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# Uniform ergodicity of the Particle Gibbs sampler

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## Abstract

The particle Gibbs (PG) sampler is a systematic way of using a particle filter within Markov chain Monte Carlo (MCMC). This results in an off-the-shelf Markov kernel on the space of state trajectories, which can be used to simulate from the full joint smoothing distribution for a state space model in an MCMC scheme. We show that the PG Markov kernel is uniformly ergodic under rather general assumptions, that we will carefully review and discuss. In particular, we provide an explicit rate of convergence which reveals that: *(i)* for fixed number of data points, the convergence rate can be made arbitrarily good by increasing the number of particles, and *(ii)* under general mixing assumptions, the convergence rate can be kept constant by increasing the number of particles superlinearly with the number of observations. We illustrate the applicability of our result by studying in detail two common state space models with non-compact state spaces.

*Keywords:* Particle Gibbs, Particle Markov chain Monte Carlo, Conditional sequential Monte Carlo, Particle smoothing, State space models

## 1 Introduction

Statistical inference in general state space hidden Markov models involves computation of the posterior distribution of a set  $X_{t:t'} := [X_t, \dots, X_{t'}]$  of hidden state variables conditionally on a record  $Y_{0:T}$  of observations, which we denote as  $\phi_{t:t'}\langle Y_{0:T} \rangle$ . Of particular interest is the so called *joint smoothing distribution* (JSD)  $\phi_{0:T}\langle Y_{0:T} \rangle$ . Any marginal or fixed-interval smoothing distribution can be obtained from the JSD by marginalization. The JSD can be expressed in closed-form only in very specific cases, principally, when the state space model is linear and Gaussian or when the state space of the hidden Markov chain is a finite set. In the vast majority of cases, nonlinearity or non-Gaussianity render analytic solutions intractable.

This limitation has led to an increase of interest in computational strategies handling more general state and measurement equations. Among these, *sequential Monte Carlo* (SMC) methods play a central role. SMC methods—in which the *sequential importance sampling* and *sampling importance resampling* methods proposed by Handschin and Mayne (1969) and Rubin

(1987), respectively, are combined—refer to a class of algorithms approximating a sequence of probability distributions, defined on a sequence of probability spaces. This is done by updating recursively a set of random *particles* with associated nonnegative importance weights. The SMC methodology has emerged as a key tool for approximating JSD flows in general state space models; see Del Moral (2004); Del Moral and Doucet (2009); Doucet and Johansen (2011) for general introductions as well as applications and theoretical results for SMC methods.

However, a well known problem with SMC methods is that the particle approximation of any marginal smoothing distribution  $\phi_{t,t}(Y_{0:T})$  becomes inaccurate for  $t \ll T$ . The reason is that the particle trajectories degenerate gradually as the interacting particle system evolves (Godsill et al., 2004; Fearnhead et al., 2010). To address this problem, several methods have been proposed; see Lindsten and Schön (2013) and the references therein. Among these methods, the recently introduced particle Markov chain Monte Carlo (PMCMC) framework, proposed in the seminal paper by Andrieu et al. (2010), plays a prominent role. PMCMC samplers make use of SMC (or variants thereof) to construct efficient, high-dimensional MCMC kernels which are reversible with respect to the JSD. These methods can then be used as components of more general sampling schemes relying on Markov kernels, for instance enabling joint state and parameter inference in general state space models. We will not discuss such composite sampling schemes in this paper, but instead focus on one of the PMCMC kernels that can be used to simulate from the JSD.

Coupling SMC and MCMC is very useful since the distribution of the state sequence given the stream of observations is generally both high-dimensional and strongly dependent, rendering the design of alternative MCMC procedures, such as single-state Gibbs samplers and Metropolis-Hastings samplers, problematic. PMCMC has already found many applications in areas such as hydrology (Vrugt et al., 2013), finance (Pitt et al., 2012), systems biology (Golightly and Wilkinson, 2011), and epidemiology (Rasmussen et al., 2011), to mention a few. Several methodological developments of the framework have also been made; see e.g. Whiteley et al. (2010); Lindsten et al. (2012); Chopin and Singh (2013); Pitt et al. (2012).

PMCMC algorithms can, broadly speaking, be grouped into two classes of methods: those based on particle independent Metropolis-Hastings (PIMH) kernels and those based on particle Gibbs (PG) kernels. The two classes of kernels are motivated in different ways and they have quite different properties. The former class, PIMH, exploits the fact that the SMC method defines an unbiased estimator of the likelihood, which is used in place of the intractable likelihood in the MH acceptance probability. This method can thus be viewed as a special case of the pseudo-marginal method introduced by Beaumont (2003); Andrieu and Roberts (2009) and later analyzed by Andrieu and Vihola (2012); Lee and Latuszynski (2012). The latter class, PG, on the other hand relies on conditioning the underlying SMC sampler on a reference trajectory to enforce the correct limiting distribution of the kernel; see Section 3. This algorithm can be interpreted as a Gibbs sampler for an extended model where the random variables generated by the SMC sampler are treated as auxiliary variables.

One of the main practical issues with PMCMC algorithms is the choice of the number,  $N$ , of particles. Using fewer particles will result in faster computations at each iteration, but can at the same time result in slower mixing of the resulting Markov kernel. For a fixed computational budget, there is a trade-off between taking the number of particles  $N$  large to get a faster mixing kernel, and to run many iterations of the MCMC sampler. Andrieu and Roberts (2009); Andrieu and Vihola (2012); Lee and Latuszynski (2012) investigate the rate of convergence of the pseudo-marginal method and characterize the approximation of the marginal algorithm by the pseudo-marginal algorithm in terms of the variability of their respective ergodic averages. Doucet et al. (2012) and Pitt et al. (2012) conclude, using partially heuristic arguments, that it is close to optimal to let  $N$  scale at least linearly with  $T$ .

The theoretical properties of the PG kernel, however, are not as well understood. Andrieu et al. (2010) establish under weak conditions that the PG kernel is  $\phi$ -irreducible and aperiodic for any  $N \geq 2$  (see Meyn and Tweedie (2009) for definitions). However, this does not provide a control for the rate of convergence of the iterates of the PG kernel to stationarity. In this work, we establish that the PG kernel is, under mild assumptions, uniformly ergodic. This interesting property has already been established in an earlier work by Chopin and Singh (2013), but we give here a more straightforward proof under weaker conditions, which in addition provides an explicit lower bound for the convergence rate.

During the preparation of this manuscript, a preprint was made available by Andrieu et al. (2013), who, independently, have found similar results as presented here. Indeed, they establish basically the same lower bound on the minorizing constant for the PG kernel (which they refer to as the iterated conditional SMC kernel), though using a different proof technique based on a “doubly conditional” SMC algorithm. There are, however, several differences between these two contributions. We focus in particular on analyzing the minorizing constant under mixing conditions for the state space model which hold very generally, even if the state space is not compact (see Section 4.3). We then study how the number of particles  $N$  should be increased with the number of observations  $T$ . We show that under weak assumptions, it suffices to increase the number of particles  $N$  as  $T^\delta$  where  $\delta \geq 1$  can be determined explicitly. This is in contrast with Andrieu et al. (2013) who, effectively, assume a compact state space; see Remark 3 and Section 4.2. On the other hand, Andrieu et al. (2013) study necessary (i.e., not only sufficient) conditions for uniform ergodicity and translate the convergence results for the PG kernel to a composite MCMC scheme for simulating both states and parameters of a state space model. Given these differences, we believe that the two contributions complement each other.

This paper is organized as follows: In Section 2 we introduce our notation, and in Section 3 we review the PG sampler and formally define the PG Markov kernel. In Section 4 we state the main results, starting with a minorization condition for the PG kernel followed by mixing conditions that allow for time uniform control of the convergence rate. In Section 5 we study, in detail, two commonly used state space models (with non-compact state spaces) to illustrate how the conditions of our results can be verified in practice. The proofs of the main theorems are postponed to Sections 6 and 7.

## 2 Notations and problem statement

Let  $(\mathsf{X}, \mathcal{X})$  and  $(\mathsf{Y}, \mathcal{Y})$  be two measurable spaces and let  $\mathcal{P}(\mathsf{X})$  be the set of all probability measures on  $(\mathsf{X}, \mathcal{X})$ . Let  $M$  be a kernel on  $(\mathsf{X}, \mathcal{X})$  and  $G$  a kernel on  $(\mathsf{X}, \mathcal{Y})$ . Assume that for all  $x \in \mathsf{X}$ ,  $G(x, \cdot)$  is dominated by some common nonnegative measure  $\kappa$  on  $(\mathsf{Y}, \mathcal{Y})$  and denote by  $g(x, \cdot)$  its Radon-Nikodym derivative, i.e., for all  $(x, y) \in \mathsf{X} \times \mathsf{Y}$ ,

$$g(x, y) = \frac{dG(x, \cdot)}{d\kappa(\cdot)}(y) .$$

Let  $\{(X_t, Y_t), t \in \mathbb{N}\}$  be a hidden Markov chain associated to the pair  $(M, G)$ . That is,  $\{(X_t, Y_t), t \in \mathbb{N}\}$  is a Markov chain with transition kernel defined by: for all  $(x, y) \in \mathsf{X} \times \mathsf{Y}$  and all  $C \in \mathcal{X} \otimes \mathcal{Y}$ ,

$$((x, y), C) \mapsto \iint_C M(x, dx') G(x', dy') .$$

The sequence  $\{X_t, t \in \mathbb{N}\}$  is usually not observed and inference should be carry out on the basis of the observations  $\{Y_t, t \in \mathbb{N}\}$  only. With  $\mu \in \mathcal{P}(\mathsf{X})$  being the initial distribution of the hidden

state process, for all  $t \geq 0$ , denote by

$$y_{0:t} \mapsto p_\mu(y_{0:t}) := \int \mu(dx_0) g(x_0, y_0) \prod_{s=1}^t M(x_{s-1}, dx_s) g(x_s, y_s),$$

the density of the observations  $Y_{0:t}$  with respect to  $\kappa^{\otimes(t+1)}$ . In what follows, we set, by abuse of notation, for all  $x \in \mathbf{X}$ ,

$$p_x(y_{0:t}) = p_{\delta_x}(y_{0:t}), \quad (1)$$

where  $\delta_x$  is the Dirac measure at  $x$ .

For all  $y \in \mathbf{Y}$ , define the (unnormalized) kernel  $Q\langle y \rangle$  on  $(\mathbf{X}, \mathcal{X})$  by

$$Q\langle y \rangle(x, A) = \int M(x, dx') g(x', y) \mathbb{1}_A(x'), \quad (2)$$

and for all  $s \leq t$  and all  $y_{s:t} \in \mathbf{Y}^{t-s+1}$ , define the kernel  $Q\langle y_{s:t} \rangle$  on  $(\mathbf{X}, \mathcal{X})$  by

$$Q\langle y_{s:t} \rangle(x, A) = Q\langle y_s \rangle Q\langle y_{s+1} \rangle \dots Q\langle y_t \rangle(x, A). \quad (3)$$

In what follows, we set by convention  $Q\langle y_{s:t} \rangle(x, A) = 1$  for all  $s > t$ . With these notations,  $p_\mu(y_{0:t}) = \mu Q\langle y_{0:t} \rangle \mathbf{1}$  where  $\mathbf{1}$  is the constant function,  $\mathbf{1}(x) = 1$  for all  $x \in \mathbf{X}$ . For all  $\mu \in \mathcal{P}(\mathbf{X})$  and for all  $0 \leq s \leq t$ , denote

$$p_\mu(y_{s:t} | y_{0:s-1}) := \begin{cases} p_\mu(y_{0:t}) / p_\mu(y_{0:s-1}), & \text{if } p_\mu(y_{0:s-1}) \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

with the convention  $p_\mu(y_{0:t} | y_{0:-1}) = p_\mu(y_{0:t})$ .

A quantity of central interest is the JSD, given by

$$\phi_{\mu,0:t}\langle y_{0:t} \rangle(D) := \frac{1}{p_\mu(y_{0:t})} \int \mu(dx_0) g(x_0, y_0) \prod_{s=1}^t M(x_{s-1}, dx_s) g(x_s, y_s) \mathbb{1}_D(x_{0:t}), \quad (4)$$

for all  $D \in \mathcal{X}^{\otimes(t+1)}$ . With  $T$  being some final time point, the PG sampler (reviewed in the subsequent section) defines a Markov kernel which is reversible with respect to  $\phi_{\mu,0:T}\langle y_{0:T} \rangle$ . Samples drawn from the PG kernel can thus be used to draw inference about the states (and/or parameters) of the state space model.

### 3 The particle Gibbs sampler

Consider first an SMC sampler targeting the sequence of JSDs defined in (4). The SMC sampler approximates  $\phi_{\mu,0:t}\langle Y_{0:t} \rangle$  by a collection of weighted samples  $\{(X_{0:t}^i, \omega_t^i)\}_{i=1}^N$ , in the sense that

$$\phi_{\mu,0:t}^N\langle Y_{0:T} \rangle(h) := \sum_{i=1}^N \frac{\omega_t^i}{\sum_{\ell=1}^N \omega_t^\ell} h(X_{0:t}^i)$$

is an estimator of  $\phi_{\mu,0:t}\langle Y_{0:t} \rangle(h)$  for a measurable function  $h : \mathbf{X}^{t+1} \rightarrow \mathbb{R}$ . These weighted samples can be generated in several different ways, see e.g. Doucet et al. (2000); Del Moral (2004); Doucet and Johansen (2011); Cappé et al. (2005) and the references therein. Here we review a basic method, though it should be noted that the PG sampler can be generalized to more advanced procedures, see Andrieu et al. (2010); Chopin and Singh (2013).

