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EIGENVALUES FOR MAXWELL'S EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS

FERRUCCIO COLOMBINI, VESSELIN PETKOV[†] AND JEFFREY RAUCH

ABSTRACT. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0$, $\gamma(x) > 0$, $\forall x \in \Gamma = \partial\Omega$. We prove that if $\gamma(x)$ is nowhere equal to 1, then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where $\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2} + \epsilon} + 1), \operatorname{Re} z < 0\}$, $\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \operatorname{Re} z < 0\}$.

1. INTRODUCTION

Suppose that $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ is an open connected domain and $\Omega := \mathbb{R}^3 \setminus \bar{K}$ is an open connected domain with C^∞ smooth boundary Γ . Consider the boundary problem

$$\begin{aligned} \partial_t E &= \operatorname{curl} B, & \partial_t B &= -\operatorname{curl} E & \text{in } \mathbb{R}_t^+ \times \Omega, \\ E_{tan} - \gamma(x)(\nu \wedge B_{tan}) &= 0 & \text{on } \mathbb{R}_t^+ \times \Gamma, \\ E(0, x) &= e_0(x), & B(0, x) &= b_0(x). \end{aligned} \tag{1.1}$$

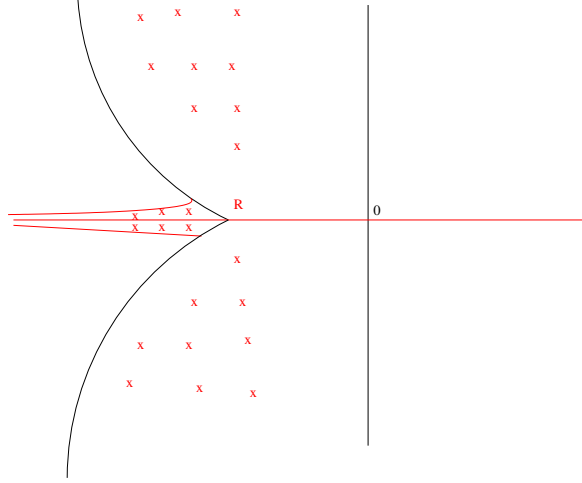
with initial data $f = (e_0, b_0) \in (L^2(\Omega))^6 = \mathcal{H}$. Here $\nu(x)$ denotes the unit outward normal to $\partial\Omega$ at $x \in \Gamma$ pointing into Ω , $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{C}^3 , $u_{tan} := u - \langle u, \nu \rangle \nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. The solution of the problem (1.1) is given by a contraction semigroup $(E, B) = V(t)f = e^{tG_b}f$, $t \geq 0$, where the generator G_b has domain $D(G_b)$ that is the closure in the graph norm of functions $u = (v, w) \in (C_{(0)}^\infty(\mathbb{R}^3))^3 \times (C_{(0)}^\infty(\mathbb{R}^3))^3$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ .

In an earlier paper [2] we proved that the spectrum of G_b in $\operatorname{Re} z < 0$ consists of isolated eigenvalues with finite multiplicity. If $G_b f = \lambda f$ with $\operatorname{Re} \lambda < 0$, the solution $u(t, x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are invisible for inverse scattering problems. It was proved [2] that if there is at least one eigenvalue λ of G_b with $\operatorname{Re} \lambda < 0$, then the wave operators W_\pm are not complete, that is $\operatorname{Ran} W_- \neq \operatorname{Ran} W_+$. Hence we cannot define the scattering operator S related to the Cauchy problem for the Maxwell system and (1.1) by the product $W_+^{-1}W_-$. For the perfect conductor boundary conditions for Maxwell's equations, the energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups. The corresponding scattering operator $S(z) : (L^2(\mathbb{S}^2))^2 \rightarrow (L^2(\mathbb{S}^2))^2$ satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C} \tag{1.2}$$

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FIGURE 1. Eigenvalues of G_b

if $S(z)$ is invertible at z . The scattering operator $S(z)$ defined in [5] is such that $S(z)$ and $S^*(z)$ are analytic in the "physical" half plane $\{z \in \mathbb{C} : \text{Im } z < 0\}$ and the above relation for conservative boundary conditions implies that $S(z)$ is invertible for $\text{Im } z > 0$. For dissipative boundary conditions the relation (1.2) in general is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \text{Im } z_0 > 0$. Lax and Phillips [5] proved that this implies that iz_0 is an eigenvalue of G_b . The analysis of the location of the eigenvalues of G_b is important for the location of the points where the kernel of $S(z)$ is not trivial.

The main result of this paper is the following (see Figure 1)

Theorem 1.1. *Assume that for all $x \in \Gamma$, $\gamma(x) \neq 1$. Then for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ there are constants $C_\epsilon > 0$ and $C_N > 0$ such that the eigenvalues of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where*

$$\Lambda_\epsilon = \{z \in \mathbb{C} : |\text{Re } z| \leq C_\epsilon(|\text{Im } z|^{1/2+\epsilon} + 1), \text{Re } z < 0\},$$

$$\mathcal{R}_N = \{z \in \mathbb{C} : |\text{Im } z| \leq C_N(|\text{Re } z| + 1)^{-N}, \text{Re } z < 0\}.$$

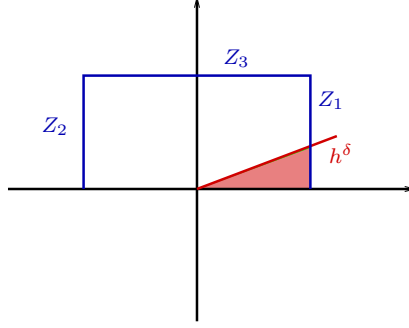
If $\text{Re } \lambda < 0$ and $G_b(E, B) = \lambda(E, B) \neq 0$, then

$$\begin{aligned} \lambda E &= \text{curl } B && \text{on } \Omega, \\ \lambda B &= -\text{curl } E && \text{on } \Omega, \\ \text{div } E &= \text{div } B = 0, && \text{on } \Omega, \\ E_{tan} - \gamma(\nu \wedge B_{tan}) &= 0 && \text{on } \Gamma. \end{aligned} \tag{1.3}$$

This implies that $u := (E, B)$ satisfies

$$\Delta u - \lambda^2 u = 0, \quad \text{on } \Omega.$$

The eigenvalues of G_b are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is nonnegative. The mapping $z \mapsto z^2$ maps the positive quadrant $\{z \in \mathbb{C} : \text{Re } z > 0, \text{Im } z > 0\}$ bijectively to the upper half space. Denote by \sqrt{z} the inverse map. The part of the

FIGURE 2. Contours $Z_1, Z_2, Z_3, \delta = 1/2 - \epsilon$

spectral domain $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0, \operatorname{Im} \lambda > 0\}$ is mapped by $\lambda = \mathbf{i}\sqrt{z}$ to the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. In $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ introduce the sets

$$\begin{aligned} Z_1 &:= \{z \in \mathbb{C} : \operatorname{Re} z = 1, h^\delta \leq \operatorname{Im} z \leq 1\}, \quad 0 < h \ll 1, \quad 0 < \delta < 1/2, \\ Z_2 &:= \{z \in \mathbb{C} : \operatorname{Re} z = -1, 0 \leq \operatorname{Im} z \leq 1\}, \\ Z_3 &:= \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \operatorname{Im} z = 1\}. \end{aligned}$$

Set $\lambda = \mathbf{i}\sqrt{z}/h$, $z \in Z_1 \cup Z_2 \cup Z_3$. To study the eigenvalues λ , $|\lambda| > R_0$, it is sufficient to consider $0 < h \ll 1$. As z runs over the rectangle in Figure 2, with $0 < h \ll 1$, λ sweeps out the large values in the intersection of left and upper half planes. The values of $z \in Z_2$ near the lower left hand corner, $z = -1$, of the rectangle go the spectral values near the negative real axis. The spectral analysis near these values in Z_2 for dissipative Maxwell's equations does not have clear analogue with the spectral problems for the wave equation with dissipative boundary conditions. In fact, for the wave equation if $0 < \gamma(x) < 1, \forall x \in \Gamma$, the eigenvalues of the generator of the corresponding semigroup are located in the domain Λ_ϵ (see Section 3, [8] and [6]). For Maxwell's equations the eigenvalues of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{R}_N$ and for $0 < \gamma(x) < 1$ and $\gamma(x) > 1$ we have the same location (see Appendix for the case $K = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$).

