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LOCATION OF EIGENVALUES FOR THE WAVE EQUATION WITH DISSIPATIVE BOUNDARY CONDITIONS

VESSELIN PETKOV

ABSTRACT. We examine the location of the eigenvalues of the generator G of a semi-group $V(t) = e^{tG}$, $t \geq 0$, related to the wave equation in an unbounded domain $\Omega \subset \mathbb{R}^d$ with dissipative boundary condition $\partial_\nu u - \gamma(x)\partial_t u = 0$ on $\Gamma = \partial\Omega$. We study two cases: (A) : $0 < \gamma(x) < 1$, $\forall x \in \Gamma$ and (B) : $1 < \gamma(x)$, $\forall x \in \Gamma$. We prove that for every $0 < \epsilon \ll 1$, the eigenvalues of G in the case (A) lie in the region $\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1), \operatorname{Re} z < 0\}$, while in the case (B) for every $0 < \epsilon \ll 1$ and every $N \in \mathbb{N}$ the eigenvalues lie in $\Lambda_\epsilon \cup \mathcal{R}_N$, where $\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \operatorname{Re} z < 0\}$.

1. INTRODUCTION

Let $K \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded domain and let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected. We suppose that the boundary Γ of Ω is C^∞ . Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x)\partial_t u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0, x) = f_0, u_t(0, x) = f_1 \end{cases} \quad (1.1)$$

with initial data $(f_1, f_2) \in H^1(\Omega) \times L^2(\Omega) = \mathcal{H}$. Here ν is the unit outward normal to Γ pointing into Ω and $\gamma(x) \geq 0$ is a C^∞ function on Γ . The solution of the problem (1.1) is given by $(u(t, x), u_t(t, x)) = V(t)f = e^{tG}f$, $t \geq 0$, where $V(t)$ is a contraction semi-group in \mathcal{H} whose generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

has a domain $D(G)$ which is the closure in the graph norm of functions $(f_1, f_2) \in C_{(0)}^\infty(\mathbb{R}^n) \times C_{(0)}^\infty(\mathbb{R}^n)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ . For d odd it is well known [5] that the spectrum of G in $\operatorname{Re} z < 0$ is formed by isolated eigenvalues with finite multiplicity. We may obtain the same result for all dimensions $d \geq 2$ by using the Dirichlet-to-Neumann map $\mathcal{N}(\lambda)$ defined at the end of Section 4. Notice that if $Gf = \lambda f$ with $f = (f_1, f_2) \neq 0$, $\operatorname{Re} \lambda < 0$ and $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ , we get

$$\begin{cases} (\Delta - \lambda^2)f_1 = 0 \text{ in } \Omega, \\ \partial_\nu f_1 - \lambda\gamma f_1 = 0 \text{ on } \Gamma \end{cases} \quad (1.2)$$

and $V(t)f = e^{\lambda t}$ has an exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they perturb the inverse scattering

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problems. Recently it was proved [2] that if we have at least one eigenvalue λ of G with $\operatorname{Re} \lambda < 0$, then the wave operators W_{\pm} related to the Cauchy problem for the wave equation and the boundary problem (1.1) are not complete, that is $\operatorname{Ran} W_{-} \neq \operatorname{Ran} W_{+}$. Hence we cannot define the scattering operator S by the product $W_{+}^{-1}W_{-}$. Notice that if the global energy is conserved in time and the unperturbed and perturbed problems are associated to unitary groups the corresponding scattering operator $S(z) : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$ satisfies the identity

$$S^{-1}(z) = S^*(\bar{z}), \quad z \in \mathbb{C}$$

if $S(z)$ is invertible at z . Since $S(z)$ and $S^*(z)$ are analytic in the "physical" half plane $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ (see [4]) the above relation implies that $S(z)$ is invertible for $\operatorname{Im} z > 0$. For dissipative boundary problems the above relation in general is not true and $S(z_0)$ may have a non trivial kernel for some $z_0, \operatorname{Im} z_0 > 0$. For odd dimensions d Lax and Phillips [5] proved that this implies that iz_0 is an eigenvalue of G . Thus the analysis of the location of the eigenvalues of G is important for the location of the points where the kernel of $S(z)$ is not trivial.

In the scattering theory of Lax-Phillips [5] the energy space can be presented as a direct sum $\mathcal{H} = D_a^- \oplus K_a \oplus D_a^+$, $a > 0$, and we have the relations

$$V(t)D_a^+ \subset D_a^+, \quad V(t)(K_a) \subset K_a \oplus D_+^a, \quad V(t)D_a^- \subset \mathcal{H}, \quad t \geq 0.$$

R. Phillips defined a system as non controllable if there exists a state $f \in K_a$ such that $V(t)f \perp D_a^+$, $t \geq 0$. This means that there exists states in the "black box" K_a which remain undetected by the scattering process. Majda [6] proved that if we have such state f , then $u(t, x) = (V(t)f)_1$ is a disappearing solution, that is there exists $T > 0$ depending on f and $u(t, x)$ vanishes for all $t \geq T > 0$. On the other hand, if $0 < \gamma(x) < 1$, $\forall x \in \Gamma$ or if $1 < \gamma(x)$, $\forall x \in \Gamma$, and the boundary is analytic there are no disappearing solutions (see [6]). Thus if $\gamma(x) \neq 1$, $\forall x \in \Gamma$, it is natural to search asymptotically disappearing solutions. From the results in [6] (see also [2]) it follows that if $f \perp D_a^-$, then f cannot be an eigenfunction of G , hence there are no outgoing eigenfunctions of G . However, we may have incoming asymptotically disappearing solutions as the following simple example shows.

Example 1.1. Consider the ball $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ and let $\gamma = \text{const} > 1$. Then the incoming solution $u = \frac{e^{-\lambda(t+r)}}{r}$, $r = |x|$ with $\lambda = (\gamma - 1)^{-1} > 0$ satisfies the boundary condition $\partial_r u - \gamma \partial_t u = 0$, for $|x| = 1$.

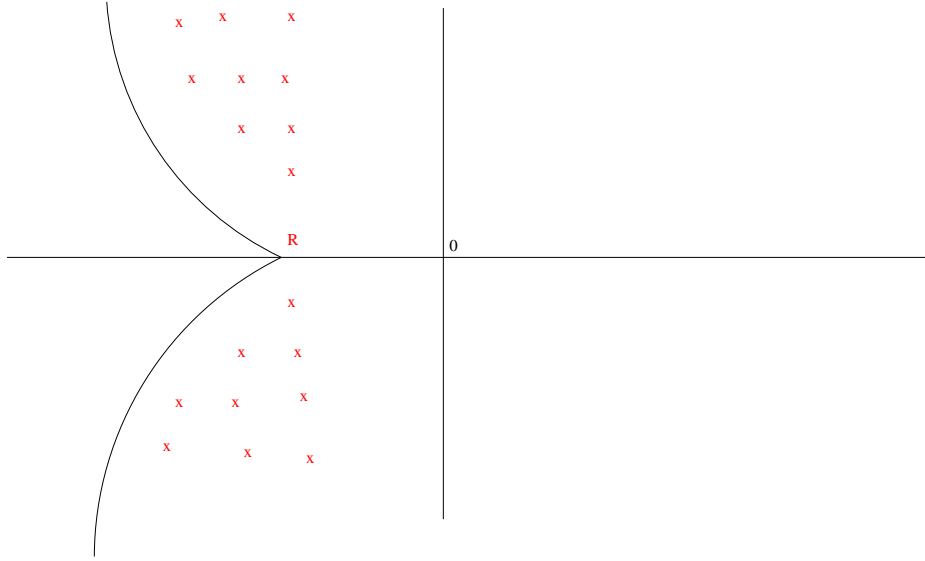
In [7] Majda examined the location of the eigenvalues of G and he proved that if $0 \leq \gamma(x) < 1$, $\forall x \in \Gamma$, the eigenvalues of G lie in the region

$$E_1 = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_1(|\operatorname{Im} z|^{3/4} + 1), \operatorname{Re} z < 0\},$$

while if $\gamma(x) \geq 1$, $\forall x \in \Gamma$, the eigenvalues of G lie in $E_1 \cup E_2$, where

$$E_2 = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_2(|\operatorname{Re} z|^{1/2} + 1), \operatorname{Re} z < 0\}.$$

In the case $\gamma(x) = 1$, $\forall x \in \Gamma$ one conjectures that for some obstacles there are no eigenvalues of G .

FIGURE 1. Eigenvalues, $0 < \gamma(x) < 1$

The purpose of this paper is to improve the above results for the location of eigenvalues. We consider two cases: (A) : $0 < \gamma(x) < 1, \forall x \in \Gamma$, (B) : $\gamma(x) > 1, \forall x \in \Gamma$. Our main result is the following

Theorem 1.2. *In the case (A) for every $\epsilon, 0 < \epsilon \ll 1$, the eigenvalues of G lie in the region*

$$\Lambda_\epsilon = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq C_\epsilon(|\operatorname{Im} z|^{\frac{1}{2}+\epsilon} + 1), \operatorname{Re} z < 0\}.$$

In the case (B) for every $\epsilon, 0 < \epsilon \ll 1$, and every $N \in \mathbb{N}$ the eigenvalues of G lie in the region $\Lambda_\epsilon \cup \mathcal{R}_N$, where

$$\mathcal{R}_N = \{z \in \mathbb{C} : |\operatorname{Im} z| \leq C_N(|\operatorname{Re} z| + 1)^{-N}, \operatorname{Re} z < 0\}.$$

For strictly convex obstacles K we prove a better result in the case (B).

Theorem 1.3. *Assume that K is strictly convex. In the case (B) there exists $R_0 > 0$ such that for every $N \in \mathbb{N}$ the eigenvalues of G lie in the region $\{z \in \mathbb{C} : |z| \leq R_0, \operatorname{Re} z < 0\} \cup \mathcal{R}_N$.*

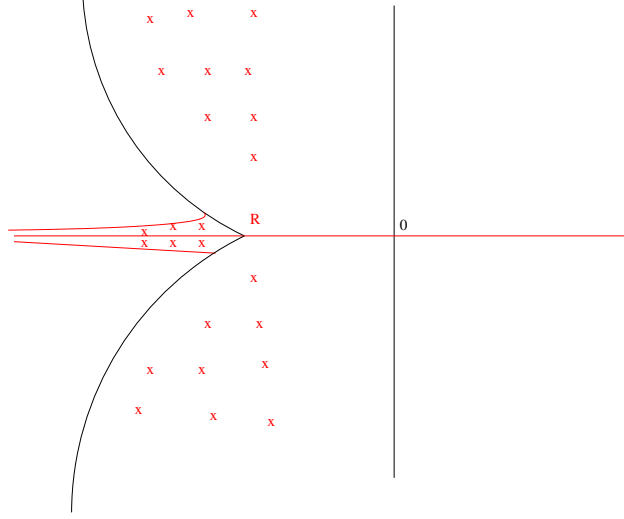
The eigenvalues of G are symmetric with respect to the real axis, so it is sufficient to examine the location of the eigenvalues whose imaginary part is non negative. Introduce in $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$ the sets

$$Z_1 = \{z \in \mathbb{C} : \operatorname{Re} z = 1, h^\delta \leq \operatorname{Im} z \leq 1\}, \quad 0 < h \ll 1, \quad 0 < \delta < 1/2,$$

$$Z_2 = \{z \in \mathbb{C} : \operatorname{Re} z = -1, 0 \leq \operatorname{Im} z \leq 1\}, \quad Z_3 = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1, \operatorname{Im} z = 1\}.$$

