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# EIGENVALUES FOR MAXWELL'S EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

FERRUCCIO COLOMBINI, VESSELIN PETKOV<sup>†</sup> AND JEFFREY RAUCH

ABSTRACT. Let  $V(t) = e^{tG_b}$ ,  $t \geq 0$ , be the semigroup generated by Maxwell's equations in an exterior domain  $\Omega \subset \mathbb{R}^3$  with dissipative boundary condition  $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0$ ,  $\gamma(x) > 0$ ,  $\forall x \in \Gamma = \partial\Omega$ . We prove that if  $\gamma(x)$  is nowhere equal to 1, then for every  $0 < \epsilon \ll 1$  and every  $N \in \mathbb{N}$  the eigenvalues of  $G_b$  lie in the region  $\Lambda_\epsilon \cup \mathcal{R}_N$ , where  $\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1), \operatorname{Re} z < 0\}$ ,  $\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \operatorname{Re} z < 0\}$ .

## 1. INTRODUCTION

Suppose that  $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$  is an open connected domain and  $\Omega := \mathbb{R}^3 \setminus \bar{K}$  is an open connected domain with  $C^\infty$  smooth boundary  $\Gamma$ . Consider the boundary problem

$$\begin{aligned} \partial_t E &= \operatorname{curl} B, & \partial_t B &= -\operatorname{curl} E \quad \text{in } \mathbb{R}_t^+ \times \Omega, \\ E_{tan} - \gamma(x)(\nu \wedge B_{tan}) &= 0 \quad \text{on } \mathbb{R}_t^+ \times \Gamma, \\ E(0, x) &= e_0(x), & B(0, x) &= b_0(x). \end{aligned} \tag{1.1}$$

with initial data  $f = (e_0, b_0) \in (L^2(\Omega))^6 = \mathcal{H}$ . Here  $\nu(x)$  denotes the unit outward normal to  $\partial\Omega$  at  $x \in \Gamma$  pointing into  $\Omega$ ,  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{C}^3$ ,  $u_{tan} := u - \langle u, \nu \rangle \nu$ , and  $\gamma(x) \in C^\infty(\Gamma)$  satisfies  $\gamma(x) > 0$  for all  $x \in \Gamma$ . The solution of the problem (1.1) is given by a contraction semigroup  $(E, B) = V(t)f = e^{tG_b}f$ ,  $t \geq 0$ , where the generator  $G_b$  has domain  $D(G_b)$  that is the closure in the graph norm of functions  $u = (v, w) \in (C_{(0)}^\infty(\mathbb{R}^3))^3 \times (C_{(0)}^\infty(\mathbb{R}^3))^3$  satisfying the boundary condition  $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$  on  $\Gamma$ .

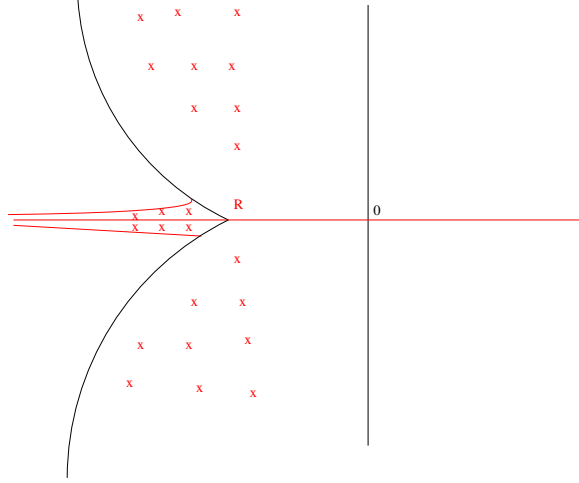
In an earlier paper [2] we proved that the spectrum of  $G_b$  in  $\operatorname{Re} z < 0$  consists of isolated eigenvalues with finite multiplicity. If  $G_b f = \lambda f$  with  $\operatorname{Re} \lambda < 0$ , the solution  $u(t, x) = V(t)f = e^{\lambda t}f(x)$  of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are invisible for inverse scattering problems. It was proved [2] that if there is at least one eigenvalue  $\lambda$  of  $G_b$  with  $\operatorname{Re} \lambda < 0$ , then the wave operators  $W_\pm$  are not complete, that is  $\operatorname{Ran} W_- \neq \operatorname{Ran} W_+$ . Hence we cannot define the scattering operator  $S$  related to the Cauchy problem for the Maxwell system and (1.1) by the product  $W_+^{-1}W_-$ . For the perfect conductor boundary conditions for Maxwell's equations, the energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups. The corresponding scattering operator  $S(z) : (L^2(\mathbb{S}^2))^2 \rightarrow (L^2(\mathbb{S}^2))^2$  satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C} \tag{1.2}$$

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FIGURE 1. Eigenvalues of  $G_b$ 

if  $S(z)$  is invertible at  $z$ . The scattering operator  $S(z)$  defined in [5] is such that  $S(z)$  and  $S^*(z)$  are analytic in the "physical" half plane  $\{z \in \mathbb{C} : \text{Im } z < 0\}$  and the above relation for conservative boundary conditions implies that  $S(z)$  is invertible for  $\text{Im } z > 0$ . For dissipative boundary conditions the relation (1.2) in general is not true and  $S(z_0)$  may have a non trivial kernel for some  $z_0, \text{Im } z_0 > 0$ . Lax and Phillips [5] proved that this implies that  $iz_0$  is an eigenvalue of  $G_b$ . The analysis of the location of the eigenvalues of  $G_b$  is important for the location of the points where the kernel of  $S(z)$  is not trivial.

The main result of this paper is the following (see Figure 1)

**Theorem 1.1.** *Assume that for all  $x \in \Gamma$ ,  $\gamma(x) \neq 1$ . Then for every  $0 < \epsilon \ll 1$  and every  $N \in \mathbb{N}$  there are constants  $C_\epsilon > 0$  and  $C_N > 0$  such that the eigenvalues of  $G_b$  lie in the region  $\Lambda_\epsilon \cup \mathcal{R}_N$ , where*

$$\Lambda_\epsilon = \{z \in \mathbb{C} : |\text{Re } z| \leq C_\epsilon(|\text{Im } z|^{1/2+\epsilon} + 1), \text{Re } z < 0\},$$

$$\mathcal{R}_N = \{z \in \mathbb{C} : |\text{Im } z| \leq C_N(|\text{Re } z| + 1)^{-N}, \text{Re } z < 0\}.$$

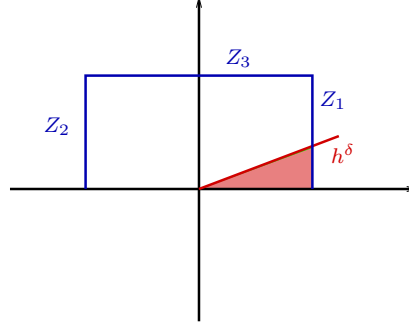
If  $\text{Re } \lambda < 0$  and  $G_b(E, B) = \lambda(E, B) \neq 0$ , then

$$\begin{aligned} \lambda E &= \text{curl } B && \text{on } \Omega, \\ \lambda B &= -\text{curl } E && \text{on } \Omega, \\ \text{div } E &= \text{div } B = 0, && \text{on } \Omega, \\ E_{tan} - \gamma(\nu \wedge B_{tan}) &= 0 && \text{on } \Gamma. \end{aligned} \tag{1.3}$$

This implies that  $u := (E, B)$  satisfies

$$\Delta u - \lambda^2 u = 0, \quad \text{on } \Omega.$$

The eigenvalues of  $G_b$  are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. The mapping  $z \mapsto z^2$  maps the positive quadrant  $\{z \in \mathbb{C} : \text{Re } z > 0, \text{Im } z > 0\}$  bijectively to the upper half space. Denote by  $\sqrt{z}$  the inverse map. The part of the

FIGURE 2. Contours  $Z_1, Z_2, Z_3, \delta = 1/2 - \epsilon$ 

spectral domain  $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0, \operatorname{Im} \lambda > 0\}$  is mapped by  $\lambda = \mathbf{i}\sqrt{z}$  to the upper half plane  $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ . In  $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$  introduce the sets