Equation (1.3) implies that on Ω each eigenfunction $u = (E, B)$ of G_b satisfies

$$\sqrt{z} E = \frac{h}{\mathbf{i}} \operatorname{curl} B, \quad \sqrt{z} B = -\frac{h}{\mathbf{i}} \operatorname{curl} E, \quad (1.4)$$

and therefore $(-h^2 \Delta - z)E = (-h^2 \Delta - z)B = 0$. For eigenfunctions $(E, B) \neq 0$, we derive a pseudodifferential system on the boundary involving $E_{tan} = E - \langle E, \nu \rangle \nu$ and $E_{nor} = \langle E, \nu \rangle$. A semi-classical analysis shows that for $z \in Z_1 \cup Z_3$ this system implies that for h small enough we have $E|_\Gamma = 0$ which yields $E = B = 0$. By scaling one concludes that the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G_b lie in the region $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \pi/4, |z| \geq R_0 > 0, \operatorname{Re} z < 0\}.$$

The strategy for the analysis of the case $z \in Z_1 \cup Z_3$ is similar to that exploited in [9] and [8]. In these papers the semi-classical Dirichlet-to-Neumann map $\mathcal{N}(z, h)$ plays a crucial role and the problem is reduced to the proof that some

h -pseudodifferential operators is elliptic in a suitable class. For the Maxwell system the pseudodifferential equation on the boundary is more complicated. Using the equation $\operatorname{div} E = 0$, yields a pseudodifferential system for E_{tan} and E_{nor} . We show that if $(E, B) \neq 0$ is an eigenfunction of G_b , then $\|E_{nor}\|_{H_h^1(\Gamma)}$ is bounded by $Ch\|E_{tan}\|_{H_h^1(\Gamma)}$. The term involving E_{nor} then plays the role of a negligible perturbation in the pseudodifferential system on the boundary and this reduces the analysis to one involving only E_{tan} . The system concerning E_{tan} has a diagonal leading term and we may apply the same arguments as those of [8] to conclude that $E_{tan} = 0$ and hence $E_{nor} = 0$.

The analysis of the case $z \in Z_2$ is more difficult since the principal symbol g of the pseudodifferential system for E_{tan} need not be elliptic at some points (see Section 3). Even where g is elliptic, if $|\operatorname{Im} z| \leq h^{1/2}$ it is difficult to estimate the norm of the difference $Op_h(g)Op_h(g^{-1}) - I$. To show that the eigenvalues of G_b lying in \mathcal{M} are in fact confined to the region \mathcal{R}_N for every $N \in \mathbb{N}$, we analyze the real part of the following scalar product in $L^2(\Gamma)$

$$Q(E_0) := \operatorname{Re}\langle (\mathcal{N}(z, h) - \sqrt{z}\gamma)E_0, E_0 \rangle_{L^2(\Gamma)}, \quad E_0 := E|_{\Gamma}.$$

We follow the approach in [9], [8] based on a Taylor expansion of $Q(E_0)$ at $z = -1$ and the fact that for $z = -1$ we have $Q(E_0) = \mathcal{O}(h^N)$, $\forall N \in \mathbb{N}$. In the Appendix we treat the case when $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ is a ball and $\gamma = \operatorname{const}$. We prove that for $\gamma \equiv 1$ the operator G has no eigenvalues in $\{\operatorname{Re} z < 0\}$, while for every $\gamma \in \mathbb{R}^+ \setminus \{1\}$ we have infinite number of real eigenvalues.

2. PSEUDODIFFERENTIAL EQUATION ON THE BOUNDARY

Introduce geodesic normal coordinates (y_1, y') on a neighborhood of a point $x_0 \in \Gamma$ as follows. For a point x , $y'(x)$ is the closest point in Γ and $y_1 = \operatorname{dist}(x, \Gamma)$. Define $\nu(x)$ to be the unit normal in the direction of increasing y_1 to the surface $y_1 = \operatorname{constant}$ through x . Thus $\nu(x)$ is an extension of the unit normal vector to a unit vector field. The boundary Γ is mapped to $y_1 = 0$ and

$$x = \alpha(y_1, y') = \beta(y') + y_1\nu(y').$$

We have

$$\frac{\partial}{\partial x_k} = \nu_k(y') \frac{\partial}{\partial y_1} + \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial}{\partial y_j}, \quad k = 1, 2, 3.$$

Moreover,

$$\sum_{k=1}^3 \nu_k(y') \frac{\partial y_j}{\partial x_k}(y_1, y') = \langle \nu, \frac{\partial y_j}{\partial x} \rangle = 0, \quad j = 1, 2, 3, \quad \text{and}$$

$$\sum_{k=1}^3 \nu_k(x) \partial_{x_k} f(x) = \partial_{y_1}(f(\alpha(y_1, y'))).$$

Since $\|\nu(x)\| = 1$, $\langle \nu, \partial_{x_j} \nu \rangle = 0$, $j = 1, 2, 3$.

A straight forward computation yields

$$\begin{aligned} \nu(x) \wedge \frac{h}{\mathbf{i}} \operatorname{curl} u(x) &= \mathbf{i}h \partial_{\nu} u_{tan} + \left(\langle D_{x_1} u, \nu \rangle, \langle D_{x_2} u, \nu \rangle, \langle D_{x_3} u, \nu \rangle \right) \Big|_{tan} \\ &= \mathbf{i}h \partial_{\nu} u_{tan} + \left(\operatorname{grad}_h \langle u, \nu \rangle \right) \Big|_{tan} - \mathbf{i}h g_0(u_{tan}), \quad x \in \Gamma, \end{aligned}$$

where

$$D_{x_j} = -\mathbf{i}h\partial_{x_j}, \quad j = 1, 2, 3, \quad \text{grad}_h f = \{D_{x_j} f\}_{j=1,2,3},$$

$$g_0(u_{tan}) = \{\langle u_{tan}, \partial_{x_j} \nu \rangle\}_{j=1,2,3}.$$

Setting $E_{nor} = \langle E, \nu \rangle$, from (1.3) one deduces

$$\nu \wedge B = -\frac{1}{\sqrt{z}} \nu \wedge \frac{h}{\mathbf{i}} \text{curl} E = \frac{1}{\sqrt{z}} D_\nu E_{tan} - \frac{1}{\sqrt{z}} \left[\left(\text{grad}_h E_{nor} \right) \Big|_{tan} - \mathbf{i}h g_0(E_{tan}) \right],$$

where $D_\nu = -\mathbf{i}h\partial_\nu$ and the boundary condition in (1.3) becomes

$$\left(D_\nu - \frac{1}{\gamma} \sqrt{z} \right) E_{tan} - \left(\text{grad}_h E_{nor} \right) \Big|_{tan} + \mathbf{i}h g_0(E_{tan}) = 0, \quad x \in \Gamma. \quad (2.1)$$

Next

$$\text{grad}_h f(x)|_{tan} = \left\{ \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} D_{y_j} f(\alpha(y_1, y')) \right\}_{k=1,2,3}$$

and for $u = (u_1, u_2, u_3) \in \mathbb{C}^3$,

$$\begin{aligned} \frac{h}{\mathbf{i}} \text{div} u(\alpha(y_1, y')) &= \langle D_{y_1} u(\alpha(y_1, y')), \nu(y') \rangle + \sum_{k=1}^3 \sum_{j=2}^3 \frac{\partial y_j}{\partial x_k} D_{y_j} u_k(\alpha(y_1, y')) \\ &= D_{y_1} \left(u_{nor}(y_1, y') \right) + \sum_{j=2}^3 D_{y_j} \left\langle u_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x} \right\rangle + h \langle u_{tan}, Z \rangle, \end{aligned}$$

where $\langle u(\alpha(y_1, y')), \nu(y') \rangle := u_{nor}(y_1, y')$ and Z depends on the second derivatives of y_j , $j = 2, 3$. Apply the operator $D_{y_1} - \frac{\sqrt{z}}{\gamma(y')}$ to $\text{div} E(\alpha(y_1, y')) = 0$ to find

$$\begin{aligned} \left(D_{y_1}^2 - \frac{\sqrt{z}}{\gamma(y')} D_{y_1} \right) E_{nor}(y_1, y') + \sum_{j=2}^3 D_{y_j} \left\langle \left(D_{y_1} - \frac{\sqrt{z}}{\gamma(y')} \right) E_{tan}(\alpha(y_1, y')), \frac{\partial y_j}{\partial x} \right\rangle \\ = h \langle \left(D_{y_1} - \frac{\sqrt{z}}{\gamma} \right) E_{tan}, Z \rangle + h \langle E_{tan}, Z_1 \rangle, \end{aligned}$$

where $\gamma(y') := \gamma(\beta(y'))$.

