



HAL
open science

Stability of Conductivities in an Inverse Problem in the Reaction-diffusion System in Electrocardiology

Bedr'Eddine Ainseba, Mostafa Mostafa Bendahmane, He Yuan

► **To cite this version:**

Bedr'Eddine Ainseba, Mostafa Mostafa Bendahmane, He Yuan. Stability of Conductivities in an Inverse Problem in the Reaction-diffusion System in Electrocardiology. *Networks and Heterogeneous Media*, 2015, 10.3934/nhm.2015.10.xx . hal-01256802

HAL Id: hal-01256802

<https://hal.science/hal-01256802>

Submitted on 22 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

STABILITY OF CONDUCTIVITIES IN AN INVERSE PROBLEM IN THE REACTION-DIFFUSION SYSTEM IN ELECTROCARDIOLOGY

BEDR'EDDINE AINSEBA AND MOSTAFA BENDAHMANE

Institut de Mathématiques de Bordeaux
UMR CNRS 5251, Université de Bordeaux
3 ter Place de la Victoire, 33076 Bordeaux cedex, France

YUAN HE

School of Mathematics and Statistics, Lanzhou University
Lanzhou, 730000, China

(Communicated by Kenneth Karlsen)

ABSTRACT. In this paper, we study the stability result for the conductivities diffusion coefficients to a strongly reaction-diffusion system modeling electrical activity in the heart. To study the problem, we establish a Carleman estimate for our system. The proof is based on the combination of a Carleman estimate and certain weight energy estimates for parabolic systems.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded connected open set whose boundary $\partial\Omega$ is regular enough. Let $T > 0$ and ω be a small nonempty subset of Ω . We will denote $(0, T) \times \Omega$ by Q_T and $(0, T) \times \partial\Omega$ by Σ_T .

To state the model of the cardiac electric activity in Ω ($\Omega \subset \mathbb{R}^3$ being the natural domain of the heart), we set $u_i = u_i(t, x)$ and $u_e = u_e(t, x)$ to represent the spacial cellular and location $x \in \Omega$ of the intracellular and extracellular electric potentials respectively. Their difference $v = u_i - u_e$ is the transmembrane potential. The anisotropic properties of the two media are modeled by intracellular and extracellular conductivity tensors $M_i(x)$ and $M_e(x)$. The surface capacitance of the membrane is represented by the constant $c_m > 0$. The transmembrane ionic current is represented by a nonlinear function $h(v)$.

The equations governing the cardiac electric activity are given by the coupled reaction-diffusion system:

$$\begin{cases} c_m \partial_t v - \operatorname{div}(M_i(x) \nabla u_i) + h(v) = f \chi_\omega, & \text{in } Q_T, \\ c_m \partial_t v + \operatorname{div}(M_e(x) \nabla u_e) + h(v) = g \chi_\omega, & \text{in } Q_T, \end{cases} \quad (1)$$

where f and g are stimulation currents applied to Ω . We complete this model with Dirichlet boundary conditions for the intra- and extracellular electric potentials

$$u_i = 0, \quad u_e = 0, \quad \text{on } \Sigma_T, \quad (2)$$

2010 *Mathematics Subject Classification.* Primary: 35K57, 35R30.

Key words and phrases. Inverse problem, stability, reaction-diffusion system, cardiac electric field, Carleman estimates.

M. Bendahmane is supported by the Moroccan project FINCOME-2014. Yuan He is supported by lzujbky-2014-22.

and with initial data for the transmembrane potential

$$v(0, x) = v_0(x), \quad x \in \Omega. \quad (3)$$

It is important to point out that realistic models describing electrical activities include a system of ODEs for computing the ionic current as a function of the transmembrane potential and a series of additional ‘‘gating variables’’, which aim to model the ionic transfer across the cell membrane.

Assume that the intra and extracellular stimulations are equal: $f\chi_\omega = g\chi_\omega$. If $M_i = \mu M_e$ for some constant $\mu \in \mathbb{R}$, then by multiplying the second equation in (1) by μ and adding it to the first equation in (1) one gets the first equation in the following parabolic-elliptic system:

$$\begin{cases} c_m \partial_t v - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v) = -h(v) + f\chi_\omega, & \text{in } Q_T, \\ \operatorname{div}(M(x) \nabla u_e) = \operatorname{div}(M_i(x) \nabla v), & \text{in } Q_T, \\ v(0, x) = v_0(x), \quad u_e(0, x) = u_{e,0}(x), & \text{in } \Omega, \\ v = 0, \quad u_e = 0, & \text{on } \Sigma_T. \end{cases} \quad (4)$$

The second equation is obtained by computing the difference of the two equations in (1). Here $M = M_i + M_e$. System (4) is known as the monodomain model.

We approximate the above model (4) by the following family of parabolic equations

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -h(v^\varepsilon) + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(0, x) = v_0(x), \quad u_e^\varepsilon(0, x) = u_{e,0}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (5)$$

ε is a fixed small constant. Since $v = u_i - u_e$ in the bidomain model, it is natural to decompose the initial condition v_0 as $v_0 = u_{i,0} - u_{e,0}$. Note that when $\varepsilon \rightarrow 0$ in (5), we obtain the classical monodomain model.

In this work, we study the stability result for the conductivities diffusion coefficients to the following linearized system of (5) with semi-initial conditions

$$\begin{cases} c_m \partial_t v^\varepsilon - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla v^\varepsilon) = -a(t, x) v^\varepsilon + f^\varepsilon \chi_\omega, & \text{in } Q_T, \\ \varepsilon \partial_t u_e^\varepsilon - \operatorname{div}(M(x) \nabla u_e^\varepsilon) = \operatorname{div}(M_i(x) \nabla v^\varepsilon), & \text{in } Q_T, \\ v^\varepsilon(\theta, x) = v_\theta(x), \quad u_e^\varepsilon(\theta, x) = u_{e,\theta}(x), & \text{in } \Omega, \\ v^\varepsilon = 0, \quad u_e^\varepsilon = 0, & \text{on } \Sigma_T, \end{cases} \quad (6)$$

where $a(t, x)$ and its derivative with respect to t exist and are bounded in Q_T . For some $\theta \in (0, T)$, the semi-initial conditions $v_\theta(x)$, $u_{e,\theta}(x)$ are sufficiently regular. The unknown conductivity tensors M and M_e are assumed to be sufficiently smooth and shall be kept independent of time t .

The existence of weak solutions of (1) is proved in [10] by the theory of evolution variational inequalities in Hilbert space. Then Bendahmane and Karlsen [2] proved the existence and uniqueness for a nonlinear version of the bidomain equations (1) by a uniformly parabolic regularization of the system and the Faedo-Galerkin method. Moreover, Bendahmane and Chaves-Silva [1] studied exact null controllability to (1) for each $\varepsilon > 0$ by establishing estimates for its dual system. To learn more about the cardiac problems, one can refer to the work of Bendahmane et al. [3, 4]. However, it is noted that there is no stability result for the inverse bidomain model.

Since the pioneer work due to A.L. Bukhgeim and M.V. Klibanov [6, 7, 8], who generalized the method of global Carleman estimates in the context of inverse problems, three fundamental issues have been successfully studied: uniqueness, stability in determining coefficients, and numerical methods [16], [13, 14, 17, 20, 23, 24, 15]).