Initially,  $\phi_{\mu,0:0}\langle Y_0 \rangle$  is approximated by importance sampling. That is, we simulate independently  $\{X_0^i\}_{i=1}^N$  from a proposal distribution:  $X_0^i \sim r_0\langle Y_0 \rangle(\cdot)$ . The samples, commonly referred to as *particles*, are then assigned importance weights,

$$\omega_0^i = w_0\langle Y_0 \rangle(X_0^i), \quad (5)$$

where  $w_0\langle Y_0 \rangle(x) = g(x, Y_0) \frac{d\mu}{dr_0\langle Y_0 \rangle}(x)$ , provided that  $r_0\langle Y_0 \rangle$  is such that  $\mu \ll r_0\langle Y_0 \rangle$ .

We proceed inductively. Denote by  $\mathcal{F}_t^N$  the filtration generated by the particles and weights up to the current time instant  $t$ :

$$\mathcal{F}_t^N := \sigma(\{(X_{0:s}^i, \omega_s^i)\}_{i=1}^N, 0 \leq s \leq t). \quad (6)$$

Assume that we have at hand a weighted sample  $\{(X_{0:t-1}^i, \omega_{t-1}^i)\}_{i=1}^N$  approximating the JSD  $\phi_{\mu,0:t-1}\langle Y_{0:t-1} \rangle$  at time  $t-1$ . This weighted sample is then propagated sequentially *forward in time*. This is done by sampling, conditionally independently given the particle history  $\mathcal{F}_{t-1}^N$ , for each particle  $i \in \{1, \dots, N\}$  an *ancestor index*  $A_t^i$  with probability

$$\mathbb{P}(A_t^i = j \mid \mathcal{F}_{t-1}^N) = \frac{\omega_{t-1}^j}{\sum_{\ell=1}^N \omega_{t-1}^\ell}, \quad j \in \{1, \dots, N\}, \quad (7)$$

and then by sampling a new particle position from the proposal kernel  $R\langle Y_t \rangle$ :

$$X_t^i \sim R\langle Y_t \rangle(X_{t-1}^{A_t^i}, \cdot). \quad (8)$$

The particle trajectories (i.e., the ancestral paths of the particles  $X_t^i$ ,  $i \in \{1, \dots, N\}$ ) are constructed sequentially by associating the current particle  $X_t^i$  with the particle trajectory of its ancestor:

$$X_{0:t}^i := (X_{0:t-1}^{A_t^i}, X_t^i). \quad (9)$$

Finally, similarly to (5) the particles are assigned importance weights given by

$$\omega_t^i = w\langle Y_t \rangle(X_{t-1}^{A_t^i}, X_t^i) := \frac{dQ\langle Y_t \rangle(X_{t-1}^{A_t^i}, \cdot)}{dR\langle Y_t \rangle(X_{t-1}^{A_t^i}, \cdot)}(X_t^i), \quad (10)$$

where  $Q\langle y \rangle$  is defined in (2) and, as before, it is assumed that  $Q\langle y \rangle(x, \cdot) \ll R\langle y \rangle(x, \cdot)$ . This results in a weighted particle system  $\{(X_{0:t}^i, \omega_t^i)\}_{i=1}^N$  targeting  $\phi_{\mu,0:t}\langle Y_{0:t} \rangle$ , completing the induction. Two classical choices for the proposal kernel  $R\langle y \rangle$  are:

$$R\langle y \rangle(x, dx') = \begin{cases} M(x, dx') & \text{bootstrap filter,} \\ \frac{M(x, dx')g(x', y)}{\int M(x, dx')g(x', y)} & \text{fully-adapted filter.} \end{cases} \quad (11)$$

Assume now that  $T$  is some final time point and that we are interested in simulating from the JSD  $\phi_{\mu,0:T}\langle Y_{0:T} \rangle$  using an MCMC procedure. For that purpose, it is required to define a Harris positive recurrent Markov kernel on the path space  $(\mathcal{X}^{T+1}, \mathcal{X}^{\otimes(T+1)})$  having the JSD  $\phi_{\mu,0:T}\langle Y_{0:T} \rangle$  as its unique invariant distribution. The PG sampler accomplishes this by making use of SMC. From an algorithmic point of view, the difference between PG and a standard SMC sampler is that in the former, one particle trajectory, denoted as  $x'_{0:T} = (x'_0, \dots, x'_T) \in \mathcal{X}^{T+1}$ , is specified *a priori*. This trajectory is used as a reference for the PG sampler, as discussed below.

The reference trajectory is taken into account by simulating only  $N-1$  particles in the usual way. The  $N$ th particle is then set deterministically according to the reference. At the

initialization, we thus simulate independently  $\{X_0^i\}_{i=1}^{N-1}$  with  $X_0^i \sim r_0 \langle Y_0 \rangle (\cdot)$  and set  $X_0^N = x'_0$ . We then compute importance weights for all particles,  $i = 1, \dots, N$ , according to (5).

Analogously, at any consecutive time point  $t$ , we sample the first  $N-1$  particles  $\{(A_t^i, X_t^i)\}_{i=1}^{N-1}$  conditionally independently given  $\mathcal{F}_{t-1}^N$  according to (7)–(8). Note that these particles will depend on the reference trajectory through the resampling step (7). The  $N$ th particle and its ancestor index are then set deterministically:  $X_t^N = x'_t$  and  $A_t^N = N$ . Finally, importance weights are then computed for all the particles according to (10). Note that, by construction, the  $N$ th particle trajectory will coincide with the reference trajectory for all  $t$ ,  $X_{0:t}^N = x'_{0:t}$ .

After a complete pass of the above procedure, a trajectory  $X_{0:T}^*$  is sampled from among the particle trajectories at time  $T$  (see (9)), with probability proportional to the importance weight  $\omega_T^i$ ,  $i \in \{1, \dots, N\}$ , i.e.

$$\mathbb{P}(X_{0:T}^* = X_{0:T}^i | \mathcal{F}_T^N) = \frac{\omega_T^i}{\sum_{\ell=1}^N \omega_T^\ell}, \quad i \in \{1, \dots, N\}. \quad (12)$$

This procedure thus associates each trajectory  $x'_{0:T} \in \mathbf{X}^{T+1}$  to a probability distribution on  $(\mathbf{X}^{T+1}, \mathcal{X}^{\otimes(T+1)})$ , defining a Markov kernel on  $(\mathbf{X}^{T+1}, \mathcal{X}^{\otimes(T+1)})$ . More specifically, this kernel is given by

$$P_{T,N}(x'_{0:T}, D) := \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_T^i \mathbb{1}_D(X_{0:T}^i)}{\sum_{i=1}^N \omega_T^i} \right], \quad (13)$$

for  $(x'_{0:T}, D) \in \mathbf{X}^{T+1} \times \mathcal{X}^{\otimes(T+1)}$ , where  $\mathbb{E}$  refers to expectation with respect to the random variables generated by the PG algorithm. We refer to  $P_{T,N}$  as the PG kernel.

As shown by Andrieu et al. (2010), the conditioning on a reference trajectory implies that the PG kernel leaves the JSD invariant:

$$\phi_{\mu,0:T} \langle Y_{0:T} \rangle (D) = \int P_{T,N}(x'_{0:T}, D) \phi_{\mu,0:T} \langle Y_{0:T} \rangle (dx'_{0:T}), \quad \forall D \in \mathcal{X}^{(T+1)}.$$

Quite remarkably, this invariance property holds for any  $N \geq 1$ .

Empirically, it has been found that the mixing of the PG kernel can be improved significantly by updating the ancestor indices  $A_t^N$  for  $t \in \{1, \dots, T\}$ , either as part of the forward recursion (Lindsten et al., 2012) or in a separate backward recursion (Whiteley et al., 2010). We shall not specifically analyze these modified PG algorithms in this work, although our uniform ergodicity result apply straightforwardly to these algorithms as well.

## 4 Main result

In this section we state the main results. First, in Section 4.1, we give a minorization condition for the PG kernel. Following this we discuss how to increase the number of particles  $N = N_T$  as a function of the number of observations  $T$  in order to obtain a non-degenerate lower-bound. We consider first a strong mixing condition and then a much weaker moment assumption in Section 4.2 and Section 4.3, respectively.

### 4.1 Minorization condition

Define the sequence of nonnegative random variables  $\{B_{t,T}\}_{t=0}^T$  by

$$B_{t,T} = \sup_{0 \leq \ell \leq T-t} \frac{|w \langle Y_t \rangle|_\infty |Q \langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty}{p_\mu(Y_{t:t+\ell} | Y_{0:t-1})} \quad (14)$$

where, by convention,  $|Q\langle Y_{t+1:t}\rangle \mathbf{1}|_\infty = 1$ .

**Theorem 1.** For all  $x'_{0:T} \in \mathcal{X}^{T+1}$  and  $D \in \mathcal{X}^{\otimes(T+1)}$ ,

$$P_{T,N}(x'_{0:T}, D) \geq \epsilon_{T,N} \phi_{\mu,0:T}\langle Y_{0:T}\rangle(D), \quad (15)$$

where

$$\epsilon_{T,N} = \prod_{t=0}^T \frac{N-1}{2B_{t,T} + N-2}. \quad (16)$$

*Proof.* The proof is postponed to Section 6. However, to provide some intuition for the result, the main ideas of the proof are outlined below.

Using the representation of the PG kernel from (13) we can write

$$P_{T,N}(x'_{0:T}, D) \geq (N-1) \mathbb{E} \left[ \frac{\omega_T^1 \mathbb{1}_D(X_{0:T}^1)}{\sum_{i=1}^N \omega_T^i} \right] \geq (N-1) \mathbb{E} \left[ \mathbb{E} \left[ \frac{\omega_T^1 \mathbb{1}_D(X_{0:T}^1)}{2|w\langle Y_t \rangle|_\infty + \sum_{i=2}^{N-1} \omega_T^i} \middle| \mathcal{F}_{T-1}^N \right] \right],$$

where, for the first inequality, we have simply discarded the  $N$ th term (corresponding to the reference particle) and used the fact that the  $N-1$  weighted particles  $\{(X_{0:T}^i, \omega_T^i)\}_{i=1}^{N-1}$  are equally distributed. For the second inequality, we bound the first and the last term of the sum in the denominator by  $|w\langle Y_t \rangle|_\infty$ . This has the effect that the random variables entering the numerator and the denominator of the expression are conditionally independent given  $\mathcal{F}_{T-1}^N$ . By convexity of  $x \mapsto 1/x$  and using Jensen's inequality we therefore obtain the bound

$$P_{T,N}(x'_{0:T}, D) \geq (N-1) \mathbb{E} \left[ \frac{\mathbb{E} [\omega_T^1 \mathbb{1}_D(X_{0:T}^1) | \mathcal{F}_{T-1}^N]}{2|w\langle Y_t \rangle|_\infty + (N-2) \mathbb{E} [\omega_T^2 | \mathcal{F}_{T-1}^N]} \right].$$

The inner conditional expectations can be computed explicitly. Principally, the result follows by repeating this procedure for time  $T-1$ , then for  $T-2$ , etc.  $\square$

**Corollary 2.** Assume that  $g(x, y) > 0$  for all  $(x, y) \in \mathcal{X} \times \mathcal{Y}$  and  $|w\langle y \rangle|_\infty < \infty$  for all  $y \in \mathcal{Y}$ . Then, for fixed  $T$ ,

$$\epsilon_{T,N} \geq 1 + \frac{1}{N-1} \sum_{t=0}^T (1 - 2B_{t,T}) + O_{\mathbb{P}}(N^{-2}),$$

and  $\lim_{N \rightarrow \infty} \epsilon_{T,N} = 1$ .

*Proof.* From the definition (16) we have

$$\epsilon_{T,N} = \exp \left\{ - \sum_{t=0}^T \ln \left( 1 + \frac{2B_{t,T} - 1}{N-1} \right) \right\} \geq \exp \left\{ \frac{1}{N-1} \sum_{t=0}^T (1 - 2B_{t,T}) \right\}.$$

For a fixed  $T$ , we thus obtain the result provided that  $B_{t,T} < \infty$  for all  $t \in \{0, \dots, T\}$ . However, the positivity of  $g$  implies that  $p_\mu(Y_{t:t+\ell} | Y_{0:t-1}) > 0$  for all  $\ell \geq 0$ , and since  $|w\langle y \rangle|_\infty < \infty$  for all  $y \in \mathcal{Y}$ , it can be easily checked that

$$|Q\langle Y_{t+1:t+\ell}\rangle \mathbf{1}|_\infty \leq \prod_{s=t+1}^{t+\ell} |w\langle Y_s \rangle|_\infty < \infty,$$

which immediately implies that  $B_{t,T} < \infty$  for all  $t \in \{0, \dots, T\}$ .  $\square$



**Remark 3.** The minorization condition of Theorem 1 is similar to Proposition 6 by Andrieu et al. (2013). However, they express the minorizing constant in terms of the expectation of a likelihood estimator with respect to the law of a “doubly conditional SMC” algorithm. They do not pursue an analysis of the effect on the minorization condition by the forgetting of the initial condition of the state space model. To obtain an explicit rate of convergence they assume, in our notation, that the triangular array of random variables  $\{B_{t,T}\}_{0 \leq t \leq T}$  is uniformly bounded for  $T \geq 0$ . This is the case, basically, only when the model satisfies strong mixing conditions, as we discuss in the subsequent section. Indeed, Andrieu et al. (2013, Proposition 14 and Lemma 17) is the same as our Proposition 5.