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{C} : \operatorname{Re} z = 1, \ h^\delta \leq \operatorname{Im} z \leq 1\}, \quad 0 < h \ll 1, \quad 0 < \delta < 1/2, \\ Z_2 &:= \{z \in \mathbb{C} : \operatorname{Re} z = -1, \ 0 \leq \operatorname{Im} z \leq 1\}, \\ Z_3 &:= \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \ \operatorname{Im} z = 1\}. \end{aligned}$$

Set  $\lambda = \mathbf{i}\sqrt{z}/h$ ,  $z \in Z_1 \cup Z_2 \cup Z_3$ . To study the eigenvalues  $\lambda$ ,  $|\lambda| > R_0$ , it is sufficient to consider  $0 < h \ll 1$ . As  $z$  runs over the rectangle in Figure 2, with  $0 < h \ll 1$ ,  $\lambda$  sweeps out the large values in the intersection of left and upper half planes. The values of  $z \in Z_2$  near the lower left hand corner,  $z = -1$ , of the rectangle go the spectral values near the negative real axis. The spectral analysis near these values in  $Z_2$  for dissipative Maxwell's equations does not have clear analogue with the spectral problems for the wave equation with dissipative boundary conditions. In fact, for the wave equation if  $0 < \gamma(x) < 1$ ,  $\forall x \in \Gamma$ , the eigenvalues of the generator of the corresponding semigroup are located in the domain  $\Lambda_\epsilon$  (see Section 3, [8] and [6]). For Maxwell's equations the eigenvalues of  $G_b$  lie in the domain  $\Lambda_\epsilon \cup \mathcal{R}_N$  and for  $0 < \gamma(x) < 1$  and  $\gamma(x) > 1$  we have the same location (see Appendix for the case  $K = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$ ).

Equation (1.3) implies that on  $\Omega$  each eigenfunction  $u = (E, B)$  of  $G_b$  satisfies

$$\sqrt{z} E = \frac{h}{\mathbf{i}} \operatorname{curl} B, \quad \sqrt{z} B = -\frac{h}{\mathbf{i}} \operatorname{curl} E, \quad (1.4)$$

and therefore  $(-h^2 \Delta - z)E = (-h^2 \Delta - z)B = 0$ . For eigenfunctions  $(E, B) \neq 0$ , we derive a pseudodifferential system on the boundary involving  $E_{tan} = E - \langle E, \nu \rangle \nu$  and  $E_{nor} = \langle E, \nu \rangle$ . A semi-classical analysis shows that for  $z \in Z_1 \cup Z_3$  this system implies that for  $h$  small enough we have  $E|_\Gamma = 0$  which yields  $E = B = 0$ . By scaling one concludes that the eigenvalues  $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$  of  $G_b$  lie in the region  $\Lambda_\epsilon \cup \mathcal{M}$ , where

$$\mathcal{M} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \pi/4, |z| \geq R_0 > 0, \operatorname{Re} z < 0\}.$$

The strategy for the analysis of the case  $z \in Z_1 \cup Z_3$  is similar to that exploited in [9] and [8]. In these papers the semi-classical Dirichlet-to-Neumann map  $\mathcal{N}(z, h)$  plays a crucial role and the problem is reduced to the proof that some

$h$ -pseudodifferential operators is elliptic in a suitable class. For the Maxwell system the pseudodifferential equation on the boundary is more complicated. Using the equation  $\operatorname{div} E = 0$ , yields a pseudodifferential system for  $E_{tan}$  and  $E_{nor}$ . We show that if  $(E, B) \neq 0$  is an eigenfunction of  $G_b$ , then  $\|E_{nor}\|_{H_h^1(\Gamma)}$  is bounded by  $Ch\|E_{tan}\|_{H_h^1(\Gamma)}$ . The term involving  $E_{nor}$  then plays the role of a negligible perturbation in the pseudodifferential system on the boundary and this reduces the analysis to one involving only  $E_{tan}$ . The system concerning  $E_{tan}$  has a diagonal leading term and we may apply the same arguments as those of [8] to conclude that  $E_{tan} = 0$  and hence  $E_{nor} = 0$ .

The analysis of the case  $z \in Z_2$  is more difficult since the principal symbol  $g$  of the pseudodifferential system for  $E_{tan}$  need not be elliptic at some points (see Section 3). Even where  $g$  is elliptic, if  $|\operatorname{Im} z| \leq h^{1/2}$  it is difficult to estimate the norm of the difference  $Op_h(g)Op_h(g^{-1}) - I$ . To show that the eigenvalues of  $G_b$  lying in  $\mathcal{M}$  are in fact confined to the region  $\mathcal{R}_N$  for every  $N \in \mathbb{N}$ , we analyze the real part of the following scalar product in  $L^2(\Gamma)$

$$Q(E_0) := \operatorname{Re}\langle (\mathcal{N}(z, h) - \sqrt{z}\gamma)E_0, E_0 \rangle_{L^2(\Gamma)}, \quad E_0 := E|_{\Gamma}.$$

We follow the approach in [9], [8] based on a Taylor expansion of  $Q(E_0)$  at  $z = -1$  and the fact that for  $z = -1$  we have  $Q(E_0) = \mathcal{O}(h^N)$ ,  $\forall N \in \mathbb{N}$ . In the Appendix we treat the case when  $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$  is a ball and  $\gamma = \operatorname{const}$ . We prove that for  $\gamma \equiv 1$  the operator  $G$  has no eigenvalues in  $\{\operatorname{Re} z < 0\}$ , while for every  $\gamma \in \mathbb{R}^+ \setminus \{1\}$  we have infinite number of real eigenvalues.

## 2. PSEUDODIFFERENTIAL EQUATION ON THE BOUNDARY

Introduce geodesic normal coordinates  $(y_1, y') \in \mathbb{R}^3$  on a neighborhood of a point  $x_0 \in \Gamma$  as follows. For a point  $x$ ,  $y'(x)$  is the closest point in  $\Gamma$  and  $y_1 = \operatorname{dist}(x, \Gamma)$ . Define  $\nu(x)$  to be the unit normal in the direction of increasing  $y_1$  to the surface  $y_1 = \operatorname{constant}$  through  $x$ . Thus  $\nu(x)$  is an extension of the unit normal vector to a unit vector field. The boundary  $\Gamma$  is mapped to  $y_1 = 0$  and

$$x = \alpha(y_1, y') = \beta(y') + y_1\nu(y').$$

We have

$$\frac{\partial}{\partial x_k} = \nu_k(y') \frac{\partial}{\partial y_1} + \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial}{\partial y_j}, \quad k = 1, 2, 3.$$

Moreover,

$$\sum_{k=1}^3 \nu_k(y') \frac{\partial y_j}{\partial x_k}(y_1, y') = \langle \nu, \frac{\partial y_j}{\partial x} \rangle = 0, \quad j = 1, 2, 3, \quad \text{and}$$

$$\sum_{k=1}^3 \nu_k(x) \partial_{x_k} f(x) = \partial_{y_1}(f(\alpha(y_1, y'))).$$

Since  $\|\nu(x)\| = 1$ ,  $\langle \nu, \partial_{x_j} \nu \rangle = 0$ ,  $j = 1, 2, 3$ .

A straight forward computation yields

$$\begin{aligned} \nu(x) \wedge \frac{h}{i} \operatorname{curl} u(x) &= i h \partial_{\nu} u_{tan} + \left( \langle D_{x_1} u, \nu \rangle, \langle D_{x_2} u, \nu \rangle, \langle D_{x_3} u, \nu \rangle \right) \Big|_{tan} \\ &= i h \partial_{\nu} u_{tan} + \left( \operatorname{grad}_h \langle u, \nu \rangle \right) \Big|_{tan} - i h g_0(u_{tan}), \quad x \in \Gamma, \end{aligned}$$

where

$$D_{x_j} = -\mathbf{i}h\partial_{x_j}, \quad j = 1, 2, 3, \quad \text{grad}_h f = \{D_{x_j} f\}_{j=1,2,3},$$

$$g_0(u_{tan}) = \{\langle u_{tan}, \partial_{x_j} \nu \rangle\}_{j=1,2,3}.$$

Setting  $E_{nor} = \langle E, \nu \rangle$ , from (1.3) one deduces

$$\nu \wedge B = -\frac{1}{\sqrt{z}} \nu \wedge \frac{h}{\mathbf{i}} \text{curl } E = \frac{1}{\sqrt{z}} D_\nu E_{tan} - \frac{1}{\sqrt{z}} \left[ \left( \text{grad}_h E_{nor} \right) \Big|_{tan} - \mathbf{i}h g_0(E_{tan}) \right],$$

where  $D_\nu = -\mathbf{i}h\partial_\nu$  and the boundary condition in (1.3) becomes

$$\left( D_\nu - \frac{1}{\gamma} \sqrt{z} \right) E_{tan} - \left( \text{grad}_h E_{nor} \right) \Big|_{tan} + \mathbf{i}h g_0(E_{tan}) = 0, \quad x \in \Gamma. \quad (2.1)$$

