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HÖLDER-CONTINUITY OF OSELEDETS SUBSPACES FOR THE KONTSEVICH-ZORICH COCYCLE

VÍTOR ARAÚJO, ALEXANDER I. BUFETOV, AND SIMION FILIP

Abstract. For Hölder cocycles over a Lipschitz base transformation, possibly non-invertible, we show that the subbundles given by the Oseledets Theorem are Hölder-continuous on compact sets of measure arbitrarily close to 1. The results extend to vector bundle automorphisms, as well as to the Kontsevich-Zorich cocycle over the Teichmüller flow on the moduli space of abelian differentials. Following a recent result of Chaika-Eskin, our results also extend to any given Teichmüller disk.

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1. Introduction

The multiplicative ergodic theorem, or Oseledets theorem [Ose68] provides the theoretical background for computation of Lyapunov exponents of a nonlinear dynamical system. These exponents define subspaces of vectors having the same exponential rate of growth under the action of a *typical* family of linear maps generated by the orbits of a measure preserving system; typical here meaning almost every point, see below for the precise statements for discrete dynamical systems and flows.

The dependence of the exponents and the corresponding subspaces on the orbit is measurable in general; strong forms of dependence being rarely studied. For the special case of an ergodic partially hyperbolic probability measure preserved by a $C^{1+\alpha}$ diffeomorphism of a compact manifold, it is known that the subspace given by the direct sum of the Oseledets subspaces corresponding to strictly negative Lyapunov exponents depend Hölder-continuously on the orbit chosen; see Brin [Bri01] and [BP07, Chpt.5, Section 3].

Here we show that all Oseledets subspaces depend Hölder-continuously on the points of hyperbolic blocks (or *regular sets*) of the system, which can be chosen to be compact sets of measure arbitrarily close to 1. We present our results for cocycles generated by discrete maps (both invertible and non-invertible) and also for vector bundle automorphisms covering flows, under some regularity conditions on the underlying dynamics.

In addition, we show that our results apply to the Kontsevich-Zorich cocycle over the Teichmüller flow on the moduli space of abelian differentials. Following a recent result of Chaika-Eskin [CE15], our results also extend to any given Teichmüller disk.

We begin, in the following subsections, by stating precisely our results in the setting of the Multiplicative Ergodic Theorem for cocycles over measure preserving Lipschitz maps; then for cocycles over measure preserving Lipschitz flows (Subsections 1.1 and 1.2); and finally stating our results in the setting of Kontsevich-Zorich cocycle over the Teichmüller flow (Subsection 1.3).

For the proof, we present some preliminary results in Section 2 together with the statements of the main technical lemmas. Then in Section 3 we prove the sufficient conditions for the Oseledets subspaces to depend Hölder continuously on points of hyperbolic blocks. Section 4 is reserved to the detailed proofs of the technical lemmas and we finally apply our results to the Kontsevich-Zorich cocycle in Section 5.

1.1. **Multiplicative ergodic theorem for cocycles.** Let $f : M \to M$ be a measurable transformation on the metric space (M, d), preserving some Borel probability measure μ, and let $A : M \to \mathrm{GL}(d, \mathbb{R})$ (or $\mathrm{GL}(d, \mathbb{C})$) be any measurable function such that $\log^+ \|A(x)\|$ is μ-integrable, where $\|\cdot\|$ is any norm in the space of real (or complex) $d \times d$ matrices and $\log^+(a) := \max\{\log a, 0\}$ for $a > 0$. The Oseledets theorem states that Lyapunov exponents

exist for the sequence $A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))\, A(x)$ for μ-almost every $x \in M$. More precisely, for μ-almost every $x \in M$ there exists $k = k(x) \in \{1, \dots, d\}$, a filtration

$$\{0\} = F_x^0 \subset F_x^1 \subset \cdots \subset F_x^{k-1} \subset F_x^k = \mathbb{R}^d (\text{or } \mathbb{C}^d),$$

and numbers $\chi_1(x) < \cdots < \chi_k(x)$ such that

$$F_x^i = \left\{ v \in \mathbb{R}^d : \lim_{n\to\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| \le \chi_i(x) \right\}, \quad \text{and}$$
$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \chi_i(x) \quad \text{for all } v \in F_x^i \setminus F_x^{i-1} \text{ and } i \in \{1, \dots, k\}$$

where $\|\cdot\|$ is a norm in $\mathbb{R}^d$ ($\mathbb{C}^d$). More generally, this conclusion holds for any vector bundle automorphism $A : \mathcal{V} \to \mathcal{V}$ over the transformation f, with $A(x) : \mathcal{V}_x \to \mathcal{V}_{f(x)}$ denoting the action of the bijective linear map between the fibers $\mathcal{V}_x$ and $\mathcal{V}_{f(x)}$. We assume that a measurable family $\{\|\cdot\|_x\}_{x\in M}$ of norms is given which enables us to define $\|A(x)\| := \sup\{\|A(x)v\|_{f(x)}/\|v\|_x : v \in \mathcal{V}_x, v \neq \vec{0}\}$.

The Lyapunov exponents $\chi_i(x)$, and their number $k(x)$, are measurable functions of x and they are constant on orbits of the transformation f. In particular, if the measure μ is ergodic then k and the χ_i are constant on a full μ-measure set of points. The subspaces F_x^i also depend measurably on the point x and are invariant under the automorphism:

$$A(x) \cdot F_x^i = F_{f(x)}^i.$$

If the transformation f is invertible one obtains a stronger conclusion, by applying the previous result also to the inverse automorphism: for $n > 0$ we set $A^{-n}(x) = A(f^{-n}(x))^{-1} \cdots A(f^{-1}(x))^{-1}$ and $A^0(x) = Id$. With this definition we associate to each x a bi-infinite sequence $(A^n(x))_{n\in\mathbb{Z}}$ of automorphisms of $\mathbb{R}^d$ ($\mathbb{C}^d$). Assuming that both $\log^+ \|A(x)\|$ and $\log^+ \|A(x)^{-1}\|$ are in $L^1(\mu)$, one gets that there exists a decomposition

$$\mathcal{V}_x = E_x^1 \oplus \cdots \oplus E_x^k,$$

defined at almost every point and such that $A(x) \cdot E_x^i = E_{f(x)}^i$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \chi_i(x) \tag{1.1}$$

for all $v \in E_x^i$ different from zero and all $i \in \{1, \dots, k\}$. Moreover the convergence in (1.1) is uniform over the set $\{v \in E_x^i : \|v\| = 1\}$ of unit vectors; see [BP07, Lemma 3.4.6] and [BP07, Theorem 3.5.10] for the non-invertible case. These *Oseledets subspaces* E_x^i are related to the subspaces F_x^i through

$$F_x^j = \oplus_{i=1}^j E_x^i. \tag{1.2}$$

Hence, $\dim E_x^i = \dim F_x^i - \dim F_x^{i-1}$ is the multiplicity of the Lyapunov exponent $\chi_i(x)$.

The angles between any two Oseledets subspaces decay sub-exponentially along orbits of f (see [BP07, Theorem 1.3.11 & Remark 3.1.8]):

$$\lim_{n\to\pm\infty}\frac{1}{n}\log\sin\text{angle}\left(\bigoplus_{i\in I}E^i_{f^n(x)},\bigoplus_{j\notin I}E^j_{f^n(x)}\right)=0$$

for any $I\subset\{1,\dots,k\}$ and almost every point, where for any given pair E,F of complementary subspaces (i.e. $E\oplus F=\mathbb{R}^d$) we set

$$\cos\text{angle}\,(E,F):=\inf\{|\langle v,w\rangle|:\|v\|=1=\|w\|,v\in E,w\in F\}\,.$$

These facts imply the regularity condition mentioned previously and, in particular,

$$\lim_{n\to\pm\infty}\frac{1}{n}\log|\det A^n(x)|=\sum_{i=1}^{k}\chi_i(x)\dim E^i_x$$

If the measure μ if f-ergodic, then the numbers $k(x),\chi_i(x)$ are constant μ-almost everywhere.

The Oseledets Multiplicative Ergodic Theorem applies, in particular, when $f:M\to M$ is a C^1 diffeomorphism on some compact manifold and $A(x)=Df_x$. Notice that the integrability conditions are automatically satisfied for any f-invariant probability measure μ, since the derivative of f and its inverse are bounded in norm; see e.g. [Ose68, Pes77, Mañ87, BP07].

In general, the dependence of E^i_x on x is measurable. Here we show that, under some mild assumptions on the cocycle and the base transformation, this dependence is in fact Hölder with probability arbitrarily close to 1. For the full stable or unstable subspace, Hölder dependence on the base point has been established by Brin [Bri01]. Our result gives Hölder dependence for each Oseledets subspace corresponding to each Lyapunov exponent. Furthermore, Theorem 5.6 establishes Hölder dependence for the Kontsevich-Zorich cocycle, while Theorem 5.7 proves a similar result in almost every direction on any Teichmüller disk.

Remark 1.1. In what follows we present the statements and proofs in the case of real vector spaces and vector bundles, but the same results hold with the same proofs in the complex case.

1.2. **Statement of the results.** Some preliminary definitions are needed. We define the distance from a vector v to a subspace E in $\mathbb{R}^d$ with a given euclidean norm $\|\cdot\|$ as

$$\text{dist}(v,E):=\min_{w\in E}\|v-w\|$$

and then the distance between two subspaces E,F of $\mathbb{R}^d$ is defined to be

$$\text{dist}(E,F):=\max\Big\{\sup_{\|v\|=1,v\in E}\text{dist}(v,F),\ \sup_{\|v\|=1,v\in F}\text{dist}(v,E)\Big\}. \tag{1.3}$$

Let M be a metric space endowed with a distance d. We say that the cocycle $A : M \to GL(d, \mathbb{R})$ is ν-Hölder-continuous if there exists a constant $H > 0$ such that

$$\|A(x) - A(y)\| \leq Hd(x, y)^{\nu}, \quad \text{for all } x, y \in M.$$

In this case we also say that A is a (H, ν)-Hölder cocycle. Note that we do not assume that the base is compact; however, the Hölder assumption is uniform on the base.