The paper by Cristofol et al. [11] obtains the stability results for reaction-diffusion system of two equations with constant coefficients using a Carleman estimate. Then Sakthivel et al. [21] established the stability results for Lotka-Volterra competition-diffusion system of three equations with variable diffusion coefficients. Our inverse stability results are new because system (6) contains a strong coupling term. The technics we shall discuss are similar to the framework using Carleman estimates for inverse problems but the obtained estimates differs from those of [24], [21] because of the strongly coupled terms.

Let $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$ be a solution of system (6) with conductivity tensors (\tilde{M}_e, \tilde{M}) and semi-initial data $(\tilde{v}_\theta^\varepsilon, \tilde{u}_{e,\theta}^\varepsilon)$. Then setting $A_1 = v^\varepsilon - \tilde{v}^\varepsilon$, $A_2 = u_e^\varepsilon - \tilde{u}_e^\varepsilon$, $g_1 = M_e - \tilde{M}_e$ and $g_2 = M - \tilde{M}$, we obtain

$$\begin{cases} c_m \partial_t A_1 - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla A_1(t, x)) = -a(t, x) A_1(t, x) + F(g_1, \nabla \tilde{v}^\varepsilon), & \text{in } Q, \\ \varepsilon \partial_t A_2 - \operatorname{div}(M(x) \nabla A_2) = \operatorname{div}(M_i(x) \nabla A_1) + G(g_2, \nabla u_e^\varepsilon), & \text{in } Q, \\ A_1(\theta, x) = A_1^\theta(x), \quad A_2(\theta, x) = A_2^\theta(x), & \text{in } \Omega, \\ A_1(t, x) = 0, \quad A_2(t, x) = 0, & \text{on } \Sigma, \end{cases} \quad (7)$$

where

$$F = \frac{\mu}{\mu+1} \operatorname{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$$

and

$$G = \operatorname{div}(g_2(x) \nabla \tilde{u}_e^\varepsilon).$$

Throughout the paper, we make the following assumptions:

Assumption 1.1. *The conductivity tensors $M_e(x)$, $M_i(x)$ and $M(x)$ are C^∞ , bounded, symmetric, semi-definite, and elliptic matrixes (there exists $\beta > 0$ such that $\sum_{i,j}^3 M_{i,j} \xi_i \xi_j \geq \beta |\xi|^2$ for all $\xi \in \mathbb{R}^3$). All their derivatives up to the third order are respectively bounded by the positive constants $\gamma_1, \gamma_2, \gamma_3$.*

Assumption 1.2. *Assume the bounded measurements $\partial_t A_1$ and $\partial_t A_2$ in $(0, T) \times \omega$ are given. Also $A_i(\theta, x)$, $\nabla A_i(\theta, x)$, $\Delta A_i(\theta, x)$ and $\nabla(\Delta A_i(\theta, x))$ for some fixed $\theta \in (0, T)$, where $i = 1, 2$ in Ω are given.*

Now the question of interest is whether we can determine the conductivity tensors M_e and M by the two measurements.

In details, let $(v^\varepsilon, u_e^\varepsilon)$ and $(\tilde{v}^\varepsilon, \tilde{u}_e^\varepsilon)$ be the solutions of the system (6) with two different conductivities. There exist a constant C with $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$, such that the following estimate holds:

$$\begin{aligned} & \int_{\Omega} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx \\ & \leq C \left(\int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\ & \quad + C \int_{\tilde{\omega}} (|M_e - \tilde{M}_e|^2 + |M - \tilde{M}|^2 + |\nabla(M_e - \tilde{M}_e)|^2 + |\nabla(M - \tilde{M})|^2) dx. \end{aligned} \quad (8)$$

2. A Carleman type estimate. In this section, we prove the Carleman estimate based on the standard technique for general parabolic equations. In order to frame a Carleman type estimate, we shall first introduce a particular type of weight functions.

2.1. Weight functions. First, we introduce weight functions for the parabolic equations given in [12].

Let $\tilde{\omega} \subset\subset \omega$ be a nonempty bounded set of Ω , and $\psi \in C^2(\bar{\Omega})$ such that

$$\psi(x) > 0, \text{ for any } x \in \Omega,$$

$$\psi(x) = 0, \text{ for any } x \in \partial\Omega,$$

$$|\nabla\psi(x)| > 0, \text{ for any } x \in \bar{\Omega} \setminus \tilde{\omega}.$$

Then we introduce another two weight functions:

$$\phi(t, x) = \frac{e^{\lambda\psi(x)}}{\beta(t)}, \quad (9)$$

$$\alpha(t, x) = \frac{e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi(x)}}{\beta(t)}, \quad (10)$$

where $\lambda > 1$, $t \in (0, T)$ and $\beta(t) = t(T - t)$. Note that the weight function α is positive, and blows up to ∞ as $t = 0$ or $t = T$. As a result, $e^{-2s\alpha}$ and $\phi e^{-2s\alpha}$ are smooth. Even they vanish when $t = 0$ or $t = T$. It can be seen that $\phi(t, x) \geq C > 0$ for all $(t, x) \in Q$, and $e^{-\epsilon\alpha}\phi^m \leq C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$.

Before proving the main estimate, we give the following estimates for the two weight functions α and ϕ . Note that throughout the paper we will denote C as a generic positive constant. After some computations, we can obtain the following estimates:

$$\begin{cases} |\phi_t| = \frac{|2t-T|}{e^{\lambda\psi}} \phi^2 \leq CT\phi^2, \\ |\alpha_t| = \frac{|2t-T|}{\beta^2} (e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi}) \leq CT\phi^2, \\ |\alpha_{tt}| = \frac{2|T^2-3tT+3t^2|}{\beta^3} (e^{2\lambda\|\psi\|_{C(\Omega)}} - e^{\lambda\psi}) \leq CT\phi^3. \end{cases} \quad (11)$$

Furthermore, we also have

$$\begin{cases} \nabla\phi = \lambda\phi\nabla\psi, \\ \nabla\alpha = -\lambda\phi\nabla\psi, \\ \phi^{-1} \leq \left(\frac{T}{2}\right)^2. \end{cases} \quad (12)$$

Refer to [12] for the details.

2.2. Main proof of a Carleman type estimate. Let us set $Q_\omega = (0, T) \times \omega$. For each positive integer m , we denote the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to m are also in $L^p(\Omega)$ with the norm denoted $\|\cdot\|_{L^p(\Omega)}$, by $W^{m,p}(\Omega)$ with $p > 1$ or $p = \infty$. When $p = 2$, we denote $W^{m,p}$ by $H^m(\Omega)$. Moreover, let $L^2(0, T; H^1(\Omega))$ be the space of all equivalent classes of square integrable functions from $(0, T)$ to $H^1(\Omega)$. For the space $L^2(0, T; L^\infty(\Omega))$, we define it in the same way.