Next

$$\text{grad}_h f(x)|_{tan} = \left\{ \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} D_{y_j} f(\alpha(y_1, y')) \right\}_{k=1,2,3}$$

and for  $u = (u_1, u_2, u_3) \in \mathbb{C}^3$ ,

$$\begin{aligned} \frac{h}{\mathbf{i}} \text{div } u(\alpha(y_1, y')) &= \langle D_{y_1} u(\alpha(y_1, y')), \nu(y') \rangle + \sum_{k=1}^3 \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} D_{y_j} u_k(\alpha(y_1, y')) \\ &= D_{y_1} \left( u_{nor}(y_1, y') \right) + \sum_{j=2}^3 D_{y_j} \left\langle u_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x} \right\rangle + h \langle u_{tan}, Z \rangle, \end{aligned}$$

where  $\langle u(\alpha(y_1, y')), \nu(y') \rangle := u_{nor}(y_1, y')$  and  $Z$  depends on the second derivatives of  $y_j$ ,  $j = 2, 3$ . Apply the operator  $D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}$  to  $\text{div } E(\alpha(y_1, y')) = 0$  to find

$$\begin{aligned} (D_{y_1}^2 - \frac{\sqrt{z}}{\gamma(y')} D_{y_1}) E_{nor}(y_1, y') + \sum_{j=2}^3 D_{y_j} \left\langle (D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}) E_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x} \right\rangle \\ = h \langle (D_{y_1} - \frac{\sqrt{z}}{\gamma}) E_{tan}, Z \rangle + h \langle E_{tan}, Z_1 \rangle, \end{aligned}$$

where  $\gamma(y') := \gamma(\beta(y'))$ .

Taking the trace  $y_1 = 0$  and applying the boundary condition (2.1), yields

$$\begin{aligned} \left( D_{y_1}^2 + \sum_{j,\mu=2}^3 \sum_{k=1}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial y_\mu}{\partial x_k} D_{y_j, y_\mu}^2 \right) E_{nor}(0, y') - \frac{\sqrt{z}}{\gamma(y')} D_{y_1} E_{nor}(0, y') \\ = h \left\langle \left( \text{grad}_h E_{nor} \right) \Big|_{tan}(0, y'), Z \right\rangle + h Q_1(E_{tan}(0, y')), \end{aligned} \quad (2.2)$$

with

$$\|Q_1(E_{tan}(0, y'))\|_{L^2(\mathbb{R}^2)} \leq C_2 \|E_{tan}(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Here  $H_h^s(\Gamma)$ ,  $s \in \mathbb{R}$ , denotes the semi-classical Sobolev spaces with norm  $\|\langle h\partial_x \rangle^s u\|_{L^2(\Gamma)}$ ,  $\langle h\partial_x \rangle = (1 + \|h\partial_x\|^2)^{1/2}$ . In the exposition below we use the spaces  $(L^2(\Gamma))^3$  and  $(H_h^s(\Gamma))^3$  of vector-valued functions but we will omit this in the notations writing simply  $L^2(\Gamma)$  and  $H_h^s(\Gamma)$ .

The operator  $-h^2 \Delta_x - z$  in the coordinates  $(y_1, y')$  has the form

$$\mathcal{P}(z, h) = D_{y_1}^2 + r(y, D_{y'}) + q_1(y, D_y) + h^2 \tilde{q} - z$$

with  $r(y, \eta') = \langle R(y)\eta', \eta' \rangle$ ,  $q_1(y, \eta) = \langle q_1(y), \eta \rangle$ . Here

$$R(y) = \left\{ \sum_{k=1}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial y_\mu}{\partial x_k} \right\}_{j,\mu=2}^3 = \left\{ \left\langle \frac{\partial y_j}{\partial x}, \frac{\partial y_\mu}{\partial x} \right\rangle \right\}_{j,\mu=2}^3$$

is a symmetric  $(2 \times 2)$  matrix and  $r(0, y', \eta') = r_0(y', \eta')$ , where  $r_0(y', \eta')$  is the principal symbol of the Laplace-Beltrami operator  $-h^2 \Delta_\Gamma$  on  $\Gamma$  equipped with the Riemannian metric induced by the Euclidean one in  $\mathbb{R}^3$ . We have

$$\left( \mathcal{P}(z, h) E_{nor} \right)(0, y') = \langle \mathcal{P}(z, h) E, \nu \rangle(0, y') + h Q_2(E(0, y')),$$

where

$$\|Q_2(E(0, y'))\|_{L^2(\mathbb{R}^2)} \leq C_2 \|E(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Since  $\mathcal{P}(z, h)E = 0$ , this lets us replace the terms with all second derivatives of  $E_{nor}$  in (2.4) by  $z E_{nor}(0, y')$  modulo terms having a factor  $h$  and containing first order derivatives of  $E_{nor}$ . This follows from the form of the matrix  $R(y)$  given above. After a multiplication by  $-\frac{\gamma(y')}{\sqrt{z}}$  the equation (2.2) yields

$$(D_{y_1} - \gamma(y')\sqrt{z})E_{nor}(0, y') = h Q_3(E(0, y')), \quad (2.3)$$

where  $Q_3(E(0, y'))$  has the same properties as  $Q_2(E(0, y'))$ .

Let  $\psi(x) \in C_0^\infty(\mathbb{R}^3)$  be a cut-off function with support in small neighborhood of  $x_0 \in \Gamma$ . Replace  $E, B$  by  $E_\psi = E\psi$ ,  $B_\psi = B\psi$ . The above analysis works for  $E_\psi$  and  $B_\psi$  with lower order terms depending on  $\psi$ . We obtain

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma \psi(x), \nu(x) \rangle = h Q_{3,\psi}(E|_\Gamma).$$

Taking a partition of unity in a neighborhood of  $\Gamma$ , yields

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma, \nu \rangle = h Q_4(E|_\Gamma), \quad \|Q_4(E|_\Gamma)\|_{L^2(\Gamma)} \leq C \|E|_\Gamma\|_{H_h^1(\Gamma)}. \quad (2.4)$$

For  $z \in Z_1 \cup Z_2 \cup Z_3$  let  $\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*\Gamma)$  be the root of the equation

$$\rho^2 + r_0(x', \xi') - z = 0$$

with  $\text{Im } \rho(x', \xi', z) > 0$ . For large  $|\xi'|$ ,

$$\rho(x', \xi', z) \sim |\xi'|, \quad \text{Im } \rho(x', \xi', z) \sim |\xi'|,$$

while for bounded  $|\xi'|$ ,

$$\text{Im } \rho(x', \xi', z) \geq \frac{h^\delta}{C}.$$

We recall some basic facts about  $h$ -pseudodifferential operators that the reader can find in [3]. Let  $X$  be a  $C^\infty$  smooth compact manifold without boundary with dimension  $d \geq 2$ . Let  $(x, \xi)$  be the coordinates in  $T^*(X)$  and let  $a(x, \xi, h) \in C^\infty(T^*(X))$ . Given  $m \in \mathbb{R}$ ,  $l \in \mathbb{R}$ ,  $\delta > 0$  and a function  $c(h) > 0$ , one denotes by  $S_\delta^{l,m}(c(h))$  the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha,\beta}(c(h))^{-l-\delta(|\alpha|+|\beta|)}(1+|\xi|)^{m-|\beta|}, \quad \forall \alpha, \forall \beta, \quad (x, \xi) \in T^*(X).$$

If  $c(h) = h$ , we denote  $S_\delta^{l,m}(c(h))$  simply by  $S_\delta^{l,m}$ . Symbols restricted to a domain where  $|\xi| \leq C$  will be denoted by  $a \in S_\delta^l(c(h))$ . The  $h$ -pseudodifferential operator

with symbol  $a(x, \xi, h)$  acts by

$$(Op_h(a)f)(x) := (2\pi h)^{-d+1} \int_{T^*X} e^{-i\langle x-y, \xi \rangle / h} a(x, \xi, h) f(y) dy d\xi.$$

For matrix valued symbols we use the same definition. This means that every element of a matrix symbol is in the class  $S_\delta^{l,m}(c(h))$ .