Theorem A. *Let $f : M \circlearrowleft$ be a Lipschitz transformation preserving an ergodic probability measure μ, and $A : M \to GL(d, \mathbb{R})$ a ν-Hölder cocycle over f such that both $\log \|A(x)\|$ and $\log \|A(x)^{-1}\|$ are μ-integrable. We denote by $\chi_1 < \cdots < \chi_k$ the distinct k Lyapunov exponents associated to this cocycle, and by F^i_x the filtration by subspaces of $\mathbb{R}^N$ corresponding to each χ_i, $i = 1, \ldots, k$, defined for μ-almost every $x \in M$.*

Then, for every $\varepsilon > 0$, there exists a compact subset Λ_ε of M, and constants $C = C(\Lambda_\varepsilon) > 0$, $\omega_i = \omega_i(\chi_1, \ldots, \chi_k) \in (0, 1), i = 1, \ldots, k$ and $\delta = \delta(\varepsilon, A, \Lambda_\varepsilon, \chi_1, \ldots, \chi_k) > 0$, such that

(1) $\mu(\Lambda_\varepsilon) \geq 1 - \varepsilon$;
(2) *for all $x, y \in \Lambda_\varepsilon$ with $d(x, y) < \delta$ we have* $\operatorname{dist}(F^i_x, F^i_y) \leq C_\varepsilon d(x, y)^{\nu\omega_i}$.

Moreover, if the map f is also invertible, then letting E^i_x be the Oseledets subspaces corresponding to each χ_i, $i = 1, \ldots, k$, which form a splitting $\mathbb{R}^n = E^1_x \oplus \cdots \oplus E^k_x$ for μ-almost every $x \in M$, we also have

(3) $\operatorname{dist}(E^i_x, E^i_y) \leq C_\varepsilon d(x, y)^{\nu\omega_i}$ *for all $x, y \in \Lambda_\varepsilon$ with $d(x, y) < \delta$.*

We say that a measurable subbundle $E = E(x)$ of $\mathbb{R}^d$ defined over a measurable subset Λ is locally α-Hölder continuous with Hölder constant $H > 0$ if, for every $x \in \Lambda$, there exists $\delta > 0$ such that

$$\operatorname{dist}(E(x), E(y)) \leq Hd(x, y)^{\alpha} \quad \text{for all} \quad x, y \in \Lambda \quad \text{with} \quad d(x, y) < \delta.$$

Remark 1.2. The definition of Hölder-continuity for measurable subbundles allows the local radius $\delta > 0$ to depend on x. This is not the case for linear cocycles, but is needed for vector bundle automorphisms; see Theorem C in the following Subsection 1.2.2.

Thus Theorem A means that the subbundles given by Oseledets' Theorem are Hölder-continuous on arbitrarily big (in measure) compact subsets of the ambient space, for Hölder cocycles over a Lipschitz base transformation.

Remark 1.3. We have $C_\varepsilon = C(\Lambda_\varepsilon) \to +\infty$ and $\delta \to 0$ as $\varepsilon \to 0+$, so that the Hölder constant becomes worse as the size of Λ_ε grows to fill a full μ-measure subset of M.

The proof shows that the dependence of the Oseledets directions on the base point at every block Λ_ε is Hölder, with Hölder exponent only dependent on the minimum gap $\varepsilon_0 = \min\{\chi_{i+i} - \chi_i : 1 \leq i < k\}$ of the Lyapunov spectrum, and Hölder constant essentially dependent on the choice of the regular block.

1.2.1. *The case of linear multiplicative cocycles over flows.* We can extend our result to the case of linear multiplicative cocycles over flows with the exact same conclusions (1), (2) and (3) of Theorem A as a corollary of the proof for invertible transformations.

We recall some common definitions. Let (X, μ) be a Lebesgue space. The measurable map $\varphi : \mathbb{R} \times X \to X$ is a *measurable flow* if $\varphi_0 = Id$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for each $t, s \in \mathbb{R}$. The flow φ preserves the measure μ if $\varphi_t := \varphi(t, \cdot)$ preserves μ for all $t \in \mathbb{R}$.

A measurable function $A : X \times \mathbb{R} \to GL(d, \mathbb{R})$ is a *linear multiplicative cocycle over* φ if $A(x, 0) = Id$ and $A(x, t + s) = A(\varphi_t(x), s)A(x, t)$ for all $s, t \in \mathbb{R}$ and for every $x \in X$.

We can also consider a measurable linear cocycle A acting on a vector bundle $\mathcal{V}$ over M, that is $A(x, t) : \mathcal{V}_x \to \mathcal{V}_{\varphi_t(x)}$ is a linear bijection for all $(x, t) \in M \times \mathbb{R}$. Taking a family of norms on the vector bundle enables us to define the norm of $A(x, t)$.

Let us take a measurable cocycle A over a flow φ which preserves a probability measure μ such that

$$\sup_{-1 \le t \le 1} \log^+ \|A(x, t)\| \in L^1(X, \mu).$$

Oseledets' Theorem ensures that for μ-almost every $x \in M$ there exists $k = k(x) \in \{1, \dots, d\}$, a filtration $\{0\} = F_x^0 \subset F_x^1 \subset \cdots \subset F_x^{k-1} \subset F_x^k = \mathcal{V}_x$, and numbers $\chi_1(x) < \cdots < \chi_k(x)$ such that

$$F_x^i = \{v \in \mathcal{V}_x : \lim_{t \to \infty} \frac{1}{t} \log \|A(x, t) \cdot v\| \le \chi_i(x)\}, \quad \text{and}$$
$$\lim_{t \to \infty} \frac{1}{t} \log \|A(x, t) \cdot v\| = \chi_i(x) \quad \text{for all } v \in F_x^i \setminus F_x^{i-1} \text{ and } i \in \{1, \dots, k\}.$$

The subspaces are invariant: $A(x, t) \cdot F_x^i = F_{\varphi_t(x)}^i$ and depend measurably on the base point $x \in X$. The function $k(x)$ and the Lyapunov exponents $\chi_i(x)$ are measurable functions, constant on orbits of the flow φ and so, if μ is ergodic, these are constant functions almost everywhere. Moreover there exists a decomposition $\mathcal{V}_x = E_x^1 \oplus \cdots \oplus E_x^k$ defined μ-almost everywhere satisfying $A(x, t) \cdot E_x^i = E_{\varphi_t(x)}^i$ and

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|A(x, t) \cdot v\| = \chi_i(x) = \chi_i, \quad 0 \ne v \in E_i(x), \quad i = 1, \dots, k. \tag{1.4}$$

As before the convergence in (1.4) is uniform over the set of unit vectors and the Oseledets subspaces E_x^i are related to the subspaces F_x^j as follows

$$F_x^j = \oplus_{i=1}^j E_x^i.$$

Now let a linear multiplicative cocycle $A(x, t)$ over a flow φ preserving a probability measure μ which is also ergodic with respect to φ be given. That is, we assume that every measurable subset A of X satisfying $\varphi_t(A) = A$ for all $t \in \mathbb{R}$ also satisfies $\mu(A) \cdot \mu(X \setminus A) = 0$. Then it is well known that there exists a denumerable subset Y of $\mathbb{R}$ such that the bijection φ_t is ergodic with respect to μ for all $t \in \mathbb{R} \setminus Y$; see e.g. [PS71].

Let us fix $0 < \tau \in \mathbb{R} \setminus Y$. Then $A^\tau : X \to GL(d, \mathbb{R}), x \in X \mapsto A(x, \tau)$ defines a cocycle over φ_τ since $A(x, 0) = Id$ and

$$A(x, n\tau) = A(\varphi_{(n-1)\tau}(x), \tau) \cdots A(\varphi_\tau(x), 1)A(x, \tau), \quad n \geq 0;$$
$$A(x, n\tau) = A(\varphi_{-n\tau}(x), -\tau) \cdots A(\varphi_{-\tau}(x), -\tau), \quad n < 0$$

where $A(\cdot, -\tau) = A(\cdot, \tau)^{-1}$ by the cocycle property.

We now note that by ergodicity, since the following limits exist, we have the equalities

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \|A(x, t)v\| = \chi_i = \lim_{n \to \pm\infty} \frac{1}{n} \log \|A(x, n\tau)v\|, \quad v \in E_i(x),\ \mu - \text{a.e. } x.$$

Therefore the Lyapunov exponents and Oseledets subspaces of the cocycle A^τ over the invertible transformation $f = \varphi_\tau$ coincide with those of the cocycle A over the flow φ.

We obtain the following result by a direct application of Theorem A to the cocycle A^τ and the transformation f. We say that a linear multiplicative cocycle $A : X \times \mathbb{R} \to X$ over a flow φ on the metric space X (which is a Lebesgue space with the Borel σ-algebra) is ν-Hölder if, for every $t \in \mathbb{R}$ there exists a constant $H = H_t > 0$ such that

$$\|A(x, t) - A(y, t)\| \leq Hd(x, y)^\nu, \quad x, y \in X.$$

Corollary B. *Let $\varphi : \mathbb{R} \times X \to X$ be a flow preserving an ergodic probability measure μ and such that $\varphi_t : X \to X$ is a Lipschitz transformation for all $t \in \mathbb{R}$. Let also $A : X \times \mathbb{R} \to GL(d, \mathbb{R})$ be a ν-Hölder linear multiplicative cocycle over φ such that $\sup_{t \in [-1,1]} \log^+ \|A(x, t)\|$ is a μ-integrable function, denote by $\chi_1 < \cdots < \chi_k$ the k distinct Lyapunov exponents associated to this cocycle, and by E^i_x the Oseledets subspaces corresponding to each χ_i, $i = 1, \ldots, k$, which form a splitting $\mathbb{R}^n = E^1_x \oplus \cdots \oplus E^k_x$ for μ-almost every $x \in X$.*

Then, for every $\varepsilon > 0$, there exists a compact subset Λ_ε of X, and constants $C = C(\Lambda_\varepsilon) > 0$, $\omega_i = \omega_i(\chi_1, \ldots, \chi_k) \in (0, 1), i = 1, \ldots, k$ and $\delta = \delta(\varepsilon, A, \Lambda_\varepsilon, \chi_1, \ldots, \chi_k) > 0$, such that

(1) *$\mu(\Lambda_\varepsilon) \geq 1 - \varepsilon$;*
(2) *$\operatorname{dist}(E^i_x, E^i_y) \leq C_\varepsilon d(x, y)^{\nu\omega_i}$ for all $x, y \in \Lambda_\varepsilon$ with $d(x, y) < \delta$.*

Remark 1.4. It is no restriction to assume that $\tau = 1$ above since we can always make a linear rescale of time, e.g., we may set $s = \tau \cdot t$ as a new time variable.