## 4.2 Strong mixing condition

We first assume a strong mixing condition for the kernel  $M$ :

(S-1) There exist positive constants  $(\sigma_-, \sigma_+)$ , a nonnegative measure  $\gamma$  and an integer  $m \in \mathbb{N}$  such that for all  $x \in \mathsf{X}$ ,

$$\sigma_- \gamma(dx') \leq M^m(x, dx') \leq \sigma_+ \gamma(dx') .$$

This condition has been introduced by Del Moral and Guionnet (1999) to establish the uniform-in-time convergence of the particle filter. This condition, which is stronger than the Doeblin condition, typically requires that the state space is compact. It is overly restrictive but is often used in the analysis of state space models because it implies a form of uniform forgetting of the initial condition of the filter, which is key to obtaining long-term stability of the particle filter.

**Proposition 4.** Assume that (S-1) holds with  $m = 1$  and that the proposal kernel is fully-adapted as defined in (11). Then, taking  $N_T \sim \lambda T$  for some  $\lambda > 0$ , we have

$$\liminf_{T \rightarrow \infty} \epsilon_{T, N_T} \geq \exp\left(\frac{1 - 2(\sigma_+/\sigma_-)^2}{\lambda}\right) > 0, \quad \mathbb{P}\text{-a.s.}$$

*Proof.* First, note that for all  $\ell \geq 1$ ,

$$|Q\langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty \leq \sigma_+ \int \gamma(dx_{t+1}) g(x_{t+1}, Y_{t+1}) Q\langle Y_{t+2:t+\ell} \rangle \mathbf{1}(x_{t+1}) \quad (17)$$

and

$$p_\mu(Y_{t:t+\ell} | Y_{0:t-1}) \geq \sigma_-^2 \int \gamma(dx_t) g(x_t, Y_t) \int \gamma(dx_{t+1}) g(x_{t+1}, Y_{t+1}) Q\langle Y_{t+2:t+\ell} \rangle \mathbf{1}(x_{t+1}) . \quad (18)$$

Now, in the fully-adapted case, we have:

$$R\langle y \rangle(x, dx') = \frac{M(x, dx') g(x', y)}{\int M(x, du) g(u, y)} ,$$

so that by the definition of  $w\langle y \rangle$ ,

$$|w\langle Y_t \rangle|_\infty = \sup_{x \in \mathsf{X}} \left| \int M(x, dx_t) g(x, Y_t) \right| \leq \sigma_+ \int \gamma(dx_t) g(x_t, Y_t) .$$

Combining this equality with (17) and (18) yields:

$$B_{t,T} = \sup_{0 \leq \ell \leq T-t} \frac{|w\langle Y_t \rangle|_\infty |Q\langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty}{p_\mu(Y_{t:t+\ell} | Y_{0:t-1})} \leq \left(\frac{\sigma_+}{\sigma_-}\right)^2 .$$

By the definition (16), we then obtain:

$$\epsilon_{T,N} \geq \prod_{t=0}^T \frac{N-1}{2B_{t,T} + N - 2} \geq \left( \frac{N-1}{N-2 + 2(\sigma_+/\sigma_-)^2} \right)^{T+1}.$$

Finally, letting  $N_T \sim \lambda T$ , we obtain

$$\liminf_{T \rightarrow \infty} \epsilon_{T,N_T} \geq \exp\left(\frac{1 - 2(\sigma_+/\sigma_-)^2}{\lambda}\right) > 0, \quad \mathbb{P}\text{-a.s.}$$

□

It is worthwhile to stress that Proposition 4 holds whatever the distribution of the observation process,  $\{Y_t, t \in \mathbb{N}\}$  is. This is a consequence of the strong mixing condition (S-1) which provides a simple result, but at the expense of an assumption which is rarely met in practice. If instead of the fully adapted case, we consider the bootstrap filter (see (11)), we may also obtain a uniform-in-time bound. However, this requires an even stronger assumption of the existence of a lower and an upper bound for the observation likelihood.

(S-2) There exists a positive constant  $\delta$ , such that for all  $y \in \mathcal{Y}$ ,

$$1 \leq \frac{\sup_{x \in \mathcal{X}} g(x, y)}{\inf_{x \in \mathcal{X}} g(x, y)} \leq \delta.$$

**Proposition 5.** *Assume that (S-1)-(S-2) hold and that the bootstrap proposal is used:  $R\langle y \rangle(x, \cdot) = M(x, \cdot)$ . Then, taking  $N_T \sim \lambda T$  for some  $\lambda > 0$ , we have*

$$\liminf_{T \rightarrow \infty} \epsilon_{T,N_T} \geq \exp\left(\frac{1 - 2\delta^m \sigma_+/\sigma_-}{\lambda}\right) > 0, \quad \mathbb{P}\text{-a.s.},$$

where  $m$  is defined in (S-1).

*Proof.* For the bootstrap filter,  $w\langle y \rangle(x, x') = g(x', y)$ . Therefore,  $|w\langle y \rangle|_\infty = \sup_{x \in \mathcal{X}} g(x, y)$ . On the other hand, for  $\ell \geq m$ ,

$$|Q\langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty \leq \sigma_+ \left( \prod_{s=t+1}^{t+m-1} \sup_{x \in \mathcal{X}} g(x, Y_s) \right) \int \gamma(dx_{t+m}) g(x_{t+m}, Y_{t+m}) Q\langle Y_{t+m+1:t+\ell} \rangle \mathbf{1}(x_{t+m}), \quad (19)$$

and

$$p_\mu(Y_{t:t+\ell} | Y_{0:t-1}) \geq \sigma_- \left( \prod_{s=t}^{t+m-1} \inf_{x \in \mathcal{X}} g(x, Y_s) \right) \int \gamma(dx_{t+m}) g(x_{t+m}, Y_{t+m}) Q\langle Y_{t+m+1:t+\ell} \rangle \mathbf{1}(x_{t+m}). \quad (20)$$

Combining (19) and (20) yields

$$B_{t,T} \leq \delta^m \frac{\sigma_+}{\sigma_-}. \quad (21)$$

The result follows as in the proof of Proposition 4. □

### 4.3 Moment assumption

Under the strong mixing condition (S-1) and the even more restrictive (S-2), we obtained non-degenerate uniform convergence bounds when the number of trajectories  $N_T$  depends linearly on the number of observations  $T$ . However, these conditions are very restrictive and hardly ever satisfied when the state space is non-compact. We now turn to the analysis of the minorization condition under a much weaker moment assumption. However, when the strong mixing assumption is relaxed, we are no longer able to obtain bounds that hold uniformly with respect to the observation sequence. Instead, we will take a probabilistic approach. In Theorem 6 below, we show that the minorizing constant will be bounded away from zero, in probability, provided that  $N_T$  is a power of  $T$ . Moreover, the result presented in this section is not restricted to the fully-adapted or the bootstrap PG kernel and may be obtained for virtually any proposal kernel.

This result holds with respect to the law of the observation process  $\{Y_t, t \in \mathbb{N}\}$ . It is therefore of interest to carry out the analysis for a parametric family of state space models  $\{(M^\theta, G^\theta), \theta \in \Theta\}$ , where  $\Theta$  is a compact subset of a Euclidean space. Informally, this allows us to analyse the ergodicity of the PG kernel, even when the algorithm is executed using a misspecified model. We consider a sequence of parameters  $\{\theta_T, T \in \mathbb{N}\}$  that become increasingly close to some “true” parameter  $\theta_*$  (in a sense that will be made precise in Theorem 6 below), converging at a rate  $1/\sqrt{T}$ . The rationale for this assumption is that we are considering the large  $T$  regime and we can therefore expect  $\theta_T$  to be close to  $\theta_*$ . We discuss this further in Remark 7 below. Note that, for a fixed observation sequence  $Y_{0:T}$  (with finite  $T$ ) we can instead appeal to Corollary 2.

(A-1) For all  $\theta \in \Theta$ , the kernel  $M^\theta$  has a unique stationary distribution denoted as  $\pi^\theta$ .

In what follows, for  $\theta \in \Theta$  we let  $\mathbb{E}_\mu^\theta$  and  $\mathbb{P}_\mu^\theta$  refer to the expectation and probability, respectively, induced on  $((X \times Y)^\mathbb{N}, (\mathcal{X} \otimes \mathcal{Y})^{\otimes \mathbb{N}})$  by a Markov chain  $\{(X_t, Y_t), t \in \mathbb{N}\}$  evolving according to the state space model  $(M^\theta, G^\theta)$  starting with  $X_0 \sim \mu$ . For simplicity, we write  $\bar{\mathbb{E}}^\theta = \mathbb{E}_{\pi^\theta}^\theta$  and  $\bar{\mathbb{P}}^\theta = \mathbb{P}_{\pi^\theta}^\theta$ . For  $1 \leq s \leq t$ , we write

$$p_{\mu,s}^\theta(y_{s:t}) = \int p_\mu^\theta(y_{0:t}) \kappa^{\otimes s}(dy_{0:s-1})$$

for the marginal probability density function of  $Y_{s:t}$  with respect to  $\kappa^{\otimes(t-s+1)}$ .

Define for all  $(t, \ell) \in \mathbb{N} \times \mathbb{N}^*$ ,

$$\tilde{B}_t^\theta \langle Y_{t:t+\ell} \rangle := \frac{|w^\theta \langle Y_t \rangle|_\infty |Q^\theta \langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty}{p_{\mu,t}^\theta(Y_{t:t+\ell})}, \quad (22)$$

$$\tilde{C}_t^\theta \langle Y_{t:t+\ell} \rangle := \frac{|w^\theta \langle Y_t \rangle|_\infty \int \lambda(dx_{t+1}) g^\theta(x_{t+1}, Y_{t+1}) Q^\theta \langle Y_{t+2:t+\ell} \rangle \mathbf{1}(x_{t+1})}{p_{\mu,t}^\theta(Y_{t:t+\ell})}, \quad (23)$$

with, by convention,

$$\tilde{B}_t^\theta \langle Y_t \rangle = |w^\theta \langle Y_t \rangle|_\infty / p_{\mu,t}^\theta(Y_t), \quad \tilde{C}_t^\theta \langle Y_{t:t+1} \rangle = |w^\theta \langle Y_t \rangle|_\infty \int \lambda(dx_{t+1}) g^\theta(x_{t+1}, Y_{t+1}) / p_{\mu,t}^\theta(Y_{t:t+1}).$$

(A-2) There exists a constant  $\sigma_+ \in \mathbb{R}^+$  and a nonnegative measure  $\lambda$  such that for all  $\theta \in \Theta$  and  $(x, A) \in X \times \mathcal{X}$ ,

$$M^\theta(x, A) \leq \sigma_+ \lambda(A).$$

Denote by  $m^\theta(x, \cdot)$  the Radon-Nikodym derivative

$$m^\theta(x, x') = \frac{dM^\theta(x, \cdot)}{d\lambda(\cdot)}(x'). \quad (24)$$

Under (A-2), the stationary distribution  $\pi^\theta$  is absolutely continuous with respect to  $\lambda$ . Furthermore, for notational simplicity it is assumed that the initial distribution  $\mu$  is absolutely continuous with respect to  $\lambda$ . By abuse of notation, we write  $\pi^\theta$  and  $\mu$  also for the corresponding density functions.

(A-3) For all  $\theta \in \Theta$  and  $(x, x', y) \in X^2 \times Y$ ,  $m^\theta(x, x') > 0$  and  $g^\theta(x, y) > 0$ .

(A-4) There exist constants  $(\ell_*, \alpha) \in \mathbb{N}^* \times (0, 1)$  such that,

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_{t:t+\ell} \rangle)^\alpha \right] < \infty, \quad \text{for all } \ell \in \{0, \dots, \ell_* - 1\}, \quad (25)$$

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+\ell_*} \rangle)^\alpha \right] < \infty. \quad (26)$$

**Theorem 6.** Assume that (A-1), (A-2), (A-3), and (A-4) hold. Let  $\theta_* \in \Theta$  and let  $\{\theta_T, T \in \mathbb{N}\}$  be a sequence of parameters such that

$$\limsup_{T \rightarrow \infty} T \bar{\mathbb{E}}^{\theta_*} \left[ \ln \left( \frac{m^{\theta_*}(X_0, X_1) g^{\theta_*}(X_1, Y_1)}{m^{\theta_T}(X_0, X_1) g^{\theta_T}(X_1, Y_1)} \right) \right] < \infty. \quad (27)$$

Furthermore, assume that

$$\bar{\mathbb{E}}^{\theta_*} \left[ \ln \left( \frac{\pi^{\theta_*}(X_0)}{\mu(X_0)} \right) \right] < \infty, \quad (28)$$

where  $\mu$  is the initial distribution used in the PG algorithm. Then, for all  $0 \leq \gamma < \alpha$  (where  $\alpha$  is defined in (A-4)) and for all sequences of integers  $\{N_T\}_{T \geq 1}$  such that  $N_T \sim T^{1/\gamma}$ , the sequence  $\{\epsilon_{T, N_T}^{-1}(\theta_T)\}_{T \geq 1}$ , defined in (16) is  $\bar{\mathbb{P}}^{\theta_*}$ -tight (bounded in probability).