Now suppose that  $a(x, \xi, h)$  satisfies the estimates

$$|\partial_x^\alpha a(x, \xi, h)| \leq c_0(h) h^{-|\alpha|/2}, \quad (x, \xi) \in T^*(X) \quad (2.5)$$

for  $|\alpha| \leq d-1$ , where  $c_0(h) > 0$  is a parameter. Then there exists a constant  $C > 0$  independent of  $h$  such that

$$\|Op_h(a)\|_{L^2(X) \rightarrow L^2(X)} \leq C c_0(h). \quad (2.6)$$

For  $0 \leq \delta < 1/2$  products of  $h$ -pseudodifferential operators are well behaved. If  $a \in S_\delta^{l_1, m_1}$ ,  $b \in S_\delta^{l_2, m_2}$  and  $s \in \mathbb{R}$ , then

$$\|Op_h(a)Op_h(b) - Op_h(ab)\|_{H^s(X) \rightarrow H^{s-m_1-m_2+1}(X)} \leq Ch^{-l_1-l_2-2\delta+1}. \quad (2.7)$$

Let  $u \in \mathbb{C}^3$  be the solution of the Dirichlet problem

$$(-h^2\Delta - z)u = 0 \quad \text{on } \Omega, \quad u = F \quad \text{on } \Gamma. \quad (2.8)$$

Introduce the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(z, h) : H_h^s(\Gamma) \ni F \longrightarrow D_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

G. Vodev [9] established for bounded domains  $K \subset \mathbb{R}^d$ ,  $d \geq 2$ , with  $C^\infty$  boundary the following approximation of the interior Dirichlet-to-Neumann map  $\mathcal{N}_{int}(z, h)$  related to (2.8), where the equation  $(-h^2\Delta - z)u = 0$  is satisfied in  $K$ .

**Theorem 2.1** ([9]). *For every  $0 < \epsilon \ll 1$  there exists  $0 < h_0(\epsilon) \ll 1$  such that for  $z \in Z_{1,\epsilon} := \{z \in Z_1, |\operatorname{Im} z| \geq h^{\frac{1}{2}-\epsilon}\}$  and  $0 < h \leq h_0(\epsilon)$  we have*

$$\|\mathcal{N}_{int}(z, h)(F) - Op_h(\rho + hb)F\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|F\|_{L^2(\Gamma)}, \quad (2.9)$$

where  $b \in S_{0,1}^0(\Gamma)$  does not depend on  $h$  and  $z$ . Moreover, (2.9) holds for  $z \in Z_2 \cup Z_3$  with  $|\operatorname{Im} z|$  replaced by 1.

With small modifications (2.9) holds for the Dirichlet-to-Neumann map  $\mathcal{N}(z, h)$  related to (2.8) (see [8]). Applying (2.9) with  $\mathcal{N}(z, h)$  and  $F = E_0 = E|_\Gamma$ , we obtain

$$\left\| \langle \mathcal{N}(z, h)E_0, \nu \rangle - \langle Op_h(\rho)E_0, \nu \rangle \right\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \quad (2.10)$$

Therefore (2.4) yields

$$\left\| \langle Op_h(\rho) - \gamma\sqrt{z}E_0, \nu \rangle - hQ_4(E_0) \right\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \quad (2.11)$$

The commutator  $[Op_h(\rho), \nu(x)]$  is a pseudodifferential operator with symbol in  $h^{1-\delta}S_\delta^{0,0}$  and so

$$\|[Op_h(\rho), \nu_k(x)]E_{nor}\|_{H_h^j(\Gamma)} \leq C_2 h^{1-\delta} \|E_{nor}\|_{H_h^j(\Gamma)}, \quad k = 1, 2, 3, \quad j = 0, 1.$$

The last estimate combined with (2.11) implies

$$\left\| (Op_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0) \right\|_{L^2(\Gamma)} \leq C_3 \left( \frac{h}{\sqrt{|\operatorname{Im} z|}} + h^{1-\delta} \right) \|E_0\|_{L^2(\Gamma)}. \quad (2.12)$$



## 3. EIGENVALUES-FREE REGIONS

For  $z \in Z_{1,\epsilon}$  we have  $\rho \in S_\delta^{0,1}$  with  $0 < \delta = 1/2 - \epsilon < 1/2$ , while for  $z \in Z_2 \cup Z_3$  we have  $\rho \in S_0^{0,1}$  (see [9]). Since  $\Gamma$  is connected one has either  $\gamma(x) > 1$  or  $0 < \gamma(z) < 1$ . We present the analysis in the case where  $0 < \gamma(x) < 1$ ,  $\forall x \in \Gamma$ . The case  $1 < \gamma(x)$  is reduced to this case at the end of the section. Clearly, there exists  $\epsilon_0 > 0$  such that

$$\epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0, \quad \forall x \in \Gamma.$$

Combing (2.4) and (2.9), yields

$$\| \langle (Op_h(\rho) - \gamma(x)\sqrt{z})E_0, \nu(x) \rangle \|_{L^2(\Gamma)} \leq C \frac{h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + C_1 h \|E_0\|_{H_h^1(\Gamma)},$$

where for  $z \in Z_2 \cup Z_3$  we can replace  $|\operatorname{Im} z|$  by 1. This estimate for  $E_0$  and the estimate for the commutator  $[Op_h(\rho), \nu_k(x)]$  imply

$$\| (Op_h(\rho) - \gamma(x)\sqrt{z})E_{nor} \|_{L^2(\Gamma)} \leq \frac{C_3 h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + C_4 h^{1-\delta} \|E_0\|_{H_h^1(\Gamma)}. \quad (3.1)$$

Let  $(x', \xi')$  be coordinates on  $T^*(\Gamma)$ . Consider the symbol

$$c(x', \xi', z) := \rho(x', \xi', z) - \gamma(x')\sqrt{z}, \quad x' \in \Gamma.$$

Following the analysis in Section 3, [8], we know that  $c$  is elliptic in the case  $0 < \gamma(x') < 1$  and if  $z \in Z_1$  we have  $c \in S_\delta^{0,1}$ ,  $|\operatorname{Im} z|c^{-1} \in S_\delta^{0,-1}$ , while if  $z \in Z_2 \cup Z_3$  one gets  $c \in S_0^{0,1}$ ,  $c^{-1} \in S_0^{0,-1}$ . This implies

$$\|Op_h(c^{-1})Op_h(c)E_{nor}\|_{H_h^1(\Gamma)} \leq \frac{C}{|\operatorname{Im} z|} \|Op_h(c)E_{nor}\|_{L^2(\Gamma)}.$$

On the other hand, according to Section 7 in [3], the symbol of the operator  $Op_h(c^{-1})Op_h(c) - I$  is given by

$$\begin{aligned} & \sum_{j=1}^N \frac{(ih)^j}{j!} \sum_{|\alpha|=j} D_{\xi'}^\alpha (c^{-1})(x', \xi') D_{y'}^\alpha c(y', \eta')|_{x'=y', \xi'=\eta'} + \tilde{b}_N(x', \xi') \\ & := b_N(x', \xi') + \tilde{b}_N(x', \xi'), \end{aligned}$$

where

$$|\partial_{x'}^\alpha \tilde{b}_N(x', \xi')| \leq C_\alpha h^{N(1-2\delta) - s_d - |\alpha|/2}.$$

Taking into account the estimates for  $c^{-1}$  and  $c$ , and applying (2.5), and (2.6) yields

$$\left\| \left( Op_h(c^{-1})Op_h(c) - I \right) E_{nor} \right\|_{H_h^j(\Gamma)} \leq C_5 \frac{h}{|\operatorname{Im} z|^2} \|E_{nor}\|_{H_h^j(\Gamma)}, \quad j = 0, 1.$$

