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# Capacity Bounds for a Class of Interference Relay Channels

Germán Bassi, Pablo Piantanida and Sheng Yang

**Abstract**—The capacity of a class of Interference Relay Channels (IRC) –the Injective Semideterministic IRC where the relay can only observe one of the sources– is investigated. We first derive a novel outer bound and two inner bounds which are based on a careful use of each of the available cooperative strategies together with the adequate interference decoding technique. The outer bound extends Telatar and Tse’s work while the inner bounds contain several known results in the literature as special cases. Our main result is the characterization of the capacity region of the Gaussian class of IRCs studied within a fixed number of bits per dimension –constant gap. The proof relies on the use of the different cooperative strategies in specific SNR regimes due to the complexity of the schemes. As a matter of fact, this issue reveals the complex nature of the Gaussian IRC where the combination of a single coding scheme for the Gaussian relay and interference channel may not lead to a good coding scheme for this problem, even when the focus is only on capacity to within a constant gap over all possible fading statistics.

**Index Terms**—Interference channel, relay channel, decode-and-forward, compress-and-forward, inner bounds, outer bound, constant gap.

## I. INTRODUCTION

CELLULAR networks have reached practical limits in many dense urban areas while data traffic and the number of users seem to be continuously increasing. Interference has become one of the most crucial problems in cellular networks where users must compete for the available resources, e.g., an improvement in terms of data rate for one of them may be detrimental to the performance of another user. Although the existence of a large amount of users in cellular networks has driven communication channels from being noise-limited to interference-limited, it can also be exploited to boost the overall network throughput by means of user cooperation.

In order to provision a new communication infrastructure, network operators are rethinking conventional cellular system topologies to consider a new paradigm called heterogeneous networks. This consists of planned macro base station (BS) deployments that typically transmit at high power overlaid with several low power nodes such as: relay and pico BSs, distributed antennas, and femto BSs. These lower power nodes are deployed to further increase the coverage of the

network, especially when terminals are far away from the macro BS. Fixed relays are infrastructure equipment that connect wirelessly to the BS and these relays aid in the signal transmission between the macro BS and the mobile users by receiving and retransmitting messages. Indeed, these relays may offer a flexible option where backhauls are not available. In order to assess the benefits of this strategy, an information-theoretic analysis of cooperation through relaying in interference-limited environments should be carried out. Nonetheless, each one of these two fundamental problems –relaying and interference– appears to be rather involved and unfortunately only partial results are available in the literature.

### A. Related Work

Perhaps the simplest model of a communication network with interference is the Interference Channel (IC), whose capacity region –even without a relay– is still an open problem. The largest known achievable rate region is due to Han and Kobayashi [1] and it is based on the idea of interference decoding via “rate-splitting” at the sources, also referred to as “Han-Kobayashi scheme”. This scheme has been shown by Etkin-Tse-Wang [2] to achieve within 1 bit per complex dimension to the capacity region of the Gaussian IC. The important feature behind the notion of “constant gap” is that it guarantees a uniform gap between the inner and the outer bound over all channel coefficients and hence all possible fading statistics. This result hinges on a new upper-bounding technique that has been later on extended to a more general class of ICs [3], also referred to as “Injective Semideterministic IC” [4].

Another challenging problem is the Relay Channel (RC), where a relay node helps the communication between a source-destination pair. Since the seminal work of Cover and El Gamal [5], which has introduced the main cooperative strategies of “decode-and-forward” (DF) and “compress-and-forward” (CF), there has been a great deal of research on this topic. Although the capacity of the RC is still unknown in general, the benefits of cooperation by relaying are rather clear by now, at least in the context of single source and/or single destination relay networks [6]. An approximation approach to general networks via deterministic channels was introduced by Avestimehr-Diggavi-Tse [7]. This approach yields a novel improvement over CF scheme –referred to as “quantize-map-and-forward” (QMF)– that achieves capacity to within a constant gap for unicast additive white Gaussian noise (AWGN) networks with an arbitrary number of relays. As a matter of fact, both DF and CF schemes can perform within the same constant gap to the capacity of the Gaussian RC, regardless of

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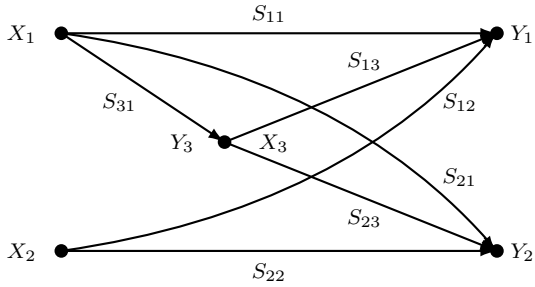


Fig. 1. The Gaussian IRC where the values  $S_{ij}$  represent the SNR between nodes  $j$  and  $i$ .

the channel parameters [7], [8] and thus of the fading statistics. More recently, Lim *et al.* [9] generalized the QMF approach to arbitrary memoryless multicast networks via the “noisy network coding” (NNC) scheme. Relay nodes based on NNC scheme send the same –long– message over many blocks of equal length and the descriptions at the relays do not require binning while their indices are non-uniquely decoded at the destination.

In wireless networks with multiple source nodes that communicate simultaneously to several destinations, “interference” becomes the central issue, and the different roles that relays can play to enhance the reliability in such scenarios are not well understood yet. In this paper, we consider the simplest scenario where interference and relaying appear together, that is the Interference Relay Channel (IRC). The problem itself is not new [10] and the research on this topic has been growing during the past years. In [11], among other works, the authors proposed inner bounds on the capacity region of the IRC based on the standard CF scheme while DF-based schemes are also studied in [12]. It is worth mentioning here that these coding schemes do not use “joint decoding” at the destination to recover all transmit messages and the compression indices. The idea of NNC was later on extended to the IRC in [13] by adding rate-splitting. Besides these works, capacity of the *physically degraded* IRC in the *strong interference* regime was determined in [14] by assuming that the relay node can only observe one of the two source encoders. Several variations of this problem have also been investigated, e.g., the cognitive IRC where the relay has non-casual knowledge of the sources’ messages was treated in [15], [16]. Additionally, the IRC with an “out-of-band relay”, i.e., the relay operates over an orthogonal band with respect to the underlying IC, was also studied in [17]–[21]. Capacity results were obtained in [21] for an IRC with oblivious relaying in which the relay is unaware of the codebook used by the source encoders.

The interference channel with cooperation at either the transmitter or receiver end, or both has also been investigated. In the extreme regimes where the relay can be thought of being collocated with the transmitters or the receivers, the IRC becomes a virtual multi-antenna IC with transmitter or receiver cooperation. The benefits of such a system have been studied in [22]. Additionally, constant-gap results regardless of channel conditions were provided in [23]–[26], while capacity results in strong interference regime were determined in [27] for the case of transmitter cooperation. Recently, in the case

of unilateral source cooperation, improved outer bounds were reported in [28].

## B. Contribution and Outline

In this paper we focus on a simplified version of the two-user IRC [29] which still captures the rather complex interplay between interference and relaying. This is the two-user IC with a relay node which can only observe one of the source encoders. Although this is not the most general two-user IRC, we shall see that it still captures the central issue of interference and relaying and hence, we seek to provide some useful insights into the understanding of this complex problem. In particular, for the class of Gaussian IRCs shown in Fig. 1, we aim at determining the underlying SNR regimes together with the adequate coding schemes and decoding technique that are needed to achieve capacity to within a *constant gap*.

Our results involve a novel outer bound for the considered class of IRCs –the Injective Semideterministic IRC– and two inner bounds based on rate-splitting and different relaying strategies (building on DF and CF schemes) with the adequate interference decoding technique. Although the use of DF and CF schemes in the context of the IRC is not new [10]–[14], our aim is to provide a set of simple but powerful enough strategies in order to characterize the capacity region of Gaussian IRCs to within a constant gap, as previously stated. In this regard, our main contributions with respect to the literature are the introduction of *partial* DF, where the relay forwards only part of the source’s message, and the use of different decoding strategies in the CF scheme which helps us obtain a compact expression of the inner bound.

The main outcome of this work is the characterization within a constant gap of the capacity of the aforementioned Gaussian IRC. We show that, for any channel realization, at least one of the proposed schemes achieves the capacity region to within a constant gap. More precisely, it is shown that when the source-to-relay channel is stronger than the source-to-destination channel full DF scheme is recommended (this regime includes the capacity result in [14, Thm. 3]). As the strength of the source-to-relay channel reduces, it is preferable to partially decode the message and thus partial DF scheme is required. Finally, when the source-to-relay channel is weaker than the interfering channel from the source to the other destination, CF scheme together with different ways of decoding is needed instead.

This paper is organized as follows. Section II presents the problem definition while the outer bound and the two inner bounds are deferred to Sections III and IV, respectively. The constant gap results are shown in Section V. Finally, all proofs are relegated to the appendices.

## Notation and Conventions

Given two integers  $i$  and  $j$ , the expression  $[i : j]$  denotes the set  $\{i, i + 1, \dots, j\}$ , whereas for real values  $a$  and  $b$ ,  $[a, b]$  denotes the closed interval between  $a$  and  $b$ . Lowercase letters such as  $x$  and  $y$  are mainly used to represent realizations of random variables, whereas capital letters such as  $X$  and  $Y$  stand for the random variables in itself. Bold capital letters

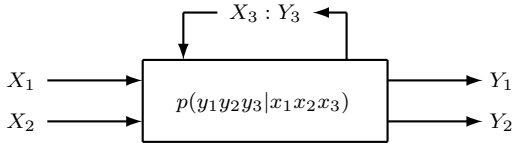


Fig. 2. Interference Relay Channel (IRC) model.

such as  $\mathbf{H}$  and  $\mathbf{Q}$  represent matrices, while calligraphic letters such as  $\mathcal{X}$  and  $\mathcal{Y}$  are reserved for sets. The probability distribution (PD) of the random vector  $X^n$ ,  $p_{X^n}(x^n)$ , is succinctly written as  $p(x^n)$  without subscript when it can be understood from the argument  $x^n$ . Given three random variables  $X$ ,  $Y$ , and  $Z$ , if its joint PD can be decomposed as  $p(xyz) = p(y)p(x|y)p(z|y)$ , then they form a Markov chain, denoted by  $X \text{---} Y \text{---} Z$ . Differential entropy is denoted by  $h(\cdot)$  and the mutual information,  $I(\cdot; \cdot)$ . The expression  $C[x] = \frac{1}{2} \log_2(1+x)$  stands for the capacity of a Gaussian channel with SNR of value  $x$ . Definitions and properties of strongly typical sequences and delta-convention are provided in Appendix A.

## II. PROBLEM DEFINITION

The IRC consists of two source encoders, two destinations and one relay node. Encoder  $k$  wishes to send a message  $\tilde{m}_k \in \mathcal{M}_{n,k} \triangleq \{1, \dots, M_{n,k}\}$  to destination  $k$ ,  $k \in \{1, 2\}$ , with the help of the relay. The IRC, depicted in Fig. 2, is modeled as a memoryless channel without feedback defined by a conditional probability distribution (PD):

$$p(y_1, y_2, y_3 | x_1, x_2, x_3) : \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \mapsto \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_3$$

where  $x_k \in \mathcal{X}_k$  and  $y_k \in \mathcal{Y}_k$ ,  $k \in \{1, 2\}$ , are the input at source  $k$  and output at destination  $k$ , respectively, whereas  $x_3 \in \mathcal{X}_3$  and  $y_3 \in \mathcal{Y}_3$  are the input and output at the relay, respectively. The relaying functions are defined as a sequence of mappings  $\{\phi_i : \mathcal{Y}_3^{i-1} \mapsto \mathcal{X}_3\}_{i=1}^n$ .

As it was previously stated, throughout the paper we deal with a specific type of IRC in which only one of the sources is connected to the relay, i.e.,

$$p(y_1, y_2, y_3 | x_1, x_2, x_3) = p(y_3 | x_1, x_3) p(y_1, y_2 | x_1, x_2, x_3, y_3). \quad (1)$$

Unless it is noted otherwise, this is a basic assumption of our model.

We also recall that a pair of rates  $(R_1, R_2)$  is said to be achievable for an IRC if for every  $\epsilon > 0$  there exists a block length  $n$  and encoders  $\text{enc}_k : \mathcal{M}_{n,k} \mapsto \mathcal{X}_k^n$ ,  $M_{n,k} \geq 2^{n(R_k - \epsilon)}$ ,  $k \in \{1, 2\}$ , and decoder  $\text{dec}_k : \mathcal{Y}_k^n \mapsto \mathcal{M}_{n,k}$ ,  $k \in \{1, 2\}$ , such that

$$\frac{1}{M_{n,1} M_{n,2}} \sum_{\tilde{m}_1, \tilde{m}_2} \mathbb{P}\{(\text{dec}_1(Y_1^n), \text{dec}_2(Y_2^n)) \neq (\tilde{m}_1, \tilde{m}_2) \mid X_1^n = \text{enc}_1(\tilde{m}_1), X_2^n = \text{enc}_2(\tilde{m}_2)\} \leq \epsilon.$$

*Definition 1 (Injective Semideterministic IRC):* In this paper, we shall focus on the class of IRCs referred to as the Injective Semideterministic IRC (IS-IRC), as shown in Fig. 3, which is an extension of that introduced in [3] for the IC. In

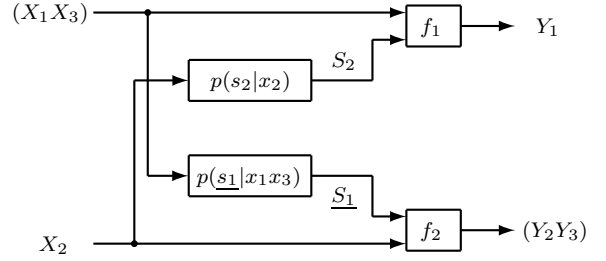


Fig. 3. Injective Semideterministic IRC (IS-IRC) model.

this model, the randomness of the channel is captured by the interference signals  $S_1$ ,  $S_2$  and  $S_3$ . For sake of clarity, we will denote the pair  $(S_1 S_3)$  as the vector  $\underline{S}_1$ .

The conditional PD of the interference signals may be decomposed as follows,  $p(s_1 s_2 | x_1 x_2 x_3) = p(s_1 | x_1 x_3) p(s_2 | x_2)$ , and the outputs of the channel are deterministic functions of  $(X_1, X_2, X_3, \underline{S}_1, S_2)$ . Specifically, we have that  $Y_1 = f_1(X_1, X_3, S_2)$ ,  $Y_2 = f'_2(X_2, S_1)$ , and  $(Y_2 Y_3) = f_2(X_2, \underline{S}_1)$ , where  $f_1$ ,  $f'_2$ , and  $f_2$  are functions that, for every  $(x_1, x_2, x_3)$ ,

$$\begin{aligned} f_1(x_1, x_3, \cdot) : \mathcal{S}_2 &\rightarrow \mathcal{Y}_1, & s_2 &\mapsto f_1(x_1, x_3, s_2), \\ f'_2(x_2, \cdot) : \mathcal{S}_1 &\rightarrow \mathcal{Y}_2, & s_1 &\mapsto f'_2(x_2, s_1), \\ f_2(x_2, \cdot) : \underline{\mathcal{S}}_1 &\rightarrow \mathcal{Y}_2 \times \mathcal{Y}_3, & \underline{s}_1 &\mapsto f_2(x_2, \underline{s}_1) \end{aligned}$$

are invertible.

*Remark 1:* Since the relay only observes the first source, its input  $X_3$  cannot depend on  $X_2$ . Therefore,  $X_3$  is regarded as desired signal at  $Y_1$  and as interference at  $Y_2$ , which motivates us to model this class of IRCs as depicted in Fig. 3. It comes as no surprise that the pair  $(X_1 X_3)$  should be taken as a whole. However, as it is shown later in the derivation of the outer bound, it is also convenient to put the pair  $(Y_2 Y_3)$  together.

A special case of the IS-IRC is the real Gaussian model, as it is shown in Fig. 1, and defined by

$$Y_1 = h_{11}X_1 + h_{12}X_2 + h_{13}X_3 + Z_1, \quad (2a)$$

$$Y_2 = h_{21}X_1 + h_{22}X_2 + h_{23}X_3 + Z_2, \quad (2b)$$

$$Y_3 = h_{31}X_1 + Z_3, \quad (2c)$$

where each noise process  $Z_k \sim \mathcal{N}(0, N_k)$ ,  $k \in \{1, 2, 3\}$ , is independent of each other, and each input has an average power constraint  $\mathbb{E}[|X_k|^2] \leq P_k$ ,  $k \in \{1, 2, 3\}$ . The link between node  $j$  and  $i$  has a fixed channel coefficient  $h_{ij}$ , and the SNR associated to it is denoted  $S_{ij} \triangleq |h_{ij}|^2 P_j / N_i$ . In this model, the interference signals are

$$\underline{S}_1 = \begin{bmatrix} S_1 \\ S_3 \end{bmatrix} = \begin{bmatrix} h_{21}X_1 + h_{23}X_3 + Z_2 \\ h_{31}X_1 + Z_3 \end{bmatrix} \text{ and } S_2 = h_{12}X_2 + Z_1. \quad (3)$$

Therefore, results for the IS-IRC can be applied straightforwardly to the Gaussian case.

## III. OUTER BOUND

In this section, we develop an outer bound for the IS-IRC model described in Section II. The model in Fig. 3 is provided to help the reader understand the genie-aided technique used in the derivation of the bounds. It would be worth to emphasize

that this model by no means assumes that the relay has previous knowledge of any message nor that  $X_3$  or  $Y_3$  are collocated with  $X_1$  or  $Y_2$  as it could be wrongly interpreted based on the aforementioned figure.

Let  $\mathcal{P}_1$  be the set of all joint PDs that can be factored as:

$$p(q)p(x_1x_3|q)p(x_2|q)p(\underline{v}_1\underline{v}_2|x_1x_2x_3q), \quad (4)$$

where  $p(\underline{v}_1\underline{v}_2|x_1x_2x_3q) = p_{S_1|X_1X_3}(\underline{v}_1|x_1x_3)p_{S_2|X_2}(\underline{v}_2|x_2)$ , i.e.,  $(\underline{V}_1\underline{V}_2)$  is a conditionally independent copy of  $(S_1S_2)$  given  $(X_1X_2X_3)$ . Let us recall that  $V_1$  represents the first component of  $\underline{V}_1$ .

*Theorem 1 (outer bound):* Given a specific  $P_1 \in \mathcal{P}_1$ , let  $\mathcal{R}_o(P_1)$  be the region of nonnegative rate pairs  $(R_1, R_2)$  satisfying

$$R_1 \leq I(X_1; Y_1Y_3|X_2X_3Q), \quad (5a)$$

$$R_1 \leq I(X_1X_3; Y_1|X_2Q), \quad (5b)$$

$$R_2 \leq I(X_2; Y_2|X_1X_3Q), \quad (5c)$$

$$R_1 + R_2 \leq I(X_1X_3; Y_1|V_1X_2Q) + I(X_1X_2X_3; Y_2|Q), \quad (5d)$$

$$R_1 + R_2 \leq I(X_1X_2X_3; Y_1|V_1Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5e)$$

$$R_1 + R_2 \leq I(X_1X_2X_3; Y_1|Q) + I(X_2; Y_2|X_1V_2X_3Q), \quad (5f)$$

$$R_1 + R_2 \leq I(X_1; Y_1Y_3|V_1X_2X_3Q) + I(X_1X_2X_3; Y_2|Q), \quad (5g)$$

$$R_1 + R_2 \leq I(X_1X_2; Y_1Y_3|V_1X_3Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5h)$$

$$R_1 + R_2 \leq I(X_1X_2; Y_1Y_3|X_3Q) + I(X_2; Y_2|X_1V_2X_3Q), \quad (5i)$$

$$R_1 + R_2 \leq I(X_1; Y_1Y_3|\underline{V}_1X_2X_3Q) + I(X_1X_2; Y_2Y_3|X_3Q), \quad (5j)$$

$$R_1 + R_2 \leq I(X_1X_2; Y_1Y_3|\underline{V}_1X_3Q) + I(X_1X_2; Y_2Y_3|V_2X_3Q), \quad (5k)$$

$$2R_1 + R_2 \leq I(X_1X_3; Y_1|V_1X_2Q) + I(X_1X_2X_3; Y_1|Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5l)$$

$$2R_1 + R_2 \leq I(X_1X_3; Y_1|V_1X_2Q) + I(X_1X_2; Y_1Y_3|X_3Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5m)$$

$$2R_1 + R_2 \leq I(X_1; Y_1Y_3|V_1X_2X_3Q) + I(X_1X_2X_3; Y_1|Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5n)$$

$$2R_1 + R_2 \leq I(X_1; Y_1Y_3|V_1X_2X_3Q) + I(X_1X_2; Y_1Y_3|X_3Q) + I(X_1X_2X_3; Y_2|V_2Q), \quad (5o)$$

$$2R_1 + R_2 \leq I(X_1; Y_1Y_3|\underline{V}_1X_2X_3Q) + I(X_1X_2X_3; Y_1|Q) + I(X_1X_2; Y_2Y_3|V_2X_3Q), \quad (5p)$$

$$2R_1 + R_2 \leq I(X_1; Y_1Y_3|\underline{V}_1X_2X_3Q) + I(X_1X_2; Y_1Y_3|X_3Q) + I(X_1X_2; Y_2Y_3|V_2X_3Q), \quad (5q)$$

$$R_1 + 2R_2 \leq I(X_1X_2X_3; Y_1|V_1Q) + I(X_2; Y_2|X_1V_2X_3Q) + I(X_1X_2X_3; Y_2|Q), \quad (5r)$$

$$R_1 + 2R_2 \leq I(X_1X_2; Y_1Y_3|V_1X_3Q) + I(X_2; Y_2|X_1V_2X_3Q) + I(X_1X_2X_3; Y_2|Q), \quad (5s)$$

$$R_1 + 2R_2 \leq I(X_1X_2; Y_1Y_3|\underline{V}_1X_3Q) + I(X_2; Y_2|X_1V_2X_3Q) + I(X_1X_2; Y_2Y_3|X_3Q). \quad (5t)$$

Then, an outer bound for the IS-IRC is defined by the union of  $\mathcal{R}_o(P_1)$  over all PDs  $P_1 \in \mathcal{P}_1$ , as decomposed in (4).