*Proof.* The proof is postponed to Section 7. □

**Remark 7.** For any  $\theta \in \Theta$ ,

$$D(\theta_* || \theta) := \bar{\mathbb{E}}^{\theta_*} \left[ \ln \left( \frac{m^{\theta_*}(X_0, X_1) g^{\theta_*}(X_1, Y_1)}{m^\theta(X_0, X_1) g^\theta(X_1, Y_1)} \right) \right],$$

is the expectation under the stationary distribution  $\pi^{\theta_*}$  of the Kullback-Leibler divergence between the conditional distribution of  $p^{\theta_*}(X_1, Y_1 | X_0)$  and  $p^\theta(X_1, Y_1 | X_0)$ . Hence,  $D(\theta_* || \theta) \geq 0$  for all  $\theta \in \Theta$  and  $D(\theta_* || \theta_*) = 0$ . Assuming that  $\theta_*$  belongs to the interior of  $\Theta$  and that the function  $\theta \mapsto D(\theta_* || \theta)$  is twice differentiable at  $\theta_*$ , a Taylor expansion at  $\theta_*$  yields

$$D(\theta_* || \theta) = \frac{1}{2}(\theta_* - \theta)^t H^{\theta_*}(\theta_* - \theta) + o(\|\theta_* - \theta\|^2),$$

where  $H^\theta$  is the Hessian of  $\theta \mapsto D(\theta_* || \theta)$ . Consequently, for regular statistical models, (27) holds provided that  $\theta_T$  converges to  $\theta_*$  at a rate  $1/\sqrt{T}$ , i.e.,

$$\theta_T = \theta_* + \varrho_T / \sqrt{T},$$

where the sequence  $\{\varrho_T, T \in \mathbb{N}\}$  is bounded:  $\sup_{T \geq 0} \|\varrho_T\| < \infty$ .

**Remark 8.** *It should be noted that our results do not cover explicitly the case when the sequence  $\{\varrho_T, T \in \mathbb{N}\}$  is stochastic. Still, we believe that our results hint at the possibility of obtaining a non-degenerate lower bound on the minorizing constant also in the stochastic case, given that  $\{\varrho_T, T \in \mathbb{N}\}$  is tight, under conditions that are much weaker than the previously considered strong mixing assumption.*

**Remark 9.** *It is interesting to note that we do not require the initial distribution  $\mu$  to be equal to  $\pi^{\theta^*}$ , but only that the Kullback-Leibler divergence (28) is bounded. Hence, we may use a quite arbitrary initial distribution and still obtain a sequence of inverse minorization constants that is tight with respect to  $\bar{\mathbb{P}}^{\theta^*}$ .*

A straightforward generalization of the above result is to let the initial distribution belong to a parametric family of distributions,  $\{\mu^\theta : \theta \in \Theta\}$ . The condition (28) should then be replaced by

$$\limsup_{T \rightarrow \infty} \bar{\mathbb{E}}^{\theta^*} \left[ \ln \left( \frac{\pi^{\theta^*}(X_0)}{\mu^{\theta_T}(X_0)} \right) \right] < \infty .$$

Allowing for the initial distribution to depend on  $\theta$  can be useful in some cases. For instance, if the stationary distribution  $\pi^\theta$  is known it may serve as a natural choice for the initial distribution used in the algorithm.

## 5 Examples

In this section we consider two examples to illustrate how the assumptions of Theorem 6 can be verified in practice. We preface the examples by a technical lemma, which will be very useful for checking the assumptions.

**Lemma 10.** *Let  $(Z, \mathcal{Z})$  be a measurable set and  $\xi$  be a measure on  $(Z, \mathcal{Z})$ . Let  $\alpha \in (0, 1)$  and let  $\varphi, \psi$  and  $q$  be nonnegative measurable functions, such that*

$$\int \psi(z)\varphi(z)\xi(dz) < \infty , \tag{29}$$

$$\int \varphi^{-\frac{\alpha}{1-\alpha}}(z)q(z)\xi(dz) < \infty . \tag{30}$$

*Then,*

$$\int \psi^\alpha(z)q^{1-\alpha}(z)\xi(dz) < \infty . \tag{31}$$

*Proof.* The result follows from Hölder's inequality:

$$\begin{aligned} \int \psi^\alpha(z)q^{1-\alpha}(z)\xi(dz) &= \int [\psi(z)\varphi(z)]^\alpha [\varphi^{-\frac{\alpha}{1-\alpha}}(z)q(z)]^{1-\alpha} \xi(dz) \\ &\leq \left( \int \psi(z)\varphi(z)\xi(dz) \right)^\alpha \left( \int \varphi^{-\frac{\alpha}{1-\alpha}}(z)q(z)\xi(dz) \right)^{1-\alpha} . \end{aligned}$$

□

## 5.1 A nonlinear model with additive measurement noise

We consider first a class of nonlinear state space models where the latent process is observed in additive noise,

$$X_{t+1} = h^\xi(X_t) + \sigma_W W_{t+1} \quad (32)$$

$$Y_t = \phi X_t + \sigma_U U_t \quad (33)$$

where  $\{W_t, t \in \mathbb{N}\}$  and  $\{U_t, t \in \mathbb{N}\}$  are two independent sequences of i.i.d. standard Gaussian random variables and  $\{h^\xi, \xi \in \Xi\}$  is a parametric family of measurable real-valued functions, where  $\Xi$  is a compact subset of a Euclidean space. We denote by  $\theta = (\xi, \phi, \sigma_U, \sigma_W)$  the parameters of the model. It is assumed that  $\theta \in \Theta$ , where  $\Theta$  is a compact subset of  $\Xi \times (0, \infty)^3$ . We assume that for all  $\xi \in \Xi$ ,  $x \mapsto h^\xi(x)$  is continuous and  $\sup_{\xi \in \Xi} \limsup_{x \rightarrow \infty} |h^\xi(x)|/|x| < 1$ . For any  $\delta > 0$ , we set  $V_\delta(x) = e^{\delta|x|}$ . It is easily seen that there exist constants  $\lambda_\delta \in (0, 1)$  and  $b_\delta < \infty$  such that

$$\sup_{\theta \in \Theta} \mathbb{E}_x^\theta [V_\delta(X_1)] \leq \lambda_\delta V_\delta(x) + b_\delta. \quad (34)$$

The Markov chain is strong Feller, Harris recurrent, all the compact sets are small, and the Markov chain admits a single invariant distribution. Therefore, (A-1) and (A-2) are satisfied. Since both the transition density and the observation density are Gaussian, (A-3) is also readily satisfied. We will thus focus on verifying the moment assumption (A-4).

First, note that

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_x^\theta [V_\delta(X_t)] \leq \lambda_\delta^t V_\delta(x) + b_\delta(1 + \lambda_\delta + \dots + \lambda_\delta^{t-1}) \leq V_\delta(x) + b_\delta/(1 - \lambda_\delta). \quad (35)$$

We assume that the initial distribution  $\mu$  is such that  $\mu(V_\delta) < \infty$ . Therefore,

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta [V_\delta(X_t)] < \infty. \quad (36)$$

Interestingly for the model (32)–(33) it is possible to use the fully adapted proposal kernel (Doucet et al., 2000) as defined in (11), for which

$$\begin{aligned} w^\theta \langle y \rangle(x, x') &= \int m^\theta(x, x'') g^\theta(x'', y) dx'' \\ &= \frac{1}{\sqrt{2\pi(\phi^2\sigma_W^2 + \sigma_U^2)}} \exp\left(-\frac{1}{2(\phi^2\sigma_W^2 + \sigma_U^2)}(y - \phi h^\xi(x))^2\right), \end{aligned} \quad (37)$$

for all  $(x, x') \in \mathbb{R} \times \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $\theta \in \Theta$ . It can be seen that, for any  $\theta \in \Theta$  and any  $y \in \mathbb{R}$ ,

$$\int_{-\infty}^{\infty} g^\theta(x, y) dx = \frac{1}{\phi}, \quad \text{and} \quad |w^\theta \langle y \rangle|_\infty \leq \frac{1}{\sqrt{2\pi(\phi^2\sigma_W^2 + \sigma_U^2)}},$$

which implies the existence of constants  $D_1$  and  $D_2$  such that

$$\sup_{\theta \in \Theta} \int_{-\infty}^{\infty} g^\theta(x, y) dx \leq D_1, \quad \text{and} \quad \sup_{\theta \in \Theta} |w^\theta \langle y \rangle|_\infty \leq D_2. \quad (38)$$

Analogous bounds hold also if we would instead consider the bootstrap proposal (see (11)).

To verify (A-4) we let  $\ell_\star = 1$  and show that,

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] < \infty, \quad \sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] < \infty, \quad (39)$$

for some (and actually any)  $\alpha \in [0, 1)$ . Consider first

$$\mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] = \int |w^\theta \langle y_t \rangle|_\infty^\alpha \{p_{\mu,t}^\theta(y_t)\}^{1-\alpha} dy_t \leq D_2^\alpha \int \{p_{\mu,t}^\theta(y_t)\}^{1-\alpha} dy_t, \quad (40)$$

where the inequality follows from (38). We apply Lemma 10 to establish a bound for the right-hand side of (40). Let  $\psi(y) = 1$  and  $\varphi(y) = 1/(1 \vee |y|^2)$ . With these definitions the first condition in (29) is satisfied. To check (30), note that

$$\varphi^{-\frac{\alpha}{1-\alpha}}(y) = (1 \vee |y|^2)^{\frac{\alpha}{1-\alpha}} \leq 1 + |y|^{2\alpha/(1-\alpha)}.$$

The integral in (30) may be expressed as

$$\int \varphi^{-\frac{\alpha}{1-\alpha}}(y_t) p_{\mu,t}^\theta(y_t) dy_t = \mathbb{E}_\mu^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_t) \right] = \mathbb{E}_\mu^\theta \left[ \mathbb{E}_{X_t}^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] \right]. \quad (41)$$

Since  $Y_0 = \phi X_0 + \sigma U_0$ , we get that for any  $x \in \mathbf{X}$ ,  $\mathbb{E}_x^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] \leq 1 + \mathbb{E} \left[ |\phi x + U|^{2\alpha/(1-\alpha)} \right]$ , where  $U$  is standard normal. This implies that there exists a constant  $D_3$  such that, for all  $x \in \mathbf{X}$  and all  $\theta \in \Theta$ ,

$$\mathbb{E}_x^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] \leq D_3(1 + |x|^{2\alpha/(1-\alpha)}). \quad (42)$$

Plugging this into (41) and using (36), this verifies the second condition in (30). Lemma 10 can thus be used to conclude that  $\mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] < \infty$  for all  $\alpha \in (0, 1)$ . Since this holds for any  $t \in \mathbb{N}$  and  $\theta \in \Theta$ , we obtain the first part of (39).

Next, we consider

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] &= \mathbb{E}_\mu^\theta \left[ \frac{|w^\theta \langle Y_t \rangle|_\infty^\alpha \left( \int g^\theta(x_{t+1}, Y_{t+1}) dx_{t+1} \right)^\alpha}{\{p_{\mu,t}^\theta(Y_{t:t+1})\}^\alpha} \right] \\ &\leq D_1^\alpha D_2^\alpha \iint \{p_{\mu,t}^\theta(y_{t:t+1})\}^{1-\alpha} dy_{t:t+1}. \end{aligned}$$

We will again make use of Lemma 10 to bound this quantity. Proceeding analogously to above, we let  $\psi(y_0, y_1) = 1$  and

$$\varphi(y_0, y_1) = \frac{1}{(y_0^2 \vee 1)(y_1^2 \vee 1)}, \quad (43)$$

for which (29) is satisfied. To check (30), we use the conditional independence of the observations given the states and (41) to get, for any  $\theta \in \Theta$ ,

$$\begin{aligned} \iint \varphi^{-\frac{\alpha}{1-\alpha}}(y_{t:t+1}) p_{\mu,t}^\theta(y_{t:t+1}) dy_{t:t+1} &= \mathbb{E}_\mu^\theta \left[ \mathbb{E}_\mu^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_{t:t+1}) \mid X_{t:t+1} \right] \right] \\ &\leq \mathbb{E}_\mu^\theta \left[ \mathbb{E}_{X_t}^\theta \left[ 1 + |Y_0|^{\frac{2\alpha}{1-\alpha}} \right] \mathbb{E}_{X_{t+1}}^\theta \left[ 1 + |Y_0|^{\frac{2\alpha}{1-\alpha}} \right] \right] \leq D_3^2 \mathbb{E}_\mu^\theta \left[ (1 + |X_t|^{\frac{2\alpha}{1-\alpha}})(1 + |X_{t+1}|^{\frac{2\alpha}{1-\alpha}}) \right]. \end{aligned}$$

From the Cauchy-Schwarz inequality we get, by using (36),

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_{t:t+1}) \right] \leq D_3^2 \sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (1 + |X_t|^{\frac{2\alpha}{1-\alpha}})^2 \right] < \infty.$$

This shows that (30) is satisfied for any  $\theta \in \Theta$  and any  $t \in \mathbb{N}$  which, by Lemma 10 implies  $\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] < \infty$  for all  $\alpha \in [0, 1)$ , verifying (A-4).