1.2.2. *The case of vector bundle automorphisms.* Now $A : \mathcal{V} \to \mathcal{V}$ is an automorphism of the d-dimensional vector bundle $\mathcal{V}$ covering $f : M \to M$. We assume that f is Lipschitz with Lipschitz constant $L > 0$, that M is a finite m-dimensional manifold and that A is *locally Hölder*. This means that

- we can find an at most denumerable locally finite open cover $(U_i, \psi_i)_{i \geq 1}$ of M together with trivializing charts of the bundle $\mathcal{V}$, that is, $\psi_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^d$ given by $w \in \mathcal{V}_x \mapsto (x, \psi_{i,x}(w))$ and $\psi_{i,x} : \mathcal{V}_x \to \mathbb{R}^d$ a linear bijection for all $x \in U_i$, whose overlaps satisfy

$$\psi_{j,i} := \psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^d \to (U_i \cap U_j) \times \mathbb{R}^d, \quad (x, v) \mapsto (x, \psi_{j,i,x}(v)) = (x, \psi_{i,x} \circ \psi_{j,x}^{-1}(v))$$

for $x \in U_i \cap U_j$ and $x \in U_i \cap U_j \mapsto \psi_{j,i,x} \in GL(d, \mathbb{R})$;

- we assume without loss of generality that the open sets of the cover of M are also domains of charts $\phi_i : U_i \to \mathbb{R}^m$ of M; so we can define on U_i a distance $d_i(x,y) = \|\phi_i(x) - \phi_i(y)\|_2$ for $x, y \in U_i$ and all possible i, where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^m$;
- since the initial open cover was locally finite, then given any point $x \in M$ we have at most finitely many chart domains $U_{i_1(x)}, \dots, U_{i_k(x)}$ containing x. Hence we can define

$$d_x(y) := \min\{d_{i_1(x)}(y,x), \dots, d_{i_k(x)}(y,x)\} \quad \text{for all} \quad y \in U_{i_1(x)} \cap \dots \cap U_{i_k(x)}.$$

  In this way, we write for $x \in M$ and $\xi > 0$

$$B(x,\xi) := \{y \in U_{i_1(x)} \cap \dots \cap U_{i_k(x)} : d_x(y) < \xi\}.$$

- likewise we define the norm $\|v\|_x := \sup\{\|\psi_{i_j(x),x}(v)\|_2 : j = 1, \dots, k\}$ for each $x \in M$, where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^d$. This is a measurable family of norms on the bundle $\mathcal{V}$ which is locally constant. Naturally we write $\|A(x)\|$ for $\sup\{\|A(x)v\|_{f(x)}/\|v\|_x : v \in \mathcal{V}_x, v \neq \vec{0}\}$ in what follows;
- for any given x and i, j such that $x \in U_j$, $f(x) \in U_i$ we assume that $A(x)$ is (H,ν)-Hölder as before on a neighborhood of x:

$$\|A(x) - A(y)\| \leq H \cdot d_x(y)^\nu, \quad \text{for all } x, y \in U_j \cap f^{-1}(U_i)$$

  where we allow $H = H_{j,i}$ to depend on i, j.

The proof of the previous Theorem A uses Lemma 2.2 which says that there exists $c_1 > 0$ such that the Hölder constant of an iterate A^n of an Hölder cocycle A is bounded by c_1^n for all $n \geq 1$. We cannot identify naturally all the subspaces $\mathcal{V}_x$ where $A(x)$ acts, so we assume that iterates A^n of the vector bundle automorphism have Hölder constant similarly bounded in the following way.

We take an ergodic f-invariant probability measure μ and the family $(U_i)_{i\geq 1}$ of open charts of the vector bundle, and assume that

- the partition $\mathcal{P}$ given by the intersection of the these sets is a $\mu - \mathrm{mod}0$ partition of M;
- there exists a full μ-measure subset Y of M such that the refined partitions $\mathcal{P}_n := \mathcal{P} \vee f^{-1}\mathcal{P} \vee \dots \vee f^{-n+1}\mathcal{P}$ *do not shrink faster than exponentially*, that is, they satisfy: there exists $0 < \xi < 1$ such that for all $x \in Y$ there is $c = c(x) > 0$ so that

$$B(x, c\xi^n) \subset \mathcal{P}_n(x) \quad \text{for all} \quad n \geq 1. \tag{1.5}$$

  In the above statement, as usually, $\mathcal{P}(x)$ denotes the atom of $\mathcal{P}$ which contains x; or $\mathcal{P}(x) = \emptyset$ if x is not contained in any atom (which can only happen for a zero μ-measure subset of points).
- the diameter of $\mathcal{P}_n$ can be made sufficiently small μ-almost everywhere, that is, there exists $\delta_0 = \delta_0(\varepsilon, A, \Lambda_\varepsilon, \chi_1, \dots, \chi_k) > 0$ and for $x \in Y$ there exists n such that $\operatorname{diam} \mathcal{P}_n(x) < \delta_0$.

- there exists $c_1 > 0$ such that for all $n \geq 1$, $x \in Y$ and $y \in \mathcal{P}_n(x)$

$$\|A^n(x) - A^n(y)\| \leq c_1^n d_x(y)^\nu. \tag{1.6}$$

Above we implicitly assume that the bound (1.6) does not depend on the choices of i, j.

We say that a vector bundle automorphism A satisfying the above properties is an *admissible* (c_1, ν)*-Hölder vector bundle automorphism with respect to* μ. With these notions we can now state our main result.

Theorem C. *Let A be an admissible (c_1, ν)-Hölder vector bundle automorphism with respect to the ergodic f-invariant probability measure μ such that both $\log \|A(x)\|$ and $\log \|A(x)^{-1}\|$ are μ-integrable. We denote by $\chi_1 < \cdots < \chi_k$ the k distinct Lyapunov exponents associated to this cocycle, and by F_x^i the filtration by subspaces of $\mathbb{R}^N$ corresponding to each χ_i, $i = 1, \ldots, k$, defined for μ-almost every $x \in M$.*

Then, for every $\varepsilon > 0$, there exists a compact subset Λ_ε of M, and constants $C = C(\Lambda_\varepsilon) > 0$, $\omega_i = \omega_i(\chi_1, \ldots, \chi_k) \in (0, 1)$, $i = 1, \ldots, k$ such that

(1) $\mu(\Lambda_\varepsilon) \geq 1 - \varepsilon$;
(2) *for all $x \in \Lambda_\varepsilon$ there exists $\delta = \delta(x) > 0$ such that for all $y \in B(x, \delta)$ we have* $\operatorname{dist}(F_x^i, F_y^i) \leq C_\varepsilon d_x(y)^{\nu\omega_i}$.

Moreover, if the map f is also invertible, then letting E_x^i be the Oseledets subspaces corresponding to each χ_i, $i = 1, \ldots, k$, which form a splitting $T_xM = E_x^1 \oplus \cdots \oplus E_x^k$ for μ-almost every $x \in M$, we also have $\delta = \delta(x) > 0$ defined for $x \in \Lambda_\varepsilon$ satisfying

(3) $\operatorname{dist}(E_x^i, E_y^i) \leq C_\varepsilon d_x(y)^{\nu\omega_i}$ *for all $x \in \Lambda_\varepsilon$, $y \in \Lambda_\varepsilon \cap B(x, \delta)$.*

Remark 1.5. Condition (1.5) can be easily obtained in many examples as follows. It is well-known that, if $\theta(x) := -\log \operatorname{dist}_x(\partial\mathcal{P}(x), x)$ is μ-integrable, then we have $(1/n) \cdot \theta \circ f^n \xrightarrow[n\to+\infty]{} 0, \mu-$a.e. Hence given $\alpha > 0$ for μ-a.e. x we can find $N(x) > 1$ such that $\operatorname{dist}_x(\partial\mathcal{P}(f^n(x)), f^n(x)) \geq e^{-\alpha n}$ for all $n \geq N(x)$. Therefore there exists $c(x) > 0$ such that $\operatorname{dist}_x(\partial\mathcal{P}(f^n(x)), f^n(x)) \geq c(x)e^{-\alpha n}$ for all $n \geq 1$.

If in addition we assume that f is locally Lipschitz, then we can find $0 < \xi < 1$ so that assumption (1.5) is satisfied.

Thus μ-integrability of θ together with a Lipschitz condition on f is enough to show that the atoms of the refined partitions $\mathcal{P}_n$ do not shrink at a rate faster than exponential.

1.2.3. *The case of a vector bundle automorphism covering a flow.* We may, analogously to the case of a vector bundle automorphism, consider a vector bundle automorphism covering a flow. We provide the relevant definitions and just note that the proof is a straightforward corollary of the previous Theorem C.

The measurable function $A : \mathcal{V} \times \mathbb{R} \to \mathcal{V}$ on a vector bundle $\mathcal{V}$ is a linear multiplicative cocycle covering a flow $\varphi : \mathbb{R} \times X \to X$ on the base manifold X of $\mathcal{V}$ if $A(x, t) : \mathcal{V}_x \to \mathcal{V}_{\varphi_t(x)}$ is a linear isomorphism satisfying the cocycle property

$$A(x, 0) = Id : \mathcal{V}_x \to \mathcal{V}_x \quad \text{and} \quad A(x, t + s) = A(\varphi_t(x), s) \cdot A(x, t), \quad x \in X, s, t \in \mathbb{R}.$$

Corollary D. *Let A be a linear multiplicative cocycle on a vector bundle covering a flow $\varphi : \mathbb{R} \times M \to M$ which preserves an ergodic probability measure μ such that* $\sup_{t\in[-1,1]} \log^+ \|A(x, t)\|$

is a μ-integrable function. We assume that the transformation $f = \varphi_1$ is Lipschitz; that μ is also f-ergodic and that $A^1 := A(\cdot, 1)$ is an admissible (c_1, ν)-Hölder vector bundle automorphism with respect to f and μ. We denote by $\chi_1 < \cdots < \chi_k$ the k distinct Lyapunov exponents associated to the cocycle A and by E^i_x the Oseledets subspaces corresponding to each χ_i, $i = 1, \ldots, k$, which form a splitting $T_xM = E^1_x \oplus \cdots \oplus E^k_x$ for μ-almost every $x \in M$.