Let A_1 be the solution of the first equation of (7) with help of using Assumption 1.1. We apply the Carleman estimate (see Theorem 6.1 in [1].) derived for the parabolic equations to the first equation in (7). For $\lambda > \lambda_0 \geq 1$, $s \leq s_0(T + T^2 + T^4)$, there exists a constant C depending on Ω , ω , ψ and β so that

$$\mathcal{I}(A_1) \leq C \left(\int_Q e^{-2s\alpha} |F|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right), \quad (13)$$

where $\tilde{\omega} \subset\subset \omega_1 \subset\subset \omega$, and

$$\begin{aligned} \mathcal{I}(A_1) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_1| + |\Delta A_1|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_1|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_1|^2 dt dx. \end{aligned} \quad (14)$$

Similarly, for $\lambda > \lambda_0 \geq 1$, $s \geq s_0(T + T^2 + T^4)$, there exists a constant C depending on Ω , ω , ψ and β satisfying

$$\begin{aligned} &\mathcal{I}(A_2) \\ &\leq C \left(\int_Q e^{-2s\alpha} (|G|^2 |\nabla(M_i \nabla A_1)|^2) dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right) \\ &\leq C \left(\int_Q e^{-2s\alpha} |G|^2 dt dx + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx + \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \right) \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx, \end{aligned} \quad (15)$$

with

$$\begin{aligned} \mathcal{I}(A_2) &= \int_Q (s\lambda\phi)^{-1} e^{-2s\alpha} (|\partial_t A_2| + |\Delta A_2|^2) dt dx + \int_Q s\lambda^2 \phi e^{-2s\alpha} |\nabla A_2|^2 dt dx \\ &\quad + s^3 \lambda^4 \int_Q e^{-2s\alpha} \phi^3 |A_2|^2 dt dx. \end{aligned} \quad (16)$$

Now coupling the above inequalities (13) and (15), we have

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq C \left(\int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right) + C \int_Q e^{-2s\alpha} |M_i \Delta A_1|^2 dt dx \\ &\quad + C \int_Q e^{-2s\alpha} |\nabla M_i \nabla A_1|^2 dt dx \end{aligned}$$

for sufficiently large $s \geq s_0(T + T^2 + T^4)$ and $\lambda \geq \lambda_0$. From the definition of \mathcal{I}_1 , also M_i and ∇M_i being bounded, we obtain

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_{\omega_1}} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right). \end{aligned} \quad (17)$$

Then it can be summarized as our desired Carleman estimate as follows.

Theorem 2.1. *Let $\psi(x)$, $\phi(t, x)$ and $\alpha(t, x)$ be defined as in the above subsection, $a(t, x)$ is a bounded function. Moreover, Assumption 1.1 holds. Then there exist λ_0 and s_0 such that for all $\lambda > \lambda_0 \geq 1$ and sufficiently large enough $s > s_0$, the following inequality is true.*

$$\begin{aligned} s\mathcal{I}(A_1) + \mathcal{I}(A_2) &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (s|F|^2 + |G|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_1|^2 dt dx \right. \\ &\quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |A_2|^2 dt dx \right), \end{aligned}$$

where $\tilde{C} > 0$ is a constant depending on Ω , T , ω , γ_2 .

3. Stability of the conductivities. In this section, we study the stability of the conductivity tensors M_e and M . Then an inequality is established which estimates g_1 , g_2 , ∇g_1 , ∇g_2 with an upper bound given by some Sobolev norms of the derivative of A_1 and A_2 over Q_ω , certain spatial derivative of $A_j(\theta, \cdot)$, $j = 1, 2$, where $\theta \in (0, T)$ makes $\frac{1}{\beta(t)}$ attain its minimum value and the Sobolev norm of g_1 , g_2 , ∇g_1 , ∇g_2 in a small space $\tilde{\omega}$.

First, we let $B_1 = \partial_t A_1$, $B_2 = \partial_t A_2$. Using this and (7), we get the following system:

$$\begin{cases} c_m \partial_t B_1 - \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla B_1(t, x)) = -\partial_t a(t, x) A_1(t, x) - a(t, x) B_1 \\ \quad + F'(g_1, \nabla \tilde{v}^\varepsilon), \text{ in } Q_T, \\ \varepsilon \partial_t B_2 - \operatorname{div}(M(x) \nabla B_2) = \operatorname{div}(M_i(x) \nabla B_1) + G'(g_2, \nabla u_e^\varepsilon), \text{ in } Q_T, \\ c_m B_1(\theta, x) = H_1^\theta(x), \quad B_2(\theta, x) = H_2^\theta(x), \text{ in } \Omega, \\ B_1(t, x) = 0, \quad B_2(t, x) = 0, \text{ on } \Sigma_T, \end{cases} \quad (18)$$

where

$$F' = \frac{\mu}{\mu+1} \operatorname{div}(g_1(x) \nabla(\partial_t \tilde{v}^\varepsilon)), \quad G' = \operatorname{div}(g_2(x) \nabla(\partial_t \tilde{u}_e^\varepsilon))$$

and

$$\begin{cases} c_m B_1(\theta, x) = \frac{\mu}{\mu+1} \operatorname{div}(M_e(x) \nabla A_1(\theta, x)) - a(\theta, x) A_1(\theta, x) + F|_{t=\theta} = H_1^\theta, \text{ in } Q_T, \\ \varepsilon B_2(\theta, x) = \operatorname{div}(M(x) \nabla A_2(\theta, x)) + \operatorname{div}(M_i(x) \nabla A_1(\theta, x)) + G|_{t=\theta} = H_2^\theta, \text{ in } Q_T, \\ A_1(t, x) = A_1(0, x) + \int_0^t B_1(s, x) ds, \text{ in } Q_T. \end{cases} \quad (19)$$

Indeed, to prove the main result here we need to impose some regularity properties as follows.

Assumption 3.1. Suppose v_θ^ε and $u_{e,\theta}^\varepsilon$ are C^3 real valued functions. Then all their derivatives up to order three are bounded and satisfy $|\nabla \psi \cdot \nabla v_\theta^\varepsilon| \geq \delta > 0$, $|\nabla \psi \cdot \nabla u_{e,\theta}^\varepsilon| \geq \delta > 0$, on $\overline{\Omega} \setminus \tilde{\omega}$, where $\tilde{\omega} \subset \subset \omega \subset \subset \Omega$.

Assumption 3.2. Suppose $(|\Delta \tilde{v}^\varepsilon|, |\Delta \tilde{u}_e^\varepsilon|)$, $(|\nabla(\Delta \tilde{v}^\varepsilon)|, |\nabla(\Delta \tilde{u}_e^\varepsilon)|)$, $(|\nabla(\partial_t \tilde{v}^\varepsilon)|, |\nabla(\partial_t \tilde{u}_e^\varepsilon)|)$ and $(|\Delta(\partial_t \tilde{v}^\varepsilon)|, |\Delta(\partial_t \tilde{u}_e^\varepsilon)|)$ are bounded by a positive constant.

Before start proving our main conclusion, we need to give the following Lemma 3.3, which will be useful in the following part. We define the following operators P_0 and Q_0 and the initial conditions on α and ϕ at $t = \theta$:

$$P_0 h = \nabla U_\theta \cdot \nabla h, \quad Q_0(e^{-s\alpha^\theta} h) = e^{-s\alpha^\theta} P_0 h \text{ and } \zeta(\theta, x) = \zeta^\theta \text{ for } \zeta = \alpha, \phi.$$

Lemma 3.3. Consider the first order partial differential operator $P_0 h = \nabla U_\theta \cdot \nabla h$, where U_θ satisfies Assumption 3.1. Then there exists a constant $C > 0$, such that for sufficiently large enough λ and s , the following result holds:

$$s^2 \lambda^2 \int_\Omega \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \leq C \left(\int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 h|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |h|^2 dx \right),$$

with $\theta \in (0, T)$ and $h \in H_0^1(\Omega)$.