Provided that  $\theta_T$  converges to  $\theta_*$  at a rate  $1/\sqrt{T}$  (see Remark 7), we may therefore apply Theorem 6 which shows that for any  $\gamma \in (0, 1)$ ,  $\{\epsilon_{T, N_T}^{-1}(\theta_T)\}_{T \geq 1}$  is tight with  $N_T \sim T^{1/\gamma}$ .

## 5.2 A stochastic volatility model

The canonical model in stochastic volatility for discrete-time data has been introduced by Taylor (1982) and worked out since then by many authors; see Hull and White (1987) and Jacquier et al. (1994) for early references and Shephard and Andersen (2009) for an up-to-date survey. In this model, the hidden volatility process,  $\{X_t, t \in \mathbb{N}\}$ , follows a first order autoregression,

$$X_{t+1} = \phi X_t + \sigma W_{t+1}, \quad (44)$$

$$Y_t = \beta \exp(X_t/2) U_t. \quad (45)$$

where  $\{W_t, t \in \mathbb{N}\}$  and  $\{U_t, t \in \mathbb{N}\}$  are white Gaussian noise with mean zero and unit variance. The error processes  $\{W_t, t \in \mathbb{N}\}$  and  $\{U_t, t \in \mathbb{N}\}$  are assumed to be mutually independent. We denote by  $\theta = (\phi, \sigma, \beta) \in \Theta$ , where  $\Theta$  is a compact subset of  $(-1, 1) \times (0, \infty)^2$ . For  $\delta > 0$ , denote by  $V_\delta(x) = e^{\delta|x|}$  and let  $\mu$  an arbitrary distribution on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , for which  $\mu(V_\delta) < \infty$ .

For this model the transition kernel and the likelihood of the observation are given by

$$m^\theta(x, x') = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x' - \phi x)^2\right) \quad \text{and} \quad (46)$$

$$g^\theta(x, y) = \frac{1}{\sqrt{2\pi\beta^2}} e^{-(x/2 + (y^2/2\beta^2)e^{-x})}, \quad (47)$$

respectively. For any  $\theta \in \Theta$ , the autoregressive process  $\{X_t, t \in \mathbb{N}\}$  has a unique stationary distribution  $\pi^\theta$ , which is Gaussian, with mean 0 and variance  $\sigma^2/(1 - \phi^2)$ . Hence, (A-1) is satisfied.

We consider the bootstrap proposal kernel as defined in (11), in which case

$$w^\theta(y)(x, x') = g^\theta(x', y), \quad \text{for all } (x, x') \in \mathbb{R} \times \mathbb{R} \text{ and } y \in \mathbb{R}. \quad (48)$$

Note that,  $|w^\theta(y)|_\infty = |g^\theta(\cdot, y)|_\infty$ . Assumptions (A-2) and (A-3) are readily satisfied. We finally check (A-4). It is easily shown that, for all  $\theta \in \Theta$ ,

$$\int_{-\infty}^{\infty} g^\theta(x, y) dx = \frac{D_1}{|y|}, \quad D_1 := \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{-u/2}}{\sqrt{u}} du \quad (49)$$

$$\sup_{x \in \mathbb{R}} g^\theta(x, y) = \frac{D_2}{|y|}, \quad D_2 := \frac{1}{\sqrt{2\pi e}}. \quad (50)$$

Similarly to Section 5.1 we will check (A-4) with  $\ell_\star = 1$ , i.e., we show that

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] < \infty, \quad \sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] < \infty, \quad (51)$$

for any  $\alpha \in (0, 1)$ . Note that we cannot expect (51) to hold with  $\alpha = 1$  since,

$$\mathbb{E}_\mu^\theta \left[ \tilde{B}_t^\theta \langle Y_t \rangle \right] = \mathbb{E}_\mu^\theta \left[ \frac{|w^\theta \langle Y_t \rangle|_\infty}{p_{\mu, t}^\theta \langle Y_t \rangle} \right] = \int \sup_{x \in \mathbb{R}} g^\theta(x, y) dy = D_2 \int \frac{1}{|y|} dy = \infty.$$

We now turn to the proof of (51). Note first that

$$\limsup_{|x| \rightarrow \infty} \sup_{\theta \in \Theta} \frac{\mathbb{E}_x^\theta [V_\delta(X_1)]}{V_\delta(x)} = 0,$$



and for any  $M < \infty$ ,  $\sup_{|x| \leq M} \sup_{\theta \in \Theta} \mathbb{E}_x^\theta [V_\delta(X_1)] < \infty$ . Therefore, there exist constants  $\lambda_\delta \in (0, 1)$  and  $b_\delta < \infty$  such that, for all  $x \in \mathbb{X}$ ,

$$\sup_{\theta \in \Theta} \mathbb{E}_x^\theta [V_\delta(X_1)] \leq \lambda_\delta V_\delta(x) + b_\delta. \quad (52)$$

Analogously to (35), this implies that, for all  $\delta > 0$ ,

$$\sup_{t \in \mathbb{N}} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta [V_\delta(X_t)] \leq \mu(V_\delta) + b_\delta / (1 - \lambda_\delta) < \infty. \quad (53)$$

Using (48) and (50), we get

$$\mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] = \int |w^\theta \langle y_t \rangle|_\infty^\alpha \{p_{\mu,t}^\theta(y_t)\}^{1-\alpha} dy_t \leq D_2^\alpha \int |y_t|^{-\alpha} \{p_{\mu,t}^\theta(y_t)\}^{1-\alpha} dy_t. \quad (54)$$

We apply Lemma 10 to establish a bound for (54). Consider the functions  $\varphi$  and  $\psi$  given by

$$\psi(y) = 1/|y|, \quad (55)$$

$$\varphi(y) = \frac{|y|^\gamma}{|y|^2 \vee 1}, \quad \text{with} \quad \frac{\gamma\alpha}{1-\alpha} < 1, \quad 0 < \gamma < 1. \quad (56)$$

With these definitions, we get

$$\int \varphi(y)\psi(y)dy = \int \frac{1}{|y|} \frac{|y|^\gamma}{|y|^2 \vee 1} dy < \infty,$$

showing that the first condition in (29) is satisfied. We now check (30):

$$\int \varphi^{-\frac{\alpha}{1-\alpha}}(y_t) p_{\mu,t}^\theta(y_t) dy_t = \mathbb{E}_\mu^\theta \left[ \mathbb{E}_{X_t}^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] \right]. \quad (57)$$

Since  $Y_0 = \beta \exp(X_0/2)U_0$  it follows that  $\mathbb{E}_x^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] = \mathbb{E} \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(\beta e^{x/2}U) \right]$  where  $U$  is standard normal. We have

$$\begin{aligned} & \mathbb{E} \left[ \left( \frac{\beta^2 e^x U^2 \vee 1}{\beta^\gamma e^{\gamma x/2} |U|^\gamma} \right)^{\frac{\alpha}{1-\alpha}} \right] \\ & \leq \mathbb{E} \left[ (\beta e^{x/2} |U|)^{-\frac{\gamma\alpha}{1-\alpha}} \mathbb{1}_{\{\beta |U| e^{x/2} \leq 1\}} \right] + \mathbb{E} \left[ (\beta^2 e^x U^2)^{\frac{\alpha}{1-\alpha}} \mathbb{1}_{\{\beta |U| e^{x/2} > 1\}} \right] \\ & \leq (\beta e^{\frac{x}{2}})^{-\frac{\gamma\alpha}{1-\alpha}} \mathbb{E} \left[ |U|^{-\frac{\gamma\alpha}{1-\alpha}} \right] + (\beta^2 e^x)^{\frac{\alpha}{1-\alpha}} \mathbb{E} \left[ |U|^{\frac{2\alpha}{1-\alpha}} \right]. \end{aligned}$$

Since  $\gamma\alpha/(1-\alpha) < 1$  it holds that  $\mathbb{E} \left[ |U|^{-\frac{\gamma\alpha}{1-\alpha}} \right] < \infty$  and, additionally,  $\mathbb{E} \left[ |U|^{\frac{2\alpha}{1-\alpha}} \right] < \infty$ . Therefore, there exist constants  $D_3 < \infty$  and  $\delta > 0$  such that, for all  $x \in \mathbb{R}$  and  $\theta \in \Theta$ ,

$$\mathbb{E}_x^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_0) \right] \leq D_3 e^{\delta|x|} = D_3 V_\delta(x). \quad (58)$$

Using (57), (58) and (53) verifies the second condition in (30). Lemma 10 can thus be used to conclude that  $\mathbb{E}_\mu^\theta \left[ (\tilde{B}_t^\theta \langle Y_t \rangle)^\alpha \right] < \infty$  for all  $\alpha \in (0, 1)$ . Since this holds for any  $t \in \mathbb{N}$  and  $\theta \in \Theta$ , we establish the first part of (51).

We will now check that, for all  $\alpha \in (0, 1)$ ,  $\mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] < \infty$ . Using (49) and (50), we get

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ (\tilde{C}_t^\theta \langle Y_{t:t+1} \rangle)^\alpha \right] &= \mathbb{E}^\theta \left[ \frac{|w^\theta \langle Y_t \rangle|_\infty^\alpha \left( \int g^\theta(x_{t+1}, Y_{t+1}) dx_{t+1} \right)^\alpha}{(p_{\mu,t}^\theta(Y_{t:t+1}))^\alpha} \right] \\ &= \iint |w^\theta \langle y_t \rangle|_\infty^\alpha \left( \int g^\theta(x_{t+1}, y_{t+1}) dx_{t+1} \right)^\alpha (p_{\mu,t}^\theta(y_{t:t+1}))^{1-\alpha} dy_{t:t+1}, \\ &\leq D_1^\alpha D_2^\alpha \iint |y_t y_{t+1}|^{-\alpha} (p_{\mu,t}^\theta(y_{t:t+1}))^{1-\alpha} dy_{t:t+1}. \end{aligned} \quad (59)$$

We use again Lemma 10 with

$$\psi(y_0, y_1) = |y_0|^{-1} |y_1|^{-1} \quad (60)$$

and

$$\varphi(y_0, y_1) = \frac{|y_0|^\gamma |y_1|^\gamma}{(y_0^2 \vee 1)(y_1^2 \vee 1)}, \quad (61)$$

with  $\gamma\alpha/(1-\alpha) < 1$  and  $\gamma \in (0, 1)$ . Note first that

$$\iint \psi(y_0, y_1) \varphi(y_0, y_1) dy_{0:1} = \iint \{(|y_0| |y_1|)^{1-\gamma} (y_0^2 \vee 1)(y_1^2 \vee 1)\}^{-1} dy_{0:1} < \infty. \quad (62)$$

Hence, (29) is satisfied. We finally check (30). Using the conditional independence of the observations given the states and (58),

$$\begin{aligned} \iint \varphi^{-\frac{\alpha}{1-\alpha}}(y_{t:t+1}) p_{\mu,t}^\theta(y_{t:t+1}) dy_{t:t+1} &= \mathbb{E}_\mu^\theta \left[ \mathbb{E}_\mu^\theta \left[ \varphi^{-\frac{\alpha}{1-\alpha}}(Y_{t:t+1}) \mid X_{t:t+1} \right] \right] \\ &= \mathbb{E}_\mu^\theta \left[ \mathbb{E}_{X_t}^\theta \left[ \left( \frac{\beta^2 e^{X_0} U^2 \vee 1}{\beta e^{\gamma X_0/2} |U|^\gamma} \right)^{\frac{\alpha}{1-\alpha}} \right] \mathbb{E}_{X_{t+1}}^\theta \left[ \left( \frac{\beta^2 e^{X_0} U^2 \vee 1}{\beta e^{\gamma X_0/2} |U|^\gamma} \right)^{\frac{\alpha}{1-\alpha}} \right] \right] \leq D_3^2 \mathbb{E}_\mu^\theta \left[ e^{\delta |X_t|} e^{\delta |X_{t+1}|} \right]. \end{aligned}$$

Using (52), we get, from the Cauchy-Schwarz inequality,

$$\mathbb{E}_\mu^\theta \left[ e^{\delta |X_t|} e^{\delta |X_{t+1}|} \right] \leq \left( \mathbb{E}_\mu^\theta \left[ e^{2\delta |X_t|} \right] \mathbb{E}_\mu^\theta \left[ e^{2\delta |X_{t+1}|} \right] \right)^{1/2},$$

Applying (53) with  $\delta$  replaced by  $2\delta$  yields (30). Using Lemma 10 thus establishes (51) and thereby, (A-4) holds.