*Proof:* See Appendix B.  $\blacksquare$

The real Gaussian model, presented in Section II, is a special case of the IS-IRC. Therefore, according to (4), the sources' inputs  $X_1$  and  $X_2$  are independent, and  $X_1$  is arbitrarily correlated to the relay's input  $X_3$ , i.e.,  $E[X_1X_2] = 0$ ,  $E[X_1X_3] = \rho\sqrt{P_1P_3}$  and  $E[X_2X_3] = 0$ . The Gaussian expression of the outer bound is readily found using the model (2) and generating the auxiliaries  $\underline{V}_1$  and  $V_2$  according to (3), but with independent noises.

The foregoing Gaussian outer bound  $\mathcal{R}_o = \bigcup_{\rho \in [-1,1]} \mathcal{R}_o(\rho)$  depends on the correlation coefficient  $\rho$  between  $X_1$  and  $X_3$  and, due to the large number of bounds, only a numerical maximization results viable. In order to obtain analytical expressions which can be used later to characterize the gap between inner and outer bounds, we establish an outer bound on  $\mathcal{R}_o$ . This outer bound is obtained by maximizing each individual rate constrain in  $\mathcal{R}_o(\rho)$  independently.

Let us define any of the bounds in  $\mathcal{R}_o(\rho)$  as  $b(\rho)$  and  $\rho_{\max}$  as the value that maximizes that particular bound. Then, it can be shown that  $b(\rho_{\max}) = b(0)$  or  $b(\rho_{\max}) \leq b(0) + \Delta$ , where  $\Delta$  is either 0.5 or 1 bit. Therefore, we can simplify the expressions in the outer bound and avoid the maximization procedure if we use uncorrelated inputs and enlarge certain bounds, as we see in the following corollary. A similar observation has also been made in [7, Appx. A] and [9, (19)].

*Corollary 1 (outer bound for the Gaussian case):* An outer bound for the Gaussian IRC is given by the set of nonnegative rate pairs  $(R_1, R_2)$  satisfying

$$R_1 \leq C[S_{11} + S_{31}], \quad (6a)$$

$$R_1 \leq C[S_{11} + S_{13}] + \frac{1}{2}, \quad (6b)$$

$$R_2 \leq C[S_{22}], \quad (6c)$$

$$R_1 + R_2 \leq C\left[\frac{S_{11} + S_{13} + \delta}{1 + S_{21} + S_{23}}\right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2}, \quad (6d)$$

$$R_1 + R_2 \leq C\left[S_{12} + \frac{S_{11} + S_{13} + \delta}{1 + S_{21} + S_{23}}\right] + C\left[S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}}\right] + \frac{1}{2}, \quad (6e)$$

$$R_1 + R_2 \leq C[S_{11} + S_{12} + S_{13}] + C\left[\frac{S_{22}}{1 + S_{12}}\right] + \frac{1}{2}, \quad (6f)$$

$$R_1 + R_2 \leq C\left[\frac{S_{11} + S_{31}}{1 + S_{21}}\right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2}, \quad (6g)$$

$$R_1 + R_2 \leq C\left[S_{12} + \frac{S_{11} + S_{31}(1 + S_{12})}{1 + S_{21}}\right] + C\left[S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}}\right] + \frac{1}{2}, \quad (6h)$$

$$R_1 + R_2 \leq C[S_{11} + S_{12} + S_{31}(1 + S_{12})] + C\left[\frac{S_{22}}{1 + S_{12}}\right], \quad (6i)$$

$$R_1 + R_2 \leq C\left[\frac{S_{11} + S_{31}}{1 + S_{21} + S_{31}}\right] + C[S_{21} + S_{22} + S_{31}(1 + S_{22})], \quad (6j)$$

$$R_1 + R_2 \leq C\left[S_{12} + \frac{S_{11} + S_{31}(1 + S_{12})}{1 + S_{21} + S_{31}}\right] + C\left[S_{21} + S_{31} + \frac{S_{22}(1 + S_{31})}{1 + S_{12}}\right], \quad (6k)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{13} + \delta}{1 + S_{21} + S_{23}} \right] + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + C[S_{11} + S_{12} + S_{13}] + 1, \quad (6l)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{13} + \delta}{1 + S_{21} + S_{23}} \right] + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + C[S_{11} + S_{12} + S_{31}(1 + S_{12})] + \frac{1}{2}, \quad (6m)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{31}}{1 + S_{21}} \right] + C[S_{11} + S_{12} + S_{13}] + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + 1, \quad (6n)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{31}}{1 + S_{21}} \right] + C[S_{11} + S_{12} + S_{31}(1 + S_{12})] + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + \frac{1}{2}, \quad (6o)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{31}}{1 + S_{21} + S_{31}} \right] + C[S_{11} + S_{12} + S_{13}] + C \left[ S_{21} + S_{31} + \frac{S_{22}(1 + S_{31})}{1 + S_{12}} \right] + \frac{1}{2}, \quad (6p)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{11} + S_{31}}{1 + S_{21} + S_{31}} \right] + C[S_{11} + S_{12} + S_{31}(1 + S_{12})] + C \left[ S_{21} + S_{31} + \frac{S_{22}(1 + S_{31})}{1 + S_{12}} \right], \quad (6q)$$

$$R_1 + 2R_2 \leq C \left[ S_{12} + \frac{S_{11} + S_{13} + \delta}{1 + S_{21} + S_{23}} \right] + C \left[ \frac{S_{22}}{1 + S_{12}} \right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2}, \quad (6r)$$

$$R_1 + 2R_2 \leq C \left[ S_{12} + \frac{S_{11} + S_{31}(1 + S_{12})}{1 + S_{21}} \right] + C \left[ \frac{S_{22}}{1 + S_{12}} \right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2}, \quad (6s)$$

$$R_1 + 2R_2 \leq C \left[ S_{12} + \frac{S_{11} + S_{31}(1 + S_{12})}{1 + S_{21} + S_{31}} \right] + C \left[ \frac{S_{22}}{1 + S_{12}} \right] + C[S_{21} + S_{22} + S_{31}(1 + S_{22})] \quad (6t)$$

where  $\delta \triangleq (\sqrt{S_{11}S_{23}} \pm \sqrt{S_{13}S_{21}})^2$ .

*Proof:* See Appendix C.  $\blacksquare$

*Remark 2:* If we define the following matrices,

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{13} \\ h_{21} & h_{23} \end{bmatrix} \text{ and } \mathbf{Q} = \frac{1}{\sqrt{N_1 N_2}} \begin{bmatrix} P_1 & 0 \\ 0 & P_3 \end{bmatrix}, \quad (7)$$

we readily see that  $\delta = \det(\mathbf{H}\mathbf{Q}\mathbf{H}^T)$ . Thus, the sign in the expression  $\delta$  depends on the sign of the channel coefficients. If there is an even number of negative coefficients in  $\mathbf{H}$ , then  $\delta = (\sqrt{S_{11}S_{23}} - \sqrt{S_{13}S_{21}})^2$ , otherwise  $\delta = (\sqrt{S_{11}S_{23}} + \sqrt{S_{13}S_{21}})^2$ .

*Remark 3:* In the strong interference regime, where each receiver can decode the interfering message completely without restricting its rate, tighter outer bounds can be derived, similarly to the IC under strong interference [4, Remark 6.9]. The sum-rates in the capacity regions under strong interference [11, Thm. 5] and [14, Thm. 2], the former with the assumption of a potent relay, i.e.,  $P_3 \rightarrow \infty$ , are tighter than the ones presented here, namely (6i), (6j), (5d), (5f), and (5g).

*Remark 4:* Outer bound sum-rates using genie-aided techniques are given in [11, Thm. 4] and [14, Thm. 4], the former extending the ‘‘useful’’ and ‘‘smart’’ genie from [30] while the latter using Kramer’s approach [31].

As it is shown in [30], the ‘‘smart’’ genie provides an outer bound that is tighter than Etkin *et al.*’s [2] under weak interference, thus, the sum-rate [11, Thm. 4] is tighter than the analogous in our region, namely, (6k). Additionally, the optimization of parameters in the sum-rate [14, Thm. 4] can potentially give tight bounds. For example, if  $d_1 = h_{21}$ ,  $d_2 = d_3 = 0$ ,  $d_4 = \sqrt{N_2}$ , and  $d_5 = h_{23}$  the genie signal  $Y_{1g}$  becomes  $V_1 = h_{21}X_1 + h_{23}X_3 + Z'_2$  and it is easy to verify that the sum-rate [14, Thm. 4] is tighter than (5e).

#### IV. INNER BOUNDS

In the following, we provide two inner bounds corresponding to two different relaying strategies, namely, DF and CF. With DF, the relay decodes the message from the only connected source (partially or completely), re-encodes it, and transmits it to both destinations. With CF, the relay compresses the received signal, and sends a compression index associated to it. A previous version of these schemes was presented in [29], but here we show a more compact expression for the CF scheme and a completely new and improved version for the DF scheme. Four main ingredients are required: rate-splitting, binning, and block-Markov coding at the sources, and backward decoding at the destinations. In the sequel, we assume the indices  $(k, j) \in \{(1, 2), (2, 1)\}$ .

In every strategy, to allow cooperation from the relay, the transmission is split in several blocks. During block  $b$ , each source  $k$  divides its message  $\tilde{m}_{kb}$  into two short messages: a common part  $m_{kb}$  and a private part  $w_{kb}$ . As in the Han-Kobayashi scheme, each receiver decodes the common part of the interfering message, hence reducing the interference.

The use of DF and CF schemes for IRCs is well-known [10]–[14], however, our goal is to derive *simple* but powerful enough strategies in order to characterize the capacity region of the IRC within a *constant gap*. The biggest obstacle to obtaining an inner bound with a manageable number of inequalities is the use of a relaying strategy jointly with rate-splitting to deal with interference. This issue may be overcome by assuming some special condition in the model, e.g., symmetric channels [10], [12] or strong interference [14], or by employing successive decoding of codewords instead of joint-decoding [11], [12]. However, we do not want to rely on these assumptions here.

Additionally, the proposed schemes have some key differences with respect to the literature. In the DF scheme, the amount of information decoded by the relay is optimized separately from the rate-splitting used to deal with interference, which can potentially improve the achievable rates. Moreover, the CF scheme presented in Section IV-B does not force both receivers to decode the compression index, unlike [11], [13], which could reduce the performance of the scheme if there is a large asymmetry among the channels.

*Remark 5:* It is worth noting that the inner bounds stated below apply to general memoryless IRCs and thus they are not limited to the IS-IRC.

### A. Decode-and-Forward

Each source sends  $B$  messages during  $B + 1$  time blocks, and the relay forwards in block  $b$  what it has decoded from the first source in the previous block. In this scheme, the *private* message of the first source is split into two parts and the relay only decodes and retransmits one of them (plus the *common* message). At the end of transmission, receiver  $k$  decodes backwardly the private message  $w_{kb}$  as well as both common messages  $m_{kb}$  and  $m_{jb}$ .

Let  $\mathcal{P}_2$  be the set of PDs that factor as

$$p(q)p(x_1x_3|q)p(x_2|q)p(v_1|x_1x_3q)p(u_1|x_1q)p(v_2|x_2q)p(v_3|x_3q). \quad (8)$$

*Theorem 2 (partial DF scheme):* Given a  $P_2 \in \mathcal{P}_2$ , let  $\mathcal{R}_{\text{p-DF}}(P_2)$  be the region of nonnegative rate pairs  $(R_1, R_2)$  satisfying

$$R_1 \leq I(U_1; Y_3|X_3Q) + I(X_1; Y_1|V_1U_1V_2X_3Q), \quad (9a)$$

$$R_1 \leq I(X_1X_3; Y_1|V_2Q), \quad (9b)$$

$$R_2 \leq I(X_2; Y_2|V_1V_3Q), \quad (9c)$$

$$R_2 \leq I(V_1X_2V_3; Y_2|Q) - I_b, \quad (9d)$$

$$R_1 + R_2 \leq I(X_1X_3; Y_1|V_1V_2V_3Q) + I(V_1X_2V_3; Y_2|Q), \quad (9e)$$

$$R_1 + R_2 \leq I(U_1; Y_3|V_1X_3Q) + I(X_1; Y_1|V_1U_1V_2X_3Q) + I(V_1X_2V_3; Y_2|Q) - I_b, \quad (9f)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|V_1V_3Q) + I(V_1X_2V_3; Y_2|V_2Q), \quad (9g)$$

$$R_1 + R_2 \leq I(U_1; Y_3|V_1X_3Q) + I(X_1V_2; Y_1|V_1U_1X_3Q) + I(V_1X_2V_3; Y_2|V_2Q) - I_b, \quad (9h)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|Q) + I(V_1X_2V_3; Y_2|V_2Q) - I_b, \quad (9i)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|Q) + I(X_2; Y_2|V_1V_2V_3Q), \quad (9j)$$

$$R_1 + R_2 \leq I(U_1; Y_3|X_3Q) + I(X_1V_2; Y_1|V_1U_1X_3Q) + I(X_2; Y_2|V_1V_2V_3Q), \quad (9k)$$

$$2R_1 + R_2 \leq I(X_1X_3; Y_1|V_1V_2V_3Q) + I(X_1V_2X_3; Y_1|Q) + I(V_1X_2V_3; Y_2|V_2Q), \quad (9l)$$

$$2R_1 + R_2 \leq I(X_1X_3; Y_1|V_1V_2V_3Q) + I(X_1V_2; Y_1|V_1U_1X_3Q) + I(U_1; Y_3|X_3Q) + I(V_1X_2V_3; Y_2|V_2Q), \quad (9m)$$

$$2R_1 + R_2 \leq I(U_1; Y_3|V_1X_3Q) + I(X_1; Y_1|V_1U_1V_2X_3Q) - I_b + I(X_1V_2X_3; Y_1|Q) + I(V_1X_2V_3; Y_2|V_2Q), \quad (9n)$$

$$R_1 + 2R_2 \leq I(X_1V_2X_3; Y_1|V_1V_3Q) + I(X_2; Y_2|V_1V_2V_3Q) + I(V_1X_2V_3; Y_2|Q), \quad (9o)$$

$$R_1 + 2R_2 \leq I(U_1; Y_3|V_1X_3Q) + I(X_1V_2; Y_1|V_1U_1X_3Q) - I_b + I(X_2; Y_2|V_1V_2V_3Q) + I(V_1X_2V_3; Y_2|Q) \quad (9p)$$

where  $I_b \triangleq I(X_3; V_1|V_3Q)$ . Then, an achievable region for the IRC is defined by the union of all rate pairs in  $\mathcal{R}_{\text{p-DF}}(P_2)$  over all joint PDs  $P_2 \in \mathcal{P}_2$ , as defined in (8).

*Proof:* The codewords  $V_2^n$  and  $X_2^n$  convey the common and full messages of the second source, respectively, with  $X_2^n$  superimposed over  $V_2^n$ . This representation follows the steps proposed in [32], due to its simplicity compared to [1], though both representations are equivalent [33].

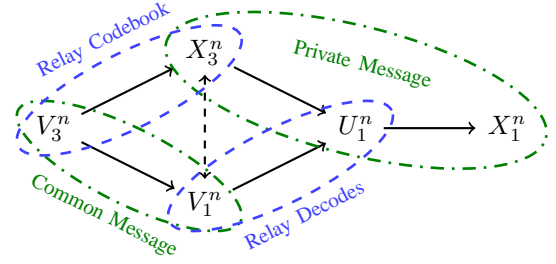


Fig. 4. Codewords of the relay and the first source. Solid arrows denote superimposed codewords while dashed arrows denote binning.

The codebook of the first source, however, is much more involved in order to allow the relay to cooperate, see Fig. 4. The scheme forces the relay to decode the common message of the first source, i.e., the codeword  $V_1^n$ , entirely but only a part of the private message. Thus, unlike the second source, an intermediate layer  $U_1^n$  is included between  $V_1^n$  and  $X_1^n$ .

The indices decoded by the relay are forwarded through superimposed codewords  $V_3^n$  and  $X_3^n$ , analogous to  $V_1^n$  and  $U_1^n$ . Coherent cooperation is achieved by superimposing  $V_1^n$  and  $U_1^n$  over  $V_3^n$  and  $X_3^n$ , respectively. An additional binning step between the codewords  $V_1^n$  and  $X_3^n$  is required to comply with (8), thus the negative term  $I_b$  in (9).

The region  $\mathcal{R}_{\text{p-DF}}$  (9) is strictly smaller than the actual partial DF region since we have purposely reduced all the bounds with  $I(V_1U_1; Y_3|X_3)$  into  $I(U_1; Y_3|X_3)$ , namely, in (9a), (9k), and (9m), in order to have a more compact expression of the whole region. See Appendix D for details. ■

If the relay is able to decode the private message of the first source completely without imposing a restriction on the achievable rate, the maximization of the previous inner bound would result in  $U_1 = X_1$ . In this case, let  $\mathcal{P}_3$  be the set of PDs which factor as

$$p(q)p(x_1x_3|q)p(x_2|q)p(v_1|x_1x_3q)p(v_2|x_2q)p(v_3|x_3q). \quad (10)$$

*Corollary 2 (full DF scheme):* Given a  $P_3 \in \mathcal{P}_3$ , let  $\mathcal{R}_{\text{f-DF}}(P_3)$  be the region of nonnegative rate pairs  $(R_1, R_2)$  satisfying

$$R_1 \leq I(X_1; Y_3|X_3Q), \quad (11a)$$

$$R_1 \leq I(X_1X_3; Y_1|V_2Q), \quad (11b)$$

$$R_2 \leq I(X_2; Y_2|V_1V_3Q), \quad (11c)$$

$$R_2 \leq I(V_1X_2V_3; Y_2|Q) - I_b, \quad (11d)$$

$$R_1 + R_2 \leq I(X_1X_3; Y_1|V_1V_2V_3Q) + I(V_1X_2V_3; Y_2|Q), \quad (11e)$$

$$R_1 + R_2 \leq I(X_1; Y_3|V_1X_3Q) + I(V_1X_2V_3; Y_2|Q) - I_b, \quad (11f)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|V_1V_3Q) + I(V_1X_2V_3; Y_2|V_2Q), \quad (11g)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|Q) + I(V_1X_2V_3; Y_2|V_2Q) - I_b, \quad (11h)$$

$$R_1 + R_2 \leq I(X_1V_2X_3; Y_1|Q) + I(X_2; Y_2|V_1V_2V_3Q), \quad (11i)$$

$$2R_1 + R_2 \leq I(X_1X_3; Y_1|V_1V_2V_3Q) + I(X_1V_2X_3; Y_1|Q) \\ + I(V_1X_2V_3; Y_2|V_2Q), \quad (11j)$$

$$2R_1 + R_2 \leq I(X_1; Y_3|V_1X_3Q) + I(X_1V_2X_3; Y_1|Q) \\ + I(V_1X_2V_3; Y_2|V_2Q) - I_b, \quad (11k)$$

$$R_1 + 2R_2 \leq I(X_1V_2X_3; Y_1|V_1V_3Q) + I(X_2; Y_2|V_1V_2V_3Q) \\ + I(V_1X_2V_3; Y_2|Q) \quad (11l)$$

where  $I_b \triangleq I(X_3; V_1|V_3Q)$ . Then, an achievable region for the IRC is defined by the union of all rate pairs in  $\mathcal{R}_{f\text{-DF}}(P_3)$  over all joint PDs  $P_3 \in \mathcal{P}_3$ , as defined in (10).