Repeating the argument in Section 3 in [8] concerning the case  $0 < \gamma(x') < 1$ , for  $z \in Z_1$  and  $0 < \delta < 1/2$ , one finds

$$\begin{aligned} \|E_{nor}\|_{H_h^1(\Gamma)} & \leq \left\| \left( Op_h(c^{-1})Op_h(c) - I \right) E_{nor} \right\|_{H_h^1(\Gamma)} + \left\| Op_h(c^{-1})Op_h(c)E_{nor} \right\|_{H_h^1(\Gamma)} \\ & \leq C_6 h^{1-2\delta} \|E_0\|_{L^2(\Gamma)} + C_5 h^{1-2\delta} \|E_{nor}\|_{H_h^1(\Gamma)} + C_7 h^{1-\delta} \|E_0\|_{H_h^1(\Gamma)}. \end{aligned} \quad (3.2)$$

Clearly,

$$\|E_0\|_{H_h^k(\Gamma)} \leq \|E_{tan}\|_{H_h^k(\Gamma)} + B_k \|E_{nor}\|_{H_h^k(\Gamma)}, \quad k \in \mathbb{N}$$

with  $B_k$  independent of  $h$ . Hence we can absorb the terms involving the norms of  $E_{nor}$  in the right hand side of (3.2) choosing  $h$  small enough, and we get

$$\|E_{nor}\|_{H_h^1(\Gamma)} \leq Ch^{1-2\delta}\|E_{tan}\|_{H_h^1(\Gamma)}. \quad (3.3)$$

The analysis of the case  $z \in Z_2 \cup Z_3$  is simpler since in the estimates above we have no coefficient  $|\operatorname{Im} z|^{-1}$  and we obtain the same result with a factor  $h$  on the right hand side of (3.3).

With a similar argument it is easy to show that

$$\|E_{nor}\|_{L^2(\Gamma)} \leq C'h^{1-2\delta}\|E_{tan}\|_{L^2(\Gamma)}. \quad (3.4)$$

In fact from (2.12) one obtains

$$\left\| Op_h(c^{-1}) \left[ (Op_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0) \right] \right\|_{L^2(\Gamma)} \leq \frac{C_8}{|\operatorname{Im} z|} \left( \frac{h}{\sqrt{|\operatorname{Im} z|}} + h^{1-\delta} \|E_0\|_{L^2(\Gamma)} \right)$$

and

$$\|Op_h(c^{-1})Q_4(E_0)\|_{L^2(\Gamma)} \leq \frac{C_9}{|\operatorname{Im} z|} \|E_0\|_{L^2(\Gamma)}.$$

Combining these estimates with the estimate of  $\|Op_h(c^{-1})Op_h(c) - I\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}$  yields (3.4).

Going back to the equation (2.1), we have

$$\begin{aligned} \left(D_\nu - \frac{1}{\gamma}\sqrt{z}\right)E &= \left(D_\nu - \gamma\sqrt{z}\right)E_{nor}\nu - \left(\frac{1}{\gamma} - \gamma\right)\sqrt{z}E_{nor}\nu \\ &\quad + \mathbf{i}hg_0(E_{tan}) + \left(\operatorname{grad}_h(E_{nor})\right)|_{\tan}, \quad x \in \Gamma. \end{aligned} \quad (3.5)$$

Notice that for the first term on the right hand side of (3.5) we can apply the equality (2.4), while for  $E_{nor}$  and  $\left(\operatorname{grad}_h(E_{nor})\right)|_{\tan}$  we have a control by the estimate (3.3). Consequently, setting  $E_0 = E|_\Gamma$ , the right hand side of (3.5) is bounded by  $Ch^{1-2\delta}\|E_0\|_{H_h^1(\Gamma)}$ . Next

$$1 < \frac{1}{1-\epsilon_0} \leq \frac{1}{\gamma(x)} \leq \frac{1}{\epsilon_0}, \quad \forall x \in \Gamma.$$

This corresponds to the case (B) examined in Section 4 of [8]. The approximation of the operator  $\mathcal{N}(z, h)$  given by (2.9) yields the estimate

$$\|(Op_h(\rho) - \frac{1}{\gamma}\sqrt{z})E_0\|_{L^2(\Gamma)} \leq C \left( \frac{h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + h^{1-2\delta} \|E_0\|_{H_h^1(\Gamma)} \right). \quad (3.6)$$

For  $z \in Z_1 \cup Z_3$  the symbol

$$d(x', \xi', z) := \rho(x', \xi', z) - \frac{1}{\gamma(x')}\sqrt{z}$$

is elliptic (see Section 4, [8]) and  $d \in S_\delta^{0,1}$ ,  $d^{-1} \in S_\delta^{0,-1}$ . Then from (3.6) we estimate  $\|E_0\|_{H_h^1(\Gamma)}$  and we obtain  $E_0 = 0$  for  $h$  small enough. This implies  $E = B = 0$ .

Now recall that we have

$$\operatorname{Re} \lambda = -\frac{\operatorname{Im} \sqrt{z}}{h}, \quad \operatorname{Im} \lambda = \frac{\operatorname{Re} \sqrt{z}}{h}.$$

Suppose that  $z \in Z_1$ . Then

$$|\operatorname{Re} \lambda| \geq C(h^{-1})^{1-\delta}, \quad |\operatorname{Im} \lambda| \leq C_1 h^{-1} \leq C_2 |\operatorname{Re} \lambda|^{\frac{1}{1-\delta}}.$$

So if

$$|\operatorname{Re} \lambda| \geq C_3 |\operatorname{Im} \lambda|^{1-\delta}, \quad \operatorname{Re} \lambda \leq -C_4 < 0,$$

there are no eigenvalues  $\lambda = \frac{i\sqrt{z}}{h}$  of  $G_b$ . In the same way we handle the case  $z \in Z_3$  and we conclude that if  $z \in Z_1 \cup Z_3$  for every  $\epsilon > 0$  the eigenvalues  $\lambda = \frac{i\sqrt{z}}{h}$  of  $G_b$  lie in the domain  $\Lambda_\epsilon \cup \mathcal{M}$ , where

$$\mathcal{M} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \pi/4, |z| \geq R_0 > 0, \operatorname{Re} z < 0\},$$

$\Lambda_\epsilon$  being the domain introduced in Theorem 1.1. Of course, if we consider the domain

$$Z_{3,\delta_0} = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \operatorname{Im} z = \delta_0 > 0\},$$

instead of  $Z_3$ , we obtain an eigenvalue-free region with  $\mathcal{M}$  replaced by

$$\mathcal{M}_{\delta_0} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \arctg \delta_0, |z| \geq R_0(\delta_0) > 0, \operatorname{Re} z < 0\}.$$

The investigation of the case  $z \in Z_2$  is more complicated since the symbol  $d$  may vanish for  $\operatorname{Im} z = 0$  and  $(x'_0, \xi'_0) \in T^*(\Gamma)$  satisfying the equation

$$\sqrt{1 + r_0(x'_0, \xi'_0)} - \frac{1}{\gamma(x'_0)} = 0.$$

To cover this case and to prove that the eigenvalues  $\lambda = \frac{i\sqrt{z}}{h}$  with  $z \in Z_2$  are confined in the domain  $\mathcal{R}_N$ ,  $\forall N \in \mathbb{N}$ , we follow the arguments in [9] and [8]. For  $z \in Z_2$  we introduce an operator  $T(z, h)$  that yields a better approximation of  $\mathcal{N}(z, h)$ . In fact,  $T(z, h)$  is defined by the construction of the semi-classical parametrix in Section 3, [9] for the problem (2.8) with  $F = E_0$ . We refer to [9] for the precise definition of  $T(z, h)$  and more details. For our exposition we need the next proposition. Since  $(\Delta - z)E = 0$ , as in [9], we obtain

**Proposition 3.1.** *For  $z \in Z_2$  and every  $N \in \mathbb{N}$  we have the estimate*

$$\|\mathcal{N}(z, h)E_0 - T(z, h)E_0\|_{H_h^1(\Gamma)} \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)} \quad (3.7)$$

with constants  $C_N, s_0 > 0$ , independent of  $E_0, h$  and  $z$ , and  $s_0$  independent of  $N$ .

*Proof of Theorem 1.1 in the case  $z \in Z_2$ .* Consider the system

$$\begin{cases} \left( D_\nu - \frac{1}{\gamma} \sqrt{z} \right) E_{tan} - \left( \operatorname{grad}_h E_{nor} \right) \Big|_{tan} + i h g_0(E_{tan}) = 0, & x \in \Gamma, \\ \operatorname{div}_h E_{tan} + \operatorname{div}_h (E_{nor} \nu) = 0, & x \in \Gamma, \end{cases} \quad (3.8)$$

where  $\operatorname{div}_h F = \sum_{k=1}^3 D_{x_k} F_k$ .