Provided that  $\theta_T$  converges to  $\theta_*$  at a rate  $1/\sqrt{T}$  (see Remark 7), we may therefore apply Theorem 6 which shows that for any  $\gamma \in (0, 1)$ ,  $\{\epsilon_{T, N_T}^{-1}(\theta_T)\}_{T \geq 1}$  is tight with  $N_T = T^{1/\gamma}$ .

## 6 Proof of Theorem 1

We will now turn to the proof of the minorization condition in Theorem 1. As in the statement of the theorem, we will not explicitly indicate any possible dependence on unknown model parameters in the notation in this section. This is done for notational convenience and is without loss of generality. Throughout this section,  $\mathbb{P}$  and  $\mathbb{E}$  refer to probability and expectation, respectively, with respect to the random variables generated by the PG algorithm. The proof is inductive and follows from a series of lemmas.

**Lemma 11.** *Let  $X \geq 0$  and  $Y > 0$  be independent random variables. Then,*

$$\mathbb{E} \left[ \frac{X}{Y} \right] \geq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}.$$

*Proof.* Since  $f(y) = 1/y$  is convex on  $y > 0$  the result follows by independence and Jensen's inequality.  $\square$

**Lemma 12.** *Let  $f$  and  $h$  be nonnegative measurable functions. For  $t \in \{0, \dots, T-1\}$ , we have*

$$\begin{aligned} \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_{t+1}^i f(X_{0:t+1}^i)}{\sum_{i=1}^N \omega_{t+1}^i h(X_{t+1}^i)} \middle| \mathcal{F}_t^N \right] \\ \geq \frac{\sum_{i=1}^N \omega_t^i \int Q\langle Y_{t+1} \rangle(X_t^i, dx_{t+1}) f(X_{0:t}^i, x_{t+1})}{\sum_{i=1}^N \omega_t^i \left[ \frac{N-2}{N-1} Q\langle Y_{t+1} \rangle h(X_t^i) + \frac{2}{N-1} \sup_{(x, x')} w\langle Y_{t+1} \rangle(x, x') h(x') \right]}, \end{aligned} \quad (63)$$

and

$$\mathbb{E} \left[ \frac{\sum_{i=0}^N \omega_0^i f(X_0^i)}{\sum_{i=0}^N \omega_0^i h(X_0^i)} \right] \geq \frac{(N-1)\mu(g(\cdot, Y_0)f(\cdot))}{(N-2)\mu(g(\cdot, Y_0)h(\cdot)) + 2 \sup_x [w\langle Y_0 \rangle(x)h(x)]}. \quad (64)$$

*Proof.* Using that

$$\omega_{t+1}^1 h(X_{t+1}^1) + \omega_{t+1}^N h(X_{t+1}^N) \leq 2 \sup_{(x, x')} w\langle Y_{t+1} \rangle(x, x') h(x'),$$

and that the weighted particles  $\{(X_{t+1}^i, \omega_{t+1}^i)\}_{i=1}^{N-1}$  are conditionally i.i.d. with respect to  $\mathcal{F}_t^N$ , we get

$$\begin{aligned} \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_{t+1}^i f(X_{0:t+1}^i)}{\sum_{i=1}^N \omega_{t+1}^i h(X_{t+1}^i)} \middle| \mathcal{F}_t^N \right] &\geq \mathbb{E} \left[ \frac{\sum_{i=1}^{N-1} \omega_{t+1}^i f(X_{0:t+1}^i)}{\sum_{i=1}^{N-1} \omega_{t+1}^i h(X_{t+1}^i)} \middle| \mathcal{F}_t^N \right] \\ &\geq (N-1) \mathbb{E} \left[ \frac{\omega_{t+1}^1 f(X_{0:t+1}^1)}{\sum_{i=2}^{N-1} \omega_{t+1}^i h(X_{t+1}^i) + 2 \sup_{(x, x')} w\langle Y_{t+1} \rangle(x, x') h(x')} \middle| \mathcal{F}_t^N \right] \\ &\geq (N-1) \frac{\mathbb{E} [\omega_{t+1}^1 f(X_{0:t+1}^1) \mid \mathcal{F}_t^N]}{\mathbb{E} \left[ \sum_{i=2}^{N-1} \omega_{t+1}^i h(X_{t+1}^i) \mid \mathcal{F}_t^N \right] + 2 \sup_{(x, x')} w\langle Y_{t+1} \rangle(x, x') h(x')}, \end{aligned} \quad (65)$$

where the last inequality follows from Lemma 11. Consider first the numerator in the right-hand side of (65). We have

$$\begin{aligned} \mathbb{E} [\omega_{t+1}^1 f(X_{0:t+1}^1) \mid \mathcal{F}_t^N] &= \frac{1}{\sum_{l=1}^N \omega_t^l} \sum_{j=1}^N \omega_t^j \int R\langle Y_{t+1} \rangle(X_t^j, dx_{t+1}) w\langle Y_{t+1} \rangle(X_t^j, x_{t+1}) f(X_{0:t}^j, x_{t+1}) \\ &= \frac{1}{\sum_{l=1}^N \omega_t^l} \sum_{j=1}^N \omega_t^j \int Q\langle Y_{t+1} \rangle(X_t^j, dx_{t+1}) f(X_{0:t}^j, x_{t+1}). \end{aligned} \quad (66)$$

We now consider the denominator in the right-hand side of (65):

$$\begin{aligned} \mathbb{E} \left[ \sum_{i=2}^{N-1} \omega_{t+1}^i h(X_{t+1}^i) \middle| \mathcal{F}_t^N \right] &= (N-2) \mathbb{E} [\omega_{t+1}^1 h(X_{t+1}^1) \mid \mathcal{F}_t^N] \\ &= (N-2) \frac{1}{\sum_{l=1}^N \omega_t^l} \sum_{j=1}^N \omega_t^j Q\langle Y_{t+1} \rangle h(X_t^j), \end{aligned}$$

where the last identity follows from (66) with  $f(x_{0:t+1}) = h(x_{t+1})$ . The proof of (63) follows.

Consider now (64). Since the particles  $\{X_0^i\}_{i=1}^{N-1}$  are i.i.d., we obtain, using again Lemma 11,

$$\begin{aligned} \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_0^i f(X_0^i)}{\sum_{i=1}^N \omega_0^i h(X_0^i)} \right] &\geq (N-1) \mathbb{E} \left[ \frac{\omega_0^1 f(X_0^1)}{\sum_{i=2}^{N-1} \omega_0^i h(X_0^i) + 2 \sup_x w \langle Y_0 \rangle(x) h(x)} \right] \\ &\geq \frac{(N-1) \mathbb{E} [\omega_0^1 f(X_0^1)]}{\mathbb{E} \left[ \sum_{i=2}^{N-1} \omega_0^i h(X_0^i) \right] + 2 \sup_x w \langle Y_0 \rangle(x) h(x)}. \end{aligned}$$

The numerator is given by

$$\mathbb{E} [\omega_0^1 f(X_0^1)] = \int r_0 \langle Y_0 \rangle(dx_0) w \langle Y_0 \rangle(x_0) f(x_0) = \int \mu(dx_0) g(x_0, Y_0) f(x_0).$$

Similarly, we get

$$\mathbb{E} \left[ \sum_{i=2}^{N-1} \omega_0^i h(X_0^i) \right] = (N-2) \mathbb{E} [\omega_0^1 h(X_0^1)] = (N-2) \int \mu(dx_0) g(x_0, Y_0) h(x_0).$$

□

Define a sequence of nonnegative scalars  $\{\beta_t\}_{t=0}^T$  by the backward recursion:  $\beta_T = |w \langle Y_T \rangle|_\infty$ , and for  $t = T-1, T-2, \dots, 0$ ,

$$\begin{aligned} \beta_t = |w \langle Y_t \rangle|_\infty &\left\{ \frac{2}{N-1} \sum_{\ell=1}^{T-t} \left( \frac{N-2}{N-1} \right)^{\ell-1} \beta_{t+\ell} |Q \langle Y_{t+1:t+\ell-1} \rangle \mathbf{1}|_\infty \right. \\ &\left. + \left( \frac{N-2}{N-1} \right)^{T-t} |Q \langle Y_{t+1:T} \rangle \mathbf{1}|_\infty \right\}. \end{aligned} \quad (67)$$

Given  $\{\beta_t\}_{t=0}^T$ , define the functions  $\{h_t\}_{t=0}^T$ ,  $h_t : \mathbf{X} \rightarrow \mathbb{R}_+$ , by the backward recursion:  $h_T = \mathbf{1}$  and, for all  $t = T-1, T-2, \dots, 0$ ,

$$h_t : x \mapsto h_t(x) = \frac{N-2}{N-1} Q \langle Y_{t+1} \rangle h_{t+1}(x) + \frac{2}{N-1} \beta_{t+1}. \quad (68)$$

By solving the backward recursion, (68) implies

$$h_t(x) = \frac{2}{N-1} \sum_{\ell=1}^{T-t} \left( \frac{N-2}{N-1} \right)^{\ell-1} \beta_{t+\ell} Q \langle Y_{t+1:t+\ell-1} \rangle \mathbf{1}(x) + \left( \frac{N-2}{N-1} \right)^{T-t} Q \langle Y_{t+1:T} \rangle \mathbf{1}(x). \quad (69)$$

For  $D \in \mathcal{X}^{\otimes(T+1)}$ , set  $f_T(x_{0:T}) = \mathbb{1}_D(x_{0:T})$  and, for  $t = T-1, T-2, \dots, 0$ ,

$$f_t(x_{0:t}) = \int Q \langle Y_{t+1} \rangle(x_t, dx_{t+1}) f_{t+1}(x_{0:t+1}), \quad (70)$$

or equivalently,

$$f_t(x_{0:t}) = \int \prod_{\ell=1}^{T-t} Q \langle Y_{t+\ell} \rangle(x_{t+\ell-1}, dx_{t+\ell}) \mathbb{1}_D(x_{0:T}). \quad (71)$$

**Lemma 13.** For any  $D \in \mathcal{X}^{\otimes(T+1)}$ ,

$$\mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_T^i \mathbb{1}_D(X_{0:T}^i)}{\sum_{i=1}^N \omega_T^i} \right] \geq \frac{(N-1)p_\mu(Y_{0:T})}{(N-2)\mu(g(\cdot, Y_0)h_0(\cdot)) + 2\beta_0} \phi_{\mu,0:T} \langle Y_{0:T} \rangle (D).$$

*Proof.* Note first that, by construction,

$$\mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_T^i \mathbb{1}_D(X_{0:T}^i)}{\sum_{i=1}^N \omega_T^i} \right] = \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_T^i f_T(X_{0:T}^i)}{\sum_{i=1}^N \omega_T^i h_T(X_T^i)} \right]. \quad (72)$$

We now show that, by backward induction, for all  $t \in \{0, \dots, T-1\}$ ,

$$\mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_{t+1}^i f_{t+1}(X_{0:t+1}^i)}{\sum_{i=1}^N \omega_{t+1}^i h_{t+1}(X_{t+1}^i)} \right] \geq \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_t^i f_t(X_{0:t}^i)}{\sum_{i=1}^N \omega_t^i h_t(X_t^i)} \right]. \quad (73)$$

To obtain (73), note first that the tower property of the conditional expectation, Lemma 12, and (70) imply

$$\begin{aligned} \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_{t+1}^i f_{t+1}(X_{0:t+1}^i)}{\sum_{i=1}^N \omega_{t+1}^i h_{t+1}(X_{t+1}^i)} \right] &= \mathbb{E} \left[ \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_{t+1}^i f_{t+1}(X_{0:t+1}^i)}{\sum_{i=1}^N \omega_{t+1}^i h_{t+1}(X_{t+1}^i)} \middle| \mathcal{F}_t^N \right] \right] \\ &\geq \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_t^i f_t(X_{0:t}^i)}{\sum_{i=1}^N \omega_t^i \left[ \frac{N-2}{N-1} Q \langle Y_{t+1} \rangle h_{t+1}(X_t^i) + \frac{2}{N-1} \sup_{(x,x')} w \langle Y_{t+1} \rangle (x, x') h_{t+1}(x') \right]} \right]. \end{aligned}$$

By the triangle inequality, it follows directly from (67) and (69) that

$$\sup_x w \langle Y_0 \rangle (x) h_0(x) \leq \beta_0, \quad (74)$$

$$\sup_{x,x'} w \langle Y_{t+1} \rangle (x, x') h_{t+1}(x') \leq \beta_{t+1}, \quad t \in \{0, \dots, T-1\}. \quad (75)$$

Combining the inequality (75) with the definition of  $h_t$  in (68) yields

$$\sum_{i=1}^N \omega_t^i \left[ \frac{N-2}{N-1} Q \langle Y_{t+1} \rangle h_{t+1}(X_t^i) + \frac{2}{N-1} \sup_{(x,x')} w \langle Y_{t+1} \rangle (x, x') h_{t+1}(x') \right] \leq \sum_{i=1}^N \omega_t^i h_t(X_t^i),$$

showing (73). Combining (73) with (72) and using Lemma 12-(64) establishes that

$$\mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_T^i \mathbb{1}_D(X_{0:T}^i)}{\sum_{i=1}^N \omega_T^i} \right] \geq \mathbb{E} \left[ \frac{\sum_{i=1}^N \omega_0^i f_0(X_0^i)}{\sum_{i=1}^N \omega_0^i h_0(X_0^i)} \right] \geq \frac{(N-1)\mu(g(\cdot, Y_0)f_0(\cdot))}{(N-2)\mu(g(\cdot, Y_0)h_0(\cdot)) + 2\beta_0},$$

where the last inequality stems from (74). The proof is completed by noting that

$$\mu(g(\cdot, Y_0)f_0(\cdot)) = \phi_{\mu,0:T} \langle Y_{0:T} \rangle (D) p_\mu(Y_{0:T}).$$

□

Finally, to prove Theorem 1 it remains to show the following.