*Proof:* The region  $\mathcal{R}_{f\text{-DF}}$  (11) is not obtained by setting  $U_1 = X_1$  in  $\mathcal{R}_{p\text{-DF}}$  (9), since some additional redundant bounds remain. To easily eliminate these bounds, one should replace  $U_1$  with  $X_1$  in the set of partial rates before applying Fourier-Motzkin elimination in the proof of Theorem 2. See Appendix E for details. ■

The keen reader can see the resemblance between the region  $\mathcal{R}_{f\text{-DF}}$  (11) and the Han-Kobayashi region [33], with the addition of bounds regarding the decoding at the relay or the presence of binning.

*Remark 6:* The capacity of the *physically degraded* IRC in the *strong interference* regime [14, Thm. 3] is achieved by the full DF scheme.

The choice of variables  $V_k = X_k$  for  $k \in [1 : 3]$  eliminates the private messages and renders the binning process unnecessary. Then, by using the strong interference condition  $I(X_1X_3; Y_1|X_2) \leq I(X_1X_3; Y_2|X_2)$ , the full DF inner bound becomes

$$R_1 \leq I(X_1; Y_3|X_3Q), \quad (12a)$$

$$R_1 \leq I(X_1X_3; Y_1|X_2Q), \quad (12b)$$

$$R_2 \leq I(X_2; Y_2|X_1X_3Q), \quad (12c)$$

$$R_1 + R_2 \leq I(X_1X_2X_3; Y_1|Q), \quad (12d)$$

$$R_1 + R_2 \leq I(X_1X_2X_3; Y_2|Q). \quad (12e)$$

The region (12) coincides with the outer bound [14, Thm. 2] by choosing  $U_1 = X_3$  and  $U_2 = X_2$ , and considering that

- 1) the relay is only able to observe the first source, i.e.,  $p(y_3|x_1x_2x_3) = p(y_3|x_1x_3)$ , and
- 2) the IRC is physically degraded, i.e., the Markov chain  $(X_1X_2) \text{---} (X_3Y_3) \text{---} (Y_1Y_2)$  holds.

In the full DF scheme, since the relay decodes the codeword  $X_1^n$  completely, there is no limit in the amount of information that can be sent as common message. However, in the partial DF scheme, we are introducing the variable  $U_1$  between  $X_1$  and  $V_1$ , effectively prohibiting  $V_1 = X_1$ . Therefore, the structure of the codebook imposes that the relay should be in a better condition to decode the common message  $V_1^n$  than the second destination. If that is not the case, we should employ the CF scheme presented in the following section.

### B. Compress-and-Forward

In this scheme, the relay does not decode any message and it only sends a compressed version of its observation. The destinations jointly decode this information with their message

and the common layer of the interference. Transmission takes place in  $B + L$  time blocks, similarly to [34], [35], and during the last  $L$  blocks, the relay repeats its message to assure a correct decoding at both destinations.

Let  $\mathcal{P}_4$  be the set of PDs that factor as

$$p(q)p(v_1x_1|q)p(v_2x_2|q)p(x_3|q)p(\hat{y}_3|x_3y_3q), \quad (13)$$

and let us define the following set of expressions

$$I_{k1} \triangleq \min\{I(X_k; Y_k\hat{Y}_3|V_kV_jX_3Q), \\ I(X_kX_3; Y_k|V_kV_jQ) - I_k\}, \quad (14a)$$

$$I_{k2} \triangleq \min\{I(X_k; Y_k\hat{Y}_3|V_jX_3Q), \\ I(X_kX_3; Y_k|V_jQ) - I_k\}, \quad (14b)$$

$$I_{k3} \triangleq \min\{I(X_kV_j; Y_k\hat{Y}_3|V_kX_3Q), \\ I(X_kV_jX_3; Y_k|V_kQ) - I_k\}, \quad (14c)$$

$$I_{k4} \triangleq \min\{I(X_kV_j; Y_k\hat{Y}_3|X_3Q), \\ I(X_kV_jX_3; Y_k|Q) - I_k\} \quad (14d)$$

where  $I_k \triangleq I(\hat{Y}_3; Y_3|X_kV_jX_3Y_kQ)$  and

$$I'_{k1} \triangleq I(X_k; Y_k|V_kV_jQ), \quad (15a)$$

$$I'_{k2} \triangleq I(X_k; Y_k|V_jQ), \quad (15b)$$

$$I'_{k3} \triangleq I(X_kV_j; Y_k|V_kQ), \quad (15c)$$

$$I'_{k4} \triangleq I(X_kV_j; Y_k|Q). \quad (15d)$$

*Theorem 3 (CF scheme):* Given a specific  $P_4 \in \mathcal{P}_4$ , let  $\mathcal{R}_{\text{CF}_0}(P_4)$  be the region of nonnegative rate pairs  $(R_1, R_2)$  that satisfy

$$R_k \leq I_{k2}, \quad (16a)$$

$$R_k + R_j \leq \min\{I_{k1} + I_{j4}, I_{k3} + I_{j3}\}, \quad (16b)$$

$$2R_k + R_j \leq I_{k1} + I_{k4} + I_{j3}, \quad (16c)$$

and  $\mathcal{R}_{\text{CF}_k}(P_4)$  defined by

$$R_k \leq I_{k2}, \quad (17a)$$

$$R_j \leq I'_{j2}, \quad (17b)$$

$$R_k + R_j \leq \min\{I_{k1} + I'_{j4}, I_{k4} + I'_{j1}, I_{k3} + I'_{j3}\}, \quad (17c)$$

$$2R_k + R_j \leq I_{k1} + I_{k4} + I'_{j3}, \quad (17d)$$

$$R_k + 2R_j \leq I_{k3} + I'_{j1} + I'_{j4}. \quad (17e)$$

Then, an achievable region for the IRC is defined by the union of  $\mathcal{R}_{\text{CF}_0}(P_4) \cup \mathcal{R}_{\text{CF}_1}(P_4) \cup \mathcal{R}_{\text{CF}_2}(P_4)$  over all joint distributions  $P_4 \in \mathcal{P}_4$ , as defined in (13).

*Proof:* Since the relay does not decode any message, the codewords  $V_k^n$  and  $X_k^n$  carry the common and full message of the present block, respectively. The variable  $X_3$  is independent of the sources' signals and is used to reconstruct the relay's observation  $Y_3$ .

Each expression  $I_{ki}$  resembles the CF inner bound for the relay channel, and when the relay is ignored it reduces to the expression  $I'_{ki}$ . The region  $\mathcal{R}_{\text{CF}_0}$  (16) is obtained when both destinations decode the compression index, whereas in region  $\mathcal{R}_{\text{CF}_k}$  (17) only destination  $k$  decodes it.

Since the compression index is sent with block-Markov coding, each destination needs to assure the correct decoding of it in each block, which results in additional bounds not



	$S_{31} < S_{21}$	$S_{31} \geq S_{21}$
$S_{31} < S_{11}$	CF	partial DF
$S_{31} \geq S_{11}$	full DF	

TABLE I

SNR REGIMES AND CORRESPONDING BEST CONSTANT-GAP STRATEGIES.

shown here. However, the union  $\mathcal{R}_{CF_0} \cup \mathcal{R}_{CF_1} \cup \mathcal{R}_{CF_2}$  after the maximization over all joint PDs provides that these bounds are redundant. See Appendix F for details. ■

*Remark 7:* The relay only generates one compression index that is decodable by both destinations, i.e., the compression rate is determined by the worst channel. It is possible, however, to improve the performance with successive refinement that is not used here because of its complexity. As we shall see in the next section, two layers of successive refinement are not needed as far as the constant gap is concerned.

*Remark 8:* If both users ignore the compression index, this strategy reduces to the Han-Kobayashi scheme, a special case of  $\mathcal{R}_{CF_0}$ . Additionally,  $\mathcal{R}_{CF_0}$  is equal to the extension of NNC [13, Thm. 1] for one relay, i.e.,  $N = 1$ .

*Remark 9:* The region  $\mathcal{R}_{CF_0}$  contains both the CF and GCF schemes presented in [11, Thm. 1 and 2]. It is easy to see that the bounds on the partial rates of the first scheme [11, (5)–(8)] are below (14) if we relax the constraint [11, (9)] to  $I(X_3; Y_j) \geq I(Y_3; \hat{Y}_3 | X_3 Y_j)$  with  $j \in \{1, 2\}$ . Additionally, relaxing  $R_0$  in [11, Thm. 2], shows that GCF<sub>1</sub> is equal to  $\mathcal{R}_{CF_0}$  with  $V_1 = V_2 = \emptyset$  and GCF<sub>2</sub> is equal to  $\mathcal{R}_{CF_0}$  with  $V_1 = X_1$  and  $V_2 = X_2$ . Therefore, the capacity results [11, Thm. 4 and 5] are achieved by the proposed CF scheme.

## V. CONSTANT GAP RESULTS AND DISCUSSION

In this section, we evaluate the gap between the achievable regions and the outer bound in the Gaussian case (Fig. 1). Then, we identify the strategies that achieve the best constant gap to the capacity region for any SNR value. This is summarized in Table I, while the value of the gap for each strategy is shown in Table II.

### A. DF Scheme Achieves Capacity to Within 1.5 Bits

Table II shows two different constant-gap values for this scheme, 1.5 bits being the larger. The difference comes from the choice of input PD used in the inner bound as we see next.

When the relay is close to the source, i.e., when  $S_{31}$  is high enough, the relay is able to decode the entire message without penalizing the rate  $R_1$ . Therefore, as mentioned in Section IV-A, the input PD verifies  $U_1 = X_1$  and the inner bound is found in Corollary 2.

*Proposition 1:* If  $S_{31} \geq S_{11}$ , the full DF scheme presented in Corollary 2 achieves capacity to within 1 bit.

*Proof:* The mentioned constant gap is quite conservative in the majority of cases since it arises from choosing a fixed input PD for the inner bound (which reduces the achievable rate) and using the loose outer bound from Corollary 1. See Appendix G for details. ■

SNR regime		CF	DF
$S_{31} < S_{21}$	$S_{31} < S_{11}$	1.32	–
	$S_{31} \geq S_{11}$	1.32	1
$S_{31} \geq S_{21}$	$S_{31} \geq S_{11}$	–	1
	$S_{31} < S_{11}$	–	1.5

TABLE II

MAXIMUM GAP IN BITS OF EACH SCHEME FOR EACH SNR REGIME.

*Remark 10:* The capacity result in [14, Thm. 3] is contained in this regime. This capacity result, which is valid for general memoryless channels, relies on three conditions, namely,

- 1) the relay can only observe one source signal;
- 2) the IRC is *physically degraded*, i.e.,  $(X_1 X_2) \ominus (X_3 Y_3) \ominus (Y_1 Y_2)$ ; and,
- 3) the IRC is under the *strong interference* regime, i.e.,  $I(X_k X_3; Y_k | X_j) \leq I(X_k X_3; Y_j | X_j)$ .

The IRC model (1) used in this work only verifies the first condition. However, if we further assume that the conditions of physically degradedness and strong interference hold, the full DF scheme presented in Corollary 2 also achieves capacity (see Remark 6). As we see next, the lack of these two assumptions imposes the 1-bit gap.

First, our Gaussian model (2) does not admit any kind of degradedness, however, if  $S_{31} \geq S_{11}$ , we can bound the corresponding term by 0.5 bits, as in (70),

$$I(X_1; Y_1 | X_2 X_3 Y_3 Q) = C \left[ \frac{S_{11}}{1 + S_{31}} \right] \leq \frac{1}{2}.$$

Second, the strong interference condition renders the rate-splitting useless, since both encoders send only common messages, and allows the development of a tighter outer bound, similar to the IC with strong interference [4, Remark 6.9]. Without common messages, not only the binning term  $I_b$  disappears but also the simplifications made in Appendix G, namely the choice of auxiliaries (67) and the uncorrelation between  $X_1$  and  $X_3$ , can be dropped. For example, as seen in Appendix G, the choice of auxiliaries (67) inflicts half a bit of gap in (71) and (72), while another half a bit of gap is due to the uncorrelation between  $X_1$  and  $X_3$  in (71) and due to the binning term  $I_b$  in (72).

Therefore, the 1-bit gap the full DF scheme presents in contrast to the capacity-achieving scheme of [14] comes from the last two conditions, which are not assumed by our model.

If the source-to-relay link is not good enough for the relay to decode the entire message, the relay should decode it partially, i.e.,  $U_1 \neq X_1$ . However, due to the structure of the codebook, the relay should still be able to decode the common message.

*Proposition 2:* If  $S_{31} \geq S_{21}$ , the partial DF scheme presented in Theorem 2 achieves capacity to within 1.5 bits.

*Proof:* Similarly to the proof of Proposition 1, we reduce the inner bound by fixing the input PD and enlarge the outer bound by choosing a subset of bounds from it. See Appendix H for details. ■

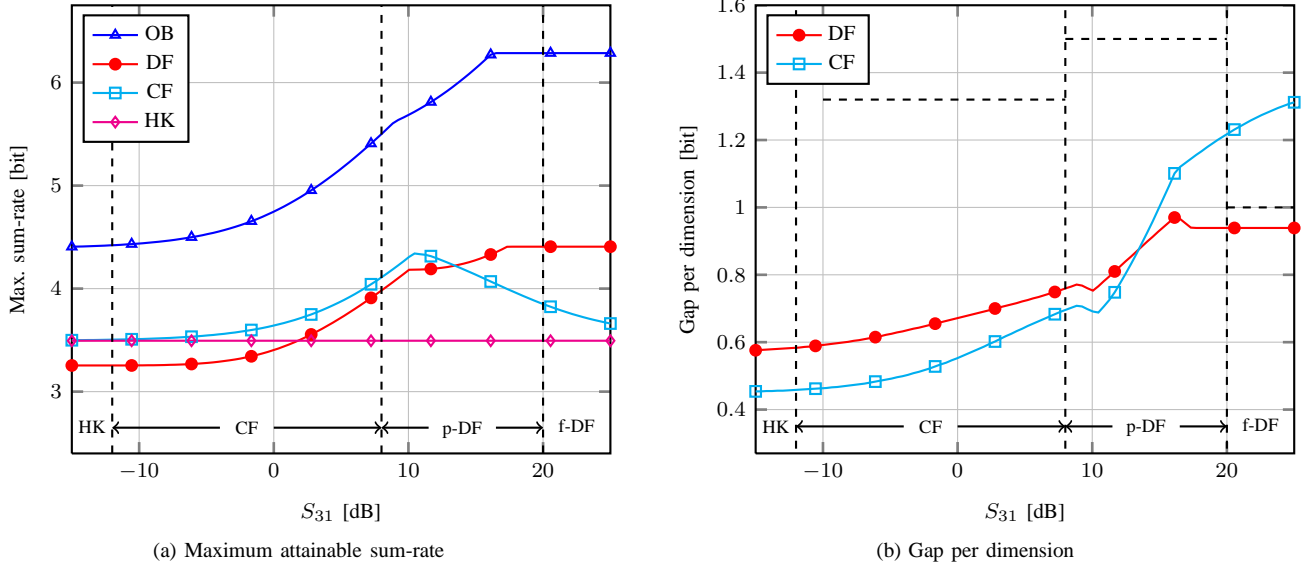


Fig. 5. Performance analysis for the Gaussian IRC (Fig. 1) with the following fixed SNRs:  $S_{11} = S_{22} = 20\text{dB}$ ,  $S_{12} = S_{21} = 8\text{dB}$ ,  $S_{13} = S_{23} = 20\text{dB}$ .

*Remark 11:* The gap between the original expression in the inner bound,  $I(V_1U_1; Y_3|X_3Q)$ , and the one used to compact the region,  $I(U_1; Y_3|X_3Q)$ , is 0.5 bit at most with the choice of auxiliaries (67) and (73) used in Appendix H. This is the cause of the larger gap for the partial DF scheme.

*Remark 12:* If  $S_{31} \geq S_{11}$  and  $S_{31} \geq S_{21}$  the DF scheme, full or partial, achieves a constant gap to capacity. Nonetheless, this regime appears in Table I as “full DF” since its gap is smaller.

### B. CF Scheme Achieves Capacity to Within 1.32 Bits

The CF scheme does not impose any condition on the sources’ codebook structure, nonetheless, a constant gap could only be found in the regime  $S_{31} \leq S_{21}$ .

*Proposition 3:* If  $S_{31} \leq S_{21}$  the CF scheme presented in Theorem 3 achieves capacity to within 1.32 bits.

*Proof:* The proof follows similar steps as the previous ones. See Appendix I for details. ■

### C. Limited Relaying Benefit

It sounds reasonable that for a really low SNR in the source-to-relay link, the use of relaying has limited benefit. In this case, it might be preferable, due to complexity, to shut the relay down and fall back to the much simpler Han-Kobayashi scheme for the IC.

*Proposition 4:* If  $S_{31} \leq S_{11}/(1+S_{12})$  and  $S_{31} \leq S_{21}/(1+S_{22})$ , the Han-Kobayashi scheme (without relay) achieves the capacity of the IS-IRC within 1 bit, i.e., relaying does not improve the achievable rate in more than 1 bit.

*Proof:* See Appendix J. ■

The two conditions over the source-to-relay link presented above can be interpreted as follows. In the first case,  $S_{31} \leq S_{11}/(1+S_{12})$  implies that, by treating the interference from source 2 as noise, destination 1 can still have a better observation on source 1’s signal than the relay does. Therefore,

the relay’s observation cannot help much for destination 1 to decode its own signal.

On the other hand,  $S_{31} \leq S_{21}/(1+S_{22})$  implies that, by treating its own signal as noise, destination 2 can still have a better observation on source 1’s signal than the relay does. Therefore, the relay’s observation cannot help much for destination 2 to learn/decode the interference from source 1.

### D. Numerical Example

To illustrate the regimes described before, we plot the maximum attainable sum-rate for the outer bound and each inner bound in Fig. 5a. Additionally, we delimit each regime with vertical dashed lines and we add the Han-Kobayashi scheme as a means of comparison. The SNR of each link in the channel remains fixed while we vary the SNR of the source-to-relay link  $S_{31}$ .

All the inner bounds present in the figure are the simplified versions used in the computation of the gap, i.e., there is no maximization of the PDs employed in them. The curve labeled DF is the maximum achievable rate attained by either the simplified inner bound of Proposition 1 or 2; the reader should refer to the appropriate appendix for details. The HK inner bound is not optimized either since we use the auxiliaries proposed in [3], but this is needed to make a fair comparison with our schemes. Moreover, Corollary 1 is the outer bound used in here.

We see that when the source-to-relay link is strong DF outperforms CF, namely in the regime labeled “f-DF”, i.e., when  $S_{31} \geq S_{11}$ . As the quality of this link degrades, CF achieves higher rates and eventually surpasses DF, mainly in the “CF” regime, i.e., when  $S_{31} < S_{21}$ . Below certain threshold in the quality of the source-to-relay link, the DF scheme even achieves lower rates than the HK scheme. The cause of this might lie in the numerous simplifications made to the scheme. However, due to the many auxiliaries present in the scheme, we did not carry out an extensive optimization of

the scheme to prove this conjecture. Finally, when the source-to-relay link is really weak, CF performs as good as the Han-Kobayashi scheme.

Another way of analyzing these curves is by looking at the gap per dimension, as in Fig. 5b. Here, the maximum theoretical gap in each regime is represented by horizontal dashed lines, and we see that they hold.

## VI. SUMMARY AND CONCLUDING REMARKS

We derived a novel outer bound and two inner bounds for a class of IRCs where the relay can only observe one of the sources. These bounds allowed us to identify the main SNR regimes of interest, and for them, we found the adequate relaying strategies that achieve capacity of the Gaussian IRC to within a constant gap regardless of the channel parameters.

While the proposed inner and outer bounds suggest the existence of different SNR regimes for the Gaussian IRC, in which different coding strategies are needed to achieve a constant gap to capacity, whether there exists a single coding scheme that achieves the constant gap in all SNR regimes is still an open question. In other words, there may be ways to improve the outer bound, the inner bounds, or both, which remains an interesting future work.

Additionally, the general IRC where the relay observe both sources is not an straightforward extension of our work. The central difficulty lies in the way of modeling the interference signals used in the injective semideterministic model and hence the derivation of an adequate outer bound. Since in the general IRC  $X_3$  can be arbitrarily correlated to both  $X_1$  and  $X_2$ , the interference signal  $S_k$  is no longer independent of the input  $X_j$ , with  $(k, j) \in \{(1, 2), (2, 1)\}$ . This, in turn, forbids us of single-letterizing the outer bound the way we did. A new technique to derive outer bounds for this problem is therefore needed, which also remains as future work.