We put $\lambda = \frac{i\sqrt{z}}{h}$ and we use the branch $0 \leq \arg z < 2\pi$ with $\operatorname{Im} \sqrt{z} > 0$ if $\operatorname{Im} z > 0$. From (1.2) we deduce that the eigenfunctions u of G satisfy the problem

$$\begin{cases} (-h^2 \Delta - z)u = 0 \text{ in } \Omega, \\ -ih \partial_\nu u - \gamma \sqrt{z} u = 0 \text{ on } \Gamma. \end{cases} \quad (1.3)$$

FIGURE 2. Eigenvalues, $1 < \gamma(x)$

The proofs of Theorems 1.2 and 1.3 are based on a semi-classical analysis of the equation

$$N_{ext}(z, h)f - \gamma\sqrt{z}f = 0, \quad (1.4)$$

where $f = u|_{x \in \Gamma}$ is the trace of an eigenfunction of G . Here

$$\mathcal{N}_{ext}(z, h)f : H_h^s(\Gamma) \ni f \longrightarrow hD_\nu u|_\Gamma \in H_h^{s-1}(\Gamma)$$

is the exterior Dirichlet-to-Neumann map, $D_\nu = -\mathbf{i}\partial_\nu$ and u is the solution of the problem

$$\begin{cases} (-h^2\Delta - z)u = 0 \text{ in } \Omega, u \in H_h^2(\Omega), \\ u|_{x \in \Gamma} = f. \end{cases} \quad (1.5)$$

In the paper we use the semi-classical Sobolev space $H_h^s(\Gamma)$, $s \in \mathbb{R}$, with norm $\|\langle hD \rangle^s u\|_{L^2(\Gamma)}$, where $\langle hD \rangle = (1 + (hD_x)^2)^{1/2}$. The purpose is to prove that if $z \in Z_1 \cup Z_2 \cup Z_3$ lies in some regions and h is small enough from (1.4) we get $f = 0$ which is not possible for an eigenfunction u . In this direction our strategy is close to that for the analysis of eigenvalues-free regions for the interior transmission eigenvalues in [14] and [15]. We apply some results for the interior Dirichlet-to-Neumann map $\mathcal{N}_{int}(z, h)$ established in [14] and [15] for bounded domains which after modifications and some constructions remain true for the exterior Dirichlet-to-Neumann map $\mathcal{N}_{ext}(z, h)$ defined above.

2. DIRICHLET-TO-NEUMANN MAP

In our exposition we apply some h -pseudodifferential operators and we are going to recall some basic facts. Let X be a C^∞ smooth compact manifold without boundary with dimension $d - 1 \geq 1$. Let (x, ξ) be the coordinates in $T^*(X)$ and let $a(x, \xi, h) \in C^\infty(T^*(X))$. Given $m \in \mathbb{R}$, $l \in \mathbb{R}$, $\delta > 0$ and a function $c(h) > 0$, one denotes by $S_\delta^{l, m}(c(h))$ the set of symbols $a(x, \xi, h)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C_{\alpha, \beta}(c(h))^{l - \delta(|\alpha| + |\beta|)} (1 + |\xi|)^{m - |\beta|}, \quad (x, \xi) \in T^*(X), \quad \forall \alpha, \forall \beta.$$

If $c(h) = 1$, we denote $S_\delta^{l,m}(c(h))$ simply by $S_\delta^{l,m}$. Also, the symbols restricted to a domain where $|\xi| \leq C$ will be denoted by $a \in S_\delta^l(c(h))$. We define the h -pseudodifferential operator

$$(Op_h(a)f)(x) = (2\pi h)^{-d+1} \int_{T^*X} e^{-i\langle x-y, \xi \rangle/h} a(x, \xi, h) f(y) dy d\xi$$

with symbol $a(x, \xi, h)$.

For the reader convenience we recall two properties of the semi-classical pseudodifferential operators $Op_h(a)$ (see Section 7 of [3] and Proposition 2.1 of [14]). Assume that $a \in T^*(X)$ satisfies the bounds

$$|\partial_x^\alpha a(x, \xi, h)| \leq c_0(h) h^{-|\alpha|/2}, \quad (x, \xi) \in T^*(X) \quad (2.1)$$

for $|\alpha| \leq d-1$, where $c_0(h) > 0$ is a parameter. Then there exists a constant $C > 0$ independent of h such that

$$\|Op_h(a)\|_{L^2(X) \rightarrow L^2(X)} \leq C c_0(h). \quad (2.2)$$

Next for $0 \leq \delta < 1/2$ we have a calculus and if $a \in S_\delta^{l_1, m_1}$, $b \in S_\delta^{l_2, m_2}$, then for $s \in \mathbb{R}$ we get

$$\|Op_h(a)Op_h(b) - Op_h(ab)\|_{H^s(X) \rightarrow H^{s-m_1-m_2+1}(X)} \leq C h^{-l_1-l_2-2\delta+1}.$$

We refer to [3] for more details concerning the calculus. The left hand side of last inequality can be estimated also in some cases when one of the symbols a or b is in a class $S_\delta^{l,m}$ with $0 \leq \delta < 1$. For the precise statements the reader should consult Proposition 2.2 in [14] and Proposition 4.2 in [15].

Let (x', ξ') be the coordinates on $T^*(\Gamma)$. Denote by $r_0(x', \xi')$ the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$ on Γ equipped with the Riemannian metric induced by the Euclidean metric in \mathbb{R}^d . For $z \in Z_1 \cup Z_2 \cup Z_3$ let

$$\rho(x', \xi', z) = \sqrt{z - r_0(x', \xi')} \in C^\infty(T^*\Gamma)$$

be the root of the equation

$$\rho^2 + r_0(x', \xi') - z = 0$$

with $\text{Im } \rho(x', \xi', z) > 0$. For large $|\xi'|$ we have $|\rho(x', \xi', z)| \sim |\xi'|$, $\text{Im } \rho(x', \xi', z) \sim |\xi'|$. Moreover, for $z \in Z_1 \cup Z_3$ we have

$$\text{Im } \rho(x', \xi', z) \geq \frac{|\text{Im } z|}{2|\rho|}, \quad |\rho| \geq \sqrt{|\text{Im } z|},$$

while for $r_0 \geq 2$, we have

$$C_1 \sqrt{r_0 + 1} \geq 2 \text{Im } \rho \geq |\rho| \geq C_2 \sqrt{r_0 + 1}.$$

For $z \in Z_2$ the last equality is true for all (x', ξ') (see Lemma 3.1 in [14]).

G. Vodev established for bounded domains $K \subset \mathbb{R}^d$, $d \geq 2$, the following approximation of the interior Dirichlet-to-Neumann map $N_{int}(z, h)$ related to the boundary problem (1.5), where the equation $(-h^2 \Delta - z)u = 0$ is satisfied in K .

Theorem 2.1 ([14]). *For every $0 < \epsilon \ll 1$ there exists $0 < h_0(\epsilon) \ll 1$ such that for $z \in Z_{1,\epsilon} := \{z \in Z_1, |\text{Im } z| \geq h^{\frac{1}{2}-\epsilon}\}$ and $0 < h \leq h_0(\epsilon)$ we have*

$$\|N_{int}(z, h)(f) - Op_h(\rho + hb)f\|_{H_h^1(\Gamma)} \leq \frac{Ch}{\sqrt{|\text{Im } z|}} \|f\|_{L^2(\Gamma)}, \quad (2.3)$$

where $b \in S_{0,1}^0(\Gamma)$ does not depend on h and z . Moreover, (2.3) holds for $z \in Z_2 \cup Z_3$ with $|\operatorname{Im} z|$ replaced by 1.

The same result remains true for unbounded domains Ω with $\mathcal{N}_{int}(z, h)$ replaced by $\mathcal{N}_{ext}(z, h)$ by modifications to the proof in [14] based on the construction of a semi-classical parametrix close to the boundary. For reader convenience we recall below some facts from [14] and we discuss some modifications which will be necessary for our exposition. Consider normal geodesic coordinates (x_1, x') in a neighborhood of a fixed point $x_0 \in \Gamma$, where locally the boundary Γ is given by $x_1 = 0$, $x_1 = \operatorname{dist}(x, \Gamma)$. Then $-h^2\Delta - z$ in these coordinates has the form

$$\mathcal{P}(z, h) = D_{x_1}^2 + r(x, D_{x'}) + hq(x, D_x) + h^2\tilde{q}(x) - z.$$

with $D_{x_1} = -ih\partial_{x_1}$, $D_{x'} = -ih\partial_{x'}$, $r(x, \xi') = \langle R(x)\xi', \xi' \rangle$, $q(x, \xi) = \langle q(x), \xi \rangle$. Here $R(x)$ is a symmetric $(d-1) \times (d-1)$ matrix with smooth real-valued entries and $r(0, x', \xi') = r_0(x', \xi')$. Let $\phi(\sigma) \in C^\infty(\mathbb{R})$ be a cut-off function such that $\phi(\sigma) = 1$ for $|\sigma| \leq 1$, $\phi(\sigma) = 0$ for $|\sigma| \geq 2$. Let $\psi(x')$ be a C^∞ cut-off function on Γ . In Section 3 in [14] for $\delta_1 > 0$ small enough one constructs a semi-classical parametrix

$$\begin{cases} \tilde{u}_\psi(x) = (2\pi h)^{-d+1} \iint e^{\frac{i}{h}\varphi(x, y', \xi', z)} \phi\left(\frac{x_1}{\delta_1}\right) \phi\left(\frac{x_1}{\delta_1 \rho_1}\right) a(x, \xi', z; h) f(y') dy' d\xi', \\ \tilde{u}_\psi|_{x_1=0} = \psi f, \end{cases} \quad (2.4)$$

where $\rho_1 = 1$ if $z \in Z_2 \cup Z_3$, $\rho_1 = |\rho|^3$ if $z \in Z_1$.

The phase $\varphi(x, y', \xi', z)$ is complex-valued and

$$\begin{aligned} \varphi &= -\langle x' - y', \xi' \rangle + \sum_{k=1}^{N-1} x_1^k \varphi_k(x', \xi', z) = -\langle x' - y', \xi' \rangle + \tilde{\varphi}, \\ a &= \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x_1^k h^j a_{k,j}(x', \xi', z), \end{aligned}$$

$N \gg 1$ being a large integer.

Moreover, $\varphi_1 = \rho$, $\operatorname{Im} \varphi \geq x_1 \operatorname{Im} \rho/2$ and the amplitude $a|_{x_1=0} = \psi(x')$. The phase φ and the amplitude a are determined so that

$$e^{-\frac{i\varphi}{h}} \mathcal{P}(z, h) e^{\frac{i\varphi}{h}} a = x_1^N A_N(x, \xi', z; h) + h^N B_N(x, \xi', z; h),$$

where A_N, B_N are smooth functions. To describe the behavior of A_N, B_N , introduce the function $\chi(x', \xi') = \phi(\delta_0 r_0(x', \xi'))$, where $0 < \delta_0 \ll 1$. Following [14], we say that a symbol $b \in C^\infty(T^*\Gamma)$ belongs to $S_{\delta_1, \delta_2}^{l_1}(\mu_1) + S_{\delta_3, \delta_4}^{l_2}(\mu_2)$ if

$$\begin{aligned} |\partial_{x'}^\alpha \partial_{\xi'}^\beta (\chi b)| &\leq C_{\alpha, \beta} |\mu_1|^{l_1 - \delta_1 |\alpha| - \delta_2 |\beta|}, \\ |\partial_{x'}^\alpha \partial_{\xi'}^\beta ((1 - \chi) b)| &\leq C_{\alpha, \beta} |\mu_2|^{l_2 - \delta_3 |\alpha| - \delta_4 |\beta|}, \quad \forall \alpha, \forall \beta. \end{aligned}$$

Therefore,

$$\partial_{x_1}^k A_N \in S_{2,2}^{2-3N-3k}(|\rho|) + S_{0,1}^2(|\rho|), \quad \partial_{x_1}^k B_N \in S_{2,2}^{3-4N-3k}(|\rho|) + S_{0,1}^{1-N}(|\rho|), \quad \forall k \in \mathbb{N}$$

uniformly with respect to z, h and $0 \leq x_1 \leq 2\delta_1 \min\{1, \rho_1\}$.