**Lemma 14.** With  $B_{t,T}$  defined as in (14), it holds that

$$(N-2)\mu(g(\cdot, Y_0)h_0(\cdot)) + 2\beta_0 \leq (N-1)p_\mu(Y_{0:T}) \left[ \prod_{t=0}^T \frac{2B_{t,T} + N - 2}{N - 1} \right]. \quad (76)$$

*Proof.* Define for  $t \in \{0, \dots, T\}$ ,

$$\alpha_t = \frac{\beta_t}{p_\mu(Y_{t:T}|Y_{0:t-1})}, \quad (77)$$

with the convention  $p_\mu(Y_{0:T}|Y_{0:-1}) = p_\mu(Y_{0:T})$ . In particular,  $\alpha_0 = \beta_0/p_\mu(Y_{0:T})$ .

Eq. (67) implies

$$\alpha_t = |w\langle Y_t \rangle|_\infty \left\{ \frac{2}{N-1} \sum_{\ell=1}^{T-t} \left( \frac{N-2}{N-1} \right)^{\ell-1} \alpha_{t+\ell} \left[ \frac{|Q\langle Y_{t+1:t+\ell-1} \rangle \mathbf{1}|_\infty p_\mu(Y_{t+\ell:T}|Y_{0:t+\ell-1})}{p_\mu(Y_{t:T}|Y_{0:t-1})} \right] + \left( \frac{N-2}{N-1} \right)^{T-t} \frac{|Q\langle Y_{t+1:T} \rangle \mathbf{1}|_\infty}{p_\mu(Y_{t:T}|Y_{0:t-1})} \right\}.$$

The identity

$$\frac{p_\mu(Y_{t+\ell:T}|Y_{0:t+\ell-1})}{p_\mu(Y_{t:T}|Y_{0:t-1})} = \frac{1}{p_\mu(Y_{t:t+\ell-1}|Y_{0:t-1})}$$

and the definition in (14) imply that

$$\alpha_t \leq B_{t,T} \left\{ \frac{2}{N-1} \sum_{\ell=1}^{T-t} \left( \frac{N-2}{N-1} \right)^{\ell-1} \alpha_{t+\ell} + \left( \frac{N-2}{N-1} \right)^{T-t} \right\}. \quad (78)$$

By a backward induction, define the sequence  $\{\tilde{\alpha}_t\}_{t=0}^T$  as follows: set  $\tilde{\alpha}_T = B_{T,T}$  and

$$\tilde{\alpha}_t = B_{t,T} \left[ \frac{2}{N-1} \sum_{\ell=1}^{T-t} \left( \frac{N-2}{N-1} \right)^{\ell-1} \tilde{\alpha}_{t+\ell} + \left( \frac{N-2}{N-1} \right)^{T-t} \right]. \quad (79)$$

Since by construction,  $\alpha_T \leq B_{T,T} = \tilde{\alpha}_T$ , an elementary backward recursion using (78) shows that,

$$\text{for all } t \in \{0, \dots, T\}, \alpha_t \leq \tilde{\alpha}_t. \quad (80)$$

However

$$\begin{aligned} \tilde{\alpha}_{t-1} &= B_{t-1,T} \left[ \frac{2}{N-1} \sum_{s=1}^{T-t+1} \left( \frac{N-2}{N-1} \right)^{\ell-1} \tilde{\alpha}_{t+\ell-1} + \left( \frac{N-2}{N-1} \right)^{T-t+1} \right] \\ &= B_{t-1,T} \left[ \frac{2}{N-1} \tilde{\alpha}_t + \frac{2}{N-1} \sum_{k=1}^{T-t} \left( \frac{N-2}{N-1} \right)^k \tilde{\alpha}_{t+k} + \left( \frac{N-2}{N-1} \right)^{T-t+1} \right] \\ &= \frac{2B_{t-1,T}}{N-1} \tilde{\alpha}_t + B_{t-1,T} \frac{N-2}{N-1} \frac{\tilde{\alpha}_t}{B_{t,T}} \\ &= \frac{B_{t-1,T}}{B_{t,T}} \left[ \frac{2B_{t,T}}{N-1} + \frac{N-2}{N-1} \right] \tilde{\alpha}_t. \end{aligned}$$

Therefore

$$\tilde{\alpha}_0 = B_{0,T} \prod_{t=1}^T \frac{2B_{t,T} + N-2}{N-1}. \quad (81)$$

Now, since

$$h_0(x) = \frac{2}{N-1} \sum_{s=1}^T \left( \frac{N-2}{N-1} \right)^{s-1} \beta_s Q\langle Y_{1:s-1} \rangle \mathbf{1}(x) + \left( \frac{N-2}{N-1} \right)^T Q\langle Y_{1:T} \rangle \mathbf{1}(x),$$

we have

$$\mu(g(\cdot, Y_0)h_0(\cdot)) = \frac{2}{N-1} \sum_{s=1}^T \left( \frac{N-2}{N-1} \right)^{s-1} \beta_s p_\mu(Y_{0:s-1}) + \left( \frac{N-2}{N-1} \right)^T p_\mu(Y_{0:T}).$$

Plugging (77) into this equation and using that

$$p_\mu(Y_{0:s-1}) = \frac{p_\mu(Y_{0:T})}{p_\mu(Y_{s:T}|Y_{0:s-1})}$$

yield

$$\mu(g(\cdot, Y_0)h_0(\cdot)) = \frac{2}{N-1} \sum_{s=1}^T \left( \frac{N-2}{N-1} \right)^{s-1} \alpha_s p_\mu(Y_{0:T}) + \left( \frac{N-2}{N-1} \right)^T p_\mu(Y_{0:T}).$$

Finally, using (80) and then (79),

$$\begin{aligned} & (N-2)\mu(g(\cdot, Y_0)h_0(\cdot)) + 2\beta_0 \\ & \leq p_\mu(Y_{0:T}) \left\{ (N-2) \left[ \frac{2}{N-1} \sum_{s=1}^T \left( \frac{N-2}{N-1} \right)^{s-1} \alpha_s + \left( \frac{N-2}{N-1} \right)^T \right] + 2\alpha_0 \right\} \\ & \leq p_\mu(Y_{0:T}) \left\{ (N-2) \left[ \frac{2}{N-1} \sum_{s=1}^T \left( \frac{N-2}{N-1} \right)^{s-1} \tilde{\alpha}_s + \left( \frac{N-2}{N-1} \right)^T \right] + 2\tilde{\alpha}_0 \right\} \\ & \leq p_\mu(Y_{0:T}) \left( (N-2) \frac{\tilde{\alpha}_0}{B_{0,T}} + 2\tilde{\alpha}_0 \right) \\ & = p_\mu(Y_{0:T}) (N-2 + 2B_{0,T}) \prod_{t=1}^T \frac{2B_{t,T} + N-2}{N-1}, \end{aligned}$$

where the last equality follows from (81). The proof follows.  $\square$

## 7 Proof of Theorem 6

Define

$$\widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle := \frac{|w^\theta \langle Y_t \rangle|_\infty |Q^\theta \langle Y_{t+1:t+\ell} \rangle \mathbf{1}|_\infty}{p_\mu^\theta(Y_{t:t+\ell}|Y_{0:t-1})}, \quad (82)$$

$$\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle := \frac{|w^\theta \langle Y_t \rangle|_\infty \int \lambda(dx_{t+1}) g^\theta(x_{t+1}, Y_{t+1}) Q^\theta \langle Y_{t+2:t+\ell} \rangle \mathbf{1}(x_{t+1})}{p_\mu^\theta(Y_{t:t+\ell}|Y_{0:t-1})}. \quad (83)$$

Note that

$$\widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle = \widetilde{B}_t^\theta \langle Y_{t:t+\ell} \rangle \frac{p_{\mu,t}^\theta(Y_{t:t+\ell})}{p_\mu^\theta(Y_{t:t+\ell}|Y_{0:t-1})} \quad \text{and} \quad \widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle = \widetilde{C}_t^\theta \langle Y_{t:t+\ell} \rangle \frac{p_{\mu,t}^\theta(Y_{t:t+\ell})}{p_\mu^\theta(Y_{t:t+\ell}|Y_{0:t-1})}, \quad (84)$$

where  $\widetilde{B}_t^\theta \langle Y_{t:t+\ell} \rangle$  and  $\widetilde{C}_t^\theta \langle Y_{t:t+\ell} \rangle$  are defined in (22) and (23), respectively.

**Lemma 15.** *For all  $\theta \in \Theta$ , the sequence  $\{\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle\}_{\ell \geq 0}$  defined in (83) is a  $(\mathbb{P}_\mu^\theta, \{\mathcal{F}_{t+\ell}\}_{\ell \geq 0})$ -martingale, where  $\mathcal{F}_t = \sigma(Y_{0:t})$ .*

*Proof.* For all  $\ell \geq 0$ ,

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ \widehat{C}_t^\theta \langle Y_{0:t+\ell+1} \rangle \middle| \mathcal{F}_{t+\ell} \right] &= |w^\theta \langle Y_t \rangle|_\infty \int \left\{ p_\mu^\theta(y_{t+\ell+1} | Y_{0:t+\ell}) \right. \\ &\quad \times \left. \frac{\int \lambda(dx_{t+1}) g^\theta(x_{t+1}, Y_{t+1}) Q^\theta \langle Y_{t+2:t+\ell} \rangle (x_{t+1}, dx_{t+\ell}) Q^\theta \langle y_{t+\ell+1} \rangle \mathbf{1}(x_{t+\ell})}{p_\mu^\theta(Y_{t:t+\ell}, y_{t+\ell+1} | Y_{0:t-1})} \kappa(dy_{t+\ell+1}) \right\}. \end{aligned}$$

Combining this identity with

$$p_\mu^\theta(Y_{t:t+\ell}, y_{t+\ell+1} | Y_{0:t-1}) = p_\mu^\theta(y_{t+\ell+1} | Y_{0:t+\ell}) p_\mu^\theta(Y_{t:t+\ell} | Y_{0:t-1}),$$

and  $\int Q^\theta \langle y_{t+\ell+1} \rangle \mathbf{1}(x_{t+\ell}) \kappa(dy_{t+\ell+1}) = M^\theta(x_{t+\ell}, \mathbf{X}) = 1$ , we obtain

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ \widehat{C}_t^\theta \langle Y_{0:t+\ell+1} \rangle \middle| \mathcal{F}_{t+\ell} \right] \\ = \frac{|w^\theta \langle Y_t \rangle|_\infty \int \lambda(dx_{t+1}) g^\theta(x_{t+1}, Y_{t+1}) Q^\theta \langle Y_{t+2:t+\ell} \rangle \mathbf{1}(x_{t+1})}{p_\mu^\theta(Y_{t:t+\ell} | Y_{0:t-1})} = \widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 16.** For all  $0 \leq \gamma < 1$  and all  $\ell \in \mathbb{N}$ ,

$$\mathbb{E}_\mu^\theta \left[ (\widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] \leq \mathbb{E}_\mu^\theta \left[ (\widetilde{B}_t^\theta \langle Y_{t:t+\ell} \rangle)^\gamma \right], \quad (85)$$

$$\mathbb{E}_\mu^\theta \left[ (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] \leq \mathbb{E}_\mu^\theta \left[ (\widetilde{C}_t^\theta \langle Y_{t:t+\ell} \rangle)^\gamma \right]. \quad (86)$$

*Proof.* Using (84), the proof of (85) and (86) follow from the inequality:

$$\mathbb{E}_\mu^\theta \left[ \frac{\psi(Y_{t:t+\ell})}{\{p_\mu^\theta(Y_{t:t+\ell} | Y_{0:t-1})\}^\gamma} \right] \leq \mathbb{E}_\mu^\theta \left[ \frac{\psi(Y_{t:t+\ell})}{\{p_{\mu,t}^\theta(Y_{t:t+\ell})\}^\gamma} \right], \quad (87)$$

which holds for any nonnegative measurable function  $\psi : \mathbf{Y}^{\ell+1} \rightarrow \mathbb{R}^+$ . We now show (87). Note first that, by applying the tower property of the conditional expectation and then the Tonelli-Fubini theorem, we get