Take the scalar product  $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$  in  $L^2(\Gamma)$  of the first equation of (3.8) and  $E_{tan}$ . Applying Green formula, it easy to see that

$$-\operatorname{Re} \langle \operatorname{grad}_h E_{nor} \Big|_{tan}, E_{tan} \rangle_{L^2(\Gamma)} = -\operatorname{Re} \langle \operatorname{div}_h E_{tan}, E_{nor} \rangle_{L^2(\Gamma)}. \quad (3.9)$$

We claim that

$$\operatorname{Im} \langle g_0(E_{tan}), E_{tan} \rangle_{L^2(\Gamma)} = 0. \quad (3.10)$$

Let  $E_{tan} = (w_1, w_2, w_3)$ . Then

$$\langle g_0(E_{tan}), E_{tan} \rangle_{\mathbb{C}^3} = \sum_{k,j=1}^3 w_k \frac{\partial \nu_k}{\partial x_j} \overline{w_j} = \frac{1}{q} \sum_{k,j=1}^3 w_k \frac{\partial V_k}{\partial x_j} \overline{w_j} = \frac{1}{q} \langle Sw, w \rangle_{\mathbb{C}^3},$$

where  $S := \{\frac{\partial V_k}{\partial x_j}\}_{k,j=1}^3$  with  $V(x) = q(x)\nu(x)$ ,  $q(x) > 0$  because  $\sum_{k=1}^3 (\partial_{x_j} q) w_k \nu_k = 0$ . Thus if the boundary is given locally by  $x_3 = G(x_1, x_2)$ , we choose  $V(x) = (-\partial_{x_1} G, -\partial_{x_2} G, 1)$  and it is obvious that  $S$  is symmetric. Therefore  $\text{Im}\langle Sw, w \rangle_{\mathbb{C}^3} = 0$  and this proves the claim. Hence (3.10) implies

$$\text{Re}[\mathbf{i}h\langle g_0(E_{tan}), E_{tan} \rangle_{L^2(\Gamma)}] = 0. \quad (3.11)$$

From the  $L^2(\Gamma)$  scalar product of the second equation in (3.8) with  $E_{nor}$ , we obtain

$$\text{Re}\langle \text{div}_h E_{tan}, E_{nor} \rangle_{L^2(\Gamma)} + \text{Re}\langle D_\nu E_{nor}, E_{nor} \rangle_{L^2(\Gamma)} = 0. \quad (3.12)$$

In fact,

$$\text{div}_h(E_{nor}\nu) = D_\nu E_{nor} - \mathbf{i}h E_{nor} \text{div } \nu$$

$$\text{and } \text{Im}\left(\text{div } \nu |E_{nor}|^2\right) = 0.$$

Taking together (3.9), (3.11) and (3.12), we conclude that

$$\begin{aligned} & \text{Re}\left[\left\langle \left(D_\nu - \frac{\sqrt{z}}{\gamma}\right)E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} + \langle D_\nu E_{nor}\nu, E_{nor}\nu \rangle_{L^2(\Gamma)}\right] \\ &= \text{Re}\left\langle D_\nu E, E \right\rangle_{L^2(\Gamma)} - \text{Re}\left\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} = 0. \end{aligned}$$

Here we have used the fact that

$$\langle D_\nu E_{tan}, E_{nor}\nu \rangle_{\mathbb{C}^3} = D_\nu \left( \langle E_{tan}, E_{nor}\nu \rangle_{\mathbb{C}^3} \right) = 0.$$

Applying Proposition 3.1 with  $E|_\Gamma = E_0$ , yields

$$\left| \text{Re}\left\langle T(z, h)E_0, E_0 \right\rangle_{L^2(\Gamma)} - \text{Re}\left\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} \right| \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)}. \quad (3.13)$$

For  $z = -1$ , as in Lemma 3.9 in [9] and Lemma 4.1 in [8], we have

$$|\text{Re}\langle T(-1, h)E_0, E_0 \rangle_{L^2(\Gamma)}| \leq C_N h^{-s_0+N} \|E_0\|_{L^2(\Gamma)}^2 = 0.$$

Consequently, by using Taylor formula for the real-valued function

$$\text{Re}\left[\left\langle T(z, h)E_0, E_0 \right\rangle_{L^2(\Gamma)} - \left\langle \frac{\sqrt{z}}{\gamma}E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)}\right],$$

we get for every  $N \in \mathbb{N}$  the estimate

$$\begin{aligned} & \left| \text{Im}\left[\left\langle \left(\frac{\partial T}{\partial z}(z_t, h)E_0, E_0\right) \right\rangle_{L^2(\Gamma)} - \left\langle \frac{\gamma_1}{2\sqrt{z_t}}E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)}\right] \right| \\ & \leq C_N \frac{h^{-s_0+N}}{|\text{Im } z|} \|E_0\|_{L^2(\Gamma)}^2, \end{aligned} \quad (3.14)$$

where  $z_t = -1 + \mathbf{i}t \text{Im } z$ ,  $0 < t < 1$ ,  $\gamma_1 = \gamma^{-1}$ .

According to Lemma 3.9 in [9], in (3.14) we can replace  $\frac{\partial T}{\partial z}(z_t, h)$  by  $Op_h(\frac{\partial \rho}{\partial z}(z_t))$  and this yields an error term bounded by  $Ch\|E_0\|_{H_h^{-1}(\Gamma)}^2$ . On the other hand,

$$\begin{aligned} & \left| \left\langle Op_h\left(\frac{\partial \rho}{\partial z}(z_t)\right)E_{tan}, E_{nor}\nu \right\rangle_{L^2(\Gamma)} + \left\langle Op_h\left(\frac{\partial \rho}{\partial z}(z_t)\right)E_{nor}, E_{tan}\nu \right\rangle_{L^2(\Gamma)} \right| \\ & \leq Ch\|E_0\|_{L^2(\Gamma)}^2 \end{aligned}$$

since the estimate (3.4) holds for  $z \in Z_2$  with factor  $h$  and  $\frac{\partial \rho}{\partial z}(z_t) \in S_0^{0,-1}$ .

Thus the problem is reduced to a lower bound of

$$\begin{aligned} J &:= \left| \operatorname{Im} \left[ \left\langle \left( \operatorname{Op}_h \left( \frac{\partial \rho}{\partial z} (z_t) \right) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} + \left\langle \operatorname{Op}_h \left( \frac{\partial \rho}{\partial z} (z_t) \right) E_{nor} \nu, E_{nor} \nu \right\rangle_{L^2(\Gamma)} \right] \right| \\ &\geq \left| \operatorname{Im} \left\langle \left( \operatorname{Op}_h \left( \frac{\partial \rho}{\partial z} (z_t) \right) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} \right| - C_1 \|E_{nor}\|_{L^2(\Gamma)}^2. \end{aligned}$$

Since  $\gamma_1(x) > 1$ ,  $\forall x \in \Gamma$ , applying the analysis of Section 4 in [8] for the scalar product involving  $E_{tan}$ , one deduces

$$\left| \operatorname{Im} \left\langle \left( \operatorname{Op}_h \left( \frac{\partial \rho}{\partial z} (z_t) \right) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} \right| \geq \eta_1 \|E_{tan}\|_{L^2(\Gamma)}^2, \quad \eta_1 > 0.$$

By using once more the estimate (3.4), for  $h$  small enough we obtain

$$J \geq \eta_1 \left( \|E_{tan}\|_{L^2(\Gamma)}^2 + \|E_{nor}\|_{L^2(\Gamma)}^2 \right) - B_0 h \|E_{tan}\|_{L^2(\Gamma)}^2 \geq \eta_2 \|E_0\|_{L^2(\Gamma)}^2, \quad 0 < \eta_2 < \eta_1.$$

Consequently, (3.14) yields

$$(\eta_2 - B_1 h) \|E_0\|_{L^2(\Gamma)}^2 \leq C_N \frac{h^{-s_0+N}}{|\operatorname{Im} z|} \|E_0\|_{L^2(\Gamma)}^2$$

and for small  $h$  we conclude that for  $z \in Z_2$  the eigenvalues  $\lambda = \frac{i\sqrt{z}}{h}$  of  $G_b$  lie in the region  $\mathcal{R}_N$ . This completes the analysis of the case  $0 < \gamma(x) < 1$ ,  $\forall x \in \Gamma$ .