Taking the trace $y_1 = 0$ and applying the boundary condition (2.1), yields

$$\begin{aligned} \left(D_{y_1}^2 + \sum_{j,\mu=2}^3 \sum_{k=1}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial y_\mu}{\partial x_k} D_{y_j, y_\mu}^2 \right) E_{nor}(0, y') - \frac{\sqrt{z}}{\gamma(y')} D_{y_1} E_{nor}(0, y') \\ = h \left\langle \left(\text{grad}_h E_{nor} \right) \Big|_{tan}(0, y'), Z \right\rangle + h Q_1(E_{tan}(0, y')), \quad (2.2) \end{aligned}$$

with

$$\|Q_1(E_{tan}(0, y'))\|_{L^2(\mathbb{R}^2)} \leq C_2 \|E_{tan}(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Here $H_h^s(\Gamma)$, $s \in \mathbb{R}$, denotes the semi-classical Sobolev spaces with norm $\|\langle h\partial_x \rangle^s u\|_{L^2(\Gamma)}$, $\langle h\partial_x \rangle = (1 + \|h\partial_x\|^2)^{1/2}$. In the exposition below we use the spaces $(L^2(\Gamma))^3$ and $(H_h^s(\Gamma))^3$ of vector-valued functions but we will omit this in the notations writing simply $L^2(\Gamma)$ and $H_h^s(\Gamma)$.

The operator $-h^2\Delta_x - z$ in the coordinates (y_1, y') has the form

$$\mathcal{P}(z, h) = D_{y_1}^2 + r(y, D_{y'}) + q_1(y, D_y) + h^2\tilde{q} - z$$

with $r(y, \eta') = \langle R(y)\eta', \eta' \rangle$, $q_1(y, \eta) = \langle q_1(y), \eta \rangle$. Here

$$R(y) = \left\{ \sum_{k=1}^3 \frac{\partial y_j}{\partial x_k} \frac{\partial y_\mu}{\partial x_k} \right\}_{j,\mu=2}^3 = \left\{ \left\langle \frac{\partial y_j}{\partial x}, \frac{\partial y_\mu}{\partial x} \right\rangle \right\}_{j,\mu=2}^3$$

is a symmetric (2×2) matrix and $r(0, y', \eta') = r_0(y', \eta')$, where $r_0(y', \eta')$ is the principal symbol of the Laplace-Beltrami operator $-h^2 \Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean one in \mathbb{R}^3 . We have

$$\left(\mathcal{P}(z, h) E_{nor} \right)(0, y') = \langle \mathcal{P}(z, h) E, \nu \rangle(0, y') + h Q_2(E(0, y')),$$

where

$$\|Q_2(E(0, y'))\|_{L^2(\mathbb{R}^2)} \leq C_2 \|E(0, y')\|_{H_h^1(\mathbb{R}^2)}.$$

Since $\mathcal{P}(z, h)E = 0$, this lets us replace the terms with all second derivatives of E_{nor} in (2.4) by $zE_{nor}(0, y')$ modulo terms having a factor h and containing first order derivatives of E_{nor} . This follows from the form of the matrix $R(y)$ given above. After a multiplication by $-\frac{\gamma(y')}{\sqrt{z}}$ the equation (2.2) yields

$$(D_{y_1} - \gamma(y')\sqrt{z})E_{nor}(0, y') = h Q_3(E(0, y')), \quad (2.3)$$

where $Q_3(E(0, y'))$ has the same properties as $Q_2(E(0, y'))$.

Let $\psi(x) \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with support in small neighborhood of $x_0 \in \Gamma$. Replace E, B by $E_\psi = E\psi$, $B_\psi = B\psi$. The above analysis works for E_ψ and B_ψ with lower order terms depending on ψ . We obtain

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma \psi(x), \nu(x) \rangle = h Q_{3,\psi}(E|_\Gamma).$$

Taking a partition of unity in a neighborhood of Γ , yields

$$\langle (D_\nu - \gamma(x)\sqrt{z})E|_\Gamma, \nu \rangle = h Q_4(E|_\Gamma), \quad \|Q_4(E|_\Gamma)\|_{L^2(\Gamma)} \leq C \|E|_\Gamma\|_{H_h^1(\Gamma)}. \quad (2.4)$$

For $z \in Z_1 \cup Z_2 \cup Z_3$ let $\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*\Gamma)$ be the root of the equation

$$\rho^2 + r_0(x', \xi') - z = 0$$

with $\text{Im } \rho(x', \xi', z) > 0$. For large $|\xi'|$,

$$\rho(x', \xi', z) \sim |\xi'|, \quad \text{Im } \rho(x', \xi', z) \sim |\xi'|,$$

while for bounded $|\xi'|$,

$$\text{Im } \rho(x', \xi', z) \geq \frac{h^\delta}{C}.$$

We recall some basic facts about h -pseudodifferential operators that the reader can find in [3]. Let X be a C^∞ smooth compact manifold without boundary with dimension $d \geq 2$. Let (x, ξ) be the coordinates in $T^*(X)$ and let $a(x, \xi, h) \in C^\infty(T^*(X))$. Given $m \in \mathbb{R}$, $l \in \mathbb{R}$, $\delta > 0$ and a function $c(h) > 0$, one denotes by $S_\delta^{l,m}(c(h))$ the set of symbols so that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha,\beta} (c(h))^{-l - \delta(|\alpha| + |\beta|)} (1 + |\xi|)^{m - |\beta|}, \quad \forall \alpha, \forall \beta, \quad (x, \xi) \in T^*(X).$$

If $c(h) = h$, we denote $S_\delta^{l,m}(c(h))$ simply by $S_\delta^{l,m}$. Symbols restricted to a domain where $|\xi| \leq C$ will be denoted by $a \in S_\delta^l(c(h))$. The h -pseudodifferential operator

with symbol $a(x, \xi, h)$ acts by

$$(Op_h(a)f)(x) := (2\pi h)^{-d+1} \int_{T^*X} e^{-i\langle x-y, \xi \rangle/h} a(x, \xi, h) f(y) dy d\xi.$$

For matrix valued symbols we use the same definition. This means that every element of a matrix symbol is in the class $S_\delta^{l,m}(c(h))$.