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ \frac{\psi(Y_{t:t+\ell})}{\{p_\mu^\theta(Y_{t:t+\ell} | Y_{0:t-1})\}^\gamma} \right] &= \mathbb{E}_\mu^\theta \left[ \mathbb{E}_\mu^\theta \left[ \frac{\psi(Y_{t:t+\ell})}{\{p_\mu^\theta(Y_{t:t+\ell} | Y_{0:t-1})\}^\gamma} \middle| Y_{0:t-1} \right] \right] \\ &= \mathbb{E}_\mu^\theta \left[ \int \psi(y_{t:t+\ell}) \{p_\mu^\theta(y_{t:t+\ell} | Y_{0:t-1})\}^{1-\gamma} \kappa^{\otimes(\ell+1)}(dy_{t:t+\ell}) \right] \\ &= \int \psi(y_{t:t+\ell}) \mathbb{E}_\mu^\theta \left[ \{p_\mu^\theta(y_{t:t+\ell} | Y_{0:t-1})\}^{1-\gamma} \right] \kappa^{\otimes(\ell+1)}(dy_{t:t+\ell}). \quad (88) \end{aligned}$$

By the Jensen identity,  $\mathbb{E}_\mu^\theta \left[ \{p_\mu^\theta(y_{t:t+\ell} | Y_{0:t-1})\}^{1-\gamma} \right] \leq \{ \mathbb{E}_\mu^\theta [p_\mu^\theta(y_{t:t+\ell} | Y_{0:t-1})] \}^{1-\gamma}$ . On the other hand,

$$\mathbb{E}_\mu^\theta [p_\mu^\theta(y_{t:t+\ell} | Y_{0:t-1})] = \int p_\mu^\theta(y_{t:t+\ell} | y_{0:t-1}) p_\mu^\theta(y_{0:t-1}) \kappa^{\otimes t}(dy_{0:t-1}) = p_{\mu,t}^\theta(y_{t:t+\ell}). \quad (89)$$

The proof of (87) follows by combining the above relations.  $\square$



**Lemma 17.** Assume (A-2) and (A-4). Then, for all  $0 \leq \gamma < \alpha$ ,

$$\sup_{t \geq 0} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta \left[ \left( \sup_{\ell \geq 0} \widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle \right)^\gamma \right] < \infty .$$

where  $\alpha$  is defined in (A-4).

*Proof.* Under (A-2), we obtain by definitions of  $\widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle$  and  $\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle$ ,

$$\sup_{\ell \geq 0} \widehat{B}_t^\theta \langle Y_{0:\ell} \rangle \leq \sum_{\ell=0}^{\ell_*-1} \widehat{B}_t^\theta \langle Y_{0:\ell} \rangle + \sup_{\ell \geq \ell_*} \widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle \leq \sum_{\ell=0}^{\ell_*-1} \widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle + \sigma_+ \sup_{\ell \geq \ell_*} \widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle ,$$

where  $\sigma_+$  and  $\ell_*$  are defined in (A-2) and (A-4), respectively. Then, by subadditivity of  $u \mapsto u^\gamma$ ,

$$\mathbb{E}_\mu^\theta \left[ \left( \sup_{\ell \geq 0} \widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle \right)^\gamma \right] \leq \sum_{\ell=0}^{\ell_*-1} \mathbb{E}_\mu^\theta \left[ (\widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] + (\sigma_+)^\gamma \mathbb{E}_\mu^\theta \left[ \sup_{\ell \geq \ell_*} (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] .$$

Applying Lemma 16 and (25), it is thus sufficient to bound

$$\mathbb{E}_\mu^\theta \left[ \sup_{\ell \geq \ell_*} (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] .$$

Since by Lemma 15,  $\{\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle\}_{k \geq 0}$  is a  $\{\mathcal{F}_{t+\ell}\}_{\ell \geq 0}$ -martingale and  $\alpha \in (0, 1)$ , we have that  $\{(\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\alpha\}_{\ell \geq 0}$  is a nonnegative  $\{\mathcal{F}_{t+\ell}\}_{\ell \geq 0}$ -supermartingale. The Doob maximal inequality then applies: for all  $a > 0$ ,

$$a \mathbb{P}_\mu^\theta \left[ \sup_{\ell \geq \ell_*} (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\alpha \geq a \mid \mathcal{F}_{t+\ell_*-1} \right] \leq \mathbb{E}_\mu^\theta \left[ (\widehat{C}_t^\theta \langle Y_{0:t+\ell_*} \rangle)^\alpha \mid \mathcal{F}_{t+\ell_*-1} \right] .$$

Take now the expectation in both sides of the previous inequality and set  $\delta = a^{\gamma/\alpha}$ . We obtain

$$\mathbb{P}_\mu^\theta \left[ \sup_{\ell \geq \ell_*} (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \geq \delta \right] \leq \delta^{-\alpha/\gamma} \mathbb{E}_\mu^\theta \left[ (\widehat{C}_t^\theta \langle Y_{0:t+\ell_*} \rangle)^\alpha \right] .$$

Combining this with the inequality  $\mathbb{E}[U] \leq 1 + \int_1^\infty \mathbb{P}[U > \delta] d\delta$  which holds for all nonnegative random variable  $U$ , we obtain under (A-4)

$$\begin{aligned} \mathbb{E}_\mu^\theta \left[ \sup_{\ell \geq \ell_*} (\widehat{C}_t^\theta \langle Y_{0:t+\ell} \rangle)^\gamma \right] &\leq 1 + \left( \int_1^\infty \delta^{-\alpha/\gamma} d\delta \right) \mathbb{E}_\mu^\theta \left[ (\widehat{C}_t^\theta \langle Y_{0:t+\ell_*} \rangle)^\alpha \right] \\ &= 1 + \frac{\gamma}{\alpha - \gamma} \mathbb{E}_\mu^\theta \left[ (\widehat{C}_t^\theta \langle Y_{0:t+\ell_*} \rangle)^\alpha \right] . \end{aligned}$$

The proof follows by applying again Lemma 16 under (A-4).  $\square$

*Proof of Theorem 6.* For simplicity we will use in this proof the notations  $\bar{p}^\theta(Y_{0:t}) := p_{\pi^\theta}^\theta(Y_{0:t})$  and  $\bar{p}^\theta(Y_{t:s} | Y_{0:t-1}) = p_{\pi^\theta}^\theta(Y_{t:s} | Y_{0:t-1})$ . First note that

$$\begin{aligned} \bar{\mathbb{P}}^{\theta_*} \left\{ \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})} > \rho \right\} &= \bar{\mathbb{P}}^{\theta_*} \left\{ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})} + \frac{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})}{\bar{p}^{\theta_*}(Y_{0:T})} - 1 > \ln \rho + \frac{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})}{\bar{p}^{\theta_*}(Y_{0:T})} - 1 \right\} \\ &\leq \bar{\mathbb{P}}^{\theta_*} \left\{ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})} + \frac{p_{\mu^{\theta_*}}^{\theta_*}(Y_{0:T})}{\bar{p}^{\theta_*}(Y_{0:T})} - 1 > \ln \rho - 1 \right\} . \end{aligned}$$

Now, since for all  $u > 0$ ,  $\ln(u) + u^{-1} - 1 \geq 0$ , we obtain for for all  $\rho > e = \exp(1)$ :

$$\begin{aligned} \bar{\mathbb{P}}^{\theta_*} \left\{ \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} > \rho \right\} &\leq \frac{1}{\ln \rho - 1} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} + \frac{p_{\mu}^{\theta_T}(Y_{0:T})}{\bar{p}^{\theta_*}(Y_{0:T})} - 1 \right] \\ &= \frac{1}{\ln \rho - 1} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right]. \end{aligned}$$

This implies that for all  $M > 0$  and all  $\rho > e$ ,

$$\begin{aligned} \bar{\mathbb{P}}^{\theta_*}(\epsilon_{T,N_T}^{-1}(\theta_T) > M) &\leq \mathbb{E}_{\mu}^{\theta_T} \left[ \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \mathbb{1}_{\left\{ \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \leq \rho \right\}} \mathbb{1}_{\{\epsilon_{T,N_T}^{-1}(\theta_T) > M\}} \right] + \bar{\mathbb{P}}^{\theta_*} \left\{ \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} > \rho \right\} \\ &\leq \rho \mathbb{P}_{\mu}^{\theta_T} \left[ \epsilon_{T,N_T}^{-1}(\theta_T) > M \right] + \frac{1}{\ln \rho - 1} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right] \\ &\leq \rho \mathbb{P}_{\mu}^{\theta_T} \left[ \epsilon_{T,N_T}^{-1}(\theta_T) > M \right] + \frac{1}{\ln \rho - 1} \left( \sup_{T \geq 0} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right] \right). \quad (90) \end{aligned}$$

We consider first the last term of the right-hand side. Note first that, by the tower property,

$$\bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(X_{0:T}|Y_{0:T})}{p_{\mu}^{\theta_T}(X_{0:T}|Y_{0:T})} \right] = \bar{\mathbb{E}}^{\theta_*} \left[ \int \dots \int \left( \ln \frac{\bar{p}^{\theta_*}(x_{0:T}|Y_{0:T})}{p_{\mu}^{\theta_T}(x_{0:T}|Y_{0:T})} \right) \bar{p}^{\theta_*}(x_{0:T}|Y_{0:T}) \prod_{i=0}^T \lambda(dx_i) \right] \geq 0$$

because this quantity is the expectation under the stationary distribution of a Kullback-Leibler divergence. This implies that

$$\bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right] \leq \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right] + \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(X_{0:T}|Y_{0:T})}{p_{\mu}^{\theta_T}(X_{0:T}|Y_{0:T})} \right] = \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(X_{0:T}, Y_{0:T})}{p_{\mu}^{\theta_T}(X_{0:T}, Y_{0:T})} \right].$$

On the other hand, using

$$\begin{aligned} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(X_{0:T}, Y_{0:T})}{p_{\mu}^{\theta_T}(X_{0:T}, Y_{0:T})} \right] \\ = \bar{\mathbb{E}}^{\theta_*} \left[ \ln \left( \frac{\pi^{\theta_*}(X_0) g^{\theta_*}(X_0, Y_0)}{\mu(X_0) g^{\theta_T}(X_0, Y_0)} \right) \right] + T \bar{\mathbb{E}}^{\theta_*} \left[ \ln \left( \frac{m^{\theta_*}(X_0, X_1) g^{\theta_*}(X_1, Y_1)}{m^{\theta_T}(X_0, X_1) g^{\theta_T}(X_1, Y_1)} \right) \right], \end{aligned}$$

we obtain, under (27) and (28), that

$$\sup_{T \geq 0} \bar{\mathbb{E}}^{\theta_*} \left[ \ln \frac{\bar{p}^{\theta_*}(Y_{0:T})}{p_{\mu}^{\theta_T}(Y_{0:T})} \right] < \infty. \quad (91)$$

Assume first that

$$\limsup_{M \rightarrow \infty} \sup_{T \geq 0} \mathbb{P}_{\mu}^{\theta_T} \left[ \epsilon_{T,N_T}^{-1}(\theta_T) > M \right] = 0. \quad (92)$$

The proof of the tightness of  $\{\epsilon_{T,N_T}^{-1}(\theta_T)\}_{T \geq 0}$  then follows by plugging (91) into (90) and by noting that (90) holds for all  $\rho > e$ , combined with (92). To complete the proof, it thus remains to show (92). Rewriting the definition (16), we obtain

$$\epsilon_{T,N_T}^{-1}(\theta_T) \leq \prod_{t=0}^T \frac{2B_t^{\theta_T} + N_T - 2}{N_T - 1} = \exp \left\{ \sum_{t=0}^T \ln \left( \frac{2B_t^{\theta_T} + N_T - 2}{N_T - 1} \right) \right\} \leq \exp \left\{ \sum_{t=0}^T \frac{2B_t^{\theta_T} - 1}{N_T - 1} \right\}.$$

where  $B_t^\theta := \sup_{\ell \geq 0} \widehat{B}_t^\theta \langle Y_{0:t+\ell} \rangle$ . This implies that for  $M > 1$ ,

$$\mathbb{P}_\mu^{\theta_T} \left[ \epsilon_{T, N_T}^{-1}(\theta_T) > M \right] \leq \mathbb{P}_\mu^{\theta_T} \left[ \sum_{t=0}^T \frac{2B_t^{\theta_T} - 1}{N_T - 1} > \ln M \right] \leq \frac{1}{(\ln M)^\gamma} \mathbb{E}_\mu^{\theta_T} \left[ \left( \sum_{t=0}^T \frac{2B_t^{\theta_T} - 1}{N_T - 1} \right)^\gamma \right].$$

The proof of (92) follows by noting that  $N_T \sim T^{1/\gamma}$  and by using

$$\mathbb{E}_\mu^{\theta_T} \left[ \left( \frac{\sum_{t=0}^T 2B_t^{\theta_T} - 1}{T^{1/\gamma}} \right)^\gamma \right] \leq \mathbb{E}_\mu^{\theta_T} \left[ \frac{\sum_{t=0}^T (2B_t^{\theta_T})^\gamma}{T} \right] \leq 2^\gamma \sup_{t \geq 0} \sup_{\theta \in \Theta} \mathbb{E}_\mu^\theta [(B_t^\theta)^\gamma] < \infty,$$

where the last inequality follows from Lemma 17. □

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