Then, for every $\varepsilon > 0$, there exists a compact subset Λ_ε of M, and constants $C = C(\Lambda_\varepsilon) > 0$, $\omega_i = \omega_i(\chi_1, \ldots, \chi_k) \in (0, 1)$, $i = 1, \ldots, k$ such that

(1) *$\mu(\Lambda_\varepsilon) \geq 1 - \varepsilon$;*
(2) *for all $x \in \Lambda_\varepsilon$ there exists $\delta = \delta(x) > 0$ such that for all $y \in \Lambda_\varepsilon \cap B(x, \delta)$ we have $\operatorname{dist}(E^i_x, E^i_y) \leq C_\varepsilon d_x(y)^{\nu\omega_i}$.*

1.3. **The Kontsevich-Zorich cocycle.** Conditions (1.5) and (1.6) are non-trivial to verify in the case of the Kontsevich-Zorich cocycle (see [Zor06] for a survey). We thus deal with the corresponding theorem separately in Section 5 (see Theorem 5.6). The result is as follows.

Theorem E. *Let $\mathcal{M}$ be an affine invariant manifold and let μ be the corresponding ergodic $\mathrm{SL}_2\,\mathbb{R}$-invariant probability measure (see [EM13]). Let E be the Kontsevich-Zorich cocycle (or any of its tensor powers) and let $\{\lambda_i\}$ be its Lyapunov exponents, with Oseledets subspaces E^i.*

Then there exists $\nu_i > 0$ such that for any $\varepsilon > 0$ there exists a compact set K_ε with $\mu(K_\varepsilon) > 1 - \varepsilon$ and such that the spaces E^i vary ν_i-Hölder continuously on K_ε.

Similar in spirit to a recent result of Chaika-Eskin [CE15], the above result in fact generalizes to individual Teichmüller disks (see Theorem 5.7).

Theorem F. *Let $x \in \mathcal{H}(\kappa)$ be a flat surface in a stratum and let E be the Kontsevich-Zorich cocycle. If E is some tensor power of the KZ cocycle, let $k \geq 1$ be the smallest order of the tensor product which contains it.*

For $g \in \mathrm{SL}_2\,\mathbb{R}$ denote by $A(g, x) : E_x \to E_{gx}$ the matrix of the cocycle E. By Theorem 1.2 in [CE15] there exist $\lambda_1 > \cdots > \lambda_k$ such that for a.e. $\theta \in S^1$ we have

(1) *There exists a decomposition $E_x = \oplus_i E^i(\theta)_x$ with $E^i(\theta)_x$ measurably varying in θ*
(2) *We have (where g^θ_t is the geodesic flow in direction θ)*

$$\lim_{t \to \pm\infty} \frac{1}{t} \log \left\| A(g^\theta_t, x) v_i \right\| = \lambda_i \quad \forall v_i \in E^i(\theta)_x$$

Let $\nu_i := \frac{1}{2k} \min\left(\log \frac{\lambda_i}{\lambda_{i+1}}, \log \frac{\lambda_{i-1}}{\lambda_i}\right) - \varepsilon'$, for arbitrarily small ε'.

Then for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq S^1$ of Lebesgue measure at least $1 - \varepsilon$ such that the subspaces $E_i(\theta)$ vary ν_i-Hölder continuously for $\theta \in K_\varepsilon$.

2. Preliminary results and definitions

In what follows we adopt the convention that for a point $x \in M$ its iterates are denoted by $x_i := f^i(x)$ for $i \in \mathbb{Z}$ if f is invertible. For non-invertible f the meaning of x_i with $i < 0$ is given by the set of ith pre-images $x_i = (f^i)^{-1}(\{x\})$.

2.1. **Regular blocks for (possibly) non-invertible base transformation.** Using the definition of Lyapunov exponents and of the subspaces of the filtration F^i_x and of the Oseledets-Pesin Reduction Theorem in [BP07, Theorem 3.5.5], we can find sets Λ^ℓ_ε where the expansion/contraction rates are bounded, as follows.

For simplicity we assume that μ is f-ergodic and so function k, the exponents $\chi_1, \dots, \chi_k$ and the dimensions $\dim F^i_x - \dim F^{i-1}_x$ for $i = 1, \dots, k$ are constant on a full μ-measure subset. We denote by $\varepsilon_0 = \min\{\chi_{i+i} - \chi_i : 1 \le i < k\}$ the minimum gap in the Lyapunov spectrum.

For any given $\ell \in \mathbb{Z}^+$ and $0 < \varepsilon < \varepsilon_0/10$ we define the *regular block* Λ^ℓ_ε as the set of points $x \in M$ such that for $i = 1, \dots, k$ and all $m, n \ge 0$

$$\|A^n(x_m)v_i\| \le \ell e^{(\chi_i(x)+\varepsilon)n+\varepsilon m}\|v_i\|, \quad \forall v_i \in A^m(x)F^i_x;$$
$$\ell^{-1}e^{(\chi_i(x)-\varepsilon)n+\varepsilon m}\|v_i\| \le \|A^n(x_m)v_i\| \le \ell e^{(\chi_i(x)+\varepsilon)n+\varepsilon m}\|v_i\|, \quad \forall v_i \in \left(A^m(x)F^{i-1}_x\right)^\perp \cap A^m(x)F^i_x,$$

where $x_m = f^m(x)$ and $(\cdot)^\perp$ denotes the orthogonal complement with respect to the inner product that defines $\|\cdot\|$.

2.2. **Regular blocks for invertible base transformation.** Again using the definition of Lyapunov exponents and Oseledets subspaces, we can find sets Λ^ℓ_ε where the expansion/contraction rates are uniform and the angles between the subspaces are bounded away from zero, as follows.

We assume that μ is f-ergodic and so the function k, the exponents $\chi_1, \dots, \chi_k$ and the dimensions $\dim E^i_x$ for $i = 1, \dots, k$ are constant on a full μ-measure subset.

We continue to write $\varepsilon_0 > 0$ for the minimum gap in the Lyapunov spectrum and for $\ell \in \mathbb{Z}^+$ and $0 < \varepsilon < \varepsilon_0/10$ we define Λ^ℓ_ε as the set of points $x \in M$ such that for $i = 1, \dots, k$ and for all $m \in \mathbb{Z}$, writing $x_m = f^m(x)$ we have

$$\ell^{-1}e^{(\chi_i(x)-\varepsilon)n-\varepsilon|m|}\|v_i\| \le \|A^n(x_m)v_i\| \le \ell e^{(\chi_i(x)+\varepsilon)n+\varepsilon|m|}\|v_i\|, \quad \forall v_i \in A^m(x)E_i(x), \quad \forall n \ge 0;$$
$$\ell^{-1}e^{(\chi_i(x)+\varepsilon)n-\varepsilon|m|}\|v_i\| \le \|A^n(x_m)v_i\| \le \ell e^{(\chi_i(x)-\varepsilon)n+\varepsilon|m|}\|v_i\|, \quad \forall v_i \in A^m(x)E_i(x), \quad \forall n \le 0; \text{ and}$$
$$\cos \operatorname{angle}\left(\bigoplus_{i\in I} E^i_{f^n(x)}, \bigoplus_{j\notin I} E^j_{f^n(x)}\right) \le 1 - \frac{e^{-\varepsilon|n|}}{\ell} \text{ for any } I \subset \{1, \dots, k\} \text{ and for all } n \in \mathbb{Z}.$$

The control of the norm along the subspaces F^i_x on Λ^ℓ_ε implies easily that there exists $L > 0$ such that

$$\|A^n(x_m)\| \le \ell L^{|n|}e^{\varepsilon m} \quad \text{for all } m, n \in \mathbb{Z} \text{ and every } x \in \Lambda^\ell_\varepsilon \tag{2.1}$$

with $L = e^{2(\chi_k - \chi_1)} \le \exp\left(4\int \log\|A(x)\|\, d\mu(x)\right)$.

From the Theorem of Oseledets we have that for any given small $\varepsilon > 0$ we can find $\ell_0 \in \mathbb{Z}^+$ such that $\mu(\Lambda^\ell_\varepsilon) > 0$ for $\ell \ge \ell_0$ and $\cup_{\ell\ge\ell_0}\Lambda^\ell_\varepsilon$ has full μ-measure in M. We note that to have $\mu(\Lambda_\varepsilon) > 1 - \varepsilon$ with smaller $\varepsilon > 0$ we must increase the value of $\ell \in \mathbb{Z}^+$.

2.3. **Consequences of the angle control on regular blocks for invertible base.** We remark that the angle condition has the following consequence: given a pair of complementary subspaces E, F with angle $(E, F) \leq 1 - 1/\ell$ for some $\ell \in \mathbb{Z}^+$, then for any vector $v + w$ with $v \in E, w \in F$ we have $||v|| \leq \ell||v + w||$ and $||w|| \leq \ell||v + w||$.