For $z \in Z_{1,0}$ and any integer $s \geq 0$, there exist $l_s, N_s > 0$ so that for $N \geq N_s$ we have the estimate (see Proposition 3.7 in [14])

$$\|\mathcal{P}(z, h) \tilde{u}_\psi\|_{H_h^s(\Omega)} \leq C_N h^{-l_s} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)}, \quad (2.5)$$

while for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\operatorname{Im} z|$ replaced by 1. Next introduce the operator

$$T_\psi(z, h)f := D_{x_1} \tilde{u}_\psi|_{x_1=0} = \overline{Op}_h(\tau_\psi)f$$

with

$$\tau_\psi = a \frac{\partial \varphi}{\partial x_1} \Big|_{x_1=0} - \mathbf{i}h \frac{\partial a}{\partial x_1} \Big|_{x_1=0} = \psi \rho - \mathbf{i}h \sum_{j=0}^{N-1} h^j a_{1,j}.$$

Let G_D be the self-adjoint realization of the operator $-\Delta$ on $L^2(\Omega)$ with Dirichlet boundary condition on Γ . Since the spectrum of G_D is the positive real axis, for $z \in Z_1$ we have the estimate

$$\left\| \left(h^2 G_D - z \right)^{-1} \right\|_{H_h^{2k}(\Omega) \rightarrow H_h^{2k}(\Omega)} \leq \frac{C_k}{|\operatorname{Im} z|}, \quad \forall k \in \mathbb{N},$$

while for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\operatorname{Im} z|$ replaced by 1. For $k = 0$ this estimate is trivial, and for $k \geq 1$ it follows from the coercive estimates for the Dirichlet problem in unbounded domains (see [4])

$$\|v\|_{H_h^{2k}(\Omega)} \leq C'_k \left(\|h^2 G_D v\|_{H_h^{2k-2}(\Omega)} + \|v\|_{H_h^{2k-2}(\Omega)} \right), \quad v \in D(G_D) \cap H_h^{2k-2}(\Omega).$$

Now let $u_\psi \in H_h^2(\Omega)$ be the solution of the problem

$$\mathcal{P}(z, h)u_\psi = 0 \text{ in } \Omega, \quad u_\psi|_\Gamma = \psi f.$$

Then

$$w_\psi := u_\psi - \tilde{u}_\psi + \left(h^2 G_D - z \right)^{-1} \mathcal{P}(z, h) \tilde{u}_\psi$$

will be a solution of $(h^2 G_D - z)w_\psi = 0$ in Ω , $w_\psi|_\Gamma = 0$. Since for $z \in Z_1 \cup Z_2 \cup Z_3$ the point z/h^2 is not in the spectrum of G_D , one deduces $w_\psi = 0$. This implies as in [14] the following

Proposition 2.2. *For $z \in Z_{1,0}$ we have the estimate*

$$\|\mathcal{N}_{ext}(z, h)u_\psi - T_\psi(z, h)f\|_{H_h^1(\Gamma)} \leq C_N h^{-s_d} \left(\frac{\sqrt{h}}{|\operatorname{Im} z|} \right)^{2N} \|f\|_{L^2(\Gamma)}, \quad \forall N \in \mathbb{N} \quad (2.6)$$

with constants $C_N, s_d > 0$, independent of f, h and z , and s_d independent of N . If $z \in Z_2 \cup Z_3$, then (2.6) holds with $|\operatorname{Im} z|$ replaced by 1.

Choose a partition of unity $\sum_{j=1}^J \psi_j(x') = 1$ on Γ and set $T(z, h) = \sum_{j=1}^J T_{\psi_j}(z, h)$. Notice that the principal symbol of $T(z, h)$ is ρ . By using Proposition 2.2 and repeating without any change the argument in Section 3 in [14], one concludes that the statement of Theorem 2.1 remains true replacing in (2.3) $\mathcal{N}_{int}(z, h)$ by $\mathcal{N}_{ext}(z, h)$.

3. EIGENVALUES-FREE REGIONS IN THE CASE (A)

In this section we suppose that $0 < \epsilon_0 \leq \gamma(x) \leq 1 - \epsilon_0$, $\epsilon_0 > 0$, $\forall x \in \Gamma$. If $u \neq 0$ is an eigenfunction of G with eigenvalue $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, then $f = u|_{x \in \Gamma} \neq 0$. Indeed, if $f = 0$ on Γ , then $u \in H^2(\Omega)$ will be eigenfunction of the Dirichlet problem in Ω and this is impossible. From (1.3) one obtains the equation (1.4).

According to Theorem 2.1 with $\mathcal{N}_{ext}(z, h)$, for $z \in Z_1$, $\delta = 1/2 - \epsilon$, we have

$$\|\overline{Op}_h(\rho)f - \sqrt{z}\gamma f\|_{L^2(\Gamma)} \leq C \frac{h}{\sqrt{|\operatorname{Im} z|}} \|f\|_{L^2(\Gamma)}, \quad (3.1)$$

where for $z \in Z_2 \cup Z_3$ the above estimate holds with $|\operatorname{Im} z|$ replaced by 1. Introduce the symbol

$$c(x', \xi', z) := \rho(x', \xi', z) - \gamma\sqrt{z}.$$

We will show that $c(x', \xi', z)$ is elliptic in a suitable class. Write

$$c(x', \xi', z) = \frac{(1 - \gamma^2)z - r_0(x', \xi')}{\rho(x', \xi', z) + \gamma\sqrt{z}}.$$

Case I. $z \in Z_1$. The symbol c is elliptic for $|\xi'|$ large enough and it remains to examine its behavior for $|\xi'| \leq C_0$. For these values of ξ' we have $|\rho + \gamma\sqrt{z}| \leq C_1$. First consider the set

$$\mathcal{F} = \{(x', \xi') : |1 - r_0(x', \xi')| \leq \frac{\epsilon_0^2}{2}\}.$$

Then $\operatorname{Re}\left((1 - \gamma^2)z - r_0\right) = 1 - r_0 - \gamma^2 \leq -\frac{\epsilon_0^2}{2}$. If $(x', \xi') \notin \mathcal{F}$, we get

$$\operatorname{Im}\left((1 - \gamma^2)z - r_0\right) = (1 - \gamma^2)\operatorname{Im} z \geq (1 - \gamma^2)h^\delta \geq \epsilon_0 h^\delta.$$

Consequently, the symbol c is elliptic and

$$\operatorname{Im}(\rho + \gamma\sqrt{z}) = \operatorname{Im} \rho + \gamma \operatorname{Im} \sqrt{z} \geq Ch^\delta.$$

Hence, for bounded $|\xi'|$ we have $|c| \geq C_3 h^\delta$, $C_3 > 0$, while for large $|\xi'|$ we have $|c| \sim |\xi'|$. Introduce the function

$$\chi(x', \xi') = \phi(\delta_0 r_0(x', \xi')), \quad 0 < \delta_0 \leq 1/2$$

and define $\mathcal{M}_1 := Z_1 \times \operatorname{supp} \chi$, $\mathcal{M}_2 := (Z_1 \times \operatorname{supp}(1 - \chi)) \cup ((Z_2 \cup Z_3) \times T^*\Gamma)$. Set $\langle \xi' \rangle = (1 + |\xi'|)^{1/2}$. It is easy to see that for $(z, x', \xi') \in \mathcal{M}_1$, $\operatorname{Im} z \neq 0$, we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho| \leq C_{\alpha, \beta} |\operatorname{Im} z|^{1/2 - |\alpha| - |\beta|}, \quad |\alpha| + |\beta| \geq 1, \quad (3.2)$$

$|\rho| \leq C$, while for $(z, x', \xi') \in \mathcal{M}_2$ we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho| \leq C_{\alpha, \beta} \langle \xi' \rangle^{1 - |\beta|}. \quad (3.3)$$

Thus, we conclude that $c = (\rho - \gamma\sqrt{z}) \in S_\delta^{0,1}$.

Now consider the symbol $c^{-1} = \frac{\rho + \gamma\sqrt{z}}{(1 - \gamma^2)z - r_0}$. Since $\rho + \gamma\sqrt{z} \in S_\delta^{0,1}$, it remains to study the properties of $g := ((1 - \gamma^2)z - r_0)^{-1}$. For $(x', \xi') \in \mathcal{F}$, we get $|(1 - \gamma^2)z - r_0| \geq \frac{\epsilon_0^2}{2} > 0$ and

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta g| \leq C_{\alpha, \beta}.$$

Therefore for $(x', \xi') \in \mathcal{F}$, we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta (c^{-1})| \leq C_{\alpha, \beta} |\operatorname{Im} z|^{1/2 - |\alpha| - |\beta|}. \quad (3.4)$$

Next for $(x', \xi') \notin \mathcal{F}$ notice that for every $0 < \delta' \ll 1$, if $|(1 - \gamma^2)z - r_0| \leq \delta'$, $\operatorname{Im} z \neq 0$, we have

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta g| \leq C_{\alpha, \beta} |\operatorname{Im} z|^{-1 - |\alpha| - |\beta|}, \quad (3.5)$$

while for $|(1 - \gamma^2)z - r_0| \geq \delta'$ we get

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta g| \leq C_{\alpha, \beta} \langle \xi' \rangle^{-2 - |\beta|}. \quad (3.6)$$

On the other hand, $(x', \xi') \notin \mathcal{F}$ yields $|1 - r_0(x', \xi')| > \frac{\epsilon_0^2}{2}$ and for $(x', \xi') \notin \mathcal{F}$ we obtain

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta \rho| \leq C_{\alpha, \beta} \langle \xi' \rangle^{1-|\alpha|-|\beta|}. \quad (3.7)$$

Thus for bounded $|\xi'|$ and $(x', \xi') \notin \mathcal{F}$, we deduce

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta (c^{-1})| \leq C_{\alpha, \beta} |\operatorname{Im} z|^{-1-|\alpha|-|\beta|}. \quad (3.8)$$

Combining this with the estimates (3.4), one concludes that $|\operatorname{Im} z| c^{-1} \in S_\delta^{0, -1}$.

Case II. $z \in Z_2$. We have

$$\operatorname{Re}((1 - \gamma^2)z - r_0) \leq -(1 - \gamma^2) \leq -\epsilon_1 < 0.$$

Consequently, c is elliptic and $c \in S_0^{0, 1}$, $c^{-1} \in S_0^{0, -1}$.

Case III. $z \in Z_3$. In this case $\operatorname{Im} z = 1$ and one has

$$|\operatorname{Im}((1 - \gamma^2)z - r_0)| = |(1 - \gamma^2)| \geq \epsilon_1 > 0.$$

This implies that $c \in S_0^{0, 1}$ is elliptic and $c^{-1} \in S_0^{0, -1}$.