To study the case  $\gamma(x) > 1$ ,  $\forall x \in \Gamma$ , we write the boundary condition in (1.1) as

$$\frac{1}{\gamma(x)} (\nu \wedge E_{tan}) - (\nu \wedge (\nu \wedge B_{tan})) = \frac{1}{\gamma(x)} (\nu \wedge E_{tan}) + B_{tan} = 0.$$

Next

$$\nu \wedge E = \frac{1}{\sqrt{z}} \nu \wedge \frac{h}{i} \operatorname{curl} B = -\frac{1}{\sqrt{z}} D_\nu B_{tan} + \frac{1}{\sqrt{z}} \left[ \left( \operatorname{grad}_h B_{nor} \right) \Big|_{tan} - i h g_0(B_{tan}) \right]$$

and one obtains

$$\left( D_\nu - \gamma(x) \sqrt{z} \right) B_{tan} - \left( \operatorname{grad}_h B_{nor} \right) \Big|_{tan} + i h g_0(B_{tan}) = 0, \quad x \in \Gamma \quad (3.15)$$

which is the same as (2.1) with  $E_{tan}$ ,  $E_{nor}$  replaced respectively by  $B_{tan}$ ,  $B_{nor}$  and  $\frac{1}{\gamma(x)}$  replaced by  $\gamma(x) > 1$ . We apply the operator  $D_{y_1} - \gamma \sqrt{z}$  to the equation  $\operatorname{div} B = 0$  and repeat without any change the above analysis concerning  $E_{tan}$ ,  $E_{nor}$ . Thus the proof of Theorem 1.1 is complete.  $\square$

*Remark 3.2.* The result of Theorem 1.1 holds for obstacles  $K = \cup_{j=1}^J K_j$ , where  $K_j$ ,  $j = 1, \dots, J$  are open connected domains with  $C^\infty$  boundary and  $K_i \cap K_j = \emptyset$ ,  $i \neq j$ . Let  $\Gamma_j = \partial K_j$ ,  $j = 1, \dots, J$ . In this case we may have  $\gamma(x) < 1$  for some obstacles  $\Gamma_j$  and  $\gamma(x) > 1$  for other ones. The proof extends with only minor modifications. The construction of the semi-classical parametrix in [9] is local and for the Dirichlet-to-Neumann map  $\mathcal{N}_j(z, h)$  related to  $\Gamma_j$  we get the estimate

$$\|\mathcal{N}_j(z, h)(F) - \operatorname{Op}_h(\rho + hb)F\|_{H_h^1(\Gamma_j)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|F\|_{L^2(\Gamma_j)}.$$

The boundary condition in (1.1) is local and we can reduce the analysis to a fixed obstacle  $K_j$ . If  $(E, B) \neq 0$  is an eigenfunction of  $G_b$ , our argument implies  $E_{tan} = 0$  for  $x \in \Gamma_j$  if  $0 < \gamma(x) < 1$  on  $\Gamma_j$  and  $B_{tan} = 0$  for  $x \in \Gamma_j$  in the case  $\gamma(x) > 1$  on  $\Gamma_j$ . By the boundary condition we get  $E_{tan} = 0$  on  $\Gamma$  and this yields  $E = B = 0$

since the Maxwell system with boundary condition  $E_{tan} = 0$  has no eigenvalues in  $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ .

#### 4. APPENDIX

In this Appendix, assume that  $\gamma > 0$  is constant. Our purpose is to study the eigenvalues of  $G_b$  in case the obstacle is equal to the ball  $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ . Setting  $\lambda = \mathbf{i}\mu$ ,  $\operatorname{Im} \mu > 0$ , an eigenfunction  $(E, B) \neq 0$  of  $G_b$  satisfies

$$\operatorname{curl} E = -\mathbf{i}\mu B, \quad \operatorname{curl} B = \mathbf{i}\mu E. \quad (4.1)$$

Replacing  $B$  by  $H = -B$  yields for  $(E, H) \in (H^2(|x| \leq 1))^6$ ,

$$\begin{cases} \operatorname{curl} E = \mathbf{i}\mu H, & \operatorname{curl} H = -\mathbf{i}\mu E, & \text{for } x \in B_3, \\ E_{tan} + \gamma(\nu \wedge H_{tan}) = 0, & \text{for } x \in \mathbb{S}^2. \end{cases} \quad (4.2)$$

Expand  $E(x), H(x)$  in the spherical functions  $Y_n^m(\omega)$ ,  $n = 0, 1, 2, \dots, |m| \leq n$ ,  $\omega \in \mathbb{S}^2$  and the modified Hankel functions  $h_n^{(1)}(z)$  of first kind. An application of Theorem 2.50 in [4] (in the notation of [4] it is necessary to replace  $\omega$  by  $\mu \in \mathbb{C} \setminus \{0\}$ ) says that the solution of the system (4.2) for  $|x| = r = 1$  has the form

$$\begin{aligned} E_{tan}(\omega) &= \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \alpha_n^m \left( h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) U_n^m(\omega) + \beta_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right], \\ H_{tan}(\omega) &= -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \beta_n^m \left( h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) U_n^m(\omega) + \mu^2 \alpha_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right]. \end{aligned}$$

Here  $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^2} Y_n^m(\omega)$  and  $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$  for  $n \in \mathbb{N}$ ,  $-n \leq m \leq n$  form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u \in (L^2(\mathbb{S}^2))^3 : \langle \nu, u \rangle = 0 \text{ on } \mathbb{S}^2\}.$$

To find a representation of  $\nu \wedge H_{tan}$ , observe that  $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$ , so

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[ \beta_n^m \left( h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) V_n^m(\omega) - \mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega) \right]$$

and the boundary condition in (4.2) is satisfied if

$$\alpha_n^m \left[ h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r)) \Big|_{r=1} - \gamma \mathbf{i}\mu h_n^{(1)}(\mu) \right] = 0, \quad \forall n \in \mathbb{N}, |m| \leq n, \quad (4.3)$$

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \left[ h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r)) \Big|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu) \right] = 0, \quad \forall n \in \mathbb{N}, |m| \leq n. \quad (4.4)$$

For  $\gamma \equiv 1$ , there are no eigenvalues.

**Proposition 4.1.** *For  $\gamma \equiv 1$  the operator  $G_b$  has no eigenvalues in  $\{\operatorname{Re} z < 0\}$ .*

*Proof.* The functions  $h_n^{(1)}(z)$  have the form (see for example [7])

$$h_n^{(1)}(x) = (-\mathbf{i})^{n+1} \frac{e^{ix}}{x} \sum_{m=0}^n \frac{\mathbf{i}^m}{m! (2x)^m} \frac{(n+m)!}{(n-m)!} = (-\mathbf{i})^{n+1} \frac{e^{ix}}{x} R_n\left(\frac{\mathbf{i}}{2x}\right)$$

with

$$R_n(z) := \sum_{m=0}^n \frac{z^m}{m!} \frac{(n+m)!}{(n-m)!} = \sum_{m=0}^n a_m z^m.$$

Therefore the term in the brackets [...] in (4.3) becomes

$$(1 - \gamma)\mathbf{i}\mu R_n\left(\frac{\mathbf{i}}{2\mu}\right) - \sum_{m=0}^n a_m m \left(\frac{\mathbf{i}}{2\mu}\right)^m.$$

Setting  $w = \frac{\mathbf{i}}{2\mu}$ , we must study for  $\operatorname{Re} w > 0$  the roots of the equation

$$g_n(w) := \frac{1-\gamma}{2w} R_n(w) + w R'_n(w) = 0. \quad (4.5)$$

For  $\gamma = 1$  one obtains  $R'_n(w) = 0$ . A result of Macdonald says that the zeros of the function  $h_n^{(1)}(z)$  lie in the half plane  $\operatorname{Im} z < 0$  (see Theorem 8.2 in [7]), hence  $R_n(w) \neq 0$  for  $\operatorname{Re} w \geq 0$ . By the theorem of Gauss-Lucas we deduce that the roots of  $R'_n(w) = 0$  lie in the convex hull of the set of the roots of  $R_n(w) = 0$ , so  $R'_n(w) \neq 0$  for  $\operatorname{Re} w > 0$ . Consequently, (4.3) and (4.4) are satisfied only for  $\alpha_n^m = \beta_n^m = 0$  and  $E_{tan} = 0$ . This implies  $E = H = 0$ .  $\square$

For the case  $\gamma \neq 1$ , there are an infinite number of real eigenvalues.