Indeed, we consider $E \oplus F$ with the given norm $||\cdot||$ and $E \times F$ with the norm $|(v, w)| := (||v||^2 + ||w||^2)^{1/2}$ for $(v, w) \in E \times F$. Then we can write for any $v \in E$ and $w \in F$ with $v + w \neq 0$ that

$$v = \lambda v_0 \quad \text{and} \quad w = \mu w_0 \text{ with } \lambda, \mu \in \mathbb{R}^+, ||v_0|| = 1 = ||w_0||; \quad \text{and}$$

$$\frac{||v + w||^2}{|v + w|^2} = \frac{||v_0 + \mu w_0/\lambda||^2}{|v_0 + \mu w_0/\lambda|^2} = \frac{1 + 2(\mu/\lambda) < v_0, w_0 > +(\mu/\lambda)^2}{1 + (\mu/\lambda)^2} = 1 + \frac{2\lambda\mu}{\lambda^2 + \mu^2} < v_0, w_0 >,$$

where $< v_0, w_0 > \in [-1, 1]$ and $0 \leq 2\lambda\mu/(\lambda^2 + \mu^2) \leq 1$. Hence we get

$$||v|| = |v| \leq |v + w| = \left(1 + \frac{2\lambda\mu}{\lambda^2 + \mu^2} < v_0, w_0 >\right)^{-1} ||v + w|| \leq \frac{1}{1 - \text{angle}\,(E, F)} ||v + w|| \leq \ell||v + w||.$$

2.4. **Hölder estimates for exponentially expanded/contracted subspaces.** Now we need the following technical results whose proofs we postpone.

We compare the definition given in (1.3) to an alternative description of the distance between subspaces E, F of $\mathbb{R}^d$. We can obtain a linear map $L : E \to E^\perp$ such that its graph $\{u + Lu : u \in E\}$ equals F and obtain that, on the one hand

$$\sup_{||u+Lu||=1, u\in E} \text{dist}(u + Lu, E) = \sup_{||u||^2+||Lu||^2=1, u\in E} \min_{w\in E} ||u + Lu - w|| = \sup_{||u||^2+||Lu||^2=1, u\in E} ||Lu||$$

$$= \sup\{||L|| \cdot ||u_0|| : u_0 \in E \text{ s.t. } ||u_0||^2 + ||Lu_0||^2 = 1 \text{ and } ||L|| = \frac{||Lu_0||}{||u_0||}\} = \frac{||L||}{\sqrt{1 + ||L||^2}}.$$

On the other hand, for $u_1 \in E$ such that $||u_1|| = 1$ and $||Lu_1|| = ||L||$

$$\sup_{||u||=1, u\in E} \text{dist}(u, F) = \sup_{||u||=1, u\in E} \min_{v\in E} ||u - (v + Lv)|| \leq ||u_1 - (u_1 - Lu_1)|| = ||Lu_1|| \leq ||L||.$$

Therefore from the definition of dist(E, F) we get

$$\frac{||L||}{\sqrt{1 + ||L||^2}} \leq \text{dist}(E, F) \leq ||L|| \tag{2.2}$$

and we may estimate (or indeed define) the distance between E and F by $||L||$.

The following simple lemma enables us to provide a rough control of the distance between subspaces which are exponentially expanded/contracted by a pair of sequences of linear maps, such that the norm of the difference between these maps is a sequence that grows at most exponentially fast.

Lemma 2.1. *(from* [BP07, Lemma 5.3.3]*) Let $(A_n)_{n\geq 1}, (B_n)_{n\geq 1}$ be two sequences of real $N \times N$ matrices such that for some $0 < \lambda < \mu$ and $C \geq 1$ there exist subspaces E, E', F, F' of $\mathbb{R}^N$ satisfying $\mathbb{R}^N = E \oplus E' = F \oplus F'$ with $d > 0$ such that*

$$u = v + w, v \in E, w \in E' \text{ or } v \in F, w \in F' \implies \max\{||v||, ||w||\} \leq d||u||, \tag{2.3}$$

and, for some fixed $n > 0$

- $\|A_n u\| \le C\lambda^n \|u\|$ *for* $u \in E$*; and* $C^{-1}\mu^n \|v\| \le \|A_n v\|$ *for* $v \in E'$*;*
- $\|B_n u\| \le C\lambda^n \|u\|$ *for* $u \in F$*; and* $C^{-1}\mu^n \|v\| \le \|B_n v\|$ *for* $v \in F'$*.*

Then for every pair $(\delta, a) \in (0,1] \times [\lambda, +\infty)$ *satisfying*

$$\left(\frac{\lambda}{a}\right)^{n+1} < \delta \quad \text{and} \quad \|A_n - B_n\| \le \delta a^n \tag{2.4}$$

we get $\operatorname{dist}(E,F) \le (2+d)C^2 \frac{\mu}{\lambda} \delta^{\log(\mu/\lambda)/\log(a/\lambda)}$.

In the previous lemma we assumed condition (2.3) instead of taking $E' = E^\perp$ and $F' = F^\perp$ as in [BP07, Lemma 5.3.3].

Now we state a result which enables us to show that the iterated cocycle A^n, of a ν-Hölder cocycle A over a Lipschitz base map f, is also ν-Hölder for every $n \ge 1$. Moreover the Hölder constant of A^n grows at most exponentially fast with n.

Lemma 2.2. *Let us assume that* $A : M \to GL(d, \mathbb{R})$ *is* ν*-Hölder with constant* $c_0 = c_0(A, \nu)$ *and there exists* $L > 1$ *such that* $f : M \circlearrowleft$ *is Lipschitz with constant* L *and* $\|A^n(x)\| \le L^{|n|}$ *for all* x *is some fixed compact* $\Lambda \subset M$ *and* $n \in \mathbb{Z}^+$*. Then for* $c_1 = \max\{e^\varepsilon, L^{1+\nu}, 1 + c_0\}$ *we have* $\|A^n(x) - A^n(y)\| \le c_1^n \operatorname{dist}(x,y)^\nu$ *for all* $x, y \in \Lambda$ *and* $n \in \mathbb{Z}^+$*.*

The previous lemmas combined show (see e.g. [BP07, Theorem 5.3.1]) that the splitting of the tangent space into the directions with positive and negative Lyapunov exponents depends Hölder continuously on the base point over every "regular block" associated to an invariant probability measure with non-zero Lyapunov exponents.

We want to extend this conclusion to all the Oseledets subspaces for any ergodic invariant probability. For that we need to study splittings of the tangent bundle into three subbundles exhibiting *distinct exponential growth* under the action of a sequence of linear operators, as we state below. This allows us to analyze the behavior of all Oseledets subspaces by associating them into a splitting with three subbundles in different combinations. Otherwise Lemma 2.1 would only allow the study of the flags $\{0\} = F_x^0 \subset F_x^1 \subset \cdots \subset F_x^{k-1} \subset F_x^k = \mathbb{R}^d$ for μ-a.e. $x \in M$.

The condition of distinct exponential growth rates encompasses the case of zero Lyapunov exponents in one statement.

The main lemma we need is the following extension of Lemma 2.1.

Lemma 2.3. *Let* $(A_n)_{n\in\mathbb{Z}}, (B_n)_{n\in\mathbb{Z}}$ *be two bi-infinite sequences of real* $N \times N$ *invertible matrices admitting* $0 < \lambda_1 < \lambda_2 < \mu_1 < \mu_2 < \sigma_1 < \sigma_2$ *and* $C, d > 0$ *such that there exist subspaces* E^*, F^*, G^* *of* $\mathbb{R}^N$ *satisfying for all* $n > 0$

- $\mathbb{R}^N = E^* \oplus F^* \oplus G^*$ *for* $* = A, B$ *and*
  - *for* $u \in E^A$*:* $C\lambda_1^n \|u\| \le \|A_n u\| \le C\lambda_2^n \|u\|$ *and* $C\lambda_2^{-n}\|u\| \le \|A_{-n}u\| \le C\lambda_1^{-n}\|u\|$*;*
  - *for* $v \in F^A$*:* $C\mu_1^n \|v\| \le \|A_n v\| \le C\mu_2^n \|v\|$ *and* $C\mu_2^{-n}\|v\| \le \|A_{-n}v\| \le C\mu_1^{-n}\|v\|$*;*
  - *for* $w \in G^A$*:* $C\sigma_1^n \|w\| \le \|A_n w\| \le C\sigma_2^n \|w\|$ *and* $C\sigma_2^{-n}\|w\| \le \|A_{-n}w\| \le C\sigma_1^{-n}\|w\|$*.*
  - *for* $u \in E^B$*:* $C\lambda_1^n \|u\| \le \|B_n u\| \le C\lambda_2^n \|u\|$ *and* $C\lambda_2^{-n}\|u\| \le \|B_{-n}u\| \le C\lambda_1^{-n}\|u\|$*;*
  - *for* $v \in F^B$*:* $C\mu_1^n \|v\| \le \|B_n v\| \le C\mu_2^n \|v\|$ *and* $C\mu_2^{-n}\|v\| \le \|B_{-n}v\| \le C\mu_1^{-n}\|v\|$*;*

– *for* $w \in G^B$: $C\sigma_1^n\|w\| \le \|B_n w\| \le C\sigma_2^n\|w\|$ *and* $C\sigma_2^{-n}\|w\| \le \|B_{-n}w\| \le C\sigma_1^{-n}\|w\|$.

- *for* $* = A, B$, *if* $u = v + w$ *and either*

$$v \in E^*, v \in F^* \oplus G^* \quad or \quad v \in F^*, w \in G^* \quad or$$
$$v \in E^* \oplus F^*, w \in G^* \quad or \quad v \in E^*, w \in F^*,$$

then $\|v\| \le d\|u\|$.

Then there exists $a > \lambda_2 + 1/\lambda_2 + \sigma_1$ *and* $\delta_0 = \delta_0(a, C, \mu_1/\lambda_2, \sigma_1/\mu_2) \in (0,1)$ *such that if, for some* $n > 0$, *we have*

$$\|A_n\| \le a^n \quad and \quad \|A_{-n}\| \le a^n \tag{2.5}$$

and for some $0 < \delta < \delta_0$, *we also have*

$$\|A_n - B_n\| \le \delta a^n \quad and \quad \|A_{-n} - B_{-n}\| \le \delta a^n \tag{2.6}$$

then the following relations are true:

$$\mathrm{dist}(E^A, E^B) \le (2+d)C^2\frac{\mu_1}{\lambda_2}\delta^\alpha, \qquad \mathrm{dist}(F^A, F^B) \le \frac{9}{2}(2+3d)^{1+\eta}C^{2(1+\eta)}\frac{\sigma_1\mu_1^\eta}{\mu_2\lambda_2^\eta}\delta^\beta \quad and$$
$$\mathrm{dist}(G^A, G^B) \le (2+d)C^2\frac{\sigma_1}{\mu_2}\delta^\gamma,$$

where $\alpha := \log(\mu_1/\lambda_2)/\log(a/\lambda_2)$, $\eta = \log(\sigma_1/\mu_2)/\log(a/\mu_2)$, $\gamma := \log(\sigma_1/\mu_2)/\log(a\sigma_1)$ *and also* $\beta = [\log(\mu_1/\lambda_2)\log(\sigma_1/\mu_2)]/[\log(a\mu_1)\log(a/\mu_2)]$.