Now suppose that $a(x, \xi, h)$ satisfies the estimates

$$|\partial_x^\alpha a(x, \xi, h)| \leq c_0(h) h^{-|\alpha|/2}, \quad (x, \xi) \in T^*(X) \quad (2.5)$$

for $|\alpha| \leq d-1$, where $c_0(h) > 0$ is a parameter. Then there exists a constant $C > 0$ independent of h such that

$$\|Op_h(a)\|_{L^2(X) \rightarrow L^2(X)} \leq C c_0(h). \quad (2.6)$$

For $0 \leq \delta < 1/2$ products of h -pseudodifferential operators are well behaved. If $a \in S_\delta^{l_1, m_1}$, $b \in S_\delta^{l_2, m_2}$ and $s \in \mathbb{R}$, then

$$\|Op_h(a)Op_h(b) - Op_h(ab)\|_{H^s(X) \rightarrow H^{s-m_1-m_2+1}(X)} \leq Ch^{-l_1-l_2-2\delta+1}. \quad (2.7)$$

Let $u \in \mathbb{C}^3$ be the solution of the Dirichlet problem

$$(-h^2\Delta - z)u = 0 \quad \text{on } \Omega, \quad u = F \quad \text{on } \Gamma. \quad (2.8)$$

Introduce the semi-classical Dirichlet-to-Neumann map

$$\mathcal{N}(z, h) : H_h^s(\Gamma) \ni F \longrightarrow D_\nu u|_\Gamma \in H_h^{s-1}(\Gamma).$$

G. Vodev [9] established for bounded domains $K \subset \mathbb{R}^d$, $d \geq 2$, with C^∞ boundary the following approximation of the interior Dirichlet-to-Neumann map $\mathcal{N}_{int}(z, h)$ related to (2.8), where the equation $(-h^2\Delta - z)u = 0$ is satisfied in K .

Theorem 2.1 ([9]). *For every $0 < \epsilon \ll 1$ there exists $0 < h_0(\epsilon) \ll 1$ such that for $z \in Z_{1,\epsilon} := \{z \in Z_1, |\operatorname{Im} z| \geq h^{\frac{1}{2}-\epsilon}\}$ and $0 < h \leq h_0(\epsilon)$ we have*

$$\|\mathcal{N}_{int}(z, h)(F) - Op_h(\rho + hb)F\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|F\|_{L^2(\Gamma)}, \quad (2.9)$$

where $b \in S_{0,1}^0(\Gamma)$ does not depend on h and z . Moreover, (2.9) holds for $z \in Z_2 \cup Z_3$ with $|\operatorname{Im} z|$ replaced by 1.

With small modifications (2.9) holds for the Dirichlet-to-Neumann map $\mathcal{N}(z, h)$ related to (2.8) (see [8]). Applying (2.9) with $\mathcal{N}(z, h)$ and $F = E_0 = E|_\Gamma$, we obtain

$$\left\| \langle \mathcal{N}(z, h)E_0, \nu \rangle - \langle Op_h(\rho)E_0, \nu \rangle \right\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \quad (2.10)$$

Therefore (2.4) yields

$$\left\| \langle Op_h(\rho) - \gamma\sqrt{z} \rangle E_0, \nu \rangle - hQ_4(E_0) \right\|_{L^2(\Gamma)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)}. \quad (2.11)$$

The commutator $[Op_h(\rho), \nu(x)]$ is a pseudodifferential operator with symbol in $h^{1-\delta}S_\delta^{0,0}$ and so

$$\|[Op_h(\rho), \nu_k(x)]E_{nor}\|_{H_h^j(\Gamma)} \leq C_2 h^{1-\delta} \|E_{nor}\|_{H_h^j(\Gamma)}, \quad k = 1, 2, 3, \quad j = 0, 1.$$

The last estimate combined with (2.11) implies

$$\left\| (Op_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0) \right\|_{L^2(\Gamma)} \leq C_3 \left(\frac{h}{\sqrt{|\operatorname{Im} z|}} + h^{1-\delta} \right) \|E_0\|_{L^2(\Gamma)}. \quad (2.12)$$

3. EIGENVALUES-FREE REGIONS

For $z \in Z_{1,\epsilon}$ we have $\rho \in S_\delta^{0,1}$ with $0 < \delta = 1/2 - \epsilon < 1/2$, while for $z \in Z_2 \cup Z_3$ we have $\rho \in S_0^{0,1}$ (see [9]). Since Γ is connected one has either $\gamma(x) > 1$ or $0 < \gamma(z) < 1$. We present the analysis in the case where $0 < \gamma(x) < 1, \forall x \in \Gamma$. The case $1 < \gamma(x)$ is reduced to this case at the end of the section. Clearly, there exists $\epsilon_0 > 0$ such that

$$\epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0, \quad \forall x \in \Gamma.$$

Combing (2.4) and (2.9), yields

$$\| \langle (Op_h(\rho) - \gamma(x)\sqrt{z})E_0, \nu(x) \rangle \|_{L^2(\Gamma)} \leq C \frac{h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + C_1 h \|E_0\|_{H_h^1(\Gamma)},$$

where for $z \in Z_2 \cup Z_3$ we can replace $|\operatorname{Im} z|$ by 1. This estimate for E_0 and the estimate for the commutator $[Op_h(\rho), \nu_k(x)]$ imply

$$\| (Op_h(\rho) - \gamma(x)\sqrt{z})E_{nor} \|_{L^2(\Gamma)} \leq \frac{C_3 h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + C_4 h^{1-\delta} \|E_0\|_{H_h^1(\Gamma)}. \quad (3.1)$$

Let (x', ξ') be coordinates on $T^*(\Gamma)$. Consider the symbol

$$c(x', \xi', z) := \rho(x', \xi', z) - \gamma(x')\sqrt{z}, \quad x' \in \Gamma.$$

Following the analysis in Section 3, [8], we know that c is elliptic in the case $0 < \gamma(x') < 1$ and if $z \in Z_1$ we have $c \in S_\delta^{0,1}$, $|\operatorname{Im} z|c^{-1} \in S_\delta^{0,-1}$, while if $z \in Z_2 \cup Z_3$ one gets $c \in S_0^{0,1}$, $c^{-1} \in S_0^{0,-1}$. This implies

$$\| Op_h(c^{-1})Op_h(c)E_{nor} \|_{H_h^1(\Gamma)} \leq \frac{C}{|\operatorname{Im} z|} \|Op_h(c)E_{nor}\|_{L^2(\Gamma)}.$$

On the other hand, according to Section 7 in [3], the symbol of the operator $Op_h(c^{-1})Op_h(c) - I$ is given by

$$\begin{aligned} & \sum_{j=1}^N \frac{(ih)^j}{j!} \sum_{|\alpha|=j} D_{\xi'}^\alpha (c^{-1})(x', \xi') D_{y'}^\alpha c(y', \eta') \Big|_{x'=y', \xi'=\eta'} + \tilde{b}_N(x', \xi') \\ & := b_N(x', \xi') + \tilde{b}_N(x', \xi'), \end{aligned}$$

where

$$|\partial_x^\alpha \tilde{b}_N(x', \xi')| \leq C_\alpha h^{N(1-2\delta) - s_\alpha - |\alpha|/2}.$$

Taking into account the estimates for c^{-1} and c , and applying (2.5), and (2.6) yields

$$\left\| \left(Op_h(c^{-1})Op_h(c) - I \right) E_{nor} \right\|_{H_h^j(\Gamma)} \leq C_5 \frac{h}{|\operatorname{Im} z|^2} \|E_{nor}\|_{H_h^j(\Gamma)}, \quad j = 0, 1.$$

Repeating the argument in Section 3 in [8] concerning the case $0 < \gamma(x') < 1$, for $z \in Z_1$ and $0 < \delta < 1/2$, one finds

$$\begin{aligned} \|E_{nor}\|_{H_h^1(\Gamma)} & \leq \left\| \left(Op_h(c^{-1})Op_h(c) - I \right) E_{nor} \right\|_{H_h^1(\Gamma)} + \left\| Op_h(c^{-1})Op_h(c)E_{nor} \right\|_{H_h^1(\Gamma)} \\ & \leq C_6 h^{1-2\delta} \|E_0\|_{L^2(\Gamma)} + C_5 h^{1-2\delta} \|E_{nor}\|_{H_h^1(\Gamma)} + C_7 h^{1-\delta} \|E_0\|_{H_h^1(\Gamma)}. \end{aligned} \quad (3.2)$$

Clearly,

$$\|E_0\|_{H_h^k(\Gamma)} \leq \|E_{tan}\|_{H_h^k(\Gamma)} + B_k \|E_{nor}\|_{H_h^k(\Gamma)}, \quad k \in \mathbb{N}$$

with B_k independent of h . Hence we can absorb the terms involving the norms of E_{nor} in the right hand side of (3.2) choosing h small enough, and we get

$$\|E_{nor}\|_{H_h^1(\Gamma)} \leq Ch^{1-2\delta}\|E_{tan}\|_{H_h^1(\Gamma)}. \quad (3.3)$$

The analysis of the case $z \in Z_2 \cup Z_3$ is simpler since in the estimates above we have no coefficient $|\operatorname{Im} z|^{-1}$ and we obtain the same result with a factor h on the right hand side of (3.3).