The analysis above shows that the operator $Op_h(c)$ is invertible for $z \in Z_1 \cup Z_2 \cup Z_3$ and $|\operatorname{Im} z| c^{-1} \in S_\delta^{0, -1}$. Hence

$$\|Op_h(c^{-1})g\|_{L^2(\Gamma)} \leq C |\operatorname{Im} z|^{-1} \|g\|_{L^2(\Gamma)}$$

and, applying (3.1), we deduce

$$\|Op_h(c^{-1})Op_h(c)f\|_{L^2(\Gamma)} \leq C_5 \frac{h}{|\operatorname{Im} z|^{3/2}} \|f\|_{L^2(\Gamma)}.$$

On the other hand, for $|\alpha_1| + |\beta_1| \geq 1$, $|\alpha_2| + |\beta_2| \geq 1$ and $|\xi'| \leq C_0$ according to (3.2), (3.4), (3.7), (3.8), for $(x', \xi') \in \mathcal{F}$ we get

$$\left| \partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} (c^{-1}(x', \xi')) \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} c(x', \xi') \right| \leq C_{\alpha_1, \beta_1, \alpha_2, \beta_2} |\operatorname{Im} z|^{1-(|\alpha_1|+|\beta_1|+|\alpha_2|+|\beta_2|)}, \quad (3.9)$$

while for $(x', \xi') \notin \mathcal{F}$ we have

$$\left| \partial_{x'}^{\alpha_1} \partial_{\xi'}^{\beta_1} (c^{-1}(x', \xi')) \partial_{x'}^{\alpha_2} \partial_{\xi'}^{\beta_2} c(x', \xi') \right| \leq C_{\alpha_1, \beta_1, \alpha_2, \beta_2} |\operatorname{Im} z|^{-1-(|\alpha_1|+|\beta_1|)}. \quad (3.10)$$

Consider the operator $Op_h(c^{-1})Op_h(c) - I$. Following Section 7 in [3], the symbol of this operator is given by

$$\begin{aligned} & \sum_{j=1}^N \frac{(ih)^j}{j!} \sum_{|\alpha|=j} D_{\xi'}^\alpha (c^{-1})(x', \xi') D_{y'}^\alpha c(y', \eta') \Big|_{x'=y', \xi'=\eta'} + \tilde{b}_N(x', \xi') \\ & = b_N(x', \xi') + \tilde{b}_N(x', \xi'), \end{aligned}$$

where

$$|\partial_{x'}^\alpha \tilde{b}_N(x', \xi')| \leq C_\alpha h^{N(1-2\delta)-s_d-|\alpha|/2}.$$

Applying (2.2), one deduces for N large enough

$$\|Op_h(\tilde{b}_N)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch.$$

On the other hand, the estimates (3.9), (3.10) yield

$$|\partial_{x'}^\alpha b_N(x', \xi')| \leq C_\alpha \frac{h}{|\operatorname{Im} z|^2} h^{-\delta|\alpha|}.$$

Thus, applying once more (2.2), one gets

$$\|Op_h(c^{-1})Op_h(c)f - f\|_{L^2(\Gamma)} \leq C_6 \frac{h}{|\operatorname{Im} z|^2} \|f\|_{L^2(\Gamma)}.$$

A combination of the above estimates implies

$$\|f\|_{L^2(\Gamma)} \leq C_7 \left(h^{1-2\delta} + h^{1-\frac{3}{2}\delta} \right) \|f\|_{L^2(\Gamma)}. \quad (3.11)$$

Since $\delta = 1/2 - \epsilon$, $0 < \epsilon \ll 1$, for $0 < h \leq h_0(\epsilon)$ small enough (3.11) yields $f = 0$.

Going back to $\lambda = \frac{i\sqrt{z}}{h}$, we have

$$\operatorname{Re} \lambda = -\frac{\operatorname{Im} \sqrt{z}}{h}, \quad \operatorname{Im} \lambda = \frac{\operatorname{Re} \sqrt{z}}{h}.$$

Suppose that $z \in Z_1$. Then

$$|\operatorname{Re} \lambda| \geq C(h^{-1})^{1-\delta}, \quad |\operatorname{Im} \lambda| \leq C_1 h^{-1} \leq C_2 |\operatorname{Re} \lambda|^{\frac{1}{1-\delta}}.$$

So if $|\operatorname{Re} \lambda| \geq C_3 |\operatorname{Im} \lambda|^{1-\delta}$, $\operatorname{Re} \lambda \leq -C_4 < 0$ there are no eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ of G . For $z \in Z_2 \cup Z_3$ there are no eigenvalues λ too if $|\lambda| \geq R_0$.

This shows that in the case (A) for every $0 < \epsilon \ll 1$ the eigenvalues of G must lie in the region Λ_ϵ defined in Theorem 1.2.

4. EIGENVALUES-FREE REGION IN THE CASE (B)

In this section we deal with the case (B). The analysis of Section 3 works only for $z \in Z_1 \cup Z_3$. Indeed for $z \in Z_1$ we have

$$\operatorname{Re}((1 - \gamma^2) - r_0) \leq (1 - \gamma^2) < -\eta_0 < 0.$$

The symbol g introduced in the previous section satisfies the estimates (3.6) and $c \in S_\delta^{0,1}$, $c^{-1} \in S_\delta^{0,-1}$. For $z \in Z_3$ we apply the same argument. Thus for $z \in Z_1 \cup Z_3$ we obtain that the eigenvalues $\lambda = \frac{i\sqrt{z}}{h}$ of G must lie in Λ_ϵ . For $z \in Z_2$ the argument exploited in the case (A) breaks down since for $\operatorname{Re} z = -1, \operatorname{Im} z = 0$ the symbol

$$i[1 + r_0(x', \xi') - \gamma(x')]$$

is not elliptic and it may vanish for some (x'_0, ξ'_0) .

In the following we suppose that $z \in Z_2$. Therefore Proposition 2.3 yields a better approximation

$$\|\mathcal{N}_{ext}(z, h)(f) - T(z, h)f\|_{H^1(\Gamma)} \leq C_N h^{-s_d+N} \|f\|_{L^2(\Gamma)}, \quad \forall N \in \mathbb{N}. \quad (4.1)$$

If $f \neq 0$ is the trace of an eigenfunction of G , from the equality (1.4) we obtain

$$|\operatorname{Re}\langle T(z, h)f - \gamma\sqrt{z}f, f \rangle_{L^2(\Gamma)}| \leq C_N h^{-s_d+N} \|f\|_{L^2(\Gamma)}.$$

Next, we write

$$\begin{aligned} \operatorname{Re}\langle (T(z, h) - \gamma\sqrt{z})f, f \rangle_{L^2(\Gamma)} &= \operatorname{Re}\langle T(-1, h)f, f \rangle_{L^2(\Gamma)} \\ &\quad - \operatorname{Im} z \operatorname{Im} \left[\left\langle \frac{\partial T}{\partial z}(z_t, h) - \gamma \frac{1}{2\sqrt{z_t}} f, f \right\rangle_{L^2(\Gamma)} \right] \end{aligned} \quad (4.2)$$

with $z_t = -1 + it \operatorname{Im} z \in Z_2$, $0 < t < 1$. The next Lemma is an analog of Lemma 3.9 in [14].

Lemma 4.1. *Let $z \in Z_2$ and let $f = u|_\Gamma$ be the trace of an eigenfunction u of G with eigenvalue $\lambda = \frac{i\sqrt{z}}{h}$. Then*

$$\left\| \frac{dT}{dz}(z, h)f - \operatorname{Op}_h\left(\frac{d\rho}{dz}(z)\right)f \right\|_{L^2(\Gamma)} \leq Ch\|f\|_{H_h^{-1}(\Gamma)} \quad (4.3)$$

with a constant $C > 0$ independent of z, h and f . Moreover,

$$|\operatorname{Re}\langle T(-1, h)f, f \rangle_{L^2(\Gamma)}| \leq C_N h^{-s_d+N} \|f\|_{L^2(\Gamma)}, \quad \forall N \in \mathbb{N}. \quad (4.4)$$

Proof. The proof of (4.3) is the same as in [14] since for $z \in Z_2$ we get

$$\sum_{j=0}^{N-1} h^j \frac{da_{1,j}}{dz} \in S_0^{0,-1}.$$

To establish (4.4), we apply Green's formula in the unbounded domain Ω . By using the notation of Section 3, set $\tilde{u} = \sum_{j=1}^J \tilde{u}_\psi$. Then $-i h \partial_\nu \tilde{u}|_\Gamma = T(z, h)f$ and for $R \gg 1$ the function \tilde{u} vanishes for $|x| \geq R$. Thus one obtains

$$\mathbf{i} \langle \Delta \tilde{u}, \tilde{u} \rangle_{L^2(\Omega)} = -\mathbf{i} \int_\Omega |\nabla \tilde{u}|^2 dx - \mathbf{i} \langle \partial_\nu \tilde{u}, \tilde{u} \rangle_{L^2(\Gamma)}.$$

Multiplying the above equality by h and taking the real part, we deduce

$$-\operatorname{Im} h \langle \Delta \tilde{u}, \tilde{u} \rangle_{L^2(\Omega)} = \operatorname{Re} \langle T(z, h)f, f \rangle_{L^2(\Gamma)}.$$

Therefore,

$$\operatorname{Re} \langle T(z, h)f, f \rangle_{L^2(\Gamma)} = -\operatorname{Im} h \langle (\Delta - h^{-2})\tilde{u}, \tilde{u} \rangle_{L^2(\Omega)} = -\operatorname{Im} h^{-1} \langle \mathcal{P}(-1, h)\tilde{u}, \tilde{u} \rangle_{L^2(\Omega)}$$

and

$$|\operatorname{Re} \langle T(z, h)f, f \rangle_{L^2(\Gamma)}| \leq h^{-1} \|\mathcal{P}(-1, h)\tilde{u}\|_{L^2(\Omega)} \|\tilde{u}\|_{L^2(\Omega)}.$$

It is easy to see that $\|\tilde{u}\|_{L^2(\Omega)} \leq Ch^{-s_d} \|f\|_{L^2(\Omega)}$ and combining this with (2.5) for $z \in Z_2$, we obtain (4.4). \square

From (4.2), (4.4) and $\operatorname{Im} z \neq 0$ we have

$$|\operatorname{Im} \langle \left(\frac{\partial T}{\partial z}(z_t, h) - \frac{\gamma}{2\sqrt{z_t}}\right)f, f \rangle_{L^2(\Gamma)}| \leq C_N \frac{h^{-s_d+N}}{|\operatorname{Im} z|} \|f\|_{L^2(\Gamma)}. \quad (4.5)$$

Consider the operator $L := \operatorname{Op}_h\left(\frac{d\rho}{dz}(z_t)\right) - \frac{\gamma}{2\sqrt{z_t}}$ and notice that

$$\left| \operatorname{Im} \langle \left(\frac{\partial T}{\partial z}(z_t, h) - \frac{\gamma}{2\sqrt{z_t}}\right)f, f \rangle_{L^2(\Gamma)} - \operatorname{Im} \langle Lf, f \rangle_{L^2(\Gamma)} \right| \leq Ch\|f\|_{L^2(\Gamma)}. \quad (4.6)$$

On the other hand,

$$\operatorname{Im} \langle Lf, f \rangle_{L^2(\Gamma)} = \frac{1}{2\mathbf{i}} \langle (L - L^*)f, f \rangle_{L^2(\Gamma)}$$

and the principal symbol of $\frac{1}{2\mathbf{i}}(L - L^*)$ has the form

$$s(x', \xi'; z) := \frac{1}{2} \operatorname{Im} \left[\frac{1}{\sqrt{-1 + \mathbf{i}t \operatorname{Im} z - r_0}} - \frac{\gamma}{\sqrt{-1 + \mathbf{i}t \operatorname{Im} z}} \right].$$

Let

$$z_t = ye^{i(\pi-\varphi)}, \quad y = \sqrt{1 + t^2 |\operatorname{Im} z|^2}, \quad |\varphi| \leq \pi/4.$$

Here and below we omit the dependence of y on t . Then $1 \leq y \leq \sqrt{2}$ and

$$\sqrt{z_t} = \sqrt{y} \sin \varphi/2 + \mathbf{i}\sqrt{y} \cos \varphi/2, \quad \operatorname{Im} \frac{1}{\sqrt{z_t}} = -\frac{\cos \varphi/2}{\sqrt{y}}.$$