Proof. Let $B_1 = e^{-s\alpha^\theta} h$, we have

$$Q_0 B_1 = e^{-s\alpha^\theta} P_0(e^{s\alpha^\theta} B_1) = P_0 B_1 + s B_1 P_0 \alpha^\theta, \quad (20)$$

$h \in H_0^1(\Omega)$. Then we take the square of both sides in (20), multiply $\frac{1}{\phi^\theta}$ and integrate by parts with respect to space variable for both sides of (20) as follows:

$$\begin{aligned}
 & \int_{\Omega} \frac{1}{\phi^\theta} (Q_0 B_1)^2 dx \\
 = & \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^\theta} (P_0 \alpha^\theta)^2 dx \\
 & + \int_{\Omega} 2s \frac{1}{\phi^\theta} B_1 (P_0 B_1) (P_0 \alpha^\theta) dx \\
 = & \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \frac{1}{\phi^\theta} \lambda^2 (\phi^\theta)^2 (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
 & - \int_{\Omega} 2\lambda s \frac{1}{\phi^\theta} B_1 (\nabla U^\theta \cdot \nabla B_1) (\nabla U^\theta \cdot \phi^\theta \nabla \psi^\theta) dx \\
 = & \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
 & - \int_{\Omega} 2\lambda s B_1 (\nabla U^\theta \cdot \nabla B_1) (\nabla U^\theta \cdot \nabla \psi^\theta) dx \\
 = & \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
 & - \int_{\Omega} \lambda s P_0 \psi^\theta \nabla U^\theta \nabla (B_1^2) dx \\
 = & \int_{\Omega} \frac{1}{\phi^\theta} (P_0 B_1)^2 dx + \int_{\Omega} s^2 B_1^2 \lambda^2 \phi^\theta (\nabla U^\theta \cdot \nabla \psi^\theta)^2 dx \\
 & + \int_{\Omega} \lambda s \nabla (P_0 \psi^\theta \nabla U^\theta) B_1^2 dx \\
 \geq & \int_{\Omega} s^2 \lambda^2 B_1^2 \phi^\theta |\nabla U^\theta \cdot \nabla \psi^\theta|^2 dx + \int_{\Omega} s \lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx \\
 \geq & s^2 \lambda^2 \delta^2 \left(\int_{\Omega} |B_1|^2 \phi^\theta dx - \int_{\tilde{\omega}} |B_1|^2 \phi^\theta dx \right) + \int_{\Omega} s \lambda \nabla (P_0 \psi^\theta \nabla U^\theta) |B_1|^2 dx,
 \end{aligned}$$

where we used Assumption 3.1 in the last step. Thus we obtain

$$\begin{aligned}
 & s^2 \lambda^2 \delta^2 \left(\int_{\Omega} |B_1|^2 \phi^\theta dx - \int_{\tilde{\omega}} |B_1|^2 \phi^\theta dx \right) \\
 \leq & \int_{\Omega} \frac{1}{\phi^\theta} |Q_0 B_1|^2 dx + \int_{\Omega} s \lambda |\nabla (P_0 \psi^\theta \nabla U^\theta)| |B_1|^2 dx.
 \end{aligned}$$

From Assumption 3.2 and (12), we have

$$\begin{aligned}
 & s^2 \lambda^2 \delta^2 \int_{\Omega} e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
 \leq & \int_{\tilde{\omega}} s^2 \lambda^2 \delta^2 e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx + C_1 T^2 \int_{\Omega} s \lambda e^{-2s\alpha^\theta} \phi^\theta |h|^2 dx \\
 & + \int_{\Omega} e^{-2s\alpha^\theta} |P_0 h|^2 \frac{1}{\phi^\theta} dx. \tag{21}
 \end{aligned}$$

Taking $\lambda \geq 1$ and $s \geq \frac{2C_1 T^2}{\delta^2}$, we conclude the proof. \square

With the help of the Lemma 3.3, we are proving the following proposition.

Proposition 1. *Let (A_1, A_2) be the solution of (7), and (B_1, B_2) be the solution of (18). Suppose all the conditions of Theorem 2.1 and Assumption 3.1 hold. Then there exists a constant $C = C(\gamma_1, \gamma_2, \delta) > 0$ such that for sufficiently large enough s and λ the following estimate is true.*

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} e^{-2s\alpha^\theta} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\ & \leq C \sum_{j=1}^9 E_j + C s^2 \lambda^2 \int_{\tilde{\omega}} e^{-2s\alpha^\theta} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx, \end{aligned} \quad (22)$$

for any $g_1, g_2 \in H_0^2(\Omega)$, where the functions E_j , are given as follows:

$$\begin{aligned} E_1 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_1^\theta|^2 dx, \\ E_2 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |B_2^\theta|^2 dx, \\ E_3 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_1^\theta|^2 dx, \\ E_4 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |\nabla B_2^\theta|^2 dx, \\ E_5 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \left(|A_1^\theta|^2 + |\nabla A_1^\theta|^2 + |\Delta A_1^\theta|^2 \right) dx, \\ E_6 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \left(|\nabla A_2^\theta|^2 + |\Delta A_2^\theta|^2 \right) dx, \\ E_7 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} \left(|A_1^\theta|^2 + \sum_{j=1}^2 \left(|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2 \right) \right. \\ & \quad \left. + |\nabla(g_1 \Delta \tilde{v}_\theta^\varepsilon)|^2 + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon)| \right) dx, \\ E_8 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_1 \Delta \tilde{v}_\theta^\varepsilon|^2 dx, \\ E_9 &= \int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |g_2 \Delta \tilde{u}_{e,\theta}^\varepsilon|^2 dx. \end{aligned}$$

Proof. Due to the value of the solutions satisfying the first equation in (18) at $t = \theta$, and $F = \operatorname{div}(g_1(x) \nabla \tilde{v}^\varepsilon)$, from (19) we obtain

$$P_0 g_1 = \nabla \tilde{v}_\theta^\varepsilon \cdot \nabla g_1 = c_m B_1^\theta + a(\theta, x) A_1^\theta - \frac{\mu}{\mu + 1} \operatorname{div}(M_\varepsilon \nabla A_1^\theta) - g_1 \Delta \tilde{v}_\theta^\varepsilon.$$

Note that we replace h by g_1 when choosing U_θ as $\tilde{v}_\theta^\varepsilon$. Therefore, inspired by Lemma 3.3, we get

$$\begin{aligned} & s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \\ & \leq C \left(\int_{\Omega} \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} |P_0 g_1|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^\theta e^{-2s\alpha^\theta} |g_1|^2 dx \right) \end{aligned}$$

$$\begin{aligned}
 &\leq C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|B_1^{\theta}|^2 + |A_1^{\theta}|^2 + \left(\frac{\mu}{\mu+1} \right)^2 (|\nabla M_e|^2 |\nabla A_1^{\theta}|^2 \right. \right. \\
 &\quad \left. \left. + |M_e|^2 |\Delta A_1^{\theta}|^2) + |g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}|^2 \right) dx + Cs^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_1|^2 dx \\
 &\leq C(\gamma_1)(E_1 + E_5 + E_8) + Cs^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_1|^2 dx. \tag{23}
 \end{aligned}$$