**Proposition 4.2.** *Assume that  $\gamma \in \mathbb{R}^+ \setminus \{1\}$  is a constant. Then  $G_b$  has an infinite number of real eigenvalues. Let  $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$ . Then all real eigenvalues  $\lambda$  with exception of the eigenvalue*

$$\lambda_1 = -\frac{2}{(\gamma_0 - 1)\left(1 + \sqrt{1 + \frac{4}{\gamma_0 - 1}}\right)}. \quad (4.6)$$

satisfy the estimate

$$\lambda \leq -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}}. \quad (4.7)$$

*Proof.* Assume first that  $\gamma > 1$ . Then  $q_n(w) = w g_n(w) = 0$  has at least one real root  $w_0 > 0$ . Indeed,  $q_n(0) = \frac{1-\gamma}{2} < 0$ ,  $q_n(w) \rightarrow +\infty$  as  $w \rightarrow +\infty$ . Choosing  $\alpha_n^{m_0} \neq 0$  for an integer  $m_0$ ,  $|m_0| \leq n$  and taking all other coefficients  $\alpha_n^m, \beta_n^m$  equal to 0, yields  $E_{tan} \neq 0$  and  $G_b$  has an eigenfunction with eigenvalue  $\lambda = -\frac{1}{2w_0} < 0$ .

It is not excluded that  $g_n(w)$  and  $g_m(w)$  for  $n \neq m$  have the same real positive root. If we assume that for  $\operatorname{Re} w > 0$  the sequence of functions  $\{g_n(w)\}_{n=1}^\infty$  has only a finite number of real roots  $w_1, \dots, w_N, w_j \in \mathbb{R}^+$ , then there exists an infinite number of functions  $g_{n_j}(w)$  having the same root which implies that we have an eigenvalue of  $G_b$  with infinite multiplicity. This is a contradiction, and the number of real eigenvalues of  $G_b$  is infinite.

It remains to establish the bound on the real eigenvalues. First, consider the case  $n = 1$ . Then one obtains the equation

$$\frac{2w^2}{2w + 1} = \frac{\gamma - 1}{2}$$

which has a positive root  $w_0 = \frac{1}{4}(\gamma - 1 + \sqrt{(\gamma - 1)^2 + 4(\gamma - 1)})$ . This yields the  $\lambda_1$  from (4.6)

Next examine the case  $n \geq 2$ . For a root  $w_0 \in \mathbb{R}^+$  one has

$$w_0 \left( w_0 \frac{R'_n(w_0)}{R_n(w_0)} \right) = \frac{\gamma - 1}{2}.$$

**Case 1.**  $w_0 \geq \frac{1}{2\sqrt{3}}$ . Then the inequality

$$\frac{\sum_{m=2}^n ma_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \geq \frac{2 \sum_{m=2}^n a_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \geq 1$$

is satisfied since  $a_2 = \frac{1}{2}(n+2)(n+1)n(n-1) \geq 12$ . Consequently,  $2w_0 \leq \gamma - 1$  and this implies that the eigenvalue  $\lambda = -\frac{1}{2w_0}$  satisfies

$$\lambda < -\frac{1}{\gamma - 1}. \quad (4.8)$$

**Case 2.**  $0 < w_0 \leq \frac{1}{2\sqrt{3}}$ . Apply the inequality

$$\frac{\sum_{m=2}^n ma_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \geq \frac{2 \sum_{m=2}^n a_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \geq 2$$

that is equivalent to

$$2[(1 - w_0)S_0 - a_1 w_0] + a_1 \geq 2$$

with  $S_0 = \sum_{m=2}^n a_m w_0^{m-1}$ . This inequality holds because

$$(1 - w_0) \sum_{m=2}^n a_m w_0^{m-1} - a_1 w_0 \geq \left(\frac{1}{2}a_2 - a_1\right)w_0, \quad a_1 = (n+1)n \geq 2,$$

and,

$$\frac{1}{2}a_2 - a_1 = \frac{1}{4}(n+2)(n+1)n(n-1) - (n+1)n = n(n+1)\left[\frac{1}{4}(n+2)(n-1) - 1\right] \geq 0.$$

Therefore,

$$2w_0^2 \leq w_0^2 \frac{\sum_{m=1}^n ma_m w_0^{m-1}}{\sum_{m=1}^n a_m w_0^m + 1} = \frac{\gamma - 1}{2}.$$

This easily yields

$$\lambda \leq -\frac{1}{\sqrt{\gamma - 1}}. \quad (4.9)$$

In the case  $0 < \gamma < 1$  one has  $1/\gamma > 1$  and we apply the above analysis to the equation (4.4). Setting  $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$  and taking into account (4.8) and (4.9), we obtain the result. This completes the proof.  $\square$

*Remark 4.3.* Proposition 4.2 yields a more precise result than that in [1] since we prove the existence of an infinite number of real eigenvalues  $G_b$  for every  $\gamma \in \mathbb{R}^+ \setminus \{1\}$ . In the case  $\gamma = \frac{1}{1+\epsilon}$ ,  $\epsilon > 0$  the eigenvalue  $\lambda_1$  has the form

$$\lambda_1 = \frac{1}{2} \left(1 - \sqrt{1 + \frac{4}{\epsilon}}\right)$$

and this result for small  $\epsilon > 0$  has been obtained in [1]. Clearly, as  $\gamma \rightarrow 1$  the real eigenvalues of  $G_b$  go to  $-\infty$ .

It is easy to see that for  $\gamma > 1$  the equation  $g_n(w) = 0$  has no complex roots. Denote by

$$z_j, \quad \operatorname{Re} z_j < 0, \quad j = 1, \dots, n, \quad n \geq 1$$



the roots of  $R_n(w) = 0$ . Suppose that  $g_n(w_0) = 0$ ,  $n \geq 1$  with  $\operatorname{Re} w_0 > 0$ ,  $\operatorname{Im} w_0 \neq 0$ . Then

$$\operatorname{Im} \left[ \frac{1-\gamma}{2w_0} + w_0 \sum_{j=1}^n \frac{1}{w_0 - z_j} \right] = 0$$

and

$$\begin{aligned} -\frac{(1-\gamma)\operatorname{Im} w_0}{2|w_0|^2} + \operatorname{Re} w_0 \left[ -\sum_{j=1}^n \frac{\operatorname{Im} w_0}{|w_0 - z_j|^2} + \sum_{j=1}^n \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2} \right] \\ + \operatorname{Im} w_0 \sum_{j=1}^n \frac{\operatorname{Re} w_0 - \operatorname{Re} z_j}{|w_0 - z_j|^2} = 0. \end{aligned} \quad (4.10)$$

On the other hand, if  $z_j$  with  $\operatorname{Im} z_j \neq 0$  is a root of  $R_n(w) = 0$ , then  $\bar{z}_j$  is also a root and

$$\begin{aligned} \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2} - \frac{\operatorname{Im} \bar{z}_j}{|w_0 - \bar{z}_j|^2} &= \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2} \left( |w_0 - \bar{z}_j|^2 - |w_0 - z_j|^2 \right) \\ &= \frac{4 \operatorname{Im} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2}. \end{aligned}$$

Equation (4.10) becomes

$$\operatorname{Im} w_0 \left[ \frac{\gamma-1}{2|w_0|^2} - \sum_{j=1}^n \frac{\operatorname{Re} z_j}{|w_0 - z_j|^2} + \sum_{\operatorname{Im} z_j > 0} \frac{4 \operatorname{Re} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2} \right] = 0. \quad (4.11)$$

The term in the brackets [...] is positive, and one concludes that  $\operatorname{Im} w_0 = 0$ .

Repeating the argument of the Appendix in [8], one can show that for  $0 < \gamma < 1$  the complex eigenvalues of  $G_b$  lie in the region

$$\left\{ z \in \mathbb{C} : |\arg z - \pi| > \pi/4, \quad \operatorname{Re} z < 0 \right\}.$$

*Remark 4.4.* We do not know if there exist non real eigenvalues for  $B_3$ .

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