In the same setting

$$-1 + \mathbf{i}t \operatorname{Im} z - r_0 = qe^{\mathbf{i}(\pi-\psi)}, \quad q = \sqrt{(1+r_0)^2 + t^2(\operatorname{Im} z)^2}, \quad |\psi| \leq \pi/4,$$

we see that

$$\operatorname{Im} \frac{1}{\sqrt{-1 + \mathbf{i}t \operatorname{Im} z - r_0}} = -\frac{\cos \psi/2}{\sqrt{q}}.$$

Therefore

$$s = \frac{1}{2\sqrt{yq}} \left(\gamma\sqrt{q} \cos \varphi/2 - \sqrt{y} \cos \psi/2 \right) = \frac{\gamma^2 q \cos^2 \varphi/2 - y \cos^2 \psi/2}{2\sqrt{yq}(\gamma\sqrt{q} \cos \varphi/2 + \sqrt{y} \cos \psi/2)}.$$

To prove that s is elliptic, it is sufficient to show that

$$\begin{aligned} \gamma^2 q(1 + \cos \varphi) - y(1 + \cos \psi) &= \gamma^2 q \left(1 + \frac{1}{y}\right) - y \left(1 + \frac{1+r_0}{q}\right) \\ &= \frac{1}{yq} \left[\gamma^2 q^2(1+y) - y^2(1+q+r_0) \right] \end{aligned}$$

is elliptic. Consider the function

$$F(r_0) = \gamma^2 \left((1+r_0)^2 + t^2 \operatorname{Im}^2 z \right) (1+y) - y^2 \left(1 + \sqrt{(1+r_0)^2 + t^2 \operatorname{Im}^2 z} + r_0 \right).$$

Clearly,

$$F(0) = (\gamma^2 - 1)(1+y)y^2 \geq \eta_1 > 0,$$

since in the case (B) we have $\gamma^2 - 1 \geq \eta_0 > 0$. Next, for $\gamma \geq 1$, $r_0 \geq 0$ we have

$$\begin{aligned} \frac{\partial F}{\partial r_0} &= 2\gamma^2(1+y)(1+r_0) - y^2 \left(1 + \frac{1+r_0}{\sqrt{(1+r_0)^2 + t^2 \operatorname{Im}^2 z}} \right) \\ &\geq 2 \left(\gamma^2(1+y)(1+r_0) - y^2 \right) \geq 2(1+y-y^2). \end{aligned}$$

On the other hand, it is clear that $1+y-y^2 > 0$ for $0 \leq y < \frac{1+\sqrt{5}}{2}$. In our case $1 \leq y \leq \sqrt{2} < \frac{1+\sqrt{5}}{2}$ and we deduce $\frac{\partial F}{\partial r_0}(r_0) > 0$ for $r_0 \geq 0$, $1 \leq y \leq \sqrt{2}$. This implies $F(r_0) > 0$ for $r_0 \geq 0$ and s is elliptic. Consequently,

$$\operatorname{Im} \langle Lf, f \rangle_{L^2(\Gamma)} \geq (\eta_2 - Ch) \|f\|_{L^2(\Gamma)}, \quad \eta_2 > 0$$

and for small h and $\|f\|_{L^2(\Gamma)} \neq 0$, $\operatorname{Im} z \neq 0$, we deduce from (4.5) and (4.6)

$$|\operatorname{Im} z| \leq C'_N h^{-s_a+N} \leq B_N h^N, \quad \forall N \in \mathbb{N}.$$

Going back to $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$, we have

$$\operatorname{Re} \sqrt{z} = \mu^{1/2} \sin \varphi/2, \quad \operatorname{Im} \sqrt{z} = \mu^{1/2} \cos \varphi/2, \quad \mu = \sqrt{1 + (\operatorname{Im} z)^2} \leq \sqrt{2}$$

and $0 \leq \sin \varphi \leq B_N h^N$. This implies for h small enough the estimate

$$|\operatorname{Im} \lambda| = \left| \frac{\operatorname{Re} \sqrt{z}}{h} \right| \leq 2^{1/4} B_N (h^{-1})^{-N+1} \leq C_N |\operatorname{Re} \lambda|^{-N}.$$

Thus for $z \in Z_2$ and every $N \in \mathbb{N}$ the eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G lie in \mathcal{R}_N and this completes the proof of Theorem 1.2. \square

The Dirichlet-to-Neumann map can be used to prove the discreteness of the spectrum of G in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Consider for $\operatorname{Re} \lambda < 0$ the map

$$\mathcal{N}(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where u is the solution of the problem

$$\begin{cases} (\Delta - \lambda^2)u = 0 \text{ in } \Omega, u \in H^2(\Omega), \\ u = f \text{ on } \Gamma. \end{cases} \quad (4.7)$$

It is well known that $\mathcal{N}(\lambda)$ is a meromorphic function and the poles of $\mathcal{N}(\lambda)$ are the resonances of the Dirichlet problem for the Laplacian (see for instance, [13]). The condition $u \in H^2(\Omega)$ implies that u is λ -incoming in the sense of Lax and Phillips (see Chapter IV in [4]). This means that $u(x)$ has a representation by the λ -incoming Green function $\gamma_\lambda(x)$ which for odd dimensions d has the behavior $\mathcal{O}\left(\frac{e^{\lambda r}}{r^{(d-1)/2}}\right)$ for $|x| = r \rightarrow +\infty$.¹ Thus $\mathcal{N}(\lambda)$ is analytic for $\operatorname{Re} \lambda < 0$. The same argument can be applied to the Neumann problem for the $\Delta - \lambda^2$ and we deduce that $\mathcal{N}^{-1}(\lambda)$ is analytic for $\operatorname{Re} \lambda < 0$ since the poles of $\mathcal{N}^{-1}(\lambda)$ are the resonances of the Neumann problem ([12]). Therefore, we may write the boundary condition in (1.3) as

$$\mathcal{N}(\lambda) \left(I - \lambda \mathcal{N}^{-1}(\lambda) \gamma \right) f_1 = 0, \operatorname{Re} \lambda < 0, x \in \Gamma.$$

The operator $\mathcal{N}(\lambda) : L^2(\Gamma) \longrightarrow H^1(\Gamma)$ is compact and by Theorem 1.1 there are points λ_0 , $\operatorname{Re} \lambda_0 < 0$, for which the operator $(I - \lambda_0 \mathcal{N}^{-1}(\lambda_0) \gamma)$ is invertible. We may apply the analytic Fredholm theorem to conclude that the spectrum of G in the open half-plane $\operatorname{Re} \lambda < 0$ is formed by isolated eigenvalues with finite multiplicities.

The eigenvalues of G could have accumulation points on $i\mathbb{R}$. For odd dimension d Lax and Phillips [5] proved that the scattering matrix $S(z)$ is invertible for $z = 0$. This leads easily to the following

Proposition 4.2. *Assume d odd. The operator G has no a sequence of eigenvalues λ_j , $\operatorname{Re} \lambda_j < 0$ such that $\lim_{j \rightarrow \infty} \lambda_j = iz_0$, $z_0 \in \mathbb{R}$.*

The proof is the same as that of Proposition 4.11 in [2]. The above proposition does not exclude the possibility to have eigenvalues λ_j with $|\operatorname{Im} \lambda_j| \rightarrow +\infty$. On the other hand, Theorem 1.3, established in the next section, implies that for strictly convex obstacles and $\gamma(x) > 1$ the imaginary part of all eigenvalues of G is bounded by a constant $R_0 > 0$ and for d odd we can apply Proposition 4.2.

5. EIGENVALUE-FREE REGION FOR STRICTLY CONVEX OBSTACLES IN THE CASE (B)

In this section we study the eigenvalues-free regions when K is a strictly convex obstacle. Let $0 < \epsilon \ll 1/2$ be a small number. Set

$$\chi_1(x', \xi') = \phi\left(\frac{1 - r_0(x', \xi')}{h^{\epsilon/2}}\right),$$

¹Notice that the definition of outgoing solutions in [13] is different from that in [4] and the resonances in [13] lie in $\operatorname{Im} z < 0$, while in [4] they are in the half-plan $\operatorname{Im} z > 0$.

where ϕ is the function introduced in Section 2. Notice that on the support of $1 - \chi_1$ we have $|1 - r_0(x', \xi')| \geq h^{\epsilon/2}$. By a modification of the construction in [14] (see also [10]) we can construct a semi-classical parametrix \tilde{u}_ψ having the form (2.4).

Then for $|1 - r_0(x', \xi')| \geq h^{\epsilon/2}$ we have $|\rho|^2 \geq h^{\epsilon/2}$ and we can improve the estimate (2.5) obtaining

$$\|\mathcal{P}(z, h)\tilde{u}_\psi\|_{H^s(\Omega)} \leq C_N h^{-l_s} \left(\frac{h}{h^{\epsilon/2} |\operatorname{Im} z|} \right)^N \|f\|_{L^2(\Gamma)}, \quad |\operatorname{Im} z| \geq h^{1-\epsilon}. \quad (5.1)$$

To do this, one repeats without changes the argument in Section 3 of [14] replacing the lower bound $|\rho|^2 \geq |\operatorname{Im} z|$ by $|\rho|^2 \geq h^{\epsilon/2}$. Consequently, the right hand side of (5.1) is estimated by $\mathcal{O}_N(h^{-l_s + N\epsilon/2})$ and this yields a semi-classical parametrix

$$\mathcal{P}(z, h)w_1 = \mathcal{O}_N(h^N), \quad w_1|_{x \in \Gamma} = \operatorname{Op}_h(1 - \chi_1)\psi f.$$

Consider a partition of unity $\chi_\delta^- + \chi_\delta^0 + \chi_\delta^+ = 1$ on $T^*(\Gamma)$, where the functions $\chi_\delta^-, \chi_\delta^0, \chi_\delta^+ \in S_{\delta,0}^0$ are with values in \mathbb{R}^+ and such that $\operatorname{supp} \chi_\delta^- \subset \{r_0 - 1 \leq -h^\delta\}$, $\operatorname{supp} \chi_\delta^+ \subset \{r_0 - 1 \geq h^\delta\}$, $\operatorname{supp} \chi_\delta^0 \subset \{|r_0 - 1| \leq 2h^\delta\}$, $\chi_\delta^0 = 1$ on $\{|r_0 - 1| \leq h^\delta\}$. Then, as in [14], [15], we obtain the following

Theorem 5.1 (Theorem 2.1, [15]). *For every $0 < \epsilon \ll 1$ there exists $h_0(\epsilon) > 0$ such that for $0 < h \leq h_0(\epsilon)$, $|\operatorname{Im} z| \geq h^{1-\epsilon}$, we have*

$$\|N_{\text{ext}}(z, h)\operatorname{Op}_h(\chi_{\epsilon/2}^-) - \operatorname{Op}_h(\rho\chi_{\epsilon/2}^-)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch^{1/2} \quad (5.2)$$

and for $|\operatorname{Im} z| \leq h^\epsilon$ we have the estimate

$$\|N_{\text{ext}}(z, h)\operatorname{Op}_h(\chi_{\epsilon/2}^+) - \operatorname{Op}_h(\rho\chi_{\epsilon/2}^+)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch^{1-\epsilon/2}. \quad (5.3)$$

Thus the problem is to get an estimate for $\|N_{\text{ext}}(z, h)\operatorname{Op}_h(\chi_{\epsilon/2}^0)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)}$. We will prove the following

Theorem 5.2. *For $h^{2/3} \leq \operatorname{Im} z \leq h^\epsilon$ we have the estimate*

$$\|N_{\text{ext}}(z, h)\operatorname{Op}_h(\chi_{\epsilon/2}^0)\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq Ch^{\epsilon/2}. \quad (5.4)$$

Remark 5.3. By the analysis in [15] we may cover the region $h^{1-\epsilon} \leq \operatorname{Im} z \leq h^\epsilon$ but the above result is sufficient for our analysis since the region $0 < \operatorname{Im} z \leq h^{2/3}$ is examined in [13].