### APPENDIX A

#### STRONGLY TYPICAL SEQUENCES AND DELTA-CONVENTION

Following [36], we use in this paper *strongly typical sets* and the so-called *Delta-Convention*. Some useful facts are recalled here. Let  $X$  and  $Y$  be random variables on some finite sets  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively. We denote by  $p_{X,Y}$  (resp.  $p_{Y|X}$ , and  $p_X$ ) the joint probability distribution of  $(X, Y)$  (resp. conditional distribution of  $Y$  given  $X$ , and marginal distribution of  $X$ ).

*Definition 2 (Number of occurrences):* For any sequence  $x^n \in \mathcal{X}^n$  and any symbol  $a \in \mathcal{X}$ , notation  $N(a|x^n)$  stands for the number of occurrences of  $a$  in  $x^n$ .

*Definition 3 (Typical sequence):* A sequence  $x^n \in \mathcal{X}^n$  is called (*strongly*)  $\delta$ -*typical* w.r.t.  $X$  (or simply *typical* if the context is clear) if

$$\left| \frac{1}{n} N(a|x^n) - p_X(a) \right| \leq \delta \quad \text{for each } a \in \mathcal{X},$$

and  $N(a|x^n) = 0$  for each  $a \in \mathcal{X}$  such that  $p_X(a) = 0$ . The set of all such sequences is denoted by  $T_\delta^n(X)$ .

*Definition 4 (Conditionally typical sequence):* Let  $x^n \in \mathcal{X}^n$ . A sequence  $y^n \in \mathcal{Y}^n$  is called (*strongly*)  $\delta$ -*typical* (w.r.t.  $Y$ ) given  $x^n$  if

$$\left| \frac{1}{n} N(a, b|x^n, y^n) - \frac{1}{n} N(a|x^n) p_{Y|X}(b|a) \right| \leq \delta$$

for each  $a \in \mathcal{X}, b \in \mathcal{Y}$ ,

and,  $N(a, b|x^n, y^n) = 0$  for each  $a \in \mathcal{X}, b \in \mathcal{Y}$  such that  $p_{Y|X}(b|a) = 0$ . The set of all such sequences is denoted by  $T_\delta^n(Y|x^n)$ .

*Delta-Convention [36]:* For any sets  $\mathcal{X}, \mathcal{Y}$ , there exists a sequence  $\{\delta_n\}_{n \in \mathbb{N}^*}$  such that the lemmas stated below hold.<sup>1</sup> From now on, typical sequences are understood with  $\delta = \delta_n$ . Typical sets are still denoted by  $T_\delta^n(\cdot)$ .

*Lemma 1 ([36, Lemma 1.2.12]):* There exists a sequence  $\eta_n \xrightarrow{n \rightarrow \infty} 0$  such that

$$p_X(T_\delta^n(X)) \geq 1 - \eta_n.$$

*Lemma 2 ([36, Lemma 1.2.13]):* There exists a sequence  $\eta_n \xrightarrow{n \rightarrow \infty} 0$  such that, for each  $x^n \in T_\delta^n(X)$ ,

$$\left| \frac{1}{n} \log \|T_\delta^n(X)\| - H(X) \right| \leq \eta_n,$$

$$\left| \frac{1}{n} \log \|T_\delta^n(Y|x^n)\| - H(Y|X) \right| \leq \eta_n.$$

*Lemma 3 (Asymptotic equipartition property):* There exists a sequence  $\eta_n \xrightarrow{n \rightarrow \infty} 0$  such that, for each  $x^n \in T_\delta^n(X)$  and each  $y^n \in T_\delta^n(Y|x^n)$ ,

$$\left| -\frac{1}{n} \log p_X(x^n) - H(X) \right| \leq \eta_n,$$

$$\left| -\frac{1}{n} \log p_{Y|X}(y^n|x^n) - H(Y|X) \right| \leq \eta_n.$$

*Lemma 4 (Joint typicality lemma [4]):* There exists a sequence  $\eta_n \xrightarrow{n \rightarrow \infty} 0$  such that

$$\left| -\frac{1}{n} \log p_Y(T_\delta^n(Y|x^n)) - I(X; Y) \right| \leq \eta_n$$

for each  $x^n \in T_\delta^n(X)$ .

*Proof:*

$$p_Y(T_\delta^n(Y|x^n)) = \sum_{y^n \in T_\delta^n(Y|x^n)} p_Y(y^n)$$

$$\stackrel{(a)}{\leq} \|T_\delta^n(Y|x^n)\| 2^{-n[H(Y) - \alpha_n]}$$

$$\stackrel{(b)}{\leq} 2^{n[H(Y|X) + \beta_n]} 2^{-n[H(Y) - \alpha_n]}$$

$$= 2^{-n[I(X; Y) - \beta_n - \alpha_n]},$$

where

- step (a) follows from the fact that  $T_\delta^n(Y|x^n) \subset T_\delta^n(Y)$  and Lemma 3, for some sequence  $\alpha_n \xrightarrow{n \rightarrow \infty} 0$ ,
- step (b) from Lemma 2, for some sequence  $\beta_n \xrightarrow{n \rightarrow \infty} 0$ .

The reverse inequality  $p_Y(T_\delta^n(Y|x^n)) \geq 2^{-n[I(X; Y) + \beta_n + \alpha_n]}$  can be proved following similar argument.  $\blacksquare$

<sup>1</sup>As a matter of fact,  $\delta_n \rightarrow 0$  and  $\sqrt{n} \delta_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

## APPENDIX B

## PROOF OF THEOREM 1 (IS-IRC OUTER BOUND)

The proof follows by using a similar approach to that developed in [3] and it was partially presented in [29], [37]. As explained before, the inputs  $X_1$  and  $X_3$  are arbitrarily correlated and they are independent of  $X_2$ . Since we are not considering noise correlation in the outputs, the interference signals  $S_1$  and  $S_2$  are therefore independent.

First, let us recall that the inputs  $X_1^n$  and  $X_2^n$  are functions of the messages  $W_1$  and  $W_2$ , each one independent of the other, and the relay's input is a deterministic function of its past observations, i.e.,  $X_{3i} = \phi_i(Y_3^{i-1})$ ,  $i \in [1 : n]$ . Then, we add two new random variables  $V_1^n$  and  $V_2^n$ , which are obtained by passing  $X_1^n$ ,  $X_2^n$  and  $X_3^n$  through the memoryless channel  $p_{S_1|X_1X_3}p_{S_2|X_2}$ .

A multi-letter outer bound on each rate can be derived using Fano's inequality, i.e.,

$$n(R_k - \epsilon_n) \leq I(X_k^n; Y_k^n),$$

where  $\epsilon_n$  denotes a sequence such that  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, we present different derivations of  $I(X_k^n; Y_k^n)$  in the sequel. We first see that

$$\begin{aligned} I(X_1^n; Y_1^n) &\leq I(X_1^n X_3^n; Y_1^n) \\ &= h(Y_1^n) - h(Y_1^n | X_1^n X_3^n) \\ &= h(Y_1^n) - h(S_2^n | X_1^n X_3^n) \end{aligned} \quad (18a)$$

$$= h(Y_1^n) - \boxed{h(S_2^n)}, \quad (18b)$$

where (18a) follows from the IS model; and in (18b) we take into account that the interference signal  $S_2^n$  is independent of the inputs  $(X_1^n X_3^n)$ . We can provide the interference  $X_2^n$ ,

$$I(X_1^n; Y_1^n) \leq I(X_1^n X_3^n; Y_1^n X_2^n), \quad (19)$$

where (19) follows from the fact that  $X_2^n$  is independent of  $(X_1^n X_3^n)$ . Also, we can augment the bound with the auxiliary  $V_1^n$ ,

$$\begin{aligned} I(X_1^n; Y_1^n) &\leq I(X_1^n X_3^n; Y_1^n V_1^n) \\ &= I(X_1^n X_3^n; V_1^n) + I(X_1^n X_3^n; Y_1^n | V_1^n) \\ &= h(V_1^n) - h(V_1^n | X_1^n X_3^n) + h(Y_1^n | V_1^n) - h(Y_1^n | X_1^n X_3^n) \end{aligned} \quad (20a)$$

$$= \boxed{h(S_1^n)} - h(Y_2^n | X_1^n X_2^n X_3^n) + h(Y_1^n | V_1^n) - \boxed{h(S_2^n)}, \quad (20b)$$

where in the fourth term of (20a) we use the Markov chain  $V_1^n \ominus (X_1^n X_3^n) \ominus (\dots)$ ; and (20b) is due to the channel property and the fact that interchanging  $V_1$  and  $S_1$  does not change the entropies in question, i.e.,  $h(V_1^n) = h(S_1^n)$  and  $h(V_1^n | X_1^n X_3^n) = h(S_1^n | X_1^n X_3^n) = h(S_1^n | X_1^n X_2^n X_3^n) = h(Y_2^n | X_1^n X_2^n X_3^n)$ . We repeat the same procedure with the auxiliary  $V_2^n$ ,

$$\begin{aligned} I(X_1^n; Y_1^n) &\leq I(X_1^n X_3^n; Y_1^n V_1^n) \\ &= I(X_1^n X_3^n; V_1^n) + I(X_1^n X_3^n; Y_1^n | V_1^n) \\ &= h(V_1^n) - h(V_1^n | X_1^n X_3^n) + h(Y_1^n | V_1^n) - h(Y_1^n | X_1^n X_3^n) \end{aligned} \quad (21a)$$

$$= \boxed{h(S_1^n)} - h(Y_2^n Y_3^n | X_1^n X_2^n X_3^n) + h(Y_1^n | V_1^n) - \boxed{h(S_2^n)}, \quad (21b)$$

where in (21a) we use the Markov chain  $V_1^n \ominus (X_1^n X_3^n) \ominus (\dots)$ ; and in (21b) we again interchange  $V_1$  and  $S_1$ , i.e.,  $h(V_1^n) = h(S_1^n)$  and  $h(V_1^n | X_1^n X_3^n) = h(S_1^n | X_1^n X_3^n) = h(S_1^n | X_1^n X_2^n X_3^n) = h(Y_2^n Y_3^n | X_1^n X_2^n X_3^n)$ . We can now increase the bound with both  $X_2^n$  and  $V_1^n$ ,

$$\begin{aligned} I(X_1^n; Y_1^n) &\leq I(X_1^n X_3^n; Y_1^n V_1^n | X_2^n) \\ &= I(X_1^n X_3^n; V_1^n | X_2^n) + I(X_1^n X_3^n; Y_1^n | V_1^n X_2^n) \\ &= h(V_1^n | X_2^n) - h(V_1^n | X_1^n X_3^n) + I(X_1^n X_3^n; Y_1^n | V_1^n X_2^n) \end{aligned} \quad (22a)$$

$$= \boxed{h(S_1^n)} - h(Y_2^n | X_1^n X_2^n X_3^n) + I(X_1^n X_3^n; Y_1^n | V_1^n X_2^n), \quad (22b)$$

where the key steps in (22a) and (22b) are the same as in (20a) and (20b). Similarly, we can derive

$$\begin{aligned} I(X_2^n; Y_2^n) &\leq \boxed{h(S_1^n)} - h(Y_2^n Y_3^n | X_1^n X_2^n X_3^n) \\ &\quad + I(X_1^n X_3^n; Y_1^n | V_1^n X_2^n). \end{aligned} \quad (23)$$

In an analogous way as (18), (19), (20), and (22), we derive similar bounds for the rate  $R_2$ ,

$$I(X_2^n; Y_2^n) \leq h(Y_2^n) - \boxed{h(S_1^n)}, \quad (24)$$

$$I(X_2^n; Y_2^n) \leq I(X_2^n; Y_2^n | X_1^n X_3^n), \quad (25)$$

$$\begin{aligned} I(X_2^n; Y_2^n) &\leq \boxed{h(S_2^n)} - h(Y_1^n | X_1^n X_2^n X_3^n) \\ &\quad + h(Y_2^n | V_2^n) - \boxed{h(S_1^n)}, \end{aligned} \quad (26)$$

$$\begin{aligned} I(X_2^n; Y_2^n) &\leq \boxed{h(S_2^n)} - h(Y_1^n | X_1^n X_2^n X_3^n) \\ &\quad + I(X_2^n; Y_2^n | X_1^n V_2^n X_3^n). \end{aligned} \quad (27)$$

Additionally, if we add the sequence  $Y_3^n$  next to  $Y_2^n$  in the first steps of the derivation of (24) and (26), we obtain

$$I(X_2^n; Y_2^n) \leq h(Y_2^n Y_3^n) - \boxed{h(S_1^n)}, \quad (28)$$

$$\begin{aligned} I(X_2^n; Y_2^n) &\leq \boxed{h(S_2^n)} - h(Y_1^n | X_1^n X_2^n X_3^n) \\ &\quad + h(Y_2^n Y_3^n | V_2^n) - \boxed{h(S_1^n)}. \end{aligned} \quad (29)$$

The use of Fano's inequality and all the possible linear combinations of the expressions (18)–(29) where the boxed terms get canceled gives rise to multi-letter bounds that can be single-letterized, as summarized in Table III. For instance, (19) and (25) allow us to find bounds on the single rates, whereas the addition of (22) and (24) gives us the sum-rate (5d),

$$\begin{aligned} n(R_1 + R_2 - \epsilon'_n) &\leq I(X_1^n; Y_1^n) + I(X_2^n; Y_2^n) \\ &\leq I(X_1^n X_3^n; Y_1^n | V_1^n X_2^n) + I(X_1^n X_2^n X_3^n; Y_2^n) \end{aligned} \quad (30a)$$

$$\leq \sum_{i=1}^n I(X_{1i} X_{3i}; Y_{1i} | V_{1i} X_{2i}) + I(X_{1i} X_{2i} X_{3i}; Y_{2i}) \quad (30b)$$

$$= n[I(X_1 X_3; Y_1 | V_1 X_2 Q) + I(X_1 X_2 X_3; Y_2 | Q)], \quad (30c)$$

where (30a) follows from the addition of (22b) and (24); (30b) is due to the chain rule of the mutual information, the fact that removing conditioning increases the entropy, and the Markov

$R_1$	(5a) (5b)	(19)* (19)
$R_2$	(5c)	(25)
$R_1 + R_2$	(5d) (5e) (5f) (5g) (5h) (5i) (5j) (5k)	(22) +(24) (20) +(26) (18) +(27) (22)*+(24) (20)*+(26) (18)*+(27) (23)*+(28) (21)*+(29)
$2R_1 + R_2$	(5l) (5m) (5n) (5o) (5p) (5q)	(22) +(18) +(26) (22) +(18)*+(26) (22)*+(18) +(26) (22)*+(18)*+(26) (23)*+(18) +(29) (23)*+(18)*+(29)
$R_1 + 2R_2$	(5r) (5s) (5t)	(20) +(27) +(24) (20)*+(27) +(24) (21)*+(27) +(28)

TABLE III  
COMBINATION OF MULTI-LETTER OUTER BOUNDS. TERMS WITH \* NEED THE ADDITION OF  $Y_3^n$ .

chain  $(Y_{1i}Y_{2i}) \ominus (X_{1i}X_{2i}X_{3i}) \ominus (\dots)$ ; and (30c) follows from the addition of the time-sharing variable  $Q$  uniformly distributed in  $[1 : n]$ .

In this way, we obtain all the bounds in (5) except for the ones with the pair  $(Y_1Y_3)$ . For them, we need to add the sequence  $Y_3^n$  next to  $Y_1^n$  before applying the chain rule in the mutual information. These terms are denoted with \* in Table III. For example, continuing from (30a) we obtain the bound (5g),

$$\begin{aligned}
& n(R_1 + R_2 - \epsilon'_n) \\
& \leq I(X_1^n X_3^n; Y_1^n Y_3^n | V_1^n X_2^n) + I(X_1^n X_2^n X_3^n; Y_2^n) \\
& \leq \sum_{i=1}^n I(X_{1i}; Y_{1i} Y_{3i} | V_{1i} X_{2i} X_{3i}) + I(X_{1i} X_{2i} X_{3i}; Y_{2i}) \quad (31a) \\
& = n[I(X_1; Y_1 Y_3 | V_1 X_2 X_3 Q) + I(X_1 X_2 X_3; Y_2 | Q)] \quad (31b)
\end{aligned}$$

where (31a) follows from the fact that  $X_{3i}$  is a function of  $Y_3^{i-1}$ .

### APPENDIX C PROOF OF COROLLARY 1

The expression of the bounds (5a)–(5c) in the Gaussian case is

$$R_1 \leq C[(1 - \rho^2)(S_{11} + S_{31})], \quad (32)$$

$$R_1 \leq C[S_{11} + S_{13} + 2\rho\sqrt{S_{11}S_{13}}], \quad (33)$$

$$R_2 \leq C[S_{22}], \quad (34)$$

where we assume the channel coefficients  $h_{11}$  and  $h_{13}$  have the same sign, otherwise, the analysis is the same by inverting the sign in  $\rho$ . For any  $|\rho| \leq 1$ , we can upper bound the previous terms as follows

$$R_1 \leq C[S_{11} + S_{31}], \quad (35)$$

$$R_1 \leq C[S_{11} + S_{13}] + \frac{1}{2}, \quad (36)$$

$$R_2 \leq C[S_{22}], \quad (37)$$

which, in turn, gives us (6a)–(6c).

All the other bounds behave similarly. If both  $X_1$  and  $X_3$  appear in the conditioning part of a mutual information, it does not depend on  $\rho$ , like (34). If only  $X_3$  appears in the conditioning, it depends on  $(1 - \rho^2)$ , like (32). Otherwise, it depends on  $2\rho\sqrt{(\cdot)}$ , like (33). In the first two situations, the expressions are maximized with its value at  $\rho = 0$ , whereas, the last one has its maximum at  $\rho = 1$ .

The bounds containing  $V_1$  in the conditioning part, but not  $X_3$ , e.g. (5d), present a more complicated behavior and it is not clear which value of  $\rho$  maximizes the bound. We analyze the sum-rate (5d) in the sequel.

Let us first define

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{13} \\ h_{21} & h_{23} \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} 1 + \rho & 0 \\ 0 & 1 - \rho \end{bmatrix}}_{\mathbf{\Lambda}} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{\mathbf{U}^T},$$

where we have normalized the sources' power and noise power. We are interested in

$$\begin{aligned}
D_0 & \triangleq \det(\mathbf{I} + \mathbf{H}\mathbf{H}^T) \\
& = \det(\mathbf{I} + \mathbf{H}\mathbf{U}\mathbf{U}^T\mathbf{H}^T) = \det(\mathbf{I} + \mathbf{G}\mathbf{G}^T), \\
D & \triangleq \det(\mathbf{I} + \mathbf{H}\mathbf{Q}\mathbf{H}^T) \\
& = \det(\mathbf{I} + \mathbf{H}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^T\mathbf{H}^T) = \det(\mathbf{I} + \mathbf{G}\mathbf{\Lambda}\mathbf{G}^T)
\end{aligned}$$

where we define  $\mathbf{G} \triangleq \mathbf{H}\mathbf{U} = [g_{ij}]_{i,j=1,2}$ . For convenience, we also define the normalized matrix  $\mathbf{V}$  such that

$$\mathbf{G} = \begin{bmatrix} \sqrt{G_1} & 0 \\ 0 & \sqrt{G_2} \end{bmatrix} \mathbf{V}, \quad G_i \triangleq g_{i1}^2 + g_{i2}^2, \quad i = 1, 2$$

where  $v_{ij} \triangleq g_{ij}/\sqrt{G_i}$ . Note that  $v_{i1}^2 + v_{i2}^2 = 1$ ,  $i = 1, 2$ . We let  $V_{ij} \triangleq v_{ij}^2$  hereafter.

