag{12}$$

where  $h_{12} = h_2 - h_1$ . Following exactly the same procedure as in [9], the differentiation of the functional  $V$  along the trajectories of system (1) leads to

$$\begin{aligned} \dot{V}(x_t, \dot{x}_t) &= \zeta^T(t) \Phi_0(h) \zeta(t) - h_1 \int_{t-h_1}^t \dot{x}^T(s) R_1 \dot{x}(s) ds \\ &\quad - h_{12} \int_{t-h_2}^{t-h_1} \dot{x}^T(s) R_2 \dot{x}(s) ds, \end{aligned} \tag{13}$$

with  $\Phi_0(h)$  given in (11) and  $\zeta(t) = [\zeta_1^T(t), \zeta_2^T(t)]^T$  with

$$\zeta_1(t) = \begin{bmatrix} x(t) \\ x(t-h_1) \\ x(t-h) \\ x(t-h_2) \end{bmatrix}, \quad \zeta_2(t) = \begin{bmatrix} \frac{1}{h_1} \int_{t-h_1}^t x^T(s) ds \\ \frac{1}{h-h_1} \int_{t-h}^{t-h_1} x^T(s) ds \\ \frac{1}{h_2-h} \int_{t-h_2}^{t-h} x^T(s) ds \end{bmatrix}.$$

Applying Lemma 1 to the two integral terms, after splitting the second integral into two parts, leads to

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t) (\Phi_0(h) - \Gamma^T \Psi(h) \Gamma) \zeta(t), \tag{14}$$

where  $\Gamma$  is given in (7) and

$$\Psi(h) = \text{diag} \left( \tilde{R}_1, \begin{bmatrix} \frac{h_{12}}{h-h_1} \tilde{R}_2 & 0 \\ * & \frac{h_{12}}{h_2-h} \tilde{R}_2 \end{bmatrix} \right).$$

Applying Lemma 2 with  $\alpha = (h - h_1)/h_{12}$ , it yields that, if there exists a matrix  $X_1, X_2$  in  $\mathbb{S}^{2n}$  and  $Y_1, Y_2$  in  $\mathbb{R}^{2n \times 2n}$  such that conditions (8) hold, then we have

$$\Psi(h) \succeq \frac{h_2 - h}{h_{12}} \Psi(h_1) + \frac{h - h_1}{h_{12}} \Psi(h_2).$$

Noting that the matrix  $\Phi_0(h)$  is affine in  $h$ , so that we can write  $\Phi_0(h) = \frac{h_2 - h}{h_{12}} \Phi_0(h_1) + \frac{h - h_1}{h_{12}} \Phi_0(h_2)$  and the following inequality holds

$$\dot{V}(x_t, \dot{x}_t) \leq \zeta^T(t) \left( \frac{h_2 - h}{h_{12}} \Phi(h_1) + \frac{h - h_1}{h_{12}} \Phi(h_2) \right) \zeta(t).$$

Therefore if the two LMIs  $\Phi(h_1) \prec 0$  and  $\Phi(h_2) \prec 0$  are satisfied, any linear combination of this two matrices is also definite negative and we can conclude that the system is asymptotically stable for all time-varying delay in the interval  $[h_1, h_2]$ .  $\diamond$

It is worth noting that the proof of Theorem 1 is very similar to the one provided in [9]. The only difference relies on the use of Lemma 2. The impact in terms of reduction of the conservatism will be exposed in the example section. In light of Remark 1, the following corollary of Theorem 1 are provided to reduce the number of decision variables.

**Corollary 1** *Assume that there exist matrices  $P$  in  $\mathbb{S}_+^{3n}$ ,  $S_1, S_2, R_1, R_2$  in  $\mathbb{S}_+^n$ ,  $X$  in  $\mathbb{S}_+^{2n}$ , and  $Y$  in  $\mathbb{R}^{2n \times 2n}$ , such that the conditions (8) and (9) hold with  $X_1 = X_2 = X$  and  $Y_1 = Y_2 = Y$ . Then system (1) is asymptotically stable for any time-varying delay  $h$  satisfying (2).*

In the previous corollary, only a symmetric matrix  $X$  and a full rank matrix  $Y$  are introduced. It has to be noticed that the only remaining difference with respect to the conditions from theorem provided in [9] is the introduction of the symmetric matrix  $X$ . We will show in the example section, only the sole introduction of this matrix leads to a notable reduction of the conservatism.