We remark that α, β and γ are positive and smaller than 1 by the choice of a and the order relations between the rates of expansion/contraction.

3. Hölder continuity of the Oseledets splitting on regular blocks

Here we prove the main theorem by combining the lemmas from Section 2 and applying them to regular sets of a (not necessarily invertible) co-cycle A over f. After that we prove the invertible case of the main theorem. Then we prove the lemmas in Section 4.

We assume the first two conditions on Lemma 2.2, i.e. $f : M \circlearrowleft$ is Lipschitz and $A : M \to GL(\mathbb{R}, d)$ is (c_0, ν)-Hölder; and check the other condition, that is, that $\|A^n(x)\|$ grows at most exponentially for $n \ge 1$. We continue under the assumption that μ is f-ergodic.

3.1. **The case of non-invertible base.** For $\ell \ge \ell_0$ and $0 < \varepsilon < \varepsilon_0$ we set $\Lambda_\varepsilon = \Lambda_\varepsilon^\ell$ such that $\mu(\Lambda_\varepsilon) > 1 - \varepsilon$, from Section 2.1, we take $x, y \in \Lambda_\varepsilon$ and for each $1 < i < k$, $n \in \mathbb{Z}^+$ we define

$$A_n = A^n(x), \quad B_n = A^n(y) \quad \text{and} \quad L = e^{2(\chi_k - \chi_1)} \quad \text{and} \quad C = \ell;$$
$$\lambda = e^{\chi_i + \varepsilon}, \mu = e^{\chi_{i+1} - \varepsilon}, \sigma = e^{\chi_k + \varepsilon} \quad \text{and} \quad d = 1;$$
$$E = F_x^i, \quad E' = \left(F_x^i\right)^\perp \quad \text{and} \quad F = F_y^i, \quad F' = \left(F_y^i\right)^\perp.$$

From the definition of regular block in the possibly non-invertible case, in Section 2.1, we have guaranteed the upper bound on the exponential growth of $\|A^n(x)\|$ for $x \in \Lambda_\varepsilon^\ell$ and $n \geq 0$.

The value $d = 1$ comes from the choice of E', F' as orthogonal complements with respect to the inner product that defines $\|\cdot\|$.

From Lemma 2.2 we can find a constant $a > c_1$ big enough such that the condition (2.4) in Lemma 2.1 is true for the choices of sequences A_n, B_n and for some $n \in \mathbb{Z}^+$ with $0 < \delta = \operatorname{dist}(x,y)^\nu < \min\{\chi_1/c_1\}$. Here χ_1 is the smallest Lyapunov exponent and c_1 is given by Lemma 2.2.

Indeed, for δ and λ chosen as above for any given $i = 2, \dots, k-1$, since $\chi_1 \leq \lambda$ we can certainly find $n > 0$ such that $(\lambda/a)^{n+1} < \delta$ and then proceed using the statements of Lemmas 2.1 and 2.2. The other assumptions on Lemma 2.1 are true by the definition of hyperbolic block in Section 2.1 together with the choices of subspaces above.

We can then apply Lemma 2.1 to conclude that for $i = 1, \dots, k-1$

$$\operatorname{dist}(F_x^i, F_y^i) \leq 3\ell^2 e^{\eta_i} \operatorname{dist}(x,y)^{\nu\omega_i} \quad \text{where} \quad \eta_i = \chi_{i+1} - \chi_i - 2\varepsilon \quad \text{and} \quad \omega_i = \frac{\eta_i}{\log a - \chi_i - \varepsilon}.$$

This completes the proof of Theorem A in the non-invertible case once we set $C_\varepsilon = \ell^2 \max_{i=1,\dots,k-1}\{e^{\eta_i}\}$.

3.2. **The case of invertible base.** Again for $\ell \geq \ell_0$ and $0 < \varepsilon < \varepsilon_0$ we set $\Lambda_\varepsilon = \Lambda_\varepsilon^\ell$ such that $\mu(\Lambda_\varepsilon) > 1 - \varepsilon$, from Section 2.2, we take $x, y \in \Lambda_\varepsilon$ and for each $1 < i < k$, $n \in \mathbb{Z}$ we define

$$\begin{aligned}
&A_n = A^n(x), \quad B_n = A^n(y) \quad \text{and} \quad L = e^{2(\chi_k - \chi_1)} \quad \text{and} \quad C = d = \ell;\\
&\lambda_1 = e^{\chi_1 - \varepsilon}, \lambda_2 = e^{\chi_{i-1} + \varepsilon}, \mu_1 = e^{\chi_i - \varepsilon}, \mu_2 = e^{\chi_i + \varepsilon}, \sigma_1 = e^{\chi_{i+1} - \varepsilon}, \sigma_2 = e^{\chi_k + \varepsilon};\\
&E^A = F_x^{i-1} = \oplus_{j=1}^{i-1} E_x^j, \quad F^A = E_x^i \quad \text{and} \quad G^A = \oplus_{j=i+1}^{k} E_x^i\\
&E^B = F_y^{i-1} = \oplus_{j=1}^{i-1} E_y^j, \quad F^B = E_y^i \quad \text{and} \quad G^B = \oplus_{j=i+1}^{k} E_y^i.
\end{aligned}$$

From the definition of regular block, in Section 2.2, we know that the assumption on exponential growth of $\|A^n(x)\|$ for $x \in \Lambda_\varepsilon^\ell$ and $n \in \mathbb{Z}$ is guaranteed; see the observation in (2.1). So from Lemma 2.2 we can find a constant $a > 0$ big enough such that condition (2.5) and item (1) in the statement of Lemma 2.3 are true for the choices of sequences A_n, B_n and subspaces above with $0 < \delta = \operatorname{dist}(x,y)^\nu < \min\{\chi_1/c_1, \delta_0\}$ and some $n \in \mathbb{Z}^+$. By the definition of the regular block Λ_ε^ℓ, and the choices of constants, all the conditions of item (2) and (3) are satisfied also. The statement of Lemma 2.3 ensures then that

$$\operatorname{dist}(E_x^i, E_y^i) \leq C_\ell e^{\chi_{i+1} - \chi_i - 2\varepsilon + \eta(\chi_i - \chi_{i-1} - 2\varepsilon)} d(x,y)^{\nu\beta} \quad \text{where} \quad \eta = \frac{\chi_{i+1} - \chi_i - 2\varepsilon}{\log a - \chi_i - \varepsilon},$$

$\beta = \eta \frac{\chi_i - \chi_{i-1} - 2\varepsilon}{\chi_i - \varepsilon + \log a}$ and $C_\ell \to \infty$ monotonically as $\ell \to +\infty$.

For the subbundle E^1 (with smallest Lyapunov exponent) we take $i = 2$ as above and consider the estimate for $\mathrm{dist}(E^A, E^B)$ from Lemma 2.3 to obtain

$$\mathrm{dist}(E^1_x, E^1_y) \le C_\ell e^{\chi_2-\chi_1-2\varepsilon} d(x,y)^{\nu\alpha} \quad \text{where} \quad \alpha = \frac{\chi_2-\chi_1-2\varepsilon}{\log a - \chi_1 - \varepsilon}.$$

Finally for the subbundle E^k (with largest Lyapunov exponent) we take $i = k-1$ as above and consider the estimate for $\mathrm{dist}(G^A, G^B)$ from Lemma 2.3 to obtain

$$\mathrm{dist}(E^k_x, E^k_y) \le C_\ell e^{\chi_k-\chi_{k-1}-2\varepsilon} d(x,y)^{\nu\gamma} \quad \text{where} \quad \gamma = \frac{\chi_k-\chi_{k-1}-2\varepsilon}{\log a + \chi_k - \varepsilon}.$$

This shows that the dependence of the Oseledets directions on the base point at every regular block is Hölder, with Hölder exponent essentially dependent on the minimum gap ε_0 of the Lyapunov spectrum, and Hölder constant essentially dependent on the choice of the regular block. The proof of Theorem A is complete once we set C_ε to equal the maximum of the factors multiplying $\mathrm{dist}(x,y)$ in the above expressions.

3.3. **The case of vector bundle automorphisms.** Now $A : \mathcal{V} \to \mathcal{V}$ is an automorphism of the d-dimensional vector bundle $\mathcal{V}$ covering the map $f : M \to M$ on the m- dimensional manifold M. We assume that f admits an ergodic f-invariant probability such that A is admissible with respect to μ.

We again start by choosing a regular block $\Lambda_\varepsilon = \Lambda^\ell_\varepsilon$ for some $0 < \varepsilon < \varepsilon_0$ and $\ell \ge \ell_0$, such that $\mu(\Lambda_\varepsilon) > 1 - \varepsilon$. We recall that for μ-a.e. x we have defined in (1.5) a positive function $c(x)$ relating the distance between x and $\mathcal{P}_n(x)$. For μ-a.e. x let $N_1 = N(x) \in \mathbb{Z}^+$ be the first integer so that $c(x) > 1/2^N$. Then we have

$$n > N_1 \implies y \in \mathcal{P}_n(x) \setminus \mathcal{P}_{n+1}(x) \quad \text{satisfies} \quad \mathrm{dist}_x(y) \ge c(x)\xi^{n+1} > \left(\frac{\xi}{2}\right)^{n+1}. \tag{3.1}$$

Let $N_2 = N_2(x) \ge N_1$ be such that $\mathrm{diam}\, \mathcal{P}_n(x) < \chi_1/c_1$ for all $n > N_2$.