Then, we can rewrite

$$\begin{aligned}
D_0 & = 1 + G_1 + G_2 + G_1 G_2 \underbrace{\det(\mathbf{V}\mathbf{V}^T)}_{\gamma} \\
D & = 1 + G_1(1 + \underbrace{(V_{11} - V_{12})\rho}_{\alpha_1}) + G_2(1 + \underbrace{(V_{21} - V_{22})\rho}_{\alpha_2}) \\
& \quad + G_1 G_2 \gamma (1 - \rho^2)
\end{aligned}$$

where  $\gamma \in [0, 1]$  and  $\alpha_1, \alpha_2 \in [-1, 1]$ . In fact,  $\gamma$  can be presented as a function of  $\alpha_1$  and  $\alpha_2$

$$\gamma = (v_{11}v_{22} - v_{21}v_{12})^2 \quad (38a)$$

$$\geq (\sqrt{V_{11}V_{22}} - \sqrt{V_{21}V_{12}})^2 \quad (38b)$$

$$= \frac{1 - \alpha_1\alpha_2}{2} - \frac{1}{2}\sqrt{(1 - \alpha_1^2)(1 - \alpha_2^2)} \triangleq \gamma_*. \quad (38c)$$

Given the sum-rate (5d),

$$\begin{aligned}
R_1 + R_2 & \leq I(X_1 X_3; Y_1 | V_1 X_2) + I(X_1 X_2 X_3; Y_2) \\
& = I(X_1 X_3; Y_1 V_1 | X_2) - I(X_1 X_3; V_1 | X_2) + I(X_1 X_2 X_3; Y_2),
\end{aligned}$$

the ultimate goal is to quantify the maximum gap between the value of this bound with and without correlation in the inputs ( $X_1 X_3$ ). In other words, we shall obtain an upper bound on

$$\frac{D}{D_0} \frac{1 + G_2}{1 + G_2(1 + \alpha_2 \rho)} \frac{1 + G_2(1 + \alpha_2 \rho) + S_{22}}{1 + G_2 + S_{22}}. \quad (39)$$

If  $S_{22} \rightarrow 0$ , the expression (39) tends to  $D/D_0$ , and since the eigenvalues of  $\mathbf{A}$  are less or equal than 2, it can be easily upper-bounded,

$$\frac{D}{D_0} = \frac{\det(\mathbf{I} + \mathbf{G}\mathbf{A}\mathbf{G}^T)}{\det(\mathbf{I} + \mathbf{G}\mathbf{G}^T)} \leq \frac{\det(\mathbf{I} + 2\mathbf{G}\mathbf{G}^T)}{\det(\mathbf{I} + \mathbf{G}\mathbf{G}^T)} \leq 2.$$

On the other hand, if  $S_{22} \rightarrow \infty$ , (39) becomes

$$\begin{aligned} \frac{D}{D_0} \frac{1 + G_2}{1 + G_2(1 + \alpha_2 \rho)} &= \frac{1 + G_1 \frac{1 + \alpha_1 \rho + G_2 \gamma (1 - \rho^2)}{1 + G_2(1 + \alpha_2 \rho)}}{1 + G_1 \frac{1 + G_2 \gamma}{1 + G_2}} \\ &= \frac{1 + G_1 A}{1 + G_1 B}. \end{aligned}$$

We observe that this function is upper-bounded by 1 when  $A \leq B$ , while it is otherwise upper-bounded by  $A/B$ . Therefore, it suffices to find an upper bound on  $A/B$  that can be rewritten as

$$\begin{aligned} \frac{A}{B} &= \frac{(1 + \alpha_1 \rho) + G_2 \gamma (1 - \rho^2) + G_2(1 + \alpha_1 \rho) + G_2^2 \gamma (1 - \rho^2)}{(1 + G_2 \gamma)(1 + G_2(1 + \alpha_2 \rho))} \\ &= (1 + \alpha_1 \rho) \frac{1 + G_2}{1 + G_2(1 + \alpha_2 \rho)} \frac{1 + G_2 \frac{\gamma(1 - \rho^2)}{1 + \alpha_1 \rho}}{1 + G_2 \gamma}. \quad (40) \end{aligned}$$

Without loss of generality, we assume that  $\rho \geq 0$ . The case when  $\rho < 0$  follows straightforwardly by simply changing both signs of  $\alpha_1$  and  $\alpha_2$ . In the following, we shall show that

$$\frac{A}{B} \leq 2.$$

First, from (40), we derive a trivial upper bound

$$\frac{A}{B} \leq (1 + \alpha_1 \rho) \max \left\{ 1, \frac{1}{1 + \alpha_2 \rho} \right\} \max \left\{ 1, \frac{1 - \rho^2}{1 + \alpha_1 \rho} \right\} \quad (41a)$$

$$= \max \left\{ 1 - \rho^2, 1 + \alpha_1 \rho, \frac{1 - \rho^2}{1 + \alpha_2 \rho}, \frac{1 + \alpha_1 \rho}{1 + \alpha_2 \rho} \right\}, \quad (41b)$$

where both maximizations in (41a) come from the monotonicity of  $\frac{1 + G_2 x}{1 + G_2 y}$  w.r.t.  $G_2$  and that it is bounded by the extreme values for  $G_2 = 0$  and  $G_2 \rightarrow \infty$ . Note that only the last term in (41b) is not always upper-bounded by 2. In the following, we focus on the case  $\frac{1 - \rho^2}{1 + \alpha_1 \rho} < 1$ , i.e.,  $\alpha_1 > -\rho$ , since the opposite would imply that the last term in (41b) is upper-bounded by the third term. In this case ( $\alpha_1 > -\rho$ ), the third term in (40), and thus  $A/B$ , is decreasing with  $\gamma$ . Therefore, the worst case in which  $A/B$  is maximized is when  $\gamma$  achieves  $\gamma_*$ . It suffices to show that

$$\sup_{G_2 \geq 0} \frac{1 + \alpha_1 \rho + G_2 (1 + \alpha_1 \rho + \gamma_* (1 - \rho^2)) + G_2^2 \gamma_* (1 - \rho^2)}{(1 + G_2 \gamma_*)(1 + G_2(1 + \alpha_2 \rho))} \leq 2,$$

$\forall (\alpha_1, \alpha_2, \rho) \in \mathcal{A}$  where we define the set  $\mathcal{A}$

$$\mathcal{A} \triangleq \{\alpha_1, \alpha_2 \in (-1, 1), \rho \in (0, 1) : \alpha_1 > \alpha_2, \alpha_1 > -\rho\}.$$

We observe that for each point at the boundary of the set  $\mathcal{A}$ , the objective function is upper-bounded by 2. Note that, in the denominator,  $\gamma_* > 0$  since  $\alpha_1 \neq \alpha_2$ , and  $1 + \alpha_2 \rho > 0$  since  $\rho < 1$ . Therefore, the objective function is the ratio between two quadratic functions in the form  $(a_0 + a_1 G_2 + a_2 G_2^2)/((1 + b_1 G_2)(1 + b_2 G_2))$  with  $a_0, a_1, a_2 \geq 0$  and  $b_1, b_2 > 0$ , that are continuous functions of  $(\alpha_1, \alpha_2, \rho)$ . Let us first assume that  $b_1 \neq b_2$ . It is readily shown that

$$f(G_2) = \frac{a_0 + a_1 G_2 + a_2 G_2^2}{(1 + b_1 G_2)(1 + b_2 G_2)} \quad (42)$$

$$= c_0 + \frac{c_1}{1 + b_1 G_2} + \frac{c_2}{1 + b_2 G_2}, \quad \forall G_2 \quad (43)$$

where  $(c_0, c_1, c_2)$  is a continuous function of  $\{\alpha_i\}$  and  $\{b_i\}$ . Then, we differentiate the function  $f(G_2)$

$$f'(G_2) = -\frac{b_1 c_1}{(1 + b_1 G_2)^2} - \frac{b_2 c_2}{(1 + b_2 G_2)^2}.$$

It is clear that there is at most one solution in  $[0, \infty]$  such that  $f'(G_2) = 0$ . If such a solution does not exist, then  $f'(G_2)$  is either strictly positive or strictly negative in  $[0, \infty]$ . In this case, both extreme values  $f(0)$  and  $f(\infty)$  are upper-bounded by 2 from (40). If such a solution does exist, it is in the following form

$$G_2^* = \frac{\beta - 1}{b_1 - b_2 \beta}, \quad \beta \triangleq \sqrt{-\frac{b_1 c_1}{b_2 c_2}}, \quad \frac{c_1}{c_2} < 0. \quad (44)$$

Note that the function  $f$  defined in (42), alternatively denoted as  $f_{b_1, b_2}$ , converges pointwise to  $f_{b, b}$  when  $b_1, b_2 \rightarrow b, \forall b > 0$ , and that  $f'_{b_1, b_2}$  converges uniformly to  $f'_{b, b}$ . Therefore, the solution (44) holds even when  $b_1 = b_2$  by taking the limit. Finally, let us define a set  $\mathcal{B}$  of  $(\alpha_1, \alpha_2, \rho)$  such that  $c_1/c_2 < 0$  and  $G_2^* \geq 0$ . It remains to show that

$$\sup_{(\alpha_1, \alpha_2, \rho) \in \mathcal{A} \cap \mathcal{B}} f(G_2^*) \leq 2. \quad (45)$$

Since  $\mathcal{A} \cap \mathcal{B}$  is a bounded set and the objective function is continuous in  $(\alpha_1, \alpha_2, \rho)$  in  $\mathcal{A} \cap \mathcal{B}$ , we can perform numerical optimization and obtain the value 2, which confirms the claim in (45).

Similar steps can be performed in every other bound containing  $V_1$  in the conditioning, which concludes the proof.

## APPENDIX D

### PROOF OF THEOREM 2 (PARTIAL DF SCHEME)

Each source transmits  $B$  messages during  $B + 1$  time blocks, each of them of length  $n$ . The messages are sent using block-Markov coding and the destinations employ backward decoding to retrieve them.

The second source splits its message  $\tilde{m}_2$  into a common message  $m_2$  and a private one  $w_2$ , with partial rates  $R_{20}$  and  $R_{22}$ , respectively, such that  $R_2 = R_{20} + R_{22}$ . On the other hand, the first source splits its message  $\tilde{m}_1$  into three parts:  $(m_1, w'_1, w''_1)$ . The relay decodes and retransmits the common message and a part of the private one, i.e.,  $(m_1, w'_1)$ , whereas the other part is only decoded by the final destination. The rate of the first user is therefore the sum of these three partial rates:  $R_1 = R_{10} + R'_{11} + R''_{11}$ .

$b = 1$	$b = 2$	$\dots$	$b = B$	$b = B + 1$
$v_3^n(1)$	$v_3^n(t_{11})$	$\dots$	$v_3^n(t_{1(B-1)})$	$v_3^n(t_{1B})$
$x_3^n(1, 1)$	$x_3^n(t_{11}, w'_{11})$	$\dots$	$x_3^n(t_{1(B-1)}, w'_{1(B-1)})$	$x_3^n(t_{1B}, w'_{1B})$
$v_1^n(1, t_{11})$	$v_1^n(t_{11}, t_{12})$	$\dots$	$v_1^n(t_{1(B-1)}, t_{1B})$	$v_1^n(t_{1B}, 1)$
$x_1^n(1, t_{11}, 1, w'_{11}, w''_{11})$	$x_1^n(t_{11}, t_{12}, w'_{11}, w'_{12}, w''_{12})$	$\dots$	$x_1^n(t_{1(B-1)}, t_{1B}, w'_{1(B-1)}, w'_{1B}, w''_{1B})$	$x_1^n(t_{1B}, 1, w'_{1B}, 1, 1)$
$v_2^n(1)$	$v_2^n(m_{21})$	$\dots$	$v_2^n(m_{2(B-1)})$	$v_2^n(m_{2B})$
$x_2^n(1, 1)$	$x_2^n(m_{21}, w_{21})$	$\dots$	$x_2^n(m_{2(B-1)}, w_{2(B-1)})$	$x_2^n(m_{2B}, w_{2B})$

TABLE IV  
CODEWORDS IN THE PROPOSED PARTIAL DF SCHEME FOR THE IRC.

### A. Code Generation

- 1) Generate the time-sharing sequence  $q^n$  where each element is independent and identically distributed (i.i.d.) according to the PD

$$p(q^n) = \prod_{i=1}^n p_Q(q_i).$$

- 2) For each sequence  $q^n$ , generate  $2^{nT_{10}}$  conditionally independent sequences  $v_3^n(t_0)$ , where  $t_0 \in [1 : 2^{nT_{10}}]$ , and distributed according to the conditional PD

$$p(v_3^n|q^n) = \prod_{i=1}^n p_{V_3|Q}(v_{3i}|q_i).$$

- 3) For each  $v_3^n(t_0)$ , generate  $2^{nR'_{11}}$  conditionally independent sequences  $x_3^n(t_0, r_0)$ , where  $r_0 \in [1 : 2^{nR'_{11}}]$ , and distributed according to the conditional PD

$$p(x_3^n|v_3^n(t_0), q^n) = \prod_{i=1}^n p_{X_3|V_3Q}(x_{3i}|v_{3i}(t_0), q_i).$$

- 4) For each  $v_3^n(t_0)$ , generate  $2^{nT_{10}}$  conditionally independent sequences  $v_1^n(t_0, t_1)$ , where  $t_1 \in [1 : 2^{nT_{10}}]$ , and distributed according to the conditional PD

$$p(v_1^n|v_3^n(t_0), q^n) = \prod_{i=1}^n p_{V_1|V_3Q}(v_{1i}|v_{3i}(t_0), q_i).$$

- 5) Partition the set  $[1 : 2^{nT_{10}}]$  into  $2^{nR_{10}}$  cells and label them  $\mathcal{T}(m_1)$ , where  $m_1 \in [1 : 2^{nR_{10}}]$ .

- 6) For every pair  $(x_3^n(t_0, r_0), v_1^n(t_0, t_1))$ , generate  $2^{nR'_{11}}$  conditionally independent sequences  $u_1^n(t_0, t_1, r_0, r_1)$ , where  $r_1 \in [1 : 2^{nR'_{11}}]$ , and distributed according to the conditional PD

$$p(u_1^n|v_1^n(t_0, t_1), x_3^n(t_0, r_0), v_3^n(t_0), q^n) = \prod_{i=1}^n p(u_{1i}|v_{1i}(t_0, t_1), x_{3i}(t_0, r_0), v_{3i}(t_0), q_i).$$

- 7) For each  $u_1^n(t_0, t_1, r_0, r_1)$ , generate  $2^{nR''_{11}}$  conditionally independent sequences  $x_1^n(t_0, t_1, r_0, r_1, r_2)$ , where  $r_2 \in [1 : 2^{nR''_{11}}]$ , and distributed according to the conditional PD

$$p(x_1^n|u_1^n(\cdot), v_1^n(t_0, t_1), x_3^n(t_0, r_0), v_3^n(t_0), q^n) = \prod_{i=1}^n p(x_{1i}|u_{1i}(\cdot), v_{1i}(t_0, t_1), x_{3i}(t_0, r_0), v_{3i}(t_0), q_i).$$

- 8) For each sequence  $q^n$ , generate  $2^{nR_{20}}$  conditionally independent sequences  $v_2^n(s_0)$ , where  $s_0 \in [1 : 2^{nR_{20}}]$ , and distributed according to the conditional PD

$$p(v_2^n|q^n) = \prod_{i=1}^n p_{V_2|Q}(v_{2i}|q_i).$$

- 9) For each  $v_2^n(s_0)$ , generate  $2^{nR_{22}}$  conditionally independent sequences  $x_2^n(s_0, s_1)$ , where  $s_1 \in [1 : 2^{nR_{22}}]$ , and distributed according to the conditional PD

$$p(x_2^n|v_2^n(s_0), q^n) = \prod_{i=1}^n p_{X_2|V_2Q}(x_{2i}|v_{2i}(s_0), q_i).$$

### B. Encoding Part

Encoding in block  $b$  proceeds as follows,

- 1) The relay already knows the indices  $(t_{1(b-1)}, w'_{1(b-1)})$  from decoding step 1 in the previous block, thus it transmits  $x_3^n(t_{1(b-1)}, w'_{1(b-1)})$ . For block  $b = 1$ , it transmits the dummy message  $x_3^n(1, 1)$ .
- 2) Encoder 1 wants to transmit  $\tilde{m}_{1b} = (m_{1b}, w'_{1b}, w''_{1b})$ , thus, it searches for an index  $t_{1b} \in \mathcal{T}(m_{1b})$  such that  $(v_1^n(t_{1(b-1)}, t_{1b}), x_3^n(t_{1(b-1)}, w'_{1(b-1)}), v_3^n(t_{1(b-1)}), q^n) \in T_\delta^n(V_1 X_3 V_3 Q)$ . The success of this step requires that

$$T_{10} - R_{10} > I_b + \delta', \quad (46)$$

where  $\delta' > 0$  is an arbitrarily small constant and  $I_b \triangleq I(X_3; V_1|V_3Q)$ . It then transmits the codeword  $x_1^n(t_{1(b-1)}, t_{1b}, w'_{1(b-1)}, w'_{1b}, w''_{1b})$ . The source sends the dummy messages  $\tilde{m}_{10} = (1, 1, 1)$  and  $\tilde{m}_{1(B+1)} = (1, 1, 1)$  known to all users at the beginning and at the end of the transmission.

- 3) Encoder 2 sends its message  $\tilde{m}_{2(b-1)} = (m_{2(b-1)}, w_{2(b-1)})$  through the codeword  $x_2^n(m_{2(b-1)}, w_{2(b-1)})$ . During block  $b = 1$ , it sends the dummy message  $x_2^n(1, 1)$ .

See Table IV for references.

### C. Decoding Part

- 1) Let  $\delta > \delta'$ . At the end of block  $b \in [1 : B]$  and assuming its past message estimates are correct, the relay looks for the unique pair of indices  $(t_{1b}, w'_{1b}) \equiv (i, j)$  such that

$$(v_3^n(t_{1(b-1)}), x_3^n(t_{1(b-1)}, w'_{1(b-1)}), v_1^n(t_{1(b-1)}, i), y_{3b}^n, q^n, u_1^n(t_{1(b-1)}, i, w'_{1(b-1)}, j)) \in T_\delta^n(V_3 X_3 V_1 U_1 Y_3 Q).$$

The probability of error becomes arbitrarily small if

$$R'_{11} < I(U_1; Y_3 | V_1 X_3 Q) - \delta, \quad (47a)$$

$$T_{10} + R'_{11} < I(V_1 U_1; Y_3 | X_3 Q) + I_b - \delta. \quad (47b)$$

- 2) Starting at the end of block  $B + 1$  and assuming its past message estimates are correct, destination 1 looks for the indices  $(t_{1(b-1)}, w'_{1(b-1)}, w''_{1b}, m_{2(b-1)}) \equiv (i, j, k, l)$  backwardly such that

$$(v_3^n(i), v_1^n(i, t_{1b}), x_3^n(i, j), u_1^n(i, t_{1b}, j, w'_{1b}), v_2^n(l), y_{1b}^n, q^n, x_1^n(i, t_{1b}, j, w'_{1b}, k)) \in T_\delta^n(V_3 V_1 X_3 U_1 X_1 V_2 Y_1 Q).$$

The probability of error becomes arbitrarily small if

$$R''_{11} < I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q) - \delta, \quad (48a)$$

$$R'_{11} + R''_{11} < I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I_b - \delta, \quad (48b)$$

$$T_{10} + R'_{11} + R''_{11} < I(X_1 X_3; Y_1 | V_2 Q) + I_b - \delta, \quad (48c)$$

$$R''_{11} + R_{20} < I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) - \delta, \quad (48d)$$

$$R'_{11} + R''_{11} + R_{20} < I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I_b - \delta, \quad (48e)$$

$$T_{10} + R'_{11} + R''_{11} + R_{20} < I(X_1 V_2 X_3; Y_1 | Q) + I_b - \delta. \quad (48f)$$

- 3) Destination 2 performs similarly, thus, it looks for the indices  $(t_{1(b-1)}, m_{2(b-1)}, w_{2(b-1)}) \equiv (i, k, l)$  backwardly such that

$$(v_3^n(i), v_1^n(i, t_{1b}), v_2^n(k), x_2^n(k, l), y_{2b}^n, q^n) \in T_\delta^n(V_3 V_1 V_2 X_2 Y_2 Q).$$

The probability of error becomes arbitrarily small if

$$R_{22} < I(X_2; Y_2 | V_1 V_2 V_3 Q) - \delta, \quad (49a)$$

$$R_{20} + R_{22} < I(X_2; Y_2 | V_1 V_3 Q) - \delta, \quad (49b)$$

$$T_{10} + R_{22} < I(V_1 X_2 V_3; Y_2 | V_2 Q) - \delta, \quad (49c)$$

$$T_{10} + R_{20} + R_{22} < I(V_1 X_2 V_3; Y_2 | Q) - \delta, \quad (49d)$$

$$T_{10} < I(V_1 V_3; Y_2 | X_2 Q) - \delta. \quad (49e)$$

*Remark 13:* If at this point we replace  $U_1$  with  $X_1$ , the region boils down to the one attained by the full DF scheme (Corollary 2). See Appendix E.

*Remark 14:* The bound (49e) represents the perfect decoding of the common layer of interference. This bound is needed, however, because of the block-Markov coding technique and the assumption that the index  $t_{1b}$  present in  $v_1^n(\cdot)$  is correct. Nonetheless, this term only appears in some of the additional bounds shown below and it does not affect the final region  $\mathcal{R}_{\text{p-DF}}$ .