## 5 Illustrative Examples

Two numerical examples from the literature will illustrate the efficiency of the proposed conditions in Theorems 1 and its corollary. Before entering into the numerical results, we point out in Table 1, the number of decision variables involved in Theorem 1 and Corollary 1 compared with the ones from existing results from the

Th.	No. of variables	Th.	No. of variables
[5]	$3.5n^2 + 2.5n$	[1]	$11.5n^2 + 3.5n$
[6]	$21n^2 + 6n$	[9]	$10.5n^2 + 3.5n$
[12]	$54.5n^2 + 9.5n$		
Th. 1	$18.5n^2 + 5.5n$	Cor. 1	$12.5n^2 + 4.5n$

Table 1

Number of decision variables involved in several conditions from the literature and in Theorem 1 and its corollary

$h_1$	0.0	0.4	0.7	1.0	2.0	3.0
[5]	1.86	1.88	1.95	2.06	2.61	3.31
[1]	1.86	1.89	1.98	2.12	2.72	3.45
[6]	2.14	2.19	2.24	2.31	2.80	3.50
[9]	2.11	2.17	2.23	2.31	2.79	3.49
[12]	2.18	2.21	2.25	2.32	2.79	3.49
Th. 1	2.21	2.25	2.28	2.34	2.80	3.49
Cor. 1	2.19	2.24	2.28	2.34	2.80	3.49

Table 2

Example 1: Admissible upper bound of  $h_2$  for various  $h_1$ .

literature. For the two next examples, we expose in Tables 2 and 3 the maximal upper-bound,  $h_2$  of the delay functions for various values of  $h_1$  obtained by solving by Theorems 1, its corollary and several recent stability conditions from literature.

There exists a large number of paper dealing with the stability analysis of such a class of system. Because of space limitations, we consider only few representative conditions from the literature. On a first side, conditions derived using Jensen's inequality ([1] and [5]), Wirtinger-based inequality ([9]), auxiliary-based inequality [6] and the recent free-matrix-based inequality ([12]). On the other hand, we also discriminate conditions that are based on Young/Moon inequality [1,12], or on the reciprocally convex combination lemma [5,9,6]. A last comment on the contribution presented in [6]. Indeed, the conditions proposed in [6] is proven to be less conservative than the Wirtinger-based inequality together with the reciprocally convex combination lemma. Therefore, it is expected that the conditions presented in [6] are less conservative than the one from Theorem 1.

**Example 1:** Consider the following much-studied linear time-delay system (1) with

$$A = \begin{bmatrix} -2.0 & 0.0 \\ 0.0 & -0.9 \end{bmatrix}, A_1 = \begin{bmatrix} -1.0 & 0.0 \\ -1.0 & -1.0 \end{bmatrix}.$$

The results obtained by solving Theorem 1 and its corollary show a clear reduction of the conservatism. Moreover, the improvements due to the use of Lemma 2 and its corollary can be seen when comparing the results obtained with [9] and the stability conditions provided in

$h_1$	0.0	0.3	0.5	0.8	1.0	2.0
[4]	0.77	0.94	1.09	1.34	1.51	2.40
[10]	0.87	1.07	1.21	1.45	1.61	2.47
[5]	1.06	1.24	1.38	1.60	1.75	2.58
[6]	1.19	1.35	1.47	1.67	1.82	2.63
[9]	1.19	1.35	1.47	1.67	1.82	2.63
[12]	1.20	1.35	1.47	1.67	1.82	2.63
Th. 1	1.20	1.35	1.47	1.67	1.82	2.63
Cor. 1	1.20	1.35	1.47	1.67	1.82	2.63

Table 3  
Example 2: admissible upper bound of  $h_2$  for various  $h_1$ .

the present paper. Indeed the only difference between these two papers is the use of the delay-dependent reciprocally convex lemma. Moreover, it is worth noting that Theorem 1 and its corollaries provide less conservative results, on this example, than other conditions from the literature except for [6] with  $h_1 = 3$ . This improvement of [6] can be explained by the use of the auxiliary function integral inequality, which is less conservative than the Wirtinger inequality. It is also worth noting that Theorem 1 and its corollaries leads in general to the same results except for small lower bounds  $h_1 = 0$  even if the computational complexities of the stability conditions are different.

**Example 2:** We consider now the linear time-delay system (1), taken from [5], with

$$A = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -2.0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.0 & 0.0 \\ -1.0 & 1.0 \end{bmatrix}.$$

For this example, Theorem 1 and its corollaries deliver the same results. Hence, we present in Table 3 the maximal allowable upper-bound of the delay obtained, for different values of  $h_1$ , by application of various conditions from the literature and the ones presented in this paper. Theorem 1 and Corollary 1 deliver the same result as in [12] and [6] (except when  $h_1 = 0$ ) but with a lower number of decision variables as shown in Table 2. This shows again, the potential of the improved reciprocally convex lemma.

## 6 Conclusions

In this paper, an improved version of the reciprocally convex lemma is provided. The novelty of this technical lemma brings a notable reduction of the conservatism of LMI stability conditions for fast-varying delay systems with a reasonable additional computational burden.

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