With a similar argument it is easy to show that

$$\|E_{nor}\|_{L^2(\Gamma)} \leq C'h^{1-2\delta}\|E_{tan}\|_{L^2(\Gamma)}. \quad (3.4)$$

In fact from (2.12) one obtains

$$\left\| \operatorname{Op}_h(c^{-1}) \left[(\operatorname{Op}_h(\rho) - \gamma\sqrt{z})E_{nor} - hQ_4(E_0) \right] \right\|_{L^2(\Gamma)} \leq \frac{C_8}{|\operatorname{Im} z|} \left(\frac{h}{\sqrt{|\operatorname{Im} z|}} + h^{1-\delta} \|E_0\|_{L^2(\Gamma)} \right)$$

and

$$\|\operatorname{Op}_h(c^{-1})Q_4(E_0)\|_{L^2(\Gamma)} \leq \frac{C_9}{|\operatorname{Im} z|} \|E_0\|_{L^2(\Gamma)}.$$

Combining these estimates with the estimate of $\|\operatorname{Op}_h(c^{-1})\operatorname{Op}_h(c) - I\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}$ yields (3.4).

Going back to the equation (2.1), we have

$$\begin{aligned} \left(D_\nu - \frac{1}{\gamma}\sqrt{z} \right) E &= \left(D_\nu - \gamma\sqrt{z} \right) E_{nor}\nu - \left(\frac{1}{\gamma} - \gamma \right) \sqrt{z} E_{nor}\nu \\ &\quad + \mathbf{i}hg_0(E_{tan}) + \left(\operatorname{grad}_h(E_{nor}) \right) \Big|_{\tan}, \quad x \in \Gamma. \end{aligned} \quad (3.5)$$

Notice that for the first term on the right hand side of (3.5) we can apply the equality (2.4), while for E_{nor} and $\left(\operatorname{grad}_h(E_{nor}) \right) \Big|_{\tan}$ we have a control by the estimate (3.3). Consequently, setting $E_0 = E|_\Gamma$, the right hand side of (3.5) is bounded by $Ch^{1-2\delta}\|E_0\|_{H_h^1(\Gamma)}$. Next

$$1 < \frac{1}{1-\epsilon_0} \leq \frac{1}{\gamma(x)} \leq \frac{1}{\epsilon_0}, \quad \forall x \in \Gamma.$$

This corresponds to the case (B) examined in Section 4 of [8]. The approximation of the operator $\mathcal{N}(z, h)$ given by (2.9) yields the estimate

$$\|(\operatorname{Op}_h(\rho) - \frac{1}{\gamma}\sqrt{z})E_0\|_{L^2(\Gamma)} \leq C \left(\frac{h}{\sqrt{|\operatorname{Im} z|}} \|E_0\|_{L^2(\Gamma)} + h^{1-2\delta} \|E_0\|_{H_h^1(\Gamma)} \right). \quad (3.6)$$

For $z \in Z_1 \cup Z_3$ the symbol

$$d(x', \xi', z) := \rho(x', \xi', z) - \frac{1}{\gamma(x')} \sqrt{z}$$

is elliptic (see Section 4, [8]) and $d \in S_\delta^{0,1}$, $d^{-1} \in S_\delta^{0,-1}$. Then from (3.6) we estimate $\|E_0\|_{H_h^1(\Gamma)}$ and we obtain $E_0 = 0$ for h small enough. This implies $E = B = 0$.

Now recall that we have

$$\operatorname{Re} \lambda = -\frac{\operatorname{Im} \sqrt{z}}{h}, \quad \operatorname{Im} \lambda = \frac{\operatorname{Re} \sqrt{z}}{h}.$$

Suppose that $z \in Z_1$. Then

$$|\operatorname{Re} \lambda| \geq C(h^{-1})^{1-\delta}, \quad |\operatorname{Im} \lambda| \leq C_1 h^{-1} \leq C_2 |\operatorname{Re} \lambda|^{\frac{1}{1-\delta}}.$$

So if

$$|\operatorname{Re} \lambda| \geq C_3 |\operatorname{Im} \lambda|^{1-\delta}, \quad \operatorname{Re} \lambda \leq -C_4 < 0,$$

there are no eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ of G_b . In the same way we handle the case $z \in Z_3$ and we conclude that if $z \in Z_1 \cup Z_3$ for every $\epsilon > 0$ the eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ of G_b lie in the domain $\Lambda_\epsilon \cup \mathcal{M}$, where

$$\mathcal{M} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \pi/4, |z| \geq R_0 > 0, \operatorname{Re} z < 0\},$$

Λ_ϵ being the domain introduced in Theorem 1.1. Of course, if we consider the domain

$$Z_{3,\delta_0} = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \operatorname{Im} z = \delta_0 > 0\},$$

instead of Z_3 , we obtain an eigenvalue-free region with \mathcal{M} replaced by

$$\mathcal{M}_{\delta_0} = \{z \in \mathbb{C} : |\arg z - \pi| \leq \arctg \delta_0, |z| \geq R_0(\delta_0) > 0, \operatorname{Re} z < 0\}.$$

The investigation of the case $z \in Z_2$ is more complicated since the symbol d may vanish for $\operatorname{Im} z = 0$ and $(x'_0, \xi'_0) \in T^*(\Gamma)$ satisfying the equation

$$\sqrt{1 + r_0(x'_0, \xi'_0)} - \frac{1}{\gamma(x'_0)} = 0.$$

To cover this case and to prove that the eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ with $z \in Z_2$ are confined in the domain \mathcal{R}_N , $\forall N \in \mathbb{N}$, we follow the arguments in [9] and [8]. For $z \in Z_2$ we introduce an operator $T(z, h)$ that yields a better approximation of $\mathcal{N}(z, h)$. In fact, $T(z, h)$ is defined by the construction of the semi-classical parametrix in Section 3, [9] for the problem (2.8) with $F = E_0$. We refer to [9] for the precise definition of $T(z, h)$ and more details. For our exposition we need the next proposition. Since $(\Delta - z)E = 0$, as in [9], we obtain

Proposition 3.1. *For $z \in Z_2$ and every $N \in \mathbb{N}$ we have the estimate*

$$\|\mathcal{N}(z, h)E_0 - T(z, h)E_0\|_{H^1_h(\Gamma)} \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)} \quad (3.7)$$

with constants $C_N, s_0 > 0$, independent of E_0, h and z , and s_0 independent of N .

Proof of Theorem 1.1 in the case $z \in Z_2$. Consider the system

$$\begin{cases} \left(D_\nu - \frac{1}{\gamma} \sqrt{z} \right) E_{tan} - \left(\operatorname{grad}_h E_{nor} \right) \Big|_{tan} + i h g_0(E_{tan}) = 0, & x \in \Gamma, \\ \operatorname{div}_h E_{tan} + \operatorname{div}_h (E_{nor} \nu) = 0, & x \in \Gamma, \end{cases} \quad (3.8)$$

where $\operatorname{div}_h F = \sum_{k=1}^3 D_{x_k} F_k$.