Similarly, from the value of the solutions satisfying the second equation in (18) at $t = \theta$, and $G = \operatorname{div}(g_2(x) \nabla \tilde{u}_{\varepsilon}^{\theta})$, we obtain

$$P_0 g_2 = \nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla g_2 = \varepsilon B_2^{\theta} - \operatorname{div}(M \nabla A_2^{\theta}) - \operatorname{div}(M_i \nabla A_1^{\theta}) - g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}.$$

It leads to

$$\begin{aligned}
 &s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \\
 &\leq C \left(\int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} |P_0 g_2|^2 dx + s^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \right) \\
 &\leq C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|B_2^{\theta}|^2 + |\nabla M|^2 |\nabla A_2^{\theta}|^2 + |M|^2 |\Delta A_2^{\theta}|^2 \right. \\
 &\quad \left. + |\nabla M_i|^2 |\nabla A_1^{\theta}|^2 + |M_i|^2 |\Delta A_1^{\theta}|^2 + |g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}|^2 \right) dx \\
 &\quad + Cs^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx \\
 &\leq C(\gamma_2, \gamma_3)(E_2 + E_5 + E_6 + E_9) + Cs^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} |g_2|^2 dx. \tag{24}
 \end{aligned}$$

On the other hand, from the expression of $P_0 g_1$, we can see that,

$$\begin{aligned}
 P_0 \nabla g_1 &= \nabla \tilde{v}_{\theta}^{\varepsilon} \cdot \nabla (\nabla g_1) = \nabla (\nabla \tilde{v}_{\theta}^{\varepsilon} \cdot \nabla g_1) - \nabla g_1 \Delta \tilde{v}_{\theta}^{\varepsilon} \\
 &= c_m \nabla B_1^{\theta} + \nabla a^{\theta} A_1^{\theta} + a^{\theta} \nabla A_1^{\theta} - \frac{\mu}{\mu+1} \Delta (M_e \nabla A_1^{\theta}) - \nabla (g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}) - \nabla g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}.
 \end{aligned}$$

Similarly, we also have

$$\begin{aligned}
 P_0 \nabla g_2 &= \nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla (\nabla g_2) = \nabla (\nabla \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \cdot \nabla g_2) - \nabla g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon} \\
 &= \varepsilon \nabla B_2^{\theta} - \Delta (M \nabla A_2^{\theta}) - \Delta (M_i \nabla A_1^{\theta}) - \nabla (g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}) - \nabla g_2 \Delta \tilde{u}_{\varepsilon, \theta}^{\varepsilon}.
 \end{aligned}$$

Using the same method to preceding estimates and Lemma 3.3, it follows that

$$\begin{aligned}
 &s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |\nabla g_1|^2 dx + s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} |\nabla g_2|^2 dx \\
 &\leq C(\gamma_1, \gamma_2, \gamma_3)(E_3 + E_4 + E_7) + Cs^2 \lambda^2 \int_{\tilde{\omega}} \phi^{\theta} e^{-2s\alpha^{\theta}} (|\nabla g_1|^2 + |\nabla g_2|^2) dx. \tag{25}
 \end{aligned}$$

Combing the above three estimates (23), (24) and (25), the proof is complete. \square

In order to prove the main conclusion, we need to get further estimations for E_j , $j = 1, 2, 3, 4$. The Carleman estimate in the previous section plays an important role in obtaining these estimations.

Lemma 3.4. *Assume all the conditions in Theorem 2.1 are satisfied. Then there exists a constant C depending only on \tilde{C} , such that for any $\lambda \geq \lambda_0$ and $s \geq s_1(\Omega, T)$, the following inequality holds:*

$$E_1 + E_2 \leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2), \quad (26)$$

where $\mathcal{E}(g_1, g_2, B_1, B_2)$ is defined as follows

$$\mathcal{E}(g_1, g_2, B_1, B_2) = \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx. \quad (27)$$

Proof. Note that $\alpha(0, x) = +\infty$. As a result of (11) and (12), we have

$$\begin{aligned} & \int_\Omega \frac{1}{\phi^\theta} e^{-2s\alpha^\theta} (|B_1^\theta|^2 + |B_2^\theta|^2) dx \\ &= \int_0^\theta \frac{\partial}{\partial t} \left(\int_\Omega \phi^{-1} e^{-2s\alpha} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dx \right) dt \\ &= \int_{Q_\theta} \phi^{-1} (-2s) \partial_t e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx - \int_{Q_\theta} \phi^{-2} \partial_t \phi e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \\ & \quad + 2 \int_{Q_\theta} \phi^{-1} e^{-2s\alpha} (2B_1 \partial_t B_1 + 2B_2 \partial_t B_2) dt dx \\ &\leq C(sT^5 + s\lambda T^8 + T^7) \int_Q \phi^3 e^{-2s\alpha} (|B_1(t, x)|^2 + |B_2(t, x)|^2) dt dx \\ & \quad + (s\lambda)^{-1} \int_Q \phi^{-1} e^{-2s\alpha} (|\partial_t B_1|^2 + |\partial_t B_2|^2) dt dx \\ &\leq C(s\mathcal{I}(B_1) + \mathcal{I}(B_2)), \end{aligned}$$

where $Q_\theta = (0, \theta) \times \Omega$, $\mathcal{I}(B_j)|_{j=1,2}$ is defined in (14) and (16), for any $s \geq C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)$ and $\lambda \geq 1$. Then using the estimate given in Theorem 2.1 to the system (18), we obtain

$$\begin{aligned} & s\mathcal{I}(B_1) + \mathcal{I}(B_2) \\ &\leq \tilde{C} \left(\int_Q e^{-2s\alpha} (s|F'|^2 + |G'|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ & \quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + \int_Q s e^{-2s\alpha} |\partial_t a(t, x)| \left(\int_0^t |B_1(s, x)| ds \right. \right. \\ & \quad \left. \left. + A_1(0, x) \right)^2 dt dx \right) \\ &\leq \tilde{C} \left(s \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^4 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_1|^2 dt dx \right. \\ & \quad \left. + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} |B_2|^2 dt dx + C_1 T s \int_Q e^{-2s\alpha} |B_1|^2 dt dx \right). \end{aligned} \quad (28)$$

Due to this term $C_1 T \int_Q e^{-2s\alpha} |B_1|^2 dt dx$ can be absorbed by $\mathcal{I}(B_1)$, $\lambda > 1$ and s being large enough, we have

$$\mathcal{I}(B_1) + \mathcal{I}(B_2) \leq \tilde{C}_1 s \lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2).$$

Thus for $s \geq s_1 = \max\{s_0, C(T^{\frac{5}{2}} + T^{\frac{7}{3}} + T^4)\}$, the proof is complete. \square

Lemma 3.5. *Let Assumption 3.1 be satisfied. Then there exists $\lambda_1 = \max\{\lambda_0, C(\gamma_1, \gamma_2, \gamma_3)\}$ and $s_2 = \max\{s_1, C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)\}$ for all $\lambda \geq \lambda_1$, $s \geq s_2$, the following inequality holds:*

$$E_3 + E_4 \leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2),$$

where $\mathcal{E}(g_1, g_2, B_1, B_2)$ is defined in (27).