3.3.1. *The case of non-invertible base.* With the previous choices, for each $1 < i < k$ and $n \in \mathbb{Z}^+$ we define $x_n = f^n(x)$ and $y_n = f^n(y)$ for $y \in \Lambda_\varepsilon \cap \mathcal{P}_{N_2}(x)$

$$A_n = \psi_{j_n,x_n} A^n(x) \psi^{-1}_{i,x}, \text{ with } x_n \in U_{j_n}, x \in U_i; \tag{3.2}$$

$$B_n = \psi_{r_n,y_n} A^n(y) \psi^{-1}_{i,y}, \text{ with } y_n \in U_{r_n}, x \in U_i; \tag{3.3}$$

and then choose $L, C, \lambda, \mu, \sigma, d, E, E', F, F'$ and also $\delta = d_x(y)^\nu$ as in Section 3.1.

There exists $n \ge N_2$ such that $y \in \mathcal{P}_n(x) \setminus \mathcal{P}_{n+1}(x)$, thus if we choose $a > c_1$ big enough, since $\lambda \ge \chi_1$, we have $(\lambda/a)^{n+1} < (\xi/2)^{(n+1)/\nu} < d_x(y)$. For this it is enough to take $a > c_1 + \chi_k \cdot (\xi/2)^{-\nu}$. This, together with the definition of hyperbolic block, ensures that the conditions of Lemma 2.1 are verified for $y \in \Lambda_\varepsilon \cap \mathcal{P}_{N_2}(x)$. Finally we note that, from (1.5), every point $y \in \Lambda_\varepsilon \cap B(x, c(x)\xi^{N_2})$ belongs to $\Lambda_\varepsilon \cap \mathcal{P}_{N_2}(x)$, so that we arrive at the same conclusion as in Section 3.1.

This completes the proof of Theorem C in the non-invertible case, if we set $\delta(x) := c(x)\xi^{N_2}$.

3.3.2. *The case of invertible base.* We again use the previous choices of $N_2 = N_2(x)$ and, for each $1 < i < k$, we set $L, C, \lambda_1, \lambda_2, \mu_1, \mu_2, \sigma_1, \sigma_2$ as in Section 3.2. This provides a value $\delta_0 = \delta_0(i)$ from Lemma 2.3. We now choose $\kappa_i = \kappa_i(x) \geq N_2$ so that $\operatorname{diam} \mathcal{P}_n(x) < \delta_0(i)$ for all $n > \kappa_i$.

For each $n \in \mathbb{Z}$, we define $x_n = f^n(x)$ and $y_n = f^n(y)$ for $y \in \Lambda_\varepsilon \cap \mathcal{P}_{\kappa_i}(x)$, and we make the same definitions as in (3.2) and (3.3).

We then take $E^A, F^A, G^A, E^B, F^B, G^B$ as in Section 3.2. From the definition of hyperbolic block in the invertible case we can verify the conditions of Lemma 2.3 with $\delta = d_x(y)^\nu$ for some positive integer n. For that we have to choose a sufficiently large as explained above.

So we obtain the same conclusions as in Section 3.2 for each $1 < i < k$ and also for $i = 1$ and $i = k$. We then set $N = N(x) = \max\{\kappa_1, \dots, \kappa_k\}$ and obtain the conclusion of Theorem C for $y \in \Lambda_\varepsilon \cap \mathcal{P}_n(x)$ with $n \geq N$.

To conclude, since $\lambda_\varepsilon \cap B(x, c(x)\xi^N)$ is contained in $\lambda_\varepsilon \cap \mathcal{P}_N(x)$, setting $\delta(x) = c(x)\xi^N$ completes the proof of Theorem C.

4. Lemmata

Now we present a proof of the technical lemmas from Section 2.

Proof of Lemma 2.1. The proof follows the one given in [BP07, Lemma 5.3.3] closely.

Let us fix $n > 0$ as in the statement. We define the following cones $Q = \{u \in \mathbb{R}^N : \|A_n u\| \leq 2C\lambda^n \|u\|\}$ and $R = \{u \in \mathbb{R}^N : \|B_n u\| \leq 2C\lambda^n \|u\|\}$. We decompose $u \in \mathbb{R}^N$ into $v + w$ with $v \in E$ and $w \in E'$. If $u \in Q$, then since $\|v\| \leq d\|u\|$ by assumption we see that

$$2C\lambda^n \|u\| \geq \|A_n u\| = \|A_n(v + w)\| \geq C^{-1}\mu^n \|w\| - C\lambda^n \|v\| \geq C^{-1}\mu^n \|w\| - C\lambda^n d\|u\|$$

implies $\|w\| \leq (2 + d)C^2 \left(\frac{\lambda}{\mu}\right)^n \|u\|$. Therefore $\operatorname{dist}(u, E) \leq (2 + d)C^2 \left(\frac{\lambda}{\mu}\right)^n \|u\|$ for all $u \in Q$.

Now for $a > \lambda$ and $\delta \in (0, 1]$ such that (2.4) is true we set $\gamma := \lambda/a \in (0, 1)$ and observe that if $u \in F$ then

$$\begin{aligned} \|A_n u\| &\leq \|B_n u\| + \|A_n - B_n\| \|u\| \leq C\lambda^n \|u\| + \delta a^n \|u\| \\ &\leq (C\lambda^n + (\gamma a)^n)\|u\| \leq 2C\lambda^n \|u\| \end{aligned}$$

and so $u \in Q$, that is, $F \subset Q$. Symmetrically we get $E \subset R$. Hence using again (2.4) we conclude that $\operatorname{dist}(E, F) \leq (2 + d)C^2(\lambda/\mu)^n \leq (2 + d)C^2 \frac{\lambda}{\mu} \delta^{\log(\lambda/\mu)/\log(\lambda/a)}$.

□

Proof of Lemma 2.2. We follow [Mañ87, proof of Lemma 13.5] (see also [BP07, Lemma 5.3.4]) and argue by induction on $n \in \mathbb{N}$ and write $x_n = f^n(x_0)$ and $y_n = f^n(y_0)$ for $n \geq 0$ and $x_0, y_0 \in \Lambda$. For $n = 1$ we have $\|A(x_0) - A(y_0)\| \leq c_0 \operatorname{dist}(x_0, y_0)^\nu$ by the Hölder assumption, for some $c_0 > 0$.

Let us assume that we can find $c > c_0$ as in the statement of the lemma for $k = 1, \dots, n$ and let us see what we need to extend the property for $k = n + 1$. Since we have a cocycle

with values in $GL(d,\mathbb{R})$, we can write

$$A^{n+1}(x_0) - A^{n+1}(y_0) = A(x_n)\cdot(A^n(x_0) - A^n(y_0)) + (A(x_n) - A(y_n))\cdot A^n(y_0)$$

adding and subtracting $A(x_n)\cdot A^n(y_0)$. We recall that $\log L > 0$ is assumed to be an upper bound for $\{n^{-1}\log\|A^n(z)\| : z \in M, n \in \mathbb{N}\}$ and write, using the induction assumption

$$\begin{aligned}\|A^{n+1}(x_0) - A^{n+1}(y_0)\| &\le \|A(x_n)\|\cdot\|A^n(x_0) - A^n(y_0)\| + \|A^n(y_0)\|\cdot\|A(x_n) - A(y_n)\| \\ &\le e^{\varepsilon n}C^n \operatorname{dist}(x_0,y_0)^\nu + L^n c_0 \operatorname{dist}(x_n,y_n)^\nu.\end{aligned}$$

Now since L is also a Lipschitz constant of f we get $\operatorname{dist}(x_n,y_n) \le L^n \operatorname{dist}(x_0,y_0)$ and so we can bound the previous expression

$$\le [(e^\varepsilon C)^n + L^n c_0 L^{n\nu}]\operatorname{dist}(x_0,y_0)^\nu = [(e^\varepsilon C)^n + c_0 L^{n(1+\nu)}]\operatorname{dist}(x_0,y_0)^\nu.$$

To complete the inductive step all we need is that

$$C^{n+1} \ge (Ce^\varepsilon)^n + c_0 L^{n(1+\nu)} \quad \text{that is} \quad C \ge \left(\frac{e^\varepsilon}{C}\right)^n + c_0\left(\frac{L^{1+\nu}}{C}\right)^n \quad \text{for all} \quad n \ge 0$$

which can easily be achieved by taking a sufficiently large $C \ge c_1 > c_0 > 0$ as in the statement of the lemma. □

Proof of Lemma 2.3. We use Lemma 2.1 applied to the splitting $E^* \oplus (F^* \oplus G^*)$ to deduce that, on the one hand

$$\operatorname{dist}(E^A,E^B) \le (2+d)C^2\frac{\mu_1}{\lambda_2}\delta^{\log(\mu_1/\lambda_2)/\log(a/\lambda_2)} = (2+d)C^2\frac{\mu_1}{\lambda_2}\delta^\alpha \tag{4.1}$$

and, on the other hand

$$\operatorname{dist}(V^A,V^B) \le (2+d)C^2\frac{\lambda_2^{-1}}{\mu_1^{-1}}\delta^{\log(\lambda_2^{-1}/\mu_1^{-1})/\log(a/\mu_1^{-1})} = \widetilde{C}\frac{\mu_1}{\lambda_2}\delta^{\log(\mu_1/\lambda_2)/\log(a\mu_1)} = \widetilde{C}\frac{\mu_1}{\lambda_2}\delta^\omega, \tag{4.2}$$

where $\widetilde{C} = (2+d)C^2$, $V^A := F^A \oplus G^A$ and $V^B := F^B \oplus G^B$ and $\omega = \log(\mu_1/\lambda_2)/\log(a\mu_1)$ for simplicity. The bound for the distance between V^A, V^B is deduced using Lemma 2.1 applied to the sequences $(A_{-n})_{n\ge0}$ and $(B_{-n})_{n\ge0}$. The hypothesis in the statement of Lemma 2.3 exactly provide the necessary conditions to apply Lemma 2.1 to these sequences of matrices.