Take the scalar product $\langle \cdot, \cdot \rangle_{L^2(\Gamma)}$ in $L^2(\Gamma)$ of the first equation of (3.8) and E_{tan} . Applying Green formula, it easy to see that

$$-\operatorname{Re} \langle \operatorname{grad}_h E_{nor} \Big|_{tan}, E_{tan} \rangle_{L^2(\Gamma)} = -\operatorname{Re} \langle \operatorname{div}_h E_{tan}, E_{nor} \rangle_{L^2(\Gamma)}. \quad (3.9)$$

We claim that

$$\operatorname{Im} \langle g_0(E_{tan}), E_{tan} \rangle_{L^2(\Gamma)} = 0. \quad (3.10)$$

Let $E_{tan} = (w_1, w_2, w_3)$. Then

$$\langle g_0(E_{tan}), E_{tan} \rangle_{\mathbb{C}^3} = \sum_{k,j=1}^3 w_k \frac{\partial \nu_k}{\partial x_j} \overline{w_j} = \frac{1}{q} \sum_{k,j=1}^3 w_k \frac{\partial V_k}{\partial x_j} \overline{w_j} = \frac{1}{q} \langle S w, w \rangle_{\mathbb{C}^3},$$

where $S := \{\frac{\partial V_k}{\partial x_j}\}_{k,j=1}^3$ with $V(x) = q(x)\nu(x)$, $q(x) > 0$ because $\sum_{k=1}^3 (\partial_{x_j} q) w_k \nu_k = 0$. Thus if the boundary is given locally by $x_3 = G(x_1, x_2)$, we choose $V(x) = (-\partial_{x_1} G, -\partial_{x_2} G, 1)$ and it is obvious that S is symmetric. Therefore $\text{Im}\langle Sw, w \rangle_{\mathbb{C}^3} = 0$ and this proves the claim. Hence (3.10) implies

$$\text{Re}[\mathbf{i}h\langle g_0(E_{tan}), E_{tan} \rangle_{L^2(\Gamma)}] = 0. \quad (3.11)$$

From the $L^2(\Gamma)$ scalar product of the second equation in (3.8) with E_{nor} , we obtain

$$\text{Re}\langle \text{div}_h E_{tan}, E_{nor} \rangle_{L^2(\Gamma)} + \text{Re}\langle D_\nu E_{nor}, E_{nor} \rangle_{L^2(\Gamma)} = 0. \quad (3.12)$$

In fact,

$$\text{div}_h(E_{nor}\nu) = D_\nu E_{nor} - \mathbf{i}h E_{nor} \text{div } \nu$$

and $\text{Im}(\text{div } \nu |E_{nor}|^2) = 0$.

Taking together (3.9), (3.11) and (3.12), we conclude that

$$\begin{aligned} & \text{Re}\left[\langle (D_\nu - \frac{\sqrt{z}}{\gamma})E_{tan}, E_{tan} \rangle_{L^2(\Gamma)} + \langle D_\nu E_{nor}\nu, E_{nor}\nu \rangle_{L^2(\Gamma)}\right] \\ &= \text{Re}\langle D_\nu E, E \rangle_{L^2(\Gamma)} - \text{Re}\langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan} \rangle_{L^2(\Gamma)} = 0. \end{aligned}$$

Here we have used the fact that

$$\langle D_\nu E_{tan}, E_{nor}\nu \rangle_{\mathbb{C}^3} = D_\nu \langle E_{tan}, E_{nor}\nu \rangle_{\mathbb{C}^3} = 0.$$

Applying Proposition 3.1 with $E|_\Gamma = E_0$, yields

$$\left| \text{Re}\langle T(z, h)E_0, E_0 \rangle_{L^2(\Gamma)} - \text{Re}\langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan} \rangle_{L^2(\Gamma)} \right| \leq C_N h^{-s_0} h^N \|E_0\|_{L^2(\Gamma)}. \quad (3.13)$$

For $z = -1$, as in Lemma 3.9 in [9] and Lemma 4.1 in [8], we have

$$|\text{Re}\langle T(-1, h)E_0, E_0 \rangle_{L^2(\Gamma)}| \leq C_N h^{-s_0+N} \|E_0\|_{L^2(\Gamma)}^2 = 0.$$

Consequently, by using Taylor formula for the real-valued function

$$\text{Re}\left[\langle T(z, h)E_0, E_0 \rangle_{L^2(\Gamma)} - \langle \frac{\sqrt{z}}{\gamma} E_{tan}, E_{tan} \rangle_{L^2(\Gamma)}\right],$$

we get for every $N \in \mathbb{N}$ the estimate

$$\begin{aligned} & \left| \text{Im}\left[\langle \frac{\partial T}{\partial z}(z_t, h)E_0, E_0 \rangle_{L^2(\Gamma)} - \langle \frac{\gamma_1}{2\sqrt{z_t}} E_{tan}, E_{tan} \rangle_{L^2(\Gamma)}\right] \right| \\ & \leq C_N \frac{h^{-s_0+N}}{|\text{Im } z|} \|E_0\|_{L^2(\Gamma)}^2, \end{aligned} \quad (3.14)$$

where $z_t = -1 + \mathbf{i}t \text{Im } z$, $0 < t < 1$, $\gamma_1 = \gamma^{-1}$.

According to Lemma 3.9 in [9], in (3.14) we can replace $\frac{\partial T}{\partial z}(z_t, h)$ by $Op_h(\frac{\partial \rho}{\partial z}(z_t))$ and this yields an error term bounded by $Ch\|E_0\|_{H_h^{-1}(\Gamma)}^2$. On the other hand,

$$\begin{aligned} & \left| \langle Op_h(\frac{\partial \rho}{\partial z}(z_t))E_{tan}, E_{nor}\nu \rangle_{L^2(\Gamma)} + \langle Op_h(\frac{\partial \rho}{\partial z}(z_t))E_{nor}, E_{tan}\nu \rangle_{L^2(\Gamma)} \right| \\ & \leq Ch\|E_0\|_{L^2(\Gamma)}^2 \end{aligned}$$

since the estimate (3.4) holds for $z \in Z_2$ with factor h and $\frac{\partial \rho}{\partial z}(z_t) \in S_0^{0,-1}$.

Thus the problem is reduced to a lower bound of

$$\begin{aligned} J &:= \left| \operatorname{Im} \left[\left\langle \left(\operatorname{Op}_h \left(\frac{\partial \rho}{\partial z} \right) (z_t) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} + \left\langle \operatorname{Op}_h \left(\frac{\partial \rho}{\partial z} \right) (z_t) E_{nor} \nu, E_{nor} \nu \right\rangle_{L^2(\Gamma)} \right] \right| \\ &\geq \left| \operatorname{Im} \left\langle \left(\operatorname{Op}_h \left(\frac{\partial \rho}{\partial z} \right) (z_t) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} \right| - C_1 \|E_{nor}\|_{L^2(\Gamma)}^2. \end{aligned}$$

Since $\gamma_1(x) > 1$, $\forall x \in \Gamma$, applying the analysis of Section 4 in [8] for the scalar product involving E_{tan} , one deduces

$$\left| \operatorname{Im} \left\langle \left(\operatorname{Op}_h \left(\frac{\partial \rho}{\partial z} \right) (z_t) - \frac{\gamma_1}{2\sqrt{z}} \right) E_{tan}, E_{tan} \right\rangle_{L^2(\Gamma)} \right| \geq \eta_1 \|E_{tan}\|_{L^2(\Gamma)}^2, \quad \eta_1 > 0.$$

By using once more the estimate (3.4), for h small enough we obtain

$$J \geq \eta_1 \left(\|E_{tan}\|_{L^2(\Gamma)}^2 + \|E_{nor}\|_{L^2(\Gamma)}^2 \right) - B_0 h \|E_{tan}\|_{L^2(\Gamma)}^2 \geq \eta_2 \|E_0\|_{L^2(\Gamma)}^2, \quad 0 < \eta_2 < \eta_1.$$

Consequently, (3.14) yields

$$(\eta_2 - B_1 h) \|E_0\|_{L^2(\Gamma)}^2 \leq C_N \frac{h^{-s_0+N}}{|\operatorname{Im} z|} \|E_0\|_{L^2(\Gamma)}^2$$

and for small h we conclude that for $z \in Z_2$ the eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ of G_b lie in the region \mathcal{R}_N . This completes the analysis of the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$.