Set for simplicity of notation $\mu = \operatorname{Im} z$. We will follow closely the construction of a semi-classical parametrix in Sections 5, 6 in [15]. The only difference is that we deal with an unbounded domain and the local form of \mathcal{P} slightly changes. For the convenience of the reader we are going to recall the result in [11]. Let $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \Gamma) < \delta\}$. Since K is strictly convex, in local normal geodesic coordinates $(x, \xi) \in T^*(\Omega_\delta)$, considered in Section 2, the principal symbol of \mathcal{P} becomes

$$p(x, \xi) = \xi_1^2 + r_0(x', \xi') + x_1 q_1(x', \xi') - 1 - \mathbf{i}\mu + \mathcal{O}(x_1^2 r_0)$$

with $0 < C_1 \leq q_1(x', \xi') \leq C_2$. Here locally in the interior of K we have $x_1 > 0$, while in the exterior of K we have $x_1 < 0$. Following [15], denote by \mathcal{R} the set of functions $a \in C^\infty(T^*(\Omega_\delta))$ satisfying with all derivatives the estimates

$$a = \mathcal{O}(x_1^\infty) + \mathcal{O}(\xi_1^\infty) + \mathcal{O}((1 - r_0)^\infty)$$

in a neighborhood of $\mathcal{K} := \{(x, \xi) : x_1 = \xi_1 = 1 - r_0 = 0\}$. It was shown in Theorem 3.1 in [11] that there exists an exact symplectic map $\chi : T^*(\Omega_\delta) \rightarrow T^*(\Omega_\delta)$ so that $\chi(x, \xi) = (y(x, \xi), \eta(x, \xi))$ satisfies

$$\begin{aligned} y_1 &= x_1 q_1(x', \xi')^{-1/3} + \mathcal{O}(x_1^2) + \mathcal{O}(x_1(1 - r_0)), \\ \eta_1 &= \xi_1 q_1(x', \xi')^{1/3} + \mathcal{O}(x_1) + \mathcal{O}(\xi_1(1 - r_0)), \\ (y', \eta') &= (x', \xi') + \mathcal{O}(x_1), \\ (p \circ \chi(x, \xi)) &= \left(q_1(x', \xi')^{2/3} + \mathcal{O}(x_1) \right) (\xi_1^2 + x_1 - \zeta(x', \xi')), \pmod{\mathcal{R}} \end{aligned}$$

in a neighborhood of \mathcal{K} with

$$\zeta(x', \xi') = \left(q_1(x', \xi')^{-2/3} + \mathcal{O}(1 - r_0) \right) (1 + \mathbf{i}\mu - r_0(x', \xi')).$$

Let $\mathcal{U} \subset T^*(\Omega_\delta)$ be a small neighborhood of \mathcal{K} . By using a h -Fourier integral operator on Ω_δ associated to the canonical relation

$$\Lambda = \{(y, \eta, x, \xi) \in T^*(\Omega_\delta) \times T^*(\Omega_\delta) : (y, \eta) = \chi(x, \xi), (x, \xi) \in \mathcal{U}\},$$

one transforms \mathcal{P} into an operator P'_0 which in the new coordinates denoted again by (x, ξ) has the form

$$P'_0 = D_{x_1}^2 + x_1 - L_1(x', D_{x'}; h) - \mathbf{i}\mu L_2(x', D_{x'}; h),$$

where $L_j(x', \xi'; h) = \sum_{k=0}^{\infty} h^k L_j^{(k)}(x', \xi')$, $j = 1, 2$, with

$$\begin{aligned} L_1^{(0)}(x', \xi') &= \left(q_1(x', \xi')^{-2/3} + \mathcal{O}(1 - r_0) \right) (1 - r_0(x', \xi')), \\ L_2^{(0)}(x', \xi') &= q_1(x', \xi')^{-2/3} + \mathcal{O}(1 - r_0). \end{aligned}$$

By a simple change of variable $t = -x_1$, we pass to the situation when the exterior of K is presented by $t > 0$. Next one applies a new symplectic transformation of the tangential variables $(x^\#, \xi^\#) = \chi^\#(x', \xi') \in T^*(\Gamma)$ so that $\xi_d^\# = -L_1^{(0)}(x', \xi')$ (see Section 2 in [15]). Therefore the operator P'_0 is transformed into

$$\tilde{P}_0 = D_t^2 - t + D_{x_d^\#} - \mathbf{i}\mu q(x^\#, D_{x^\#}) + \mathcal{Q}(x^\#, D_{x^\#}; \mu, h), \quad (5.5)$$

where $q(x^\#, \xi^\#) > 0$, $q \in S_0^0$ in a neighborhood of $\xi_d^\# = 0$ and

$$\mathcal{Q} = \sum_{k=1}^{\infty} h^k Q_k(x^\#, \xi^\#; \mu).$$

The only difference with [15] is the sign $(-)$ in front of t in the form of \tilde{P}_0 .

For simplicity of the notations we denote the coordinates $(x^\#, \xi^\#)$ by (y, η) and consider the operator

$$P_0 = D_t^2 - t + D_{y_d} - \mathbf{i}\mu q(y, D_y) + h\tilde{q}(y, D_y; \mu, h)$$

with $0 < C_1 \leq q(y, \eta) \leq C_2$, $q \in S_0^0$, $\tilde{q} \in S_0^0$. Notice that we have the term $-\mathbf{i}\mu q(y, \eta)$ with $\mu > 0$, while in [15] the model operator involves $\mathbf{i}\mu q(y, \eta)$ since the sign of μ is not important for the argument in Sections 5, 6 of [15].

First we will treat the situation examined in Section 6 in [15] when $\mu > 0$ and η_d satisfy the conditions

$$\mu \sqrt{\mu + |\eta_d|} \geq h^{1-\epsilon}, \quad (5.6)$$

$$\mu + |\eta_d| \leq \mathcal{O}(h^\epsilon). \quad (5.7)$$

Clearly, if $h^{2/3-\epsilon} \leq \mu \leq h^\epsilon$, the condition (5.6) holds. The same is true also if $h^{2/3} \leq \mu \leq h^{2/3-\epsilon}$ and $|\eta_d| \geq h^{1/2-\epsilon}$. Introduce the function

$$\Phi_1(\eta_d) = \begin{cases} \phi\left(\frac{\eta_d}{h^\epsilon}\right), & \text{if } \mu \geq h^{2/3-\epsilon}, \\ \left(1 - \phi\left(\frac{\eta_d}{h^{1/2-\epsilon}}\right)\right)\phi\left(\frac{\eta_d}{h^\epsilon}\right), & \text{if } h^{2/3} \leq \mu < h^{2/3-\epsilon}, \end{cases}$$

where ϕ is the function introduced in Section 2. Let ρ be the solution of the equation

$$\rho^2 + \eta_d - \mathbf{i}\mu q(y, \eta) = 0$$

with $\text{Im } \rho > 0$. With a minor modifications of the argument in Section 6 in [15] we may construct a parametrix $\tilde{u}_1 = \text{Op}_h(A(t))f$, where

$$A(t) = \phi\left(\frac{t}{\delta_1|\rho|^2}\right)a(t, y, \eta; \mu, h)e^{\frac{\mathbf{i}\varphi(t, y, \eta; \mu)}{h}}$$

and $\delta_1 > 0$ is small enough. We take φ and a in the form

$$\varphi = \sum_{k=1}^M t^k \varphi_k, \quad a = \sum_{0 \leq k+\nu \leq M} h^k t^\nu a_{k,\nu},$$

where $M \gg 1$ and φ_k and $a_{k,\nu}$ do not depend on t . We choose $a_{0,0} = \Phi_1(\eta_d)$, $a_{k,0} = 0$ for $k \geq 1$. We have the identity

$$\begin{aligned} & e^{-\mathbf{i}\varphi/h}(D_t^2 - t + \eta_d - \mu q(y, \eta) - \mathbf{i}h\partial_{y_d})(e^{\mathbf{i}\varphi/h}a) \\ &= -2\mathbf{i}h\partial_t\varphi\partial_t a - h^2\partial_t^2 a - \mathbf{i}h\partial_{y_d} a + ((\partial_t\varphi)^2 + \partial_{y_d}\varphi - t - \rho^2)a \\ &= -2\mathbf{i}h \sum_{0 \leq k+\nu \leq 2M-2} h^k t^\nu \sum_{j=0}^{\nu} (j+1)(\nu+1-j)\varphi_{\nu+1-j} a_{k,j+1} \\ &- h \sum_{0 \leq k+\nu \leq M-1} (\nu+1)(\nu+2)h^k t^\nu a_{k-1,\nu+2} - \mathbf{i}h \sum_{0 \leq k+\nu \leq M} h^k t^\nu \partial_{y_d} a_{k,\nu} + ((\partial_t\varphi)^2 + \partial_{y_d}\varphi - t - \rho^2)a. \end{aligned}$$

The phase φ satisfies the eikonal equation

$$(\partial_t\varphi)^2 + \partial_{y_d}\varphi - t - \rho^2 - \mathbf{i}\mu \sum_{|\alpha|=1}^M (\partial_\eta^\alpha q)g_\alpha(\varphi) = R_M(t),$$

with $g_\alpha(\varphi) = \frac{1}{|\alpha|} \prod_{j=1}^{n-1} (\partial_{y_j}\varphi)^{\alpha_j}$ and $R_M(t) = \mathcal{O}(t^M)$. We choose $\varphi_1 = \rho$ and one determines φ_k , $k \geq 2$, from the equation

$$\sum_{k+j=K} (k+1)(j+1)\varphi_{k+1}\varphi_{j+1} + \partial_{y_d}\varphi_K + \epsilon_K = F(\varphi_1, \dots, \varphi_K)$$

with $\epsilon_1 = -1$, $\epsilon_K = 0$ for $K \geq 2$. Next the functions $a_{k,\nu}$ are determined from the equations

$$\begin{aligned} & 2\mathbf{i} \sum_{j=0}^{\nu} (j+1)(\nu+1-j)\varphi_{\nu+1-j} a_{k,j+1} + (\nu+1)(\nu+2)a_{k-1,\nu+2} + \mathbf{i}\partial_{y_d} a_{k,\nu} \\ &= \sum_{|\alpha|=0}^M \sum_{k'=0}^k \sum_{\nu'=0}^{\nu} b_{\alpha,k',\nu,\nu'} \partial_y^\alpha a_{k',\nu'}. \end{aligned}$$

Therefore Lemma 6.1, 6.2, 6.3, 6.4 in [15] hold without any change since the sign before t in the form of P_0 is not involved. Thus, as in Section 6 of [15], for a neighborhood Y of a point in \mathbb{R}^{d-1} obtain

Proposition 5.4. *Assume (5.6) and (5.7) fulfilled for $\eta \in \text{supp } \Phi_1$. Then for all $s \geq 0$ we have the estimates*

$$\|P_0 \tilde{u}_1\|_{H^s(\mathbb{R}^+ \times Y)} \leq C_{s,M} h^{M\epsilon/2} \|f\|_{L^2(Y)}, \quad (5.8)$$

$$\|D_t \tilde{u}_1|_{t=0}\|_{L^2(Y)} \leq Ch^\epsilon \|f\|_{L^2(Y)}. \quad (5.9)$$