After running Fourier-Motzkin elimination (FME) to the set (46)–(49) and letting  $n \rightarrow \infty$ , we obtain the region  $\mathcal{R}_{\text{p-DF}}(P_2)$  (9) with the term  $I(V_1 U_1; Y_3 | X_3 Q)$  instead of  $I(U_1; Y_3 | X_3 Q)$  in (9a), (9k), and (9m), plus four additional bounds

$$R_1 < I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(V_1 V_3; Y_2 | X_2 Q), \quad (50a)$$

$$R_1 < I(U_1; Y_3 | V_1 X_3 Q) + I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q) + I(V_1 V_3; Y_2 | X_2 Q) - I_b, \quad (50b)$$

$$R_2 < I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q), \quad (50c)$$

$$R_2 < I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q) - I_b. \quad (50d)$$

These bounds on the single rates arise from the decoding of the common message of the interference at the interfered receiver. It is reasonable to assume that the maximizing PD will render these bounds *inactive*, i.e., if the single rates are penalized due to the large amount of common information, another PD with *less* common information will increase the achievable rate.

In order to eliminate the bounds (50) – a necessary condition to later compare to the outer bound – we proceed in a similar way as [33, Lemma 2]. First, let us define, for a given PD  $p \in \mathcal{P}_2$ , the region  $\mathcal{R}_{\text{p-DF}}^o(p)$  as the *original* region after FME, i.e., the region  $\mathcal{R}_{\text{p-DF}}(p)$  (9) with the term  $I(V_1 U_1; Y_3 | X_3 Q)$  instead of  $I(U_1; Y_3 | X_3 Q)$  plus the four bounds (50).

Second, we define  $\mathcal{R}_{\text{p-DF}}^{c_1}(p)$  as the region  $\mathcal{R}_{\text{p-DF}}^o(p)$  without bounds (50c) and (50d). For this reason, it is easy to see that  $\mathcal{R}_{\text{p-DF}}^o(p) \subseteq \mathcal{R}_{\text{p-DF}}^{c_1}(p)$ . On the other hand, when either (50c) or (50d) is active in  $\mathcal{R}_{\text{p-DF}}^o(p)$ , then  $\mathcal{R}_{\text{p-DF}}^o(p^{**})$  with  $p^{**} = \sum_{v_2} p$  attains higher rates than  $\mathcal{R}_{\text{p-DF}}^{c_1}(p)$ . The PD  $p^{**}$  is the marginal of  $p$  w.r.t.  $V_2$ , therefore, effectively eliminating the common message from the second source. In summary,  $\mathcal{R}_{\text{p-DF}}^{c_1}(p) \subseteq \mathcal{R}_{\text{p-DF}}^o(p) \cup \mathcal{R}_{\text{p-DF}}^o(p^{**})$ . After maximizing over all joint PDs, we obtain  $\mathcal{R}_{\text{p-DF}}^{c_1} = \mathcal{R}_{\text{p-DF}}^o$ , thus (50c) and (50d) are redundant.

Third, we reduce the achievable region  $\mathcal{R}_{\text{p-DF}}^{c_1}(p)$  by replacing the terms  $I(V_1 U_1; Y_3 | X_3 Q)$  with  $I(U_1; Y_3 | X_3 Q)$ , let us call this new reduced region  $\mathcal{R}_{\text{p-DF}}^{c_2}(p)$ . We define the region  $\mathcal{R}_{\text{p-DF}}(p)$  based on  $\mathcal{R}_{\text{p-DF}}^{c_2}(p)$  and eliminate the bounds (50a) and (50b) from it. After this, it is easy to prove that both  $\mathcal{R}_{\text{p-DF}}^{c_2}(p) \subseteq \mathcal{R}_{\text{p-DF}}(p)$  and  $\mathcal{R}_{\text{p-DF}}(p) \subseteq \mathcal{R}_{\text{p-DF}}^{c_2}(p) \cup \mathcal{R}_{\text{p-DF}}^{c_2}(p^*)$ , with  $p^* = \sum_{v_1 v_3} p$ , hold. Therefore, after the maximization, we obtain  $\mathcal{R}_{\text{p-DF}} = \mathcal{R}_{\text{p-DF}}^{c_2}$ .

It is worth mentioning that the region  $\mathcal{R}_{\text{p-DF}}(p)$  is not the optimal one for partial DF because of the aforementioned reduction, i.e.  $\mathcal{R}_{\text{p-DF}} = \mathcal{R}_{\text{p-DF}}^{c_2} \subseteq \mathcal{R}_{\text{p-DF}}^{c_1} = \mathcal{R}_{\text{p-DF}}^o$ . However, as we see later, this loss does not prevent us from obtaining a constant-gap result.

## APPENDIX E

### PROOF OF COROLLARY 2 (FULL DF SCHEME)

Since  $U_1 = X_1$ , the first source does not split its private message in two, i.e.,  $R''_{11} = 0$  and  $R_1 = R_{10} + R'_{11}$ . The codebook generation, encoding and decoding is carried out as in the partial DF scheme.

After running Fourier-Motzkin elimination to the set (46)–(49) and letting  $n \rightarrow \infty$ , we obtain the region  $\mathcal{R}_{\text{f-DF}}(P_3)$  (11), plus three additional bounds

$$R_1 < I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(V_1 V_3; Y_2 | X_2 Q), \quad (51a)$$

$$R_1 < I(X_1; Y_3 | V_1 X_3 Q) + I(V_1 V_3; Y_2 | X_2 Q) - I_b, \quad (51b)$$

$$R_2 < I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q) + I_b. \quad (51c)$$

As in the partial DF scheme, these bounds are redundant when maximized over all possible PDs. Let us define  $\mathcal{R}_{\text{f-DF}}^o(P_3)$  as the *original* region after FME. Then, it is clear that for a given PD  $p \in \mathcal{P}_3$ ,  $\mathcal{R}_{\text{f-DF}}^o(p) \subseteq \mathcal{R}_{\text{f-DF}}(p)$ , because of the presence of (51).

When either (51a) or (51b) is active in  $\mathcal{R}_{\text{f-DF}}^o(p)$ , then  $\mathcal{R}_{\text{f-DF}}^o(p^*)$  with  $p^* = \sum_{v_1 v_3} p$  attains higher rates than  $\mathcal{R}_{\text{f-DF}}(p)$ . Similarly, when (51c) is active,  $\mathcal{R}_{\text{f-DF}}^o(p^{**})$  with



$b = 1$	$b = 2$	$\dots$	$b = B$	$b = B + 1$	$\dots$	$b = B + L$
$v_1^n(m_{11})$	$v_1^n(m_{12})$	$\dots$	$v_1^n(m_{1B})$	$v_1^n(1)$	$\dots$	$v_1^n(1)$
$x_1^n(m_{11}, w_{11})$	$x_1^n(m_{12}, w_{12})$	$\dots$	$x_1^n(m_{1B}, w_{1B})$	$x_1^n(1, 1)$	$\dots$	$x_1^n(1, 1)$
$v_2^n(m_{21})$	$v_2^n(m_{22})$	$\dots$	$v_2^n(m_{2B})$	$v_2^n(1)$	$\dots$	$v_2^n(1)$
$x_2^n(m_{21}, w_{21})$	$x_2^n(m_{22}, w_{22})$	$\dots$	$x_2^n(m_{2B}, w_{2B})$	$x_2^n(1, 1)$	$\dots$	$x_2^n(1, 1)$
$\hat{y}_3^n(1, l_1)$	$\hat{y}_3^n(l_1, l_2)$	$\dots$	$\hat{y}_3^n(l_{B-1}, l_B)$	$\emptyset$	$\dots$	$\emptyset$
$x_3^n(1)$	$x_3^n(l_1)$	$\dots$	$x_3^n(l_{B-1})$	$x_3^n(l_B)$	$\dots$	$x_3^n(l_B)$

TABLE V  
CODEWORDS IN THE PROPOSED CF SCHEME FOR THE IRC.

$p^{**} = \sum_{v_2} p$  outperforms  $\mathcal{R}_{f\text{-DF}}(p)$ . Succinctly,  $\mathcal{R}_{f\text{-DF}}(p) \subseteq \mathcal{R}_{f\text{-DF}}^o(p) \cup \mathcal{R}_{f\text{-DF}}^o(p^*) \cup \mathcal{R}_{f\text{-DF}}^o(p^{**})$ .

Therefore, after maximizing over all possible PDs,  $\mathcal{R}_{f\text{-DF}} = \mathcal{R}_{f\text{-DF}}^o$ , which renders (51) redundant.

#### APPENDIX F PROOF OF THEOREM 3 (CF SCHEME)

As before, each source  $k \in \{1, 2\}$  splits its message  $\tilde{m}_k$  into a common message  $m_k$  and a private one  $w_k$ , each with partial rate  $R_{k0}$  and  $R_{kk}$ , respectively, such that  $R_k = R_{k0} + R_{kk}$ . But now, each source transmits  $B$  messages during  $B+L$  time blocks, each of them of length  $n$ . During these additional  $L$  time blocks, the relay repeats the same compression index to ensure a correct decoding at each destination [34], [35].

##### A. Code Generation

- 1) Generate the time-sharing sequence  $q^n$  where each element is independent and identically distributed (i.i.d.) according to the PD

$$p(q^n) = \prod_{i=1}^n p_Q(q_i).$$

- 2) For each source  $k \in \{1, 2\}$  and the sequence  $q^n$ , generate  $2^{nR_{k0}}$  conditionally independent sequences  $v_k^n(m_k)$ , where  $m_k \in [1 : 2^{nR_{k0}}]$ , and distributed according to the conditional PD

$$p(v_k^n | q^n) = \prod_{i=1}^n p_{V_k | Q}(v_{ki} | q_i).$$

- 3) For each source  $k \in \{1, 2\}$  and for each  $v_k^n(m_k)$ , generate  $2^{nR_{kk}}$  conditionally independent sequences  $x_k^n(m_k, w_k)$ , where  $w_k \in [1 : 2^{nR_{kk}}]$ , and distributed according to the conditional PD

$$p(x_k^n | v_k^n(m_k), q^n) = \prod_{i=1}^n p_{X_k | V_k Q}(x_{ki} | v_{ki}(m_k), q_i).$$

- 4) For the sequence  $q^n$ , generate  $2^{n\hat{R}}$  conditionally independent sequences  $x_3^n(l_1)$ , where  $l_1 \in [1 : 2^{n\hat{R}}]$  for  $\hat{R} = I(\hat{Y}_3; Y_3 | X_3 Q) + \delta'$ , and distributed according to the conditional PD

$$p(x_3^n | q^n) = \prod_{i=1}^n p_{X_3 | Q}(x_{3i} | q_i).$$

- 5) For the sequence  $q^n$  and each  $x_3^n(l_1)$ , generate  $2^{n\hat{R}}$  conditionally independent sequences  $\hat{y}_3^n(l_1, l_2)$ , where  $l_2 \in [1 : 2^{n\hat{R}}]$ , and distributed according to the conditional PD

$$p(\hat{y}_3^n | x_3^n(l_1), q^n) = \prod_{i=1}^n p_{\hat{Y}_3 | X_3 Q}(\hat{y}_{3i} | x_{3i}(l_1), q_i).$$

##### B. Encoding Part

Encoding in block  $b$  proceeds as follows,

- 1) Each source  $k \in \{1, 2\}$  uses its present message  $\tilde{m}_{kb}$  to choose the codeword it transmits,  $x_k^n(m_{kb}, w_{kb})$  for blocks  $b \in [1 : B]$ . During blocks  $b \in [B+1 : B+L]$ , the sources send the dummy message  $\tilde{m}_{kb} = 1$  known to all users.
- 2) At the end of block  $b \in [1 : B]$ , the relay looks for at least one index  $l_b$ , with  $l_0 = 1$  s.t.  $(x_3^n(l_{b-1}), \hat{y}_3^n(l_{b-1}, l_b), y_{3b}^n, q^n) \in T_{\delta'}^n(X_3 \hat{Y}_3 Y_3 Q)$ . The probability of finding such  $l_b$  goes to one as  $n$  approaches infinity. It then transmits  $x_3^n(l_b)$  in the next time block. Moreover, for blocks  $b \in [B+1 : B+L]$ , the last compression index  $l_B$  is repeated.

See Table V for references.

##### C. Decoding Part

- 1) Destination 1 decodes the compression index in two steps. First, it looks for the unique index  $l_B \equiv l$  such that,  $\forall b \in [B+1 : B+L]$ ,

$$(v_1^n(1), x_1^n(1, 1), v_2^n(1), x_3^n(l), y_{1b}^n, q^n) \in T_{\delta}^n(V_1 X_1 V_2 X_3 Y_1 Q).$$

For a finite but sufficiently large  $L$ , the probability of incorrectly decoding  $l_B$  can be made arbitrarily small.

- 2) After finding  $l_B$ , destination 1 looks for the indices  $(m_{1b}, w_{1b}, m_{2b}, l_{b-1}) \equiv (i, j, k, l)$  for  $b \in [1 : B]$  such that

$$(v_1^n(i), x_1^n(i, j), v_2^n(k), x_3^n(l), \hat{y}_3^n(l, l_b), y_{1b}^n, q^n) \in T_{\delta}^n(V_1 X_1 V_2 X_3 \hat{Y}_3 Y_1 Q).$$

The probability of error can be made arbitrarily small provided that,

$$R_{11} < I_{11} - \delta, \quad (52a)$$

$$R_{10} + R_{11} < I_{12} - \delta, \quad (52b)$$

$$R_{20} + R_{11} < I_{13} - \delta, \quad (52c)$$

$$R_{10} + R_{11} + R_{20} < I_{14} - \delta, \quad (52d)$$

$$R_{20} < I(V_2 X_3; Y_1 | X_1 Q) - I_1 - \delta, \quad (52e)$$

$$I_1 < I(X_3; Y_1 | X_1 V_2 Q) - \delta \quad (52f)$$

where  $I_1 \triangleq I(\hat{Y}_3; Y_3 | X_1 V_2 X_3 Y_1 Q) + \delta'$  and

$$I_{11} \triangleq \min\{I(X_1; Y_1 \hat{Y}_3 | V_1 V_2 X_3 Q), I(X_1 X_3; Y_1 | V_1 V_2 Q) - I_1\}$$

$$I_{12} \triangleq \min\{I(X_1; Y_1 \hat{Y}_3 | V_2 X_3 Q), I(X_1 X_3; Y_1 | V_2 Q) - I_1\}$$

$$I_{13} \triangleq \min\{I(X_1 V_2; Y_1 \hat{Y}_3 | V_1 X_3 Q), I(X_1 V_2 X_3; Y_1 | V_1 Q) - I_1\}$$

$$I_{14} \triangleq \min\{I(X_1 V_2; Y_1 \hat{Y}_3 | X_3 Q), I(X_1 V_2 X_3; Y_1 | Q) - I_1\}.$$

- 3) If destination 1 ignores the compression index, it looks for the indices  $(m_{1b}, w_{1b}, m_{2b}) \equiv (i, j, k)$  for  $b \in [1 : B]$  such that

$$(v_1^n(i), x_1^n(i, j), v_2^n(k), y_1^n(j), q^n) \in T_\delta^n(V_1 X_1 V_2 Y_1 Q).$$

The probability of error can be made arbitrarily small provided that,

$$R_{11} < I(X_1; Y_1 | V_1 V_2 Q) - \delta, \quad (53a)$$

$$R_{10} + R_{11} < I(X_1; Y_1 | V_2 Q) - \delta, \quad (53b)$$

$$R_{20} + R_{11} < I(X_1 V_2; Y_1 | V_1 Q) - \delta, \quad (53c)$$

$$R_{10} + R_{11} + R_{20} < I(X_1 V_2; Y_1 | Q) - \delta. \quad (53d)$$

- 4) Destination 2 performs similarly, and all the above inequalities hold by swapping the indices 1 and 2.

It is noteworthy that the bound in the rate of the interfering common message (52e), i.e.,  $R_{j0} \leq I(V_j X_3; Y_k | X_k Q) - I_k$ , is a by-product of the CF scheme. Although the error in decoding the index of the interfering common message is normally not taken into account in the IC, this bound is needed in order to assure that the compression index  $l_b$  is the right one at time  $b$ . Nonetheless, both the bound (52e) and (52f) are redundant as we see next.

When (52e) does not hold, (52c) and (52d) become:

$$\begin{aligned} R_{11} &< I(X_1 V_2 X_3; Y_1 | V_1 Q) - I(V_2 X_3; Y_1 | X_1 Q) \\ &= I(X_1; Y_1 | V_1 Q), \end{aligned} \quad (54a)$$

$$\begin{aligned} R_{10} + R_{11} &< I(X_1 V_2 X_3; Y_1 | Q) - I(V_2 X_3; Y_1 | X_1 Q) \\ &= I(X_1; Y_1 | Q). \end{aligned} \quad (54b)$$

This is included in the region (53) for the special case  $V_2 = \emptyset$ .

Moreover, if (52f) does not hold, the first five bounds of (52) become:

$$\begin{aligned} R_{11} &< I(X_1 X_3; Y_1 | V_1 V_2 Q) - I_1 \\ &< I(X_1 X_3; Y_1 | V_1 V_2 Q) - I(X_3; Y_1 | X_1 V_2 Q) \\ &= I(X_1; Y_1 | V_1 V_2 Q), \end{aligned} \quad (55a)$$

$$R_{10} + R_{11} < I(X_1; Y_1 | V_2 Q), \quad (55b)$$

$$R_{20} + R_{11} < I(X_1 V_2; Y_1 | V_1 Q), \quad (55c)$$

$$R_{10} + R_{11} + R_{20} < I(X_1 V_2; Y_1 | V_1 Q), \quad (55d)$$

$$R_{20} < I(V_2; Y_1 | X_1 Q). \quad (55e)$$

This region is also included in (53). Therefore, when either condition (52e) or (52f) does not hold for a given distribution, the region (52) is included inside (53), i.e., destination 1 should ignore the relay to achieve higher rates. Since the final region

is the union over all possible PDs of (52) and (53) for both users, we can drop (52e) and (52f) because they do not affect the final region after the maximization. This result can be seen as an extension of [35].

Before running Fourier-Motzkin elimination to this system, we shall make some clarifications. First, let us define  $\mathcal{R}_{\text{CF}_3}(P_4)$  as the region obtained with the distribution  $P_4$  when both users ignore the compression index, i.e., the Han-Kobayashi inner bound. The regions  $\mathcal{R}_{\text{CF}_1}(P_4)$  and  $\mathcal{R}_{\text{CF}_2}(P_4)$  are the ones obtained when only the first or second user decodes the relay's message, respectively.  $\mathcal{R}_{\text{CF}_0}(P_4)$  corresponds to the region when both users decode the compression index.

Second, even though the expressions  $I_{ki}$  look rather complex, there exists an ordering between them analogous to  $I'_{ki}$  that allows us to reduce the number of bounds. In other words, the following inequalities hold,

$$I_{k1} \leq I_{k2} \leq I_{k4} \text{ and } I_{k1} \leq I_{k3} \leq I_{k4}. \quad (56)$$

To check this, take each term of  $I_{11}$  and  $I_{12}$  separately

$$\begin{aligned} I_{11} &\leq I(X_1; Y_1 \hat{Y}_3 | V_1 V_2 X_3 Q) \\ &= h(Y_1 \hat{Y}_3 | V_1 V_2 X_3 Q) - h(Y_1 \hat{Y}_3 | X_1 V_2 X_3 Q), \end{aligned} \quad (57a)$$

$$\begin{aligned} I_{11} &\leq I(X_1 X_3; Y_1 | V_1 V_2 Q) - I_1 \\ &= h(Y_1 | V_1 V_2 Q) - h(Y_1 | X_1 V_2 X_3 Q) - I_1, \end{aligned} \quad (57b)$$

$$\begin{aligned} I_{12} &\leq I(X_1; Y_1 \hat{Y}_3 | V_2 X_3 Q) \\ &= h(Y_1 \hat{Y}_3 | V_2 X_3 Q) - h(Y_1 \hat{Y}_3 | X_1 V_2 X_3 Q), \end{aligned} \quad (57c)$$

$$\begin{aligned} I_{12} &\leq I(X_1 X_3; Y_1 | V_2 Q) - I_1 \\ &= h(Y_1 | V_2 Q) - h(Y_1 | X_1 V_2 X_3 Q) - I_1. \end{aligned} \quad (57d)$$

Since conditioning reduces entropy, we have that (57a)  $\leq$  (57c) and (57b)  $\leq$  (57d), which leads to  $I_{11} \leq I_{12}$ . The same reasoning applies for the other  $I_{ki}$  in (56).