Proof. First, we define

$$\pi(B_1) := e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1).$$

We multiply the first equation in (18) by $\pi(B_1)$ and we integrate over Q_θ , the result is

$$\int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) = \int_{Q_\theta} \pi(B_1) \left(\frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F'(g_1, \nabla \tilde{v}^\varepsilon) \right) dt dx. \quad (29)$$

We divide (29) into left and right sides integrals to estimate separately. Firstly, we integrate the left side integral by parts, and get

$$\begin{aligned} & - \int_{Q_\theta} c_m \partial_t B_1 \pi(B_1) dt dx \\ &= - \int_{Q_\theta} c_m \partial_t B_1 e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) dt dx \\ &= \int_{Q_\theta} c_m \partial_t B_1 \nabla(e^{-2s\alpha} \phi^{-1}) M_e \nabla B_1 dt dx + \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \quad (30)$$

Note that $|\nabla(\phi^{-1} e^{-2s\alpha})| \leq s\lambda e^{-2s\alpha}$ for $s \geq CT^2$. Thus we have

$$\begin{aligned} \mathcal{J}_1 &\leq s\lambda (C \|M_e\|_{L^\infty(\Omega)}^2) s\lambda \int_{Q_\theta} \phi e^{-2s\alpha} |\nabla B_1|^2 dt dx + (s\lambda)^{-1} \int_{Q_\theta} \phi^{-1} e^{-2s\alpha} |\partial_t B_1|^2 dt dx \\ &\leq s\lambda \mathcal{I}(B_1) \end{aligned} \quad (31)$$

for any $s \geq C(\gamma_2)T^2$. Integrating by parts with respect to time in \mathcal{J}_2 , we have

$$\begin{aligned} \mathcal{J}_2 &= \frac{1}{2} \int_{Q_\theta} c_m \partial_t (|\nabla B_1|^2) e^{-2s\alpha} \phi^{-1} M_e dt dx \\ &= -\frac{1}{2} \int_{Q_\theta} c_m |\nabla B_1|^2 \partial_t (e^{-2s\alpha} \phi^{-1}) M_e dt dx \\ &\quad + \frac{1}{2} \int_{\Omega} (c_m |\nabla B_1|^2 e^{-2s\alpha} \phi^{-1} M_e) |_{t=\theta} dx. \end{aligned} \quad (32)$$

Here,

$$\begin{aligned} |\partial_t (e^{-2s\alpha} \phi^{-1})| &= |e^{-2s\alpha} \phi^{-2} \phi_t + e^{-2s\alpha} \phi^{-1} (-2s) \alpha_t| \\ &= |e^{-2s\alpha} \phi^{-1} (\phi^{-1} \phi_t - 2s\alpha_t)| \\ &\leq |e^{-2s\alpha} \phi^{-1}| \left(\frac{T^2}{4} + 2s \right) CT \phi^2 \\ &\leq Cs\lambda^2 \phi e^{-2s\alpha}, \end{aligned}$$

for $\lambda > 1$ and $s \geq CT^3$. Therefore,

$$\mathcal{J}_2 \geq \frac{1}{2} \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2) |_{t=\theta} dx - Cs\lambda^2 \int_{Q_\theta} c_m \phi e^{-2s\alpha} M_e |\nabla B_1|^2 dt dx. \quad (33)$$

Now coming to the right side integrals of (29), we have

$$\begin{aligned} & \int_{Q_\theta} \pi(B_1) \left(\frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) - \partial_t a A_1 - a B_1 + F' \right) dt dx \\ &= \int_{Q_\theta} \pi(B_1) F' dt dx + \int_{Q_\theta} \pi(B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\ & \quad - \int_{Q_\theta} \pi(B_1) \partial_t a \left(\int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx - \int_{Q_\theta} \pi(B_1) a B_1 dt dx \\ &= \sum_{j=1}^4 \mathcal{K}_j. \end{aligned} \quad (34)$$

Then we estimate the above integrals one by one. Applying the Cauchy inequality, we get the following estimates for $\mathcal{K}_{j=1,2}$.

$$\begin{aligned} \mathcal{K}_1 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) F' dt dx \\ &= \int_{Q_\theta} e^{-s\alpha} \phi^{-\frac{1}{2}} F' (e^{-s\alpha} \phi^{-\frac{1}{2}} \nabla M_e \nabla B_1 + e^{-s\alpha} \phi^{-\frac{1}{2}} M_e \Delta B_1) dt dx \\ &\leq \int_{Q_\theta} CT^2 e^{-2s\alpha} |F'|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx \\ & \quad + \int_{Q_\theta} CT^4 \phi e^{-2s\alpha} |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\ &\leq s\lambda^2 (\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx), \end{aligned} \quad (35)$$

and

$$\begin{aligned} \mathcal{K}_2 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \frac{\mu}{\mu+1} \operatorname{div}(M_e \nabla B_1) dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla(M_e \nabla B_1)|^2 dt dx \\ &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} |\nabla M_e \nabla B_1 + M_e \Delta B_1|^2 dt dx \\ &\leq \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \frac{\mu}{\mu+1} \left(2|\nabla M_e|^2 |\nabla B_1|^2 + 2|M_e|^2 |\Delta B_1|^2 \right) dt dx \\ &\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\ &\leq s\lambda^2 \mathcal{I}(B_1), \end{aligned} \quad (36)$$

where $\lambda \geq 1$ and $s \geq C(\gamma_1)T^4$. Next, we estimate the integral \mathcal{K}_3 , and obtain

$$\mathcal{K}_3 = - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M_e \nabla B_1) \partial_t a(t, x) \left(\int_0^t B_1(s, x) ds + A_1(0, x) \right) dt dx$$

$$\begin{aligned}
 &\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e \Delta B_1 + \nabla M_e \nabla B_1|^2 dt dx + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
 &\leq C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_e|^2 |\Delta B_1|^2 dt dx + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi |\nabla M_e|^2 |\nabla B_1|^2 dt dx \\
 &\quad + CT^8 \int_{Q_\theta} e^{-2s\alpha} \phi^3 \int_0^t |B_1(s, x)|^2 ds dt dx \\
 &\leq s\lambda^2 \mathcal{I}(B_1). \tag{37}
 \end{aligned}$$

Similarly, we have

$$\mathcal{K}_4 = - \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} a(t, x) B_1 \nabla (M_e \nabla B_1 + M_e \Delta B_1) dt dx \leq s\lambda^2 \mathcal{I}(B_1). \tag{38}$$

Using the assumptions on the conductivity M_e and substituting the inequalities (35)-(38) into (29), we get

$$-\mathcal{J}_1 - \mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j \leq s\lambda^2 \left(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right),$$

which means

$$-\mathcal{J}_2 \leq \sum_{j=1}^4 \mathcal{K}_j + \mathcal{J}_1 \leq s\lambda^2 \left(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right) + s\lambda \mathcal{I}(B_1).$$

Substituting (33) to the above inequality, we have

$$\begin{aligned}
 &\left| \int_{\Omega} (c_m e^{-2s\alpha} \phi^{-1} M_e |\nabla B_1|^2) |_{t=\theta} dx \right| \tag{39} \\
 &\leq |\mathcal{J}_1| + \sum_{j=1}^4 \mathcal{K}_j \\
 &\leq s\lambda^2 \left(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right) + s\lambda \mathcal{I}(B_1),
 \end{aligned}$$

which leads to

$$\left| \int_{\Omega} e^{-2s\alpha^\theta} (\phi^\theta)^{-1} |\nabla B_1^\theta|^2 dx \right| \leq s\lambda^2 \left(\mathcal{I}(B_1) + \int_{Q_\theta} e^{-2s\alpha} |F'|^2 dt dx \right). \tag{40}$$