To cover the region $h^{2/3} \leq \text{Im } \mu \leq h^\epsilon$, it remains to study the case when $h^{2/3} \leq \mu < h^{2/3-\epsilon}$ and $|\eta_d| \leq h^{1/2-\epsilon}$. For these values of μ and η_d the condition

$$\mu(\mu + |\eta_d|) \leq h^{1+\epsilon} \quad (5.10)$$

is satisfied. We will construct a parametrix for the problem

$$\begin{cases} P_0 u = 0 \text{ in } \mathbb{R}^+ \times Y, \\ u = f_2 \text{ on } Y \end{cases} \quad (5.11)$$

with $f_2 = O_{ph} \left(\phi \left(\frac{\eta_d}{h^{1/2-\epsilon}} \right) \right) f + \mathcal{O}(h^\infty) f$, $f \in L^2(Y)$. For the construction we need some estimates for the Airy function $A(z) = Ai(e^{i2\pi/3} z)$. Here $Ai(z)$ is the Airy function defined for $s \in \mathbb{R}$ by

$$Ai(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(st+t^3/3)} dt.$$

In the following the branch $-\pi < \arg z < \pi$ will be used and $z^{1/2} = |z|^{1/2} e^{i \arg z/2}$. Notice also that

$$\text{Re } \sqrt{z} \geq \frac{|\text{Im } z|}{2|z|^{1/2}}, \quad \text{Im } \sqrt{z} = \frac{\text{Im } z}{2 \text{Re } \sqrt{z}}.$$

The function $A(z)$ satisfies the equalities

$$(\partial_z^2 - z)A^{(k)}(z) = kA^{(k-1)}(z), \quad k \in \mathbb{N}, \quad (5.12)$$

where $A^{(k)}(z) = \frac{d^k A(z)}{dz^k}$. It is well known (see [9], [8]) that $A(z)$ has for $|\arg z - \frac{\pi}{3}| \geq \delta > 0$ the representation

$$A(z) = \Xi(\omega z) \exp\left(\frac{2}{3}i(-z)^{3/2}\right),$$

where $\omega = e^{2\pi i/3}$ and

$$\Xi(z) \sim z^{-1/4} \sum_{j=0}^{\infty} a_j z^{-3j/2}, \quad a_0 = \frac{1}{4}\pi^{-3/4}, \quad |z| \rightarrow \infty.$$

In the same domain in \mathbb{C} one has also an asymptotic expansion for the derivatives of $A(z)$ by taking in the above expansion differentiation term by term (see [9]). Introduce the function

$$F(z) = \frac{A'(z)}{A(z)}.$$

Then for $|\arg z - \pi/3| \geq \delta > 0$ we have

$$F(z) = z^{1/2} \sum_{k=0}^{\infty} b_k z^{-k}, \quad |z| \gg 1, \quad b_0 \neq 0.$$

For large $|z|$ and $\text{Im } z < 0$ we have the estimate $|F(z)| \leq C|z|^{1/2}$, while for bounded $|z|$ and $\text{Im } z < 0$ one obtains $|F(z)| \leq C_1$. Consequently,

$$|F(z)| \leq C_0(|z| + 1)^{1/2}, \quad \text{Im } z < 0.$$

For the derivatives $F^{(k)}(z) = \frac{\partial^k F}{\partial z^k}(z)$ (see Chapter 5 in [8]) we get the following

Lemma 5.5. *For $\text{Im } z < 0$ and every integer $k \geq 0$ we have the estimate*

$$|F^{(k)}(z)| \leq C_k(|z| + 1)^{1/2-k}. \quad (5.13)$$

Given an integer $k \geq 0$, set $\Phi_0(z) = 1$,

$$\Phi_k(z) := A(z)\partial_z^k(A(z)^{-1}) = \partial_z \Phi_{k-1}(z) - F(z)\Phi_{k-1}(z), \quad k \geq 1.$$

Taking the derivatives in the above equality and using (5.13), by induction in k one obtains

Lemma 5.6. *For $\text{Im } z < 0$ and all integers $k \geq 1, l \geq 0$, we have the bound*

$$|\partial_z^l \Phi_k(z)| \leq C_{k,l}(|z| + 1)^{\frac{k}{2}-l}. \quad (5.14)$$

For $t \geq 0$ and $\text{Im } z < 0$, set

$$\Psi_k(t, z) := \frac{A^{(k)}(-t + z)}{A(z)}, \quad \Psi_k^{(l)}(t, z) := \partial_z^l \Psi_k(t, z).$$

The next Lemma is an analogue of Lemma 3.3 in [15].

Lemma 5.7. *For $\text{Im } z < 0$ and all integers $k \geq 0, l \geq 0$, we have the estimate*

$$|\Psi_k^{(l)}(0, z)| \leq C_k |\text{Im } z|^{-l} (|z|^{1/2} + 1)^k. \quad (5.15)$$

For $0 < t \leq |z|$, $\text{Im } z < 0$ and all integers $k \geq 0, l \geq 0$, we have

$$|\Psi_k^{(l)}(t, z)| \leq C_{k,l} |\text{Im } z|^{-l} (|z|^{1/2} + |\text{Im } z|^{-1}) (|z|^{1/2} + 1)^k, \quad (5.16)$$

while for $|t| \geq |z|$ one obtains

$$|\Psi_k^{(l)}(t, z)| \leq C_{k,l} |\text{Im } z|^{-l} (|z|^{1/2} + |\text{Im } z|^{-1}) (t^{1/2} + |\text{Im } z|^{-1})^k e^{-t^{1/2}|\text{Im } z|/4}. \quad (5.17)$$

Proof. Since $\Psi(t, z)$ is analytic for $\text{Im } z < 0$, it is sufficient to establish the above estimates for $l = 0$ and to apply Cauchy formula for the derivatives (see Section 3 in [15]). Taking into account (5.12), (5.13), by induction in k one deduces

$$|A^{(k)}(z)| \leq C_k (|z|^{1/2} + 1)^k |A(z)|$$

hence

$$|\Psi_k(t, z)| \leq C_k (t^{1/2} + |z|^{1/2} + 1)^k |\Psi_0(t, z)|. \quad (5.18)$$

Thus it is sufficient to estimate $|\Psi_0(t, z)|$. The representation of $A(z)$ with phase $\exp\left(\frac{2}{3}\mathbf{i}(-z)^{3/2}\right)$ mentioned above holds for $\text{Im } z < 0$. Hence

$$\begin{aligned} \left| \frac{A(-t+z)}{A(z)} \right| &\leq \left| \frac{\Xi(\omega(-t+z))}{\Xi(\omega z)} \right| \exp\left(-\text{Im} \frac{2}{3} \left((t-z)^{3/2} - (-z)^{3/2} \right)\right) \\ &= \left| \frac{\Xi(\omega(-t+z))}{\Xi(\omega z)} \right| e^{-\varphi}. \end{aligned}$$

It clear that $|\Xi(\omega(-t+z))| \leq c_0$. For $|z| \leq C$, $C \gg 1$ we have

$$\left| \left(\Xi(\omega z) \right)^{-1} \right| \leq C_1 \leq C_2 |\text{Im } z|^{-1},$$

while for $|z| \geq C$ we have

$$\left| \left(\Xi(\omega z) \right)^{-1} \right| \leq C_3 |z|^{1/4} \leq C_3 |z|^{1/2}.$$

Thus $\left| \frac{\Xi(\omega(-t+z))}{\Xi(\omega z)} \right| \leq C(|z|^{1/2} + |\operatorname{Im} z|^{-1})$.

Next, we get

$$\begin{aligned} \varphi &= \frac{2}{3} \operatorname{Im}(t-z)^{3/2} - \frac{2}{3} \operatorname{Im}(-z)^{3/2} = \int_0^t \operatorname{Im}(\tau-z)^{1/2} d\tau \\ &= - \int_0^t \frac{\operatorname{Im} z}{2 \operatorname{Re}((\tau-z)^{1/2})} d\tau \geq \frac{t |\operatorname{Im} z|}{2(t^{1/2} + |z|^{1/2})} \end{aligned}$$

and this shows that for $t > 0$ we have $\varphi > 0$. For $|t| \leq |z|$ the estimate (5.18) implies (5.16). For $|t| \geq |z|$ we have

$$\frac{t |\operatorname{Im} z|}{2(t^{1/2} + |z|^{1/2})} \geq \frac{t^{1/2} |\operatorname{Im} z|}{4}$$

and

$$t^{k/2} e^{-t^{1/2} |\operatorname{Im} z|/4} \leq C_k |\operatorname{Im} z|^{-k}. \quad (5.19)$$

If $|\operatorname{Im} z| \leq 1$ we have $1 \leq |\operatorname{Im} z|^{-1}$, while if $|\operatorname{Im} z| > 1$, we get $t \geq |z| > 1$. Hence from (5.19) and (5.18) we deduce (5.17). \square

For $h^{2/3} \leq \mu \leq h^{2/3-\epsilon}$ we will construct a parametrix for (5.11) repeating without any change the construction in Section 5 of [15]. The parametrix has the form $\tilde{u}_2 = \phi(t/h^\epsilon) \operatorname{Op}_h(\mathcal{A}(t))g$, where $g \in L^2(Y)$ can be determined as in Section 5, [15]. Here

$$\begin{aligned} \mathcal{A}(t) &= \sum_{k=0}^M a_k(y, \eta; h, \mu) \psi_k(t, y; h, \mu), \\ \psi_k &= h^{k/3} \Psi_k \left(-th^{-2/3}, (\eta_d - \mathbf{i}\mu q(y, \eta))h^{-2/3} \right), \end{aligned}$$

$M \gg 1$ is an arbitrary integer, $a_0 = \phi(\frac{\eta_d}{h^{1/2-\epsilon}})$. Next a_k , $k \geq 1$, are independent on t and they can be determined from the equality

$$\begin{aligned} (k+1)a_{k+1} &= -\mathbf{i}\partial_{y_d} a_k + \mu h^{-2/3} \partial_{y_d} q F(\eta_d - \mathbf{i}\mu q(y, \eta)) h^{-2/3} a_k - \mu h^{-1} \partial_{y_d} a_{k-1} \\ &\quad + \sum_{l=0}^k \sum_{|\alpha|=0}^k \left(b_{k,l,\alpha}^{(1)} + b_{k,l,\alpha}^{(2)} \right) \partial_y^\alpha a_k. \end{aligned}$$

We have

$$\begin{aligned} P_0 \operatorname{Op}_h(\mathcal{A}(t)) &= \operatorname{Op}_h \left((D_t^2 - t + \eta_d - \mathbf{i}\mu q(y, \eta) - \mathbf{i}h\partial_{y_d}) \mathcal{A}(t) \right) \\ &\quad - \mathbf{i}\mu q(y, D_y) \operatorname{Op}_h(\mathcal{A}(t)) + \mathbf{i}\mu \operatorname{Op}_h(q\mathcal{A}(t)) + h\tilde{q}(y, D_y) \operatorname{Op}_h(\mathcal{A}(t)). \end{aligned}$$

On the other hand, (5.12) implies the equality

$$\begin{aligned} (D_t^2 - t + \eta_d - \mathbf{i}\mu q(y, \eta)) \Psi_k \left(-th^{-2/3}, (\eta_d - \mathbf{i}\mu q(y, \eta))h^{-2/3} \right) \\ = -kh^{-2/3} \Psi_{k-1} \left(-th^{-2/3}, (\eta_d - \mathbf{i}\mu q(y, \eta))h^{-2/3} \right) \end{aligned}$$

and

$$(D_t^2 - t + \eta_d - \mathbf{i}\mu q(y, \eta)) \mathcal{A}(t) = -h \sum_{k=0}^{M-1} (k+1) a_{k+1} \psi_k.$$

Next the construction of the parametrix goes without any changes as in Section 5 in [15] applying Lemmas 5.5, 5.6 and 5.7 instead of Lemmas 3.1, 3.2 and 3.3 in [15]. Thus as an analogue of Theorem 5.7 in [15] we get the following

Proposition 5.8. *For all $s \geq 0$, we have the bounds*

$$\|P_0 \tilde{u}_2\|_{H^s(\mathbb{R}^+ \times Y)} \leq C_{s,M} h^{M\epsilon/2} \|f\|_{L^2(Y)}, \quad (5.20)$$

$$\|\tilde{u}_2|_{t=0} - Op_h\left(\phi(\eta_a \mu/h^{1+\epsilon})\right) f\|_{L^2(Y)} \leq \mathcal{O}(h^\infty) \|f\|_{L^2(Y)}, \quad (5.21)$$

$$\|D_t \tilde{u}_2|_{t=0}\|_{L^2(Y)} \leq Ch^\epsilon \|f\|_{L^2(Y)}. \quad (5.22)$$

Combining Proposition 5.4 and Proposition 5.8, we obtain, as in [15], Theorem 5.2.