To study the case $\gamma(x) > 1$, $\forall x \in \Gamma$, we write the boundary condition in (1.1) as

$$\frac{1}{\gamma(x)} (\nu \wedge E_{tan}) - (\nu \wedge (\nu \wedge B_{tan})) = \frac{1}{\gamma(x)} (\nu \wedge E_{tan}) + B_{tan} = 0.$$

Next

$$\nu \wedge E = \frac{1}{\sqrt{z}} \nu \wedge \frac{h}{\mathbf{i}} \operatorname{curl} B = -\frac{1}{\sqrt{z}} D_\nu B_{tan} + \frac{1}{\sqrt{z}} \left[\left(\operatorname{grad}_h B_{nor} \right) \Big|_{tan} - \mathbf{i} h g_0(B_{tan}) \right]$$

and one obtains

$$\left(D_\nu - \gamma(x) \sqrt{z} \right) B_{tan} - \left(\operatorname{grad}_h B_{nor} \right) \Big|_{tan} + \mathbf{i} h g_0(B_{tan}) = 0, \quad x \in \Gamma \quad (3.15)$$

which is the same as (2.1) with E_{tan} , E_{nor} replaced respectively by B_{tan} , B_{nor} and $\frac{1}{\gamma(x)}$ replaced by $\gamma(x) > 1$. We apply the operator $D_{y_1} - \gamma\sqrt{z}$ to the equation $\operatorname{div} B = 0$ and repeat without any change the above analysis concerning E_{tan} , E_{nor} . Thus the proof of Theorem 1.1 is complete. \square

Remark 3.2. The result of Theorem 1.1 holds for obstacles $K = \cup_{j=1}^J K_j$, where K_j , $j = 1, \dots, J$ are open connected domains with C^∞ boundary and $K_i \cap K_j = \emptyset$, $i \neq j$. Let $\Gamma_j = \partial K_j$, $j = 1, \dots, J$. In this case we may have $\gamma(x) < 1$ for some obstacles Γ_j and $\gamma(x) > 1$ for other ones. The proof extends with only minor modifications. The construction of the semi-classical parametrix in [9] is local and for the Dirichlet-to-Neumann map $\mathcal{N}_j(z, h)$ related to Γ_j we get the estimate

$$\|\mathcal{N}_j(z, h)(F) - \operatorname{Op}_h(\rho + hb)F\|_{H_h^1(\Gamma_j)} \leq \frac{Ch}{\sqrt{|\operatorname{Im} z|}} \|F\|_{L^2(\Gamma_j)}.$$

The boundary condition in (1.1) is local and we can reduce the analysis to a fixed obstacle K_j . If $(E, B) \neq 0$ is an eigenfunction of G_b , our argument implies $E_{tan} = 0$ for $x \in \Gamma_j$ if $0 < \gamma(x) < 1$ on Γ_j and $B_{tan} = 0$ for $x \in \Gamma_j$ in the case $\gamma(x) > 1$ on Γ_j . By the boundary condition we get $E_{tan} = 0$ on Γ and this yields $E = B = 0$

since the Maxwell system with boundary condition $E_{tan} = 0$ has no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.

4. APPENDIX

In this Appendix, assume that $\gamma > 0$ is constant. Our purpose is to study the eigenvalues of G_b in case the obstacle is equal to the ball $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$. Setting $\lambda = \mathbf{i}\mu$, $\operatorname{Im} \mu > 0$, an eigenfunction $(E, B) \neq 0$ of G_b satisfies

$$\operatorname{curl} E = -\mathbf{i}\mu B, \quad \operatorname{curl} B = \mathbf{i}\mu E. \quad (4.1)$$

Replacing B by $H = -B$ yields for $(E, H) \in (H^2(|x| \leq 1))^6$,

$$\begin{cases} \operatorname{curl} E = \mathbf{i}\mu H, & \operatorname{curl} H = -\mathbf{i}\mu E, & \text{for } x \in B_3, \\ E_{tan} + \gamma(\nu \wedge H_{tan}) = 0, & \text{for } x \in \mathbb{S}^2. \end{cases} \quad (4.2)$$

Expand $E(x), H(x)$ in the spherical functions $Y_n^m(\omega)$, $n = 0, 1, 2, \dots, |m| \leq n, \omega \in \mathbb{S}^2$ and the modified Hankel functions $h_n^{(1)}(z)$ of first kind. An application of Theorem 2.50 in [4] (in the notation of [4] it is necessary to replace ω by $\mu \in \mathbb{C} \setminus \{0\}$) says that the solution of the system (4.2) for $|x| = r = 1$ has the form

$$\begin{aligned} E_{tan}(\omega) &= \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\alpha_n^m \left(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) U_n^m(\omega) + \beta_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right], \\ H_{tan}(\omega) &= -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\beta_n^m \left(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) U_n^m(\omega) + \mu^2 \alpha_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right]. \end{aligned}$$

Here $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^2} Y_n^m(\omega)$ and $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$ for $n \in \mathbb{N}, -n \leq m \leq n$ form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u \in (L^2(\mathbb{S}^2))^3 : \langle \nu, u \rangle = 0 \text{ on } \mathbb{S}^2\}.$$

To find a representation of $\nu \wedge H_{tan}$, observe that $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$, so

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\beta_n^m \left(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r) \Big|_{r=1} \right) V_n^m(\omega) - \mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega) \right]$$

and the boundary condition in (4.2) is satisfied if

$$\alpha_n^m \left[h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r)) \Big|_{r=1} - \gamma \mathbf{i}\mu h_n^{(1)}(\mu) \right] = 0, \quad \forall n \in \mathbb{N}, |m| \leq n, \quad (4.3)$$

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \left[h_n^{(1)}(\mu) + \frac{d}{dr} (h_n^{(1)}(\mu r)) \Big|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu) \right] = 0, \quad \forall n \in \mathbb{N}, |m| \leq n. \quad (4.4)$$

For $\gamma \equiv 1$, there are no eigenvalues.

Proposition 4.1. *For $\gamma \equiv 1$ the operator G_b has no eigenvalues in $\{\operatorname{Re} z < 0\}$.*

Proof. The functions $h_n^{(1)}(z)$ have the form (see for example [7])

$$h_n^{(1)}(x) = (-\mathbf{i})^{n+1} \frac{e^{ix}}{x} \sum_{m=0}^n \frac{\mathbf{i}^m}{m! (2x)^m} \frac{(n+m)!}{(n-m)!} = (-\mathbf{i})^{n+1} \frac{e^{ix}}{x} R_n \left(\frac{\mathbf{i}}{2x} \right)$$

with

$$R_n(z) := \sum_{m=0}^n \frac{z^m}{m!} \frac{(n+m)!}{(n-m)!} = \sum_{m=0}^n a_m z^m.$$

Therefore the term in the brackets [...] in (4.3) becomes

$$(1 - \gamma)\mathbf{i}\mu R_n\left(\frac{\mathbf{i}}{2\mu}\right) - \sum_{m=0}^n a_m m \left(\frac{\mathbf{i}}{2\mu}\right)^m.$$

Setting $w = \frac{\mathbf{i}}{2\mu}$, we must study for $\operatorname{Re} w > 0$ the roots of the equation

$$g_n(w) := \frac{1 - \gamma}{2w} R_n(w) + w R_n'(w) = 0. \quad (4.5)$$

For $\gamma = 1$ one obtains $R_n'(w) = 0$. A result of Macdonald says that the zeros of the function $h_n^{(1)}(z)$ lie in the half plane $\operatorname{Im} z < 0$ (see Theorem 8.2 in [7]), hence $R_n(w) \neq 0$ for $\operatorname{Re} w \geq 0$. By the theorem of Gauss-Lucas we deduce that the roots of $R_n'(w) = 0$ lie in the convex hull of the set of the roots of $R_n(w) = 0$, so $R_n'(w) \neq 0$ for $\operatorname{Re} w > 0$. Consequently, (4.3) and (4.4) are satisfied only for $\alpha_n^m = \beta_n^m = 0$ and $E_{tan} = 0$. This implies $E = H = 0$. \square

For the case $\gamma \neq 1$, there are an infinite number of real eigenvalues.