1) *Final Region  $\mathcal{R}_{\text{CF}_3}$* : After running FME to the system composed by (53) and its symmetric one for the second user, and letting  $n \rightarrow \infty$ , we obtain the region  $\mathcal{R}_{\text{CF}_3}^o(p)$ :

$$\begin{aligned} R_k &\leq \min\{I'_{k2}, I'_{k1} + I'_{j3}\}, \\ R_k + R_j &\leq \min\{I'_{k1} + I'_{j4}, I'_{k3} + I'_{j3}\}, \\ 2R_k + R_j &\leq I'_{k1} + I'_{k4} + I'_{j3}. \end{aligned}$$

This region has two redundant bounds as shown in [33]:

$$R_1 \leq I(X_1; Y_1 | V_1 V_2 Q) + I(V_1 X_2; Y_2 | V_2 Q), \quad (58a)$$

$$R_2 \leq I(X_1 V_2; Y_1 | V_1 Q) + I(X_2; Y_2 | V_1 V_2 Q). \quad (58b)$$

If we define  $\mathcal{R}_{\text{CF}_3}^c(p)$  as the compact version of the original region  $\mathcal{R}_{\text{CF}_3}^o(p)$ , i.e., without the two redundant bounds, we can readily see that  $\mathcal{R}_{\text{CF}_3}^o(p) \subseteq \mathcal{R}_{\text{CF}_3}^c(p)$  for a given distribution  $p \in \mathcal{P}_4$  since  $\mathcal{R}_{\text{CF}_3}^c(p)$  has fewer bounds.

If a pair of rates  $(R_1, R_2)$  belongs to  $\mathcal{R}_{\text{CF}_3}^c(p)$  but not to  $\mathcal{R}_{\text{CF}_3}^o(p)$ , it is because (58) does not hold. Let us first assume that

$$R_1 > I(X_1; Y_1 | V_1 V_2 Q) + I(V_1 X_2; Y_2 | V_2 Q).$$

With this condition,  $\mathcal{R}_{\text{CF}_3}^c(p)$  becomes:

$$R_1 \leq I(X_1; Y_1 | V_2 Q),$$

$$R_2 \leq I(V_2; Y_2 | Q),$$

$$R_1 + R_2 \leq I(X_1 V_2; Y_1 | Q),$$

together with some additional bounds. We may compare this region with  $\mathcal{R}_{\text{CF}_3}^o(p^*)$ , where  $p^* = \sum_{v_1} p$ ,

$$\begin{aligned} R_1 &\leq I(X_1; Y_1|V_2Q), \\ R_2 &\leq I(X_2; Y_2|Q), \\ R_1 + R_2 &\leq I(X_1V_2; Y_1|Q) + I(X_2; Y_2|V_2Q). \end{aligned}$$

It is clear that, when (58a) is violated,  $\mathcal{R}_{\text{CF}_3}^c(p) \subseteq \mathcal{R}_{\text{CF}_3}^o(p^*)$ .

Similarly, if (58b) does not hold, we see that  $\mathcal{R}_{\text{CF}_3}^c(p) \subseteq \mathcal{R}_{\text{CF}_3}^o(p^{**})$ , where  $p^{**} = \sum_{v_2} p$ . Therefore, in the general case,

$$\mathcal{R}_{\text{CF}_3}^c(p) \subseteq \mathcal{R}_{\text{CF}_3}^o(p) \cup \mathcal{R}_{\text{CF}_3}^o(p^*) \cup \mathcal{R}_{\text{CF}_3}^o(p^{**}).$$

Since we have already shown that  $\mathcal{R}_{\text{CF}_3}^o(p) \subseteq \mathcal{R}_{\text{CF}_3}^c(p)$ , when maximizing over all joint PDs, we have that  $\mathcal{R}_{\text{CF}_3}^o = \mathcal{R}_{\text{CF}_3}^c$ .

2) *Final Regions  $\mathcal{R}_{\text{CF}_1}$  and  $\mathcal{R}_{\text{CF}_2}$* : Now, we go to  $\mathcal{R}_{\text{CF}_1}^o(p)$ , where only the first user decodes the compression index. In this case, the region that is obtained after running FME is:

$$\begin{aligned} R_1 &\leq \min\{I_{12}, I_{11} + I'_{23}\}, \\ R_2 &\leq \min\{I'_{22}, I_{13} + I'_{21}\}, \\ R_1 + R_2 &\leq \min\{I_{11} + I'_{24}, I_{14} + I'_{21}, I_{13} + I'_{23}\}, \\ 2R_1 + R_2 &\leq I_{11} + I_{14} + I'_{23}, \\ R_1 + 2R_2 &\leq I_{13} + I'_{21} + I'_{24}. \end{aligned}$$

Here, we have another two redundant bounds:

$$R_1 \leq I_{11} + I(V_1X_2; Y_2|V_2Q), \quad (59a)$$

$$R_2 \leq I_{13} + I(X_2; Y_2|V_1V_2Q). \quad (59b)$$

Once again, for a given distribution  $p \in \mathcal{P}_4$ , we define  $\mathcal{R}_{\text{CF}_1}^o(p)$  as the original region with all the bounds and  $\mathcal{R}_{\text{CF}_1}^c(p)$  as the compact one without the redundant bounds. Since  $\mathcal{R}_{\text{CF}_1}^c(p)$  has fewer bounds, we can readily see that  $\mathcal{R}_{\text{CF}_1}^o(p) \subseteq \mathcal{R}_{\text{CF}_1}^c(p)$ .

If (59a) does not hold,  $\mathcal{R}_{\text{CF}_1}^c(p)$  becomes:

$$\begin{aligned} R_1 &\leq I_{12}, \\ R_2 &\leq I(V_2; Y_2|Q), \\ R_1 + R_2 &\leq I_{14}, \end{aligned}$$

together with some additional bounds. We may compare this region with  $\mathcal{R}_{\text{CF}_1}^o(p^*)$ , where  $p^* = \sum_{v_1} p$ ,

$$\begin{aligned} R_1 &\leq I_{12}, \\ R_2 &\leq I(X_2; Y_2|Q), \\ R_1 + R_2 &\leq I_{14} + I(X_2; Y_2|V_2Q). \end{aligned}$$

As we see, when (59a) is violated,  $\mathcal{R}_{\text{CF}_1}^c(p) \subseteq \mathcal{R}_{\text{CF}_1}^o(p^*)$ .

Since this region is not symmetric, we also need to see what happens when (59b) does not hold. In this case,  $\mathcal{R}_{\text{CF}_1}^c(p)$  becomes:

$$\begin{aligned} R_1 &\leq I_{14} - I_{13}, \quad (60a) \\ R_2 &\leq I(X_2; Y_2|V_1Q), \quad (60b) \\ R_1 + R_2 &\leq I(V_1X_2; Y_2|Q), \quad (60c) \end{aligned}$$

together with some additional bounds. Now, let us take  $p^{**} = \sum_{v_2} p$  and calculate  $\mathcal{R}_{\text{CF}_1}^o(p^{**})$ :

$$R_1 \leq I_{14}^*, \quad (61a)$$

$$R_2 \leq I(X_2; Y_2|V_1Q), \quad (61b)$$

$$R_2 \leq I_{13}^* + I(X_2; Y_2|V_1Q), \quad (61c)$$

$$R_1 + R_2 \leq I_{13}^* + I(V_1X_2; Y_2|Q) \quad (61d)$$

where

$$\begin{aligned} I_{13}^* &\triangleq \min\{I(X_1; Y_1\hat{Y}_3|V_1X_3Q), \\ &\quad I(X_1X_3; Y_1|V_1Q) - I(Y_3; \hat{Y}_3|X_1X_3Y_1Q)\}, \\ I_{14}^* &\triangleq \min\{I(X_1; Y_1\hat{Y}_3|X_3Q), \\ &\quad I(X_1X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|X_1X_3Y_1Q)\}. \end{aligned}$$

We shall recall that the PD  $p$  is such that the rates  $R_1$  and  $R_2$  are nonnegative in  $\mathcal{R}_{\text{CF}_1}^c(p)$ . However, this does not mean that  $I_{13}^*$  or  $I_{14}^*$  should be positive since they depend on  $p^{**}$ . If any of the two expressions is negative,  $\mathcal{R}_{\text{CF}_1}^c(p) \not\subseteq \mathcal{R}_{\text{CF}_1}^o(p^{**})$ , which is not what we are looking for. We first assume that both quantities are positive.

Let us define with a subscript  $a$  and  $b$  the first and second term of the minimums in the expressions  $I_{ki}$ , respectively. Then, if  $I_{13} = I_{13a}$ , the first rate in  $\mathcal{R}_{\text{CF}_1}^c(p)$  becomes:

$$\begin{aligned} R_1 &\leq I_{14a} - I_{13a} = I(V_1; Y_1\hat{Y}_3|X_3Q) \leq I_{14a}^*, \quad (62a) \\ R_1 &\leq I_{14b} - I_{13a} \\ &= I(X_1V_2X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|X_1V_2X_3Y_1Q) \\ &\quad - I(X_1V_2; Y_1\hat{Y}_3|V_1X_3Q) \\ &= I(X_1V_2X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|X_1V_2X_3Y_1Q) \\ &\quad - I(X_1V_2; Y_1|V_1X_3Q) - I(X_1V_2; \hat{Y}_3|V_1X_3Y_1Q) \\ &= I(V_1X_3; Y_1|Q) - I(X_1V_2Y_3; \hat{Y}_3|V_1X_3Y_1Q) \\ &= I(V_1X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|V_1X_3Y_1Q) \leq I_{14b}^* \quad (62b) \end{aligned}$$

where in the last step we take into account that  $\hat{Y}_3 \ominus (X_3Y_3Q) \ominus (X_1V_2)$ . On the other hand, if  $I_{13} = I_{13b}$ , the first rate in  $\mathcal{R}_{\text{CF}_1}^c(p)$  becomes:

$$R_1 \leq I_{14b} - I_{13b} = I(V_1; Y_1|Q) \leq I_{14a}^*. \quad (63)$$

Also, in  $\mathcal{R}_{\text{CF}_1}^o(p^{**})$ :

$$\begin{aligned} R_1 &\leq I_{14b}^* = I(X_1X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|X_1X_3Y_1Q) \\ &= I(V_1; Y_1|Q) + I_{13b}^*. \quad (64) \end{aligned}$$

If we assume that  $I_{13}^* \geq 0$ , (62b) and (64) assure us that  $I_{14}^* \geq 0$ . Putting (60) through (64) together, we have shown that  $\mathcal{R}_{\text{CF}_1}^c(p) \subseteq \mathcal{R}_{\text{CF}_1}^o(p^{**})$ . However, if  $I_{13}^* < 0$  we shall consider the case where the first user also ignores the compression index, i.e.  $\mathcal{R}_{\text{CF}_3}^o(p^{**})$ ,

$$R_1 \leq I(X_1; Y_1|Q), \quad (65a)$$

$$R_2 \leq I(X_2; Y_2|V_1Q), \quad (65b)$$

$$R_1 + R_2 \leq I(X_1; Y_1|V_1Q) + I(V_1X_2; Y_2|Q). \quad (65c)$$

The region in (60) looks smaller than (65), with the exception of the rate  $R_1$  that we analyze in the sequel. If  $I_{13} = I_{13a}$ , in (60a) we have that,

$$\begin{aligned} R_1 &\leq I_{14} - I_{13} = \min\{I_{14a}, I_{14b}\} - I_{13a} \leq I_{14b} - I_{13a} \\ &= I(V_1X_3; Y_1|Q) - I(Y_3; \hat{Y}_3|V_1X_3Y_1Q) \quad (66a) \end{aligned}$$

$$\begin{aligned} &= I(V_1X_3; Y_1|Q) - I(X_1; \hat{Y}_3|V_1X_3Y_1Q) \\ &\quad - I(Y_3; \hat{Y}_3|X_1X_3Y_1Q) \quad (66b) \end{aligned}$$

$$\begin{aligned} &< I(V_1 X_3; Y_1 | Q) - I(X_1; \hat{Y}_3 | V_1 X_3 Y_1 Q) \\ &\quad - I(X_1 X_3; Y_1 | V_1 Q) \end{aligned} \quad (66c)$$

$$\begin{aligned} &\leq I(X_1 X_3; Y_1 | Q) - I(X_1; \hat{Y}_3 | V_1 X_3 Y_1 Q) \\ &\quad - I(X_1 X_3; Y_1 | V_1 Q) \\ &= I(V_1; Y_1 | Q) - I(X_1; \hat{Y}_3 | V_1 X_3 Y_1 Q) \\ &\leq I(V_1; Y_1 | Q), \end{aligned} \quad (66d)$$

where (66a) comes from (62b), (66b) is due to the Markov chain  $\hat{Y}_3 \ominus (X_3 Y_3 Q) \ominus X_1$ , and (66c) is due to the assumption  $I_{13}^* < 0$ , i.e.  $I(X_1 X_3; Y_1 | V_1 Q) < I(Y_3; \hat{Y}_3 | X_1 X_3 Y_1 Q)$ .

On the other hand, if  $I_{13} = I_{13b}$ , we have already shown in (63) that  $R_1 \leq I(V_1; Y_1 | Q)$ . Therefore, if  $I_{13}^* < 0$ , the region  $\mathcal{R}_{CF_3}^o(p^{**})$  is larger than  $\mathcal{R}_{CF_1}^c(p)$  when  $R_2 > I_{13} + I(X_2; Y_2 | V_1 V_2 Q)$ . To sum up, in the general case,

$$\mathcal{R}_{CF_1}^c(p) \subseteq \mathcal{R}_{CF_1}^o(p) \cup \mathcal{R}_{CF_1}^o(p^*) \cup \mathcal{R}_{CF_1}^o(p^{**}) \cup \mathcal{R}_{CF_3}^o(p^{**}),$$

and since  $\mathcal{R}_{CF_1}^o(p) \subseteq \mathcal{R}_{CF_1}^c(p)$ , if we maximize over all joint possible joint distributions we obtain  $\mathcal{R}_{CF_1}^c \cup \mathcal{R}_{CF_3}^c = \mathcal{R}_{CF_1}^o \cup \mathcal{R}_{CF_3}^o$ .

The symmetric region  $\mathcal{R}_{CF_2}^o(p)$  where only the second user decodes the compression index behaves similarly. We can redo the whole proof by simply swapping the subindices 1 and 2. Consequently, if we maximize over all joint possible joint distributions we have that  $\mathcal{R}_{CF_2}^o \cup \mathcal{R}_{CF_3}^o = \mathcal{R}_{CF_2}^c \cup \mathcal{R}_{CF_3}^c$ .

3) *Final Region  $\mathcal{R}_{CF_0}$* : Finally, when both users decode the compression index, the region we obtain after running FME is,

$$\begin{aligned} R_k &\leq \min\{I_{k2}, I_{k1} + I_{j3}\}, \\ R_k + R_j &\leq \min\{I_{k1} + I_{j4}, I_{k3} + I_{j3}\}, \\ 2R_k + R_j &\leq I_{k1} + I_{k4} + I_{j3} \end{aligned}$$

where the redundant terms are

$$\begin{aligned} R_1 &\leq I_{11} + I_{23}, \\ R_2 &\leq I_{13} + I_{21}. \end{aligned}$$

We omit the complete proof for this region since it follows the same steps as the previous ones. The conclusion here is that the region  $\mathcal{R}_{CF_0}^c(p)$ , the one without the redundant terms, is larger than  $\mathcal{R}_{CF_0}^o(p)$ , and also,

$$\begin{aligned} \mathcal{R}_{CF_0}^c(p) &\subseteq \mathcal{R}_{CF_0}^o(p) \cup \mathcal{R}_{CF_0}^o(p^*) \cup \mathcal{R}_{CF_1}^o(p^*) \cup \\ &\quad \mathcal{R}_{CF_0}^o(p^{**}) \cup \mathcal{R}_{CF_2}^o(p^{**}). \end{aligned}$$

Therefore, if we maximize over all possible joint distributions we have

$$\mathcal{R}_{CF_0}^c \cup \mathcal{R}_{CF_1}^c \cup \mathcal{R}_{CF_2}^c \cup \mathcal{R}_{CF_3}^c = \mathcal{R}_{CF_0}^o \cup \mathcal{R}_{CF_1}^o \cup \mathcal{R}_{CF_2}^o \cup \mathcal{R}_{CF_3}^o.$$

Since the region  $\mathcal{R}_{CF_3}$  is a special case of  $\mathcal{R}_{CF_0}$  in the maximization, we can eliminate it. The final region without redundant terms is (16) when both destinations decode the compression index, and the region (17) when one of them ignores it.

## APPENDIX G

### PROOF OF PROPOSITION 1 (FULL DF CONSTANT GAP)

The comparison between the full DF inner bound (11) and the outer bound is complex mainly due to the different PDs in each bound and the presence of the binning terms. However, as we see next, we can propose some simplifications to help us calculate the difference between the bounds.

First, let us assume the following set of auxiliary random variables,

$$V_1 = h_{21}X_1 + h_{23}X_3 + Z'_2, \quad (67a)$$

$$V_2 = h_{12}X_2 + Z'_1, \quad (67b)$$

$$V_3 = \frac{h_{23}}{\sqrt{1 + S_{21}}}X_3 + Z''_2 \quad (67c)$$

where  $S_{21} \triangleq |h_{21}|^2 P_1 / N_2$ , and  $Z'_k$  and  $Z''_k$  are independent copies of  $Z_k$ . This choice fulfills the Markov chains in (10). Nonetheless, since it is a particular choice of variables, the region might be smaller than the optimal one.

Second, let us assume that  $X_1$  and  $X_3$  are independent. Then, the binning term becomes upper-bounded regardless of the channel coefficients,

$$I_b = C \left[ \frac{S_{23}}{1 + S_{21} + S_{23}} \right] \leq \frac{1}{2} \text{ bit.}$$

We can reduce the achievable region (11) if we add  $-I_b$  to (11c) and (11i) which render (11d) and (11h) redundant. We further shrink the region by replacing  $-I_b$  with  $-\frac{1}{2}$  which gives us,

$$R_1 \leq I(X_1; Y_3 | X_3 Q) \quad (68a)$$

$$R_1 \leq I(X_1 X_3; Y_1 | V_2 Q) \quad (68b)$$

$$R_2 \leq I(X_2; Y_2 | V_1 V_3 Q) - \frac{1}{2} \quad (68c)$$

$$R_1 + R_2 \leq I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(V_1 X_2 V_3; Y_2 | Q) \quad (68d)$$

$$R_1 + R_2 \leq I(X_1; Y_3 | V_1 X_3 Q) + I(V_1 X_2 V_3; Y_2 | Q) - \frac{1}{2} \quad (68e)$$

$$R_1 + R_2 \leq I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q) \quad (68f)$$

$$R_1 + R_2 \leq I(X_1 V_2 X_3; Y_1 | Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q) - \frac{1}{2} \quad (68g)$$

$$\begin{aligned} 2R_1 + R_2 &\leq I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(X_1 V_2 X_3; Y_1 | Q) \\ &\quad + I(V_1 X_2 V_3; Y_2 | V_2 Q) \end{aligned} \quad (68h)$$

$$\begin{aligned} 2R_1 + R_2 &\leq I(X_1; Y_3 | V_1 X_3 Q) + I(X_1 V_2 X_3; Y_1 | Q) \\ &\quad + I(V_1 X_2 V_3; Y_2 | V_2 Q) - \frac{1}{2} \end{aligned} \quad (68i)$$

$$\begin{aligned} R_1 + 2R_2 &\leq I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q) \\ &\quad + I(V_1 X_2 V_3; Y_2 | Q) \end{aligned} \quad (68j)$$

These bounds look similar to the following subset of the outer bound (5): (5a)–(5g), (5l), (5n), and (5r), which allows us to compare them. However, as the PDs present in the inner and outer bounds are different, we compare the expression of each bound in the Gaussian case since they only depend on the SNRs of the links.