Next we multiply the second equation of (18) by $\xi(B_2) := e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2)$, and integrate over Q_θ to get

$$\begin{aligned}
 &\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \varepsilon \partial_t B_2 dt dx \\
 &= \int_{Q_\theta} \xi(B_2) \nabla(M \nabla B_2) dt dx + \int_{Q_\theta} \xi(B_2) \nabla(M_i \nabla B_1) dt dx + \int_{Q_\theta} \xi(B_2) G' dt dx \\
 &= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla(M \nabla B_2)|^2 dt dx + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx \\
 &\quad + \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) G' dt dx.
 \end{aligned}$$

We estimate

$$\int_{Q_\theta} e^{-2s\alpha} \phi^{-1} \nabla(M \nabla B_2) \nabla(M_i \nabla B_1) dt dx$$

$$\begin{aligned}
&= \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} (\nabla M \nabla B_2 + M \Delta B_2) \nabla (M_i \nabla B_1) dt dx \\
&\leq \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M \nabla B_2 + M \Delta B_2|^2 dt dx \\
&\quad + \frac{1}{2} \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i \nabla B_1 + M_i \Delta B_1|^2 dt dx \\
&\leq CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M|^2 |\nabla B_2|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M|^2 |\Delta B_2|^2 dt dx \\
&\quad + CT^4 \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |\nabla M_i|^2 |\nabla B_1|^2 dt dx + C \int_{Q_\theta} e^{-2s\alpha} \phi^{-1} |M_i|^2 |\Delta B_1|^2 dt dx \\
&\leq s\lambda^2 \mathcal{I}(B_1) + s\lambda^2 \mathcal{I}(B_2). \tag{41}
\end{aligned}$$

Continuing the similar computation as the preceding estimates and using Assumption 1.1, we obtain

$$\begin{aligned}
& \left| \int_{\Omega} e^{-2s\alpha^\theta} (\phi^\theta)^{-1} |\nabla B_2^\theta|^2 dx \right| \\
&\leq s\lambda^2 (\mathcal{I}(B_1) + \mathcal{I}(B_2)) + \int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx \\
&\leq Cs\lambda^2 \left(\int_{Q_\theta} e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + s^3 \lambda^4 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx \right), \tag{42}
\end{aligned}$$

for any $s \geq C(\gamma_1, \gamma_2, \gamma_3)(T + T^2 + T^4)$ and $\lambda \geq C(\gamma_1, \gamma_2, \gamma_3)$. Thus combining the estimates (40) and (42), we obtain the conclusion. \square

Now we shall give the main result of the stability estimate of the conductivities in (6) based on the preceding lemmas and proposition.

Theorem 3.6. *Let (A_1, A_2) be the solution of (7). Suppose all the assumptions of Theorem 2.1 hold and $g_1, g_2 \in H_0^2(\Omega)$. In addition, suppose Assumption 3.1 and 3.2 are also satisfied. Then there exists a constant C with $C(\Omega, \omega, T, \gamma_1, \gamma_2, \gamma_3) > 0$, such that for sufficiently large $\lambda \geq \lambda_0 \geq 1$ and $s \geq s_4$, the following estimate holds:*

$$\begin{aligned}
& \int_{\Omega} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
&\leq C \left(\int_{Q_\omega} (|\partial_t A_1|^2 + |\partial_t A_2|^2) dt dx + \int_{\Omega} |A_1^\theta|^2 + \sum_{j=1}^2 (|\nabla A_j^\theta|^2 + |\Delta A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)|^2) dx \right) \\
&\quad + C \int_{\tilde{\omega}} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx. \tag{43}
\end{aligned}$$

Proof. Substituting the results in Lemma 3.4 and Lemma 3.5 into the inequality in Proposition 1, one obtains

$$\begin{aligned}
& s^2 \lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} (|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2) dx \\
&\leq Cs\lambda^2 \mathcal{E}(g_1, g_2, B_1, B_2) + C \sum_{j=5}^9 E_j(\theta) \\
&\leq Cs\lambda^2 \int_Q e^{-2s\alpha} (|F'|^2 + |G'|^2) dt dx + Cs^4 \lambda^6 \int_{Q_\omega} \phi^3 e^{-2s\alpha} (|B_1|^2 + |B_2|^2) dt dx
\end{aligned}$$

$$\begin{aligned}
 & + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|\nabla A_1^{\theta}|^2 + |\Delta A_1^{\theta}|^2 + |A_1^{\theta}|^2 \right) dx + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|\nabla A_2^{\theta}|^2 + |\Delta A_2^{\theta}|^2 \right) dx \\
 & + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|g_1 \Delta \tilde{v}_{\theta}^{\varepsilon}|^2 + |g_2 \Delta \tilde{u}_{e,\theta}^{\varepsilon}|^2 \right) dx + C \int_{\tilde{\omega}} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\
 & + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|A_1^{\theta}|^2 + \sum_{j=1}^2 \left(|\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) + |\nabla(g_1 \Delta \tilde{v}_{\theta}^{\varepsilon})|^2 \right. \\
 & \left. + |\nabla(g_2 \Delta \tilde{u}_{e,\theta}^{\varepsilon})|^2 \right) dx \\
 \leq & C s \lambda^2 \int_Q e^{-2s\alpha} \left(|F'|^2 + |G'|^2 \right) dt dx + C s^4 \lambda^6 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha} \left(|B_1|^2 + |B_2|^2 \right) dt dx \\
 & + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|A_1^{\theta}|^2 + \sum_{j=1}^2 \left(|\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) \right) dx \\
 & + C \int_{\tilde{\omega}} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx,
 \end{aligned}$$

for large enough $s \geq s_3 = \max\{CT^2, s_2\}$ and $\lambda \geq \lambda_1$. Now for convenience, we set $R_1(t, x) = \nabla \tilde{v}^{\varepsilon}(t, x)$ and $R_2(t, x) = \nabla \tilde{u}_e^{\varepsilon}(t, x)$. Then from the regularity of the solutions $(\tilde{v}^{\varepsilon}(t, x), \tilde{u}_e^{\varepsilon}(t, x))$, we deduce that there exist $l_j \in L^2(0, T)$, $j = 1, 2, 3, 4$,

$$|\partial_t R_j(t, x)| \leq l_j(t) |R_j^{\theta}|, \quad j = 1, 2,$$

$$|\partial_t \nabla R_1(t, x)| \leq l_3(t) |\nabla R_1^{\theta}|,$$

$$|\partial_t \nabla R_2(t, x)| \leq l_3(t) |\nabla R_2^{\theta}|,$$

for any $(t, x) \in Q$, and the functions $l_j \in L^2(0, T)$, implying $\int_0^T |l_j|^2 dt \leq N < \infty$, $j = 1, 2, 3, 4$. Then we show

$$F' = \partial_t (\nabla(g_1 \nabla \tilde{v}^{\varepsilon})) = \nabla g_1 \partial_t R_1 + g_1 \partial_t \nabla R_1,$$

$$G' = \partial_t (\nabla(g_2 \nabla \tilde{u}_e^{\varepsilon})) = \nabla g_2 \partial_t R_2 + g_2 \partial_t \nabla R_2.$$