Proposition 4.2. *Assume that $\gamma \in \mathbb{R}^+ \setminus \{1\}$ is a constant. Then G_b has an infinite number of real eigenvalues. Let $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. Then all real eigenvalues λ with exception of the eigenvalue*

$$\lambda_1 = -\frac{2}{(\gamma_0 - 1)\left(1 + \sqrt{1 + \frac{4}{\gamma_0 - 1}}\right)}. \quad (4.6)$$

satisfy the estimate

$$\lambda \leq -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}}. \quad (4.7)$$

Proof. Assume first that $\gamma > 1$. Then $q_n(w) = w g_n(w) = 0$ has at least one real root $w_0 > 0$. Indeed, $q_n(0) = \frac{1 - \gamma}{2} < 0$, $q_n(w) \rightarrow +\infty$ as $w \rightarrow +\infty$. Choosing $\alpha_n^{m_0} \neq 0$ for an integer m_0 , $|m_0| \leq n$ and taking all other coefficients α_n^m, β_n^m equal to 0, yields $E_{tan} \neq 0$ and G_b has an eigenfunction with eigenvalue $\lambda = -\frac{1}{2w_0} < 0$.

It is not excluded that $g_n(w)$ and $g_m(w)$ for $n \neq m$ have the same real positive root. If we assume that for $\operatorname{Re} w > 0$ the sequence of functions $\{g_n(w)\}_{n=1}^{\infty}$ has only a finite number of real roots $w_1, \dots, w_N, w_j \in \mathbb{R}^+$, then there exists an infinite number of functions $g_{n_j}(w)$ having the same root which implies that we have an eigenvalue of G_b with infinite multiplicity. This is a contradiction, and the number of real eigenvalues of G_b is infinite.

It remains to establish the bound on the real eigenvalues. First, consider the case $n = 1$. Then one obtains the equation

$$\frac{2w^2}{2w + 1} = \frac{\gamma - 1}{2}$$

which has a positive root $w_0 = \frac{1}{4}\left(\gamma - 1 + \sqrt{(\gamma - 1)^2 + 4(\gamma - 1)}\right)$. This yields the λ_1 from (4.6)

Next examine the case $n \geq 2$. For a root $w_0 \in \mathbb{R}^+$ one has

$$w_0 \left(w_0 \frac{R_n'(w_0)}{R_n(w_0)} \right) = \frac{\gamma - 1}{2}.$$

Case 1. $w_0 \geq \frac{1}{2\sqrt{3}}$. Then the inequality

$$\frac{\sum_{m=2}^n ma_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \geq \frac{2 \sum_{m=2}^n a_m w_0^m + a_1 w_0}{\sum_{m=2}^n a_m w_0^m + a_1 w_0 + 1} \geq 1$$

is satisfied since $a_2 = \frac{1}{2}(n+2)(n+1)n(n-1) \geq 12$. Consequently, $2w_0 \leq \gamma - 1$ and this implies that the eigenvalue $\lambda = -\frac{1}{2w_0}$ satisfies

$$\lambda < -\frac{1}{\gamma - 1}. \quad (4.8)$$

Case 2. $0 < w_0 \leq \frac{1}{2\sqrt{3}}$. Apply the inequality

$$\frac{\sum_{m=2}^n ma_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \geq \frac{2 \sum_{m=2}^n a_m w_0^{m-1} + a_1}{w_0 \sum_{m=2}^n a_m w_0^{m-1} + a_1 w_0 + 1} \geq 2$$

that is equivalent to

$$2[(1-w_0)S_0 - a_1 w_0] + a_1 \geq 2$$

with $S_0 = \sum_{m=2}^n a_m w_0^{m-1}$. This inequality holds because

$$(1-w_0) \sum_{m=2}^n a_m w_0^{m-1} - a_1 w_0 \geq \left(\frac{1}{2}a_2 - a_1\right)w_0, \quad a_1 = (n+1)n \geq 2,$$

and,

$$\frac{1}{2}a_2 - a_1 = \frac{1}{4}(n+2)(n+1)n(n-1) - (n+1)n = n(n+1) \left[\frac{1}{4}(n+2)(n-1) - 1 \right] \geq 0.$$

Therefore,

$$2w_0^2 \leq w_0^2 \frac{\sum_{m=1}^n ma_m w_0^{m-1}}{\sum_{m=1}^n a_m w_0^m + 1} = \frac{\gamma - 1}{2}.$$

This easily yields

$$\lambda \leq -\frac{1}{\sqrt{\gamma - 1}}. \quad (4.9)$$

In the case $0 < \gamma < 1$ one has $1/\gamma > 1$ and we apply the above analysis to the equation (4.4). Setting $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$ and taking into account (4.8) and (4.9), we obtain the result. This completes the proof. \square

Remark 4.3. Proposition 4.2 yields a more precise result than that in [1] since we prove the existence of an infinite number of real eigenvalues G_b for every $\gamma \in \mathbb{R}^+ \setminus \{1\}$. In the case $\gamma = \frac{1}{1+\epsilon}$, $\epsilon > 0$ the eigenvalue λ_1 has the form

$$\lambda_1 = \frac{1}{2} \left(1 - \sqrt{1 + \frac{4}{\epsilon}} \right)$$

and this result for small $\epsilon > 0$ has been obtained in [1]. Clearly, as $\gamma \rightarrow 1$ the real eigenvalues of G_b go to $-\infty$.

It is easy to see that for $\gamma > 1$ the equation $g_n(w) = 0$ has no complex roots. Denote by

$$z_j, \quad \operatorname{Re} z_j < 0, \quad j = 1, \dots, n, \quad n \geq 1$$

the roots of $R_n(w) = 0$. Suppose that $g_n(w_0) = 0$, $n \geq 1$ with $\operatorname{Re} w_0 > 0$, $\operatorname{Im} w_0 \neq 0$. Then

$$\operatorname{Im} \left[\frac{1-\gamma}{2w_0} + w_0 \sum_{j=1}^n \frac{1}{w_0 - z_j} \right] = 0$$

and

$$\begin{aligned} -\frac{(1-\gamma)\operatorname{Im} w_0}{2|w_0|^2} + \operatorname{Re} w_0 \left[-\sum_{j=1}^n \frac{\operatorname{Im} w_0}{|w_0 - z_j|^2} + \sum_{j=1}^n \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2} \right] \\ + \operatorname{Im} w_0 \sum_{j=1}^n \frac{\operatorname{Re} w_0 - \operatorname{Re} z_j}{|w_0 - z_j|^2} = 0. \end{aligned} \quad (4.10)$$

On the other hand, if z_j with $\operatorname{Im} z_j \neq 0$ is a root of $R_n(w) = 0$, then \bar{z}_j is also a root and

$$\begin{aligned} \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2} - \frac{\operatorname{Im} z_j}{|w_0 - \bar{z}_j|^2} &= \frac{\operatorname{Im} z_j}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2} \left(|w_0 - \bar{z}_j|^2 - |w_0 - z_j|^2 \right) \\ &= \frac{4 \operatorname{Im} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2}. \end{aligned}$$

Equation (4.10) becomes

$$\operatorname{Im} w_0 \left[\frac{\gamma-1}{2|w_0|^2} - \sum_{j=1}^n \frac{\operatorname{Re} z_j}{|w_0 - z_j|^2} + \sum_{\operatorname{Im} z_j > 0} \frac{4 \operatorname{Re} w_0 (\operatorname{Im} z_j)^2}{|w_0 - z_j|^2 |w_0 - \bar{z}_j|^2} \right] = 0. \quad (4.11)$$

The term in the brackets [...] is positive, and one concludes that $\operatorname{Im} w_0 = 0$.

Repeating the argument of the Appendix in [8], one can show that for $0 < \gamma < 1$ the complex eigenvalues of G_b lie in the region

$$\left\{ z \in \mathbb{C} : |\arg z - \pi| > \pi/4, \quad \operatorname{Re} z < 0 \right\}.$$

Remark 4.4. We do not know if there exist non real eigenvalues for B_3 .

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