After this preparation we pass to the analysis of an eigenvalues-free region when

$$\operatorname{Re} z = 1, \quad h^{2/3} \leq \operatorname{Im} z \leq h^\epsilon, \quad 0 < \epsilon \ll 1.$$

Let $\rho = \sqrt{1 - r_0 + \mathbf{i} \operatorname{Im} z}$. As in the previous section, we examine the equation

$$\mathcal{N}_{ext}(z, h)(f) - \sqrt{z}\gamma f = 0.$$

Consider the partition of the unity $\chi_{\epsilon/2}^+ + \chi_{\epsilon/2}^0 + \chi_{\epsilon/2}^- = 1$ on $T^*(\Gamma)$ introduced in the beginning of this section. Applying Theorem 5.2, we have

$$\|\mathcal{N}_{ext}(z, h)(1 - \chi_{\epsilon/2}^0) f - \sqrt{z}\gamma f\|_{L^2(\Gamma)} \leq Ch^{\epsilon/2} \|f\|_{L^2(\Gamma)}.$$

Taking into account Theorem 5.1 for the operators $N(z, h)\chi_{\epsilon/2}^\pm$, one deduces

$$\|Op_h\left(\rho(\chi_{\epsilon/2}^+ + \chi_{\epsilon/2}^-) - \sqrt{z}\gamma\right) f\|_{L^2(\Gamma)} \leq C_1 h^{\epsilon/2} \|f\|_{L^2(\Gamma)}. \quad (5.23)$$

We write

$$g_1 := \rho(\chi_{\epsilon/2}^+ + \chi_{\epsilon/2}^-) - \sqrt{z}\gamma = \frac{\rho^2 \left[(\chi_{\epsilon/2}^+)^2 + (\chi_{\epsilon/2}^-)^2 \right] - z\gamma^2}{\rho(\chi_{\epsilon/2}^+ + \chi_{\epsilon/2}^-) + \sqrt{z}\gamma}.$$

Clearly,

$$\operatorname{Re} \left(\rho^2 \left[(\chi_{\epsilon/2}^+)^2 + (\chi_{\epsilon/2}^-)^2 \right] - z\gamma^2 \right) = (1 - r_0) \left[(\chi_{\epsilon/2}^+)^2 + (\chi_{\epsilon/2}^-)^2 \right] - \gamma^2 \leq -\eta_0 < 0$$

since $1 - r_0 \leq 1$, $\operatorname{supp} \chi_{\epsilon/2}^+ \cap \operatorname{supp} \chi_{\epsilon/2}^- = \emptyset$ and $1 - \gamma^2 \leq -\eta_0$. Thus for bounded $|\xi'|$ we have $|g_1| \geq \eta_2 > 0$, while for $|\xi'| \gg 1$ we get $|g_1| \sim |\xi'|$. To estimate g_1^{-1} , it is necessary to estimate only $\rho(\chi_{\epsilon/2}^+ + \chi_{\epsilon/2}^-) + \sqrt{z}\gamma$ and one deduces

$$|\partial_x^\alpha \partial_{\xi'}^\beta (g_1^{-1})| \leq C_{\alpha,\beta} h^{-\frac{\epsilon}{2}(1/2 + |\alpha| + |\beta|)} (1 + |\xi'|)^{1 - |\beta|}.$$

The same estimates holds for g_1 , hence $g_1 \in S_{\epsilon/2}^{\epsilon/4, 1}$, $g_1^{-1} \in S_{\epsilon/2}^{\epsilon/4, -1}$ and

$$\|(Op_h(g_1^{-1})Op_h(g_1) - I)f\|_{L^2(\Gamma)} \leq Ch^{1-\epsilon} \|f\|_{L^2(\Gamma)}.$$

Combining this with (5.23), for small h we conclude as in Section 4, that $f = 0$.

It remains to study the case

$$z \in \mathcal{D} = \{z \in \mathbb{C} : z = 1 + \mathbf{i} \operatorname{Im} z, \quad 0 < \operatorname{Im} z \leq h^{2/3}\}.$$

The Dirichlet problem for $-h^2 \Delta - z$ with $z = 1 + \mathbf{i} \operatorname{Im} h^{2/3} w$, $|w| \leq C_0$, has been investigated by Sjöstrand in Chapters 9 and 10 in [13] (see also [1]). For $0 \leq w \leq 1$

this covers the region \mathcal{D} . In [13] the exterior Dirichlet-to-Neumann map \mathcal{N}_{ext} is defined with respect to the outgoing solution of the problem (1.5). Notice that for $\text{Im } z > 0$ the outgoing solutions are in $H_h^2(\Omega)$, so the exterior Dirichlet-to-Neumann map in [13] coincides with that defined in Section 2. We need to recall some results in Chapter 10 of [13]. The operator $\mathcal{N}_{ext}(z, h)$ is a h -pseudodifferential operator with symbol $n_{ext}(x', \xi', h)$. Introduce the glancing set

$$\mathcal{G} = \{(x', \xi') \in T^*(\Gamma) : r_0(x', \xi') = 1\}.$$

We have $\gamma(x) \geq 1 + \eta_3 > 1$, $\forall x \in \Gamma$. Choose a small number δ_0 , $0 < \delta_0 < \eta_3/2$. Then for $|r_0(x', \xi') - 1| \geq \delta_0$ the symbol n_{ext} satisfies the estimates

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta n_{ext}(x', \xi', h)| \leq C_{\alpha, \beta} \langle \xi' \rangle^{1-|\beta|}, \quad \forall \alpha, \forall \beta,$$

while for $|r_0(x', \xi') - 1| \leq 2\delta_0$ we have the estimates

$$|\partial_{x'}^\alpha \partial_{\xi'}^\beta n_{ext}(x', \xi', h)| \leq C_{\alpha, \beta} (h^{2/3} + |r_0 - 1|)^{\frac{1}{2} - \beta_d}, \quad \forall \alpha, \forall \beta$$

if $r_0(x', \xi') - 1$ is transformed into ξ_d by a tangential Fourier integral operator as it was mentioned in the beginning of this section. From the estimates near \mathcal{G} it follows that for small $0 < h \leq h_0(\delta_0)$ we have a bound

$$\left\| \mathcal{N}_{ext}(z, h) \phi\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right) \right\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C(h^{1/3} + \delta_0^{1/2})$$

with a constant $C > 0$ independent on h and δ_0 . Let $f \neq 0$ be the trace on Γ of an eigenfunction of G . Consider the equality

$$\begin{aligned} & -\text{Re} \left\langle \mathcal{N}_{ext}(z, h) \left[1 - \phi\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right) \right] f, f \right\rangle_{L^2(\Gamma)} + \text{Re} \langle \sqrt{z} \gamma f, f \rangle_{L^2(\Gamma)} \\ & = \text{Re} \langle \mathcal{N}_{ext}(z, h) \phi\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right) f, f \rangle_{L^2(\Gamma)}. \end{aligned} \quad (5.24)$$

The above estimate shows that the right hand side in (5.24) is bounded by $C_1(h^{1/3} + \delta_0^{1/2}) \|f\|_{L^2(\Gamma)}^2$.

Introduce two functions $\psi_\pm(\sigma) \in C^\infty(\mathbb{R} : [0, 1])$ such that $\psi_+(\sigma) = 0$ for $\sigma \leq 1/2$, $\psi_+(\sigma) = 1$ for $\sigma \geq 1$, $\psi_-(\sigma) = \psi_+(-\sigma)$. We write

$$\mathcal{N}_{ext}(z, h) \left[1 - \phi\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right) \right] = \mathcal{N}_{ext}(z, h) \chi_+ + \mathcal{N}_{ext}(z, h) \chi_-,$$

where

$$\chi_\pm(x', \xi') = \left[1 - \phi\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right) \right] \psi_\pm\left(\frac{1 - r_0(x', \xi')}{\delta_0}\right)$$

have support in $\{(x', \xi') : 1 - r_0(x', \xi') \geq \delta_0/2\}$ and $\{(x', \xi') : 1 - r_0(x', \xi') \leq -\delta_0/2\}$, respectively. The principal symbols n_\pm of $\mathcal{N}_{ext}(z, h) \chi_\pm$ have the form

$$n_\pm = \left(\sqrt{1 - r_0 + \mathbf{i} \text{Im } z} \right) \chi_\pm$$

and

$$\text{Re} \langle \mathcal{N}_{ext}(z, h) \chi_\pm f, f \rangle_{L^2(\Gamma)} = \langle \text{Op}_h(\text{Re}(n_\pm)) f, f \rangle_{L^2(\Gamma)} + \mathcal{O}(h) \|f\|_{L^2(\Gamma)}^2.$$

On the other hand,

$$|\text{Re } n_+| = \chi_+ |\text{Re} \sqrt{1 - r_0 + \mathbf{i} \text{Im } z}| \leq (1 + h^{2/3})^{1/2}.$$

In the same way for the principal symbol n_- of $\mathcal{N}_{ext}(z, h)\chi_-$ we get

$$|\operatorname{Re} n_-| = \chi_- |\operatorname{Re} \sqrt{1 - r_0 + \mathbf{i} \operatorname{Im} z}| \leq y^{1/2} \sin \frac{\psi}{2},$$

where $1 - r_0 + \mathbf{i} \operatorname{Im} z = ye^{i(\pi-\psi)}$, $y > 0$, $0 < \psi \ll 1$. Next

$$y^{1/2} \sin \frac{\psi}{2} = y^{1/2} \sqrt{\frac{1 - \cos \psi}{2}} = \frac{1}{\sqrt{2}} \sqrt{y - (r_0 - 1)}.$$

On the support of χ_- we have $0 < r_0 - 1 < y \leq (r_0 - 1) + h^{2/3}$, and this implies $y^{1/2} \sin \frac{\psi}{2} \leq \frac{1}{\sqrt{2}} h^{1/3}$. Combining the above estimates, we conclude that

$$-\operatorname{Re} \left\langle \mathcal{N}_{ext}(z, h) \left[1 - \phi \left(\frac{1 - r_0(x', \xi')}{\delta_0} \right) \right] f, f \right\rangle_{L^2(\Gamma)} \geq -(1 + C_1 h^{1/3}) \|f\|_{L^2(\Gamma)}^2. \quad (5.25)$$

Let $\sqrt{z} = v + \mathbf{i}w$. Then $v^2 = 1 + w^2 \geq 1$ yields $\operatorname{Re} \sqrt{z} = v \geq 1$. Consequently,

$$\operatorname{Re} \langle \sqrt{z} \gamma f, f \rangle_{L^2(\Gamma)} \geq (1 + \eta_3) \|f\|_{L^2(\Gamma)}^2.$$

From this estimate and (5.25) one deduces that the left hand side of (5.24) is greater than $(\eta_3 - C_1 h^{1/3}) \|f\|_{L^2(\Gamma)}^2$. For small h and small δ_0 (depending on η_3) we obtain a contradiction with the estimate of the right hand side of (5.24). Finally, if $\operatorname{Re} z = 1$, $0 < \operatorname{Im} z \leq h^{2/3}$ with $0 < h \leq h_0(\eta_3)$ there are no eigenvalues $\lambda = \frac{\mathbf{i}\sqrt{z}}{h}$ of G . Combining this with the result of Section 4, completes the proof of Theorem 1.3.

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