The reduced region (68) for the Gaussian case is,

$$R_1 \leq C[S_{31}] \quad (69a)$$

$$R_1 \leq C[G_2(S_{11} + S_{13})] \quad (69b)$$

$$R_2 \leq C[G_1 S_{22}] - \frac{1}{2} \quad (69c)$$

$$R_1 + R_2 \leq C \left[ G_2 \frac{S_{11} + S_{13} + \delta + S_{11} S_{23} / (1 + S_{21})}{1 + S_{21} + 2S_{23}} \right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2} \log_2 G_1 \quad (69d)$$

$$R_1 + R_2 \leq C \left[ \frac{S_{31}}{1 + S_{21}} \right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2} \log_2 G_1 - \frac{1}{2} \quad (69e)$$

$$R_1 + R_2 \leq C \left[ S_{12} + \frac{S_{11} + S_{13} + \delta + S_{11} S_{23} / (1 + S_{21})}{1 + S_{21} + 2S_{23}} \right] + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + \frac{1}{2} \log_2 G_1 G_2 \quad (69f)$$

$$R_1 + R_2 \leq C[S_{11} + S_{12} + S_{13}] + C \left[ G_1 \frac{S_{22}}{1 + S_{12}} \right] + \frac{1}{2} \log_2 G_2 - \frac{1}{2} \quad (69g)$$

$$2R_1 + R_2 \leq C \left[ G_2 \frac{S_{11} + S_{13} + \delta + S_{11} S_{23} / (1 + S_{21})}{1 + S_{21} + 2S_{23}} \right] + C[S_{11} + S_{12} + S_{13}] + \frac{1}{2} \log_2 G_1 G_2 + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] \quad (69h)$$

$$2R_1 + R_2 \leq C \left[ \frac{S_{31}}{1 + S_{21}} \right] + C[S_{11} + S_{12} + S_{13}] - \frac{1}{2} + C \left[ S_{21} + S_{23} + \frac{S_{22}}{1 + S_{12}} \right] + \frac{1}{2} \log_2 G_1 G_2 \quad (69i)$$

$$R_1 + 2R_2 \leq C \left[ S_{12} + \frac{S_{11} + S_{13} + \delta + S_{11} S_{23} / (1 + S_{21})}{1 + S_{21} + 2S_{23}} \right] + C \left[ G_1 \frac{S_{22}}{1 + S_{12}} \right] + C[S_{21} + S_{22} + S_{23}] + \frac{1}{2} \log_2 G_1 G_2, \quad (69j)$$

where

$$\delta \triangleq \left( \sqrt{S_{11} S_{23}} \pm \sqrt{S_{13} S_{21}} \right)^2, \\ G_1 \triangleq \frac{1 + 2S_{21} + 2S_{23} + S_{21}^2 + 2S_{21} S_{23}}{1 + 3S_{21} + 3S_{23} + 2S_{21}^2 + 4S_{21} S_{23}}, \\ G_2 \triangleq \frac{1 + S_{12}}{1 + 2S_{12}}.$$

To illustrate the procedure for bounding the gap, we show the single-rate gaps in the sequel. Consider,

$$\Delta_{R_1} = (6a) - (69a) \\ = C[S_{11} + S_{31}] - C[S_{31}] \\ = C \left[ \frac{S_{11}}{1 + S_{31}} \right] \leq \frac{1}{2}, \quad (70)$$

where the last inequality is due to  $S_{31} \geq S_{11}$ , otherwise, the

gap would be unbounded. Additionally,

$$\Delta_{R_1} = (6b) - (69b) \\ = C[S_{11} + S_{13}] + \frac{1}{2} - C[G_2(S_{11} + S_{13})] \\ \leq \frac{1}{2} - \frac{1}{2} \log_2 G_2 \leq 1, \quad (71)$$

where the last two inequalities are due to  $\frac{1}{2} \leq G_2 \leq 1$ . For  $R_2$  we have,

$$\Delta_{R_2} = (6c) - (69c) \\ = C[S_{22}] - C[G_1 S_{22}] + \frac{1}{2} \\ \leq \frac{1}{2} - \frac{1}{2} \log_2 G_1 \leq 1, \quad (72)$$

where the last two inequalities are due to  $\frac{1}{2} \leq G_1 \leq 1$ . In summary, if we compare the appropriate pair of bounds and we assume  $S_{31} \geq S_{11}$ , we obtain the following gaps

$$\Delta_{R_1} \leq \frac{1}{2}, \quad \Delta_{R_1 + R_2} \leq 2, \\ \Delta_{R_1} \leq 1, \quad \Delta_{R_1 + R_2} \leq 2, \\ \Delta_{R_2} \leq 1, \quad \Delta_{2R_1 + R_2} \leq 3, \\ \Delta_{R_1 + R_2} \leq 2, \quad \Delta_{2R_1 + R_2} \leq 3, \\ \Delta_{R_1 + R_2} \leq 2, \quad \Delta_{R_1 + 2R_2} \leq \frac{5}{2}.$$

Therefore, the gap between the outer bound and the full DF inner bound, when  $S_{31} \geq S_{11}$ , is 1 bit per real dimension at most.

## APPENDIX H

### PROOF OF PROPOSITION 2 (PARTIAL DF CONSTANT GAP)

The analysis of the gap for the partial DF scheme follows similar steps as for the full DF scheme. We enlarge the set of auxiliary random variables used in Appendix G with

$$U_1 = h_{31} X_1 + Z'_3. \quad (73)$$

Then, we reduce the achievable region using the assumptions of independence between  $X_1$  and  $X_3$  and the upper bound in the binning term, which gives us,

$$R_1 \leq I(U_1; Y_3 | X_3 Q) + I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q), \quad (74a)$$

$$R_1 \leq I(X_1 X_3; Y_1 | V_2 Q), \quad (74b)$$

$$R_2 \leq I(X_2; Y_2 | V_1 V_3 Q) - \frac{1}{2}, \quad (74c)$$

$$R_1 + R_2 \leq I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(V_1 X_2 V_3; Y_2 | Q), \quad (74d)$$

$$R_1 + R_2 \leq I(U_1; Y_3 | V_1 X_3 Q) + I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q) \\ + I(V_1 X_2 V_3; Y_2 | Q) - \frac{1}{2}, \quad (74e)$$

$$R_1 + R_2 \leq I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q), \quad (74f)$$

$$R_1 + R_2 \leq I(U_1; Y_3 | V_1 X_3 Q) + I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) \\ + I(V_1 X_2 V_3; Y_2 | V_2 Q) - \frac{1}{2}, \quad (74g)$$

$$R_1 + R_2 \leq I(X_1 V_2 X_3; Y_1 | Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q) - \frac{1}{2}, \quad (74h)$$

$$R_1 + R_2 \leq I(U_1; Y_3 | X_3 Q) + I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q), \quad (74i)$$

$$2R_1 + R_2 \leq I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(X_1 V_2 X_3; Y_1 | Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q), \quad (74j)$$

$$2R_1 + R_2 \leq I(X_1 X_3; Y_1 | V_1 V_2 V_3 Q) + I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) + I(U_1; Y_3 | X_3 Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q), \quad (74k)$$

$$2R_1 + R_2 \leq I(U_1; Y_3 | V_1 X_3 Q) + I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q) - \frac{1}{2} + I(X_1 V_2 X_3; Y_1 | Q) + I(V_1 X_2 V_3; Y_2 | V_2 Q), \quad (74l)$$

$$R_1 + 2R_2 \leq I(X_1 V_2 X_3; Y_1 | V_1 V_3 Q) + I(X_2; Y_2 | V_1 V_2 V_3 Q) + I(V_1 X_2 V_3; Y_2 | Q), \quad (74m)$$

$$R_1 + 2R_2 \leq I(U_1; Y_3 | V_1 X_3 Q) + I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) - \frac{1}{2} + I(X_2; Y_2 | V_1 V_2 V_3 Q) + I(V_1 X_2 V_3; Y_2 | Q). \quad (74n)$$

We can compare these bounds with a larger subset of the outer bound (5): (5a)–(5i), (5l)–(5n), and (5r)–(5s).

Half of the bounds in (74) are the same as in (68), while the other half –composed by the bounds (74a), (74e), (74g), (74i), (74k), (74l), and (74n)– have the following new terms:

$$I(U_1; Y_3 | X_3 Q) = C[S_{31}] + \frac{1}{2} \log_2 G_{31},$$

$$I(U_1; Y_3 | V_1 X_3 Q) = C \left[ \frac{S_{31}}{1 + S_{21}} \right] + \frac{1}{2} \log_2 G_{32},$$

$$I(X_1; Y_1 | V_1 U_1 V_2 X_3 Q) = C \left[ G_2 \frac{S_{11}}{1 + S_{21} + S_{31}} \right],$$

$$I(X_1 V_2; Y_1 | V_1 U_1 X_3 Q) = C \left[ S_{12} + \frac{S_{11}}{1 + S_{21} + S_{31}} \right] + \frac{1}{2} \log_2 G_2$$

where

$$G_{31} \triangleq \frac{1 + S_{31}}{1 + 2S_{31}}, \quad \text{and} \quad G_{32} \triangleq \frac{1 + S_{21} + S_{31}}{1 + S_{21} + 2S_{31}}.$$

Let us analyze only one of the gaps that change,

$$\begin{aligned} \Delta_{R_1} &= (6a) - (74a) = C[S_{11} + S_{31}] - C[S_{31}] \\ &\quad - \frac{1}{2} \log_2 G_{31} - C \left[ G_2 \frac{S_{11}}{1 + S_{21} + S_{31}} \right] \\ &\leq C \left[ \frac{S_{21}}{1 + S_{31}} \right] - \frac{1}{2} \log_2 G_{31} G_2 \leq \frac{3}{2}, \end{aligned} \quad (75)$$

where the last inequality is due to  $S_{31} \geq S_{21}$ , otherwise, the gap would be unbounded.

The gap between each pair of bounds in the inner and outer bound is,

$$\begin{aligned} \Delta_{R_1} &\leq \frac{3}{2}, & \Delta_{R_1+R_2} &\leq 2, \\ \Delta_{R_1} &\leq 1, & \Delta_{R_1+R_2} &\leq 2, \\ \Delta_{R_2} &\leq 1, & \Delta_{2R_1+R_2} &\leq 3, \\ \Delta_{R_1+R_2} &\leq 2, & \Delta_{2R_1+R_2} &\leq \frac{7}{2}, \\ \Delta_{R_1+R_2} &\leq \frac{5}{2}, & \Delta_{2R_1+R_2} &\leq \frac{7}{2}, \\ \Delta_{R_1+R_2} &\leq 2, & \Delta_{R_1+2R_2} &\leq \frac{5}{2}, \end{aligned}$$

$$\Delta_{R_1+R_2} \leq \frac{5}{2}, \quad \Delta_{R_1+2R_2} \leq 3.$$

In the previous calculations we assumed that  $S_{31} \geq S_{21}$ . Therefore, under this condition, the gap between the outer bound and the partial DF inner bound is 1.5 bits per real dimension at most.

## APPENDIX I

### PROOF OF PROPOSITION 3 (CF CONSTANT GAP)

In this section, we show the constant gap result for the CF inner bound. As with the previous two schemes, we propose some simplifications to help in the analysis which, at the same time, reduce the region. First, we only take the region  $\mathcal{R}_{CF_0}$  (16) into account. This means that we force both end users to decode the compression index when we have already stated in the proof of the scheme that sometimes is better to ignore this message.

Second, the compressed channel observation of the relay is obtained by adding an independent Gaussian noise  $Z \sim \mathcal{N}(0, N)$  to its channel output,

$$\hat{Y}_3 = Y_3 + Z.$$

Third, the random variables used in the scheme have the following structure. Given the independent random variables  $V_1, V_2, X'_1$ , and  $X'_2$ , all distributed according to  $\mathcal{N}(0, 1)$ , we construct  $X_1$  and  $X_2$  as follows:

$$\begin{aligned} X_1 &= \sqrt{\alpha_1 P_1} V_1 + \sqrt{\bar{\alpha}_1 P_1} X'_1, \\ X_2 &= \sqrt{\alpha_2 P_2} V_2 + \sqrt{\bar{\alpha}_2 P_2} X'_2 \end{aligned}$$

where  $\alpha_i \in [0, 1]$  and  $\bar{\alpha}_i \triangleq 1 - \alpha_i$ . Furthermore, inspired by [2] and taking into account the presence of the relay's compressed channel output, we choose the fixed power split strategy

$$\begin{aligned} \bar{\alpha}_1 \left( 1 + S_{21} + \frac{S_{31}}{1 + N} \right) &= 1, \\ \bar{\alpha}_2 (1 + S_{12}) &= 1. \end{aligned}$$

The expression of the bounds (14) in the Gaussian case, where we have assumed  $N_3 = 1$  for simplicity, can be found at the bottom of next page.

We start by calculating the gap for the single rate  $R_1 \leq I_{12a}$  with the bound (5a) from the outer bound:

$$\begin{aligned} \Delta_{R_1} &= I(X_1; Y_1 Y_3 | X_2 X_3 Q) - I(X_1; Y_1 \hat{Y}_3 | V_2 X_3 Q) \\ &\leq \frac{1}{2} \log_2 \{ 1 + S_{11} + S_{31} \} \\ &\quad - \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + S_{11}/2) + S_{31}}{1 + N} \right\} \quad (76a) \\ &= \frac{1}{2} \log_2 \left\{ 1 + \frac{(1 + N)S_{11}/2 + NS_{31}}{(1 + N)(1 + S_{11}/2) + S_{31}} \right\} \\ &\leq \begin{cases} \frac{1}{2} + C \left[ \frac{N}{1 + N} \right] & \text{if } S_{31} < S_{11} \\ \log_2 \frac{3}{2} + C[N] & \text{if } S_{31} \geq S_{11} \end{cases} \quad (76b) \end{aligned}$$

where in (76a) we have reduced the expression of the inner bound by adding  $(1 + N)\bar{\alpha}_2$  in the denominator and then, we apply the fixed power split strategy; and (76b) is obtained by eliminating either  $(1 + N)(1 + S_{11}/2)$  or  $S_{31}$  from the denominator and taking into account that  $S_{31} \leq S_{11}$ .

Next, we compare  $R_1 \leq I_{12b}$  with the bound (5b):

$$\begin{aligned} \Delta_{R_1} &= I(X_1 X_3; Y_1 | X_2 Q) - [I(X_1 X_3; Y_1 | V_2 Q) - I_1] \\ &\leq \frac{1}{2} \log_2 \{1 + S_{11} + S_{13}\} + \frac{1}{2} \\ &\quad - \frac{1}{2} \log_2 \left\{ \frac{N(1 + S_{11} + S_{13})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\} \end{aligned} \quad (77a)$$

$$\begin{aligned} &\leq \frac{1}{2} + \frac{1}{2} \log_2 \left\{ \frac{2(1 + N)}{N} \right\} \\ &= 1 + C \left[ \frac{1}{N} \right] \end{aligned} \quad (77b)$$

where in (77a) we have already reduced the expression of the inner bound by eliminating the term  $\bar{\alpha}_2 S_{12}$ . If  $S_{31} < S_{11}$ , the gap for  $R_1$  is dominated by (77b), since it is always greater than (76b), otherwise, the gap is the maximum of both.

Upper bounds on the gap of single rates and sum-rates can be derived using the expressions from the outer bound (5a)–(5c), (5f)–(5k), (5n)–(5q), and (5s)–(5t), and the assumption  $S_{31} < S_{21}$  is needed for the gap to be bounded. These upper bounds on the gap were analyzed numerically, due to their complexity, and after cumbersome calculations the largest gap comes from the sum-rate:

$$\begin{aligned} \Delta_{R_1+R_2} &\leq \min\{(5h), (5k)\} - [I_{13} + I_{23}] \\ &\leq \max\{(5k) - [I_{13b} + I_{23a}], (5h) - [I_{13b} + I_{23b}]\} \\ &\leq 1 + C \left[ \frac{1}{N} \right] + \max \left\{ C[N] + C \left[ \frac{1 + 2N}{2 + N} \right], 1 + C \left[ \frac{1}{N} \right] \right\}. \end{aligned}$$

The value of  $N$  that minimizes this gap is  $N \approx 1.81$ , with the gap per real dimension being approximately 1.32 bits.

## APPENDIX J

### PROOF OF PROPOSITION 4 (LIMITED RELAYING BENEFIT)

Let us define  $\mathcal{R}_{o'}$  ( $P_1$ ) as the outer bound region composed by the bounds (5a), (5c), (5i)–(5k), (5q), and (5t). This new outer bound is analogous to the outer bound presented by Telatar and Tse [3] with the addition of the *antenna*  $Y_3$ .

If the quality of the source-to-relay link is really low, this extra antenna does not provide much information and thus, both outer bounds should be within a constant gap. Since the gap between Han-Kobayashi's inner bound and Telatar-Tse's outer bound is half a bit, it follows that Han-Kobayashi scheme is within a constant gap to our outer bound under the aforementioned conditions.

We only show one of these gaps here, but all of them can be derived similarly. The expression for (5j) in the Gaussian case, i.e., (6j), is

$$\begin{aligned} (R_1 + R_2)_{IS-IRC} &= I(X_1; Y_1 Y_3 | V_1 X_2 X_3) + I(X_1 X_2; Y_2 Y_3 | X_3) \\ &\leq C \left[ \frac{S_{11} + S_{31}}{1 + S_{21} + S_{31}} \right] + C[S_{21} + S_{22} + S_{31}(1 + S_{22})], \end{aligned} \quad (78)$$

while the analogous bound in Telatar-Tse's outer bound is

$$\begin{aligned} (R_1 + R_2)_{IC} &= I(X_1; Y_1 | V_1 X_2) + I(X_1 X_2; Y_2) \\ &= C \left[ \frac{S_{11}}{1 + S_{21}} \right] + C[S_{21} + S_{22}]. \end{aligned} \quad (79)$$

Then, we calculate the gap between (78) and (79)

$$\begin{aligned} \Delta_{ob} &= (R_1 + R_2)_{IS-IRC} - (R_1 + R_2)_{IC} \\ &= C \left[ \frac{2S_{31}}{1 + S_{11} + S_{21}} \right] - C \left[ \frac{S_{31}}{1 + S_{21}} \right] + C \left[ \frac{S_{31}}{1 + \frac{S_{21}}{1 + S_{22}}} \right] \\ &\leq C \left[ \frac{2S_{31}}{1 + S_{11} + S_{21}} \right] + C \left[ \frac{S_{31}}{1 + \frac{S_{21}}{1 + S_{22}}} \right]. \end{aligned}$$

The gap in this sum-rate can be upper bounded by 1 bit given that  $S_{31} \leq S_{11}$  and  $S_{31} \leq S_{21}/(1 + S_{22})$ . Further analysis of the other bounds assures that the gap between outer bounds is half a bit per rate if  $S_{31} \leq S_{11}/(1 + S_{12})$  and  $S_{31} \leq S_{21}/(1 + S_{22})$  hold. Therefore, the use of the relay can improve the rate by at most 1 bit per real dimension compared to the Han-Kobayashi scheme without the relay.

$$\begin{aligned} I_{11} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + \bar{\alpha}_1 S_{11} + \bar{\alpha}_2 S_{12}) + \bar{\alpha}_1 S_{31}(1 + \bar{\alpha}_2 S_{12})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + \bar{\alpha}_1 S_{11} + \bar{\alpha}_2 S_{12} + S_{13})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\} \right\}, \\ I_{12} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + S_{11} + \bar{\alpha}_2 S_{12}) + S_{31}(1 + \bar{\alpha}_2 S_{12})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + S_{11} + \bar{\alpha}_2 S_{12} + S_{13})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\} \right\}, \\ I_{13} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + \bar{\alpha}_1 S_{11} + S_{12}) + \bar{\alpha}_1 S_{31}(1 + S_{12})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + \bar{\alpha}_1 S_{11} + S_{12} + S_{13})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\} \right\}, \\ I_{14} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + S_{11} + S_{12}) + S_{31}(1 + S_{12})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + S_{11} + S_{12} + S_{13})}{(1 + N)(1 + \bar{\alpha}_2 S_{12})} \right\} \right\}, \\ I_{21} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + \bar{\alpha}_1 S_{21} + \bar{\alpha}_2 S_{22}) + \bar{\alpha}_1 S_{31}(1 + \bar{\alpha}_2 S_{22})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + \bar{\alpha}_1 S_{21} + \bar{\alpha}_2 S_{22} + S_{23})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\} \right\}, \\ I_{22} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + \bar{\alpha}_1 S_{21} + S_{22}) + \bar{\alpha}_1 S_{31}(1 + S_{22})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + \bar{\alpha}_1 S_{21} + S_{22} + S_{23})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\} \right\}, \\ I_{23} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + S_{21} + \bar{\alpha}_2 S_{22}) + S_{31}(1 + \bar{\alpha}_2 S_{22})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + S_{21} + \bar{\alpha}_2 S_{22} + S_{23})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\} \right\}, \\ I_{24} &= \min \left\{ \frac{1}{2} \log_2 \left\{ \frac{(1 + N)(1 + S_{21} + S_{22}) + S_{31}(1 + S_{22})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\}, \frac{1}{2} \log_2 \left\{ \frac{N(1 + S_{21} + S_{22} + S_{23})}{(1 + N)(1 + \bar{\alpha}_1 S_{21}) + \bar{\alpha}_1 S_{31}} \right\} \right\}. \end{aligned}$$

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