Observe that from the definition of α , we get easily $e^{-2s\alpha(t,x)} \leq e^{-2s\alpha^{\theta}}$ for all $(t, x) \in Q$. This implies

$$\begin{aligned}
 & s^2 \lambda^2 \int_{\Omega} \phi^{\theta} e^{-2s\alpha^{\theta}} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\
 \leq & C s^4 \lambda^6 \int_{Q_{\omega}} \phi^3 e^{-2s\alpha} \left(|B_1|^2 + |B_2|^2 \right) dt dx + C \int_{\tilde{\omega}} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx, \\
 & + C \int_{\Omega} \frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \left(|A_1^{\theta}|^2 + \sum_{j=1}^2 \left(|\Delta A_j^{\theta}|^2 + |\nabla A_j^{\theta}|^2 + |\nabla(\Delta A_j^{\theta})| \right) \right) dx,
 \end{aligned}$$

for sufficiently large $s \geq s_4 = \max\{CT^2 N, s_3\}$. From the properties of α and ϕ , there exist e_0 and e_1 such that

$$\inf_{x \in \Omega} \left(\frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \right) \geq e_0 > 0,$$

$$\sup_{x \in \Omega} \left(\frac{1}{\phi^{\theta}} e^{-2s\alpha^{\theta}} \right) \leq e_1 < \infty.$$

Furthermore, $e^{-\epsilon\alpha}\phi^m \leq C < \infty$ for all $\epsilon > 0$ and $m \in \mathbb{R}$ in Q_ω . Thus we obtain

$$\begin{aligned} & s^2\lambda^2 \int_{\Omega} \phi^\theta e^{-2s\alpha^\theta} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx \\ & \leq C s^4 \lambda^6 \int_{Q_\omega} \left(|B_1|^2 + |B_2|^2 \right) dt dx + C \int_{\tilde{\omega}} \left(|g_1|^2 + |g_2|^2 + |\nabla g_1|^2 + |\nabla g_2|^2 \right) dx, \\ & + C \int_{\Omega} \left(|A_1^\theta|^2 + \sum_{j=1}^2 \left(|\Delta A_j^\theta|^2 + |\nabla A_j^\theta|^2 + |\nabla(\Delta A_j^\theta)| \right) \right) dx, \end{aligned}$$

Then we fix the parameters s, λ as $s = s_4, \lambda = \lambda_1$. This concludes the proof of the theorem. \square

REFERENCES

- [1] M. Bendahmane and F. W. Chaves-Silva, Controllability of a degenerating reaction-diffusion system in electrocardiology, to appear in *SIAM Journal on Control and Optimization*, [arXiv:1106.1788](https://arxiv.org/abs/1106.1788).
- [2] M. Bendahmane and K. H. Karlsen, [Analysis of a class of degenerate reaction-diffusion systems and the bidomain model of cardiac tissue](#), *Netw. Heterog. Media*, **1** (2006), 185–218.
- [3] M. Bendahmane and K. H. Karlsen, [Convergence of a finite volume scheme for the bidomain model of cardiac tissue](#), *Appl. Numer. Math.*, **59** (2009), 2266–2284.
- [4] M. Bendahmane, R. Bürger and R. Ruiz Baier, [A finite volume scheme for cardiac propagation in media with isotropic conductivities](#), *Math. Comp. Simul.*, **80** (2010), 1821–1840.
- [5] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
- [6] A. L. Bukhgeim, Carleman estimates for Volterra operators and uniqueness of inverse problems, in *Non-classical Problems of Mathematical Physics*, Computing Center of Siberian Branch of Soviet Academy of Sciences, Novosibirsk, 1981, 56–64.
- [7] A. L. Bukhgeim, *Introduction to the Theory of Inverse Problems*, VSP, Utrecht, 2000.
- [8] A. L. Bukhgeim and M. V. Klibanov, Uniqueness in the large class of multidimensional inverse problems, (*Russian*) *Dokl. Akad. Nauk SSSR*, **260** (1981), 269–272.
- [9] K. C. Chang, *Methods in Nonlinear Analysis*, Springer-Verlag Berlin Heidelberg, Netherlands, 2005.
- [10] P. Colli Franzone and G. Savaré, Degenerate evolution systems modeling the cardiac electric field at micro- and macroscopic level, in *Evolution Equations, Semigroups and Functional Analysis* (Milano, 2000), Progr. Nonlinear Differential Equations Appl., 50, Birkhäuser, Basel, 2002, 49–78.
- [11] M. Cristofol, P. Gaitan and H. Ramoul, [Inverse problems for a \$2 \times 2\$ reaction-diffusion system using a Carleman estimate with one observation](#), *Inverse Problems*, **22** (2006), 1561–1573.
- [12] A. V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations*, Lecture Notes Series, 34, RIM Seoul National University, Korea, 1996.
- [13] O. Yu. Imanuvilov, M. Yamamoto, [Lipschitz stability in inverse problems by Carleman estimates](#), *Inverse Problems*, **14** (1998), 1229–1245.
- [14] O. Yu. Imanuvilov and M. Yamamoto, [Carleman estimates for the non-stationary Lamé system and the application to an inverse problem](#), *ESAIM, COCV*, **11** (2005), 1–56.
- [15] V. Isakov, [Carleman estimates and applications to inverse problems](#), *Milan J. Math.*, **72** (2004), 249–271.
- [16] M. V. Klibanov, Carleman estimates and inverse problems in the last two decades, in *Surveys on Solution Methods for Inverse Problems*, Springer, Vienna, 2000, 119–146.
- [17] M. V. Klibanov and M. Yamamoto, [Lipschitz stability of an inverse problem for an acoustic equation](#), *Appl. Anal.*, **85** (2006), 515–538.
- [18] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.
- [19] G. Lebeau and L. Robbiano, [Contrôle exact de l'équation de la chaleur](#), *Comm. Partial Differential Equations*, **20** (1995), 335–356.
- [20] J.-P. Puel and M. Yamamoto, [On a global estimate in a linear inverse hyperbolic problem](#), *Inverse Problems*, **12** (1996), 995–1002.

- [21] K. Sakthivel, N. Baranibalan, J.-H. Kim and K. Balachandran, [Stability of diffusion coefficients in an inverse problem for the Lotka-Volterra competition system](#), *Acta Appl. Math.*, **111** (2010), 129–147.
- [22] Z. Q. Wu, J. X. Yin and C. P. Wang, *Elliptic and Parabolic Equations*, World Scientific Publishing Co. Pte. Ltd, 2003.
- [23] M. Yamamoto, [Uniqueness and stability in multidimensional hyperbolic inverse problems](#), *J. Math. Pures Appl.*, **78** (1999), 65–98.
- [24] G. Yuan and M. Yamamoto, [Lipshitz stability in the determination of the principal part of a parabolic equation](#), *ESAIM: Control Optim. Calc. Var.*, **15** (2009), 525–554.

Received November 2013; revised March 2015.

E-mail address: bedreddine.ainseba@u-bordeaux.fr

E-mail address: mostafa.bendahmane@u-bordeaux.fr

E-mail address: hey@lzu.edu.cn