We can also apply Lemma 2.1 for the sequences $(A_{-n})_{n\ge0}$ and $(B_{-n})_{n\ge0}$ but now with the splittings $U^A \oplus G^A$ and $U^B \oplus G^B$ with $U^A := E^A \oplus F^A$ and $U^B := E^B \oplus F^B$. We obtain

$$\operatorname{dist}(G^A,G^B) \le \widetilde{C}\frac{\mu_2^{-1}}{\sigma_1^{-1}}\delta^{\log(\mu_2^{-1}/\sigma_1^{-1})/\log(a/\sigma_1^{-1})} = \widetilde{C}\frac{\sigma_1}{\mu_2}\delta^{\log(\sigma_1/\mu_2)/\log(a\sigma_1)} = \widetilde{C}\frac{\sigma_1}{\mu_2}\delta^\gamma. \tag{4.3}$$

So we can find a linear map $L : V^A \to (V^A)^\perp$ such that its graph equals V^B, that is

$$V^B = \{u + Lu : u \in V^A\} \quad \text{and} \quad \frac{\|L\|}{\sqrt{1+\|L\|^2}} \le \operatorname{dist}(V^A,V^B) \le \|L\| \tag{4.4}$$

by the estimate obtained in (2.2).

This allows us to identify V^A with V^B via the isomorphism

$$\phi := I + L : V^A \to V^B, u \mapsto u + Lu$$

and then pass the splitting $F^B \oplus G^B$ of V^B to a corresponding splitting $\hat{F}^B \oplus \hat{G}^B$ of V^A with $\hat{F}^B = \phi^{-1}F^B, \hat{G}^B = \phi^{-1}G^B$.

Now we can compare the distance between the pair $F^A, \hat{F}^B$ using the action of the sequences of matrices A_n and $\hat{B}_n = \phi^{-1}B_n\phi$ and Lemma 2.1 again. To finish we need to estimate

$$\operatorname{dist}(F^A, F^B) \leq \operatorname{dist}(F^A, \hat{F}^B) + \operatorname{dist}(\hat{F}^B, F^B). \tag{4.5}$$

We first consider the estimation of $\operatorname{dist}(F^A, \hat{F}^B)$. We already have the right conditions for the action of A_n over the splitting $F^A \oplus G^A$. As for $\hat{B}_n$ it is easy to see that

$$\frac{1}{\|\phi\|}\|B_n \mid F^B\| \leq \|\hat{B}_n \mid \hat{F}^B\| = \|\phi^{-1}B_n \mid F^B\| \leq \|\phi^{-1}\| \cdot \|B_n \mid F^B\|$$

and since the image of L is orthogonal to the domain, we have $\|\phi(v)\| \geq \|v\|$ for all v, thus

$$\|\phi^{-1}\| \leq 1 \quad \text{and} \quad \|\phi\| = \|I + L\| \leq 1 + \|L\|$$

we obtain a control of the action of $\hat{B}_n$ on $\hat{F}^B$ in a similar way to what we had for B_n on F^B, were the constant C is replaced by $C(1 + \|L\|)$ on the upper bound and $C(1 - \|L\|)$ on the lower bound, as long as we take $\operatorname{dist}(V^A, V^B)$ close enough to zero by letting δ_0 be small enough since the beginning. An analogous estimate holds for $\hat{B}_n \mid \hat{G}_B$.

We now estimate the angle between $\hat{F}^B$ and $\hat{G}^B$, that is, the new value which plays the role of d in Lemma 2.1.

We assume that for all $v \in F^B, w \in G^B$ we have $\|v\| \leq d\|v + w\|$. If we now take $v \in \hat{F}^B, w \in \hat{G}^B$, then $\phi(v) \in F^B$ and $\phi(w) \in G^B$ and so $\|\phi(v)\| \leq d\|\phi(v + w)\|$. This implies $\|\phi^{-1}\|^{-1}\|v\| \leq d\|\phi\|\|v + w\|$ so we obtain

$$\|v\| \leq d\|\phi\|\|\phi^{-1}\|\|v + w\| \leq \tau(d, L)\|v + w\| \quad \text{where} \quad \tau(d, L) := d\frac{1 + \|L\|}{1 - \|L\|}.$$

The relation (4.4) between $\|L\|$ and $\operatorname{dist}(V^A, V^B)$ enables us to deduce

$$\frac{1 + \operatorname{dist}(V^A, V^B)}{1 - \operatorname{dist}(V^A, V^B)} \leq \frac{1 + \|L\|}{1 - \|L\|} \leq \frac{1 + \operatorname{dist}(V^A, V^B)\Big(1 - \operatorname{dist}(V^A, V^B)^2\Big)^{-1/2}}{1 - \operatorname{dist}(V^A, V^B)\Big(1 - \operatorname{dist}(V^A, V^B)^2\Big)^{-1/2}}$$

and so $\tau(d, L)$ is a function of $\operatorname{dist}(V^A, V^B)$. Moreover $\tau(d, L)$ goes to d as $\operatorname{dist}(V^A, V^B)$ goes to zero. In particular, if $0 < \|L\| < 1/2$ we get $\tau(d, L) \leq 3d$.

We finally check the condition on the distance between A_n and $\hat{B}_n$

$$\begin{aligned}\|A_n - \hat{B}_n\| &\le \|A_n - \phi^{-1}A_n\| + \|\phi^{-1}A_n - \phi^{-1}A_n\phi\| + \|\phi^{-1}A_n\phi - \hat{B}_n\| \\ &\le \|I-\phi\|\|A_n\| + \|\phi^{-1}\|\|A_n\|\|I-\phi\| + \|\phi^{-1}\|\|A_n - B_n\|\|\phi\| \\ &\le \frac{a^{|n|}\|L\|}{1-\|L\|} + \frac{a^{|n|}\|L\|}{(1-\|L\|)^2} + \delta a^{|n|}\frac{\|L\|(1+\|L\|)}{1-\|L\|} = a^n\|L\|\omega(L)\end{aligned}$$

since $\|I-\phi^{-1}\| = \|I-(I+L)^{-1}\| \le \frac{\|L\|}{1-\|L\|}$, where $\|L\|$ is essentially $\mathrm{dist}(V^A, V^B)$ according to (4.4), and $\omega(L)$ can be made as close to 1 as we need by letting $\|L\|$ (or $\mathrm{dist}(V^A, V^B)$) be close enough to zero. Hence by taking δ small enough since the beginning we can ensure that $\|L\| \in (0, 1/2)$ and $\|L\|\omega(L) \in (0, 1/2)$.

We can now apply Lemma 2.1 to the pairs F^A, G^A and $\hat{F}^B, \hat{G}^B$ to conclude

$$\mathrm{dist}(F^A, \hat{F}^B) \le (2+\tau(d,L))(C(1+\|L\|))^2\frac{\sigma_1}{\mu_2}(\|L\|\omega(L))^{\log(\sigma_1/\mu_2)/\log(a/\mu_2)}.$$

For $0 < \delta < \delta_0 < 1$ with δ_0 small enough, depending on $C, \mu_1/\lambda_2$ and a, we obtain that $\|L\|\omega(L) \le 2\,\mathrm{dist}(V^A, V^B)$. Thus, setting $\eta = \log(\sigma_1/\mu_2)/\log(a/\mu_2)$ for simplicity

$$\mathrm{dist}(F^A, \hat{F}^B) \le (2+3d)(\frac{3}{2}C)^2 \cdot 2^\eta \frac{\sigma_1}{\mu_2}\,\mathrm{dist}(V^A, V^B)^\eta \le \frac{9}{4}C^2(2+3d)\tilde{C}^\eta\frac{\sigma_1\mu_1^\eta}{\mu_2\lambda_2^\eta}\delta^{\omega\eta}. \tag{4.6}$$

To finish we estimate $\mathrm{dist}(\hat{F}^B, F^B)$.

We already have that $\mathrm{dist}(\phi^{-1}F^B, F^B)$ is comparable to $\|\phi^{-1}\|$ by (2.2). We remark that $\phi^{-1} = (I+L)^{-1} = I + \sum_{k>0}(-L)^k = I + \hat{L}$, where $\hat{L} : V^A \to (V^A)^\perp$ and $\|\hat{L}\| \le \|L\|(1-\|L\|)^{-1} \le \|L\|\omega(L)$. Hence

$$\mathrm{dist}(\hat{F}^B, F^B) \le \|\hat{L}\| \le \|L\|\omega(L) \le 2\,\mathrm{dist}(V^A, V^B) \tag{4.7}$$

and we finally complete (4.5) using (4.6) and (4.2)

$$\mathrm{dist}(F^A, F^B) \le \frac{9}{4}C^2(2+3d)\tilde{C}^\eta\frac{\sigma_1\mu_1^\eta}{\mu_2\lambda_2^\eta}\delta^{\omega\eta} + \widetilde{C}\frac{\mu_1}{\lambda_2}\delta^\omega \le \frac{9}{2}(2+3d)^{1+\eta}C^{2(1+\eta)}\frac{\sigma_1\mu_1^\eta}{\mu_2\lambda_2^\eta}\delta^{\omega\eta} \tag{4.8}$$

for $\delta_0 \in (0,1)$ small enough. Putting the inequalities (4.1), (4.3) and (4.8) together, we arrive to the conclusion of the lemma. □

5. The Kontsevich-Zorich cocycle

In this section, we prove the theorem about Hölder continuity of the Oseledets subspaces for the Kontsevich-Zoricqh cocycle.

We first provide some general lemmas to deal with topologically non-trivial cocycles. The first is a different point of view on Lemma 2.1; the focus is on norms rather than matrices. The second allows one to intersect Oseledets filtrations, thus reducing the necessary estimates. Finally, the third allows one to reduce vector bundle cocycles to matrix ones.

In the last part of the section, we combine these constructions with the norm constructed by Avila, Gouezel and Yoccoz [AGY06, AG13]. This gives the main results for the Kontsevich-Zorich cocycle.

5.1. **Some more preliminary lemmas.** We consider Euclidean inner products on $\mathbb{R}^d$, which we call metrics below. Given two such, say h_1 and h_2, define the distance between them as

$$\operatorname{dist}(h_1, h_2) := \log \sup_{\substack{\|v_1\|_{h_1}=1 \\ \|v_2\|_{h_2}=1}} \left\{ \|v_1\|_{h_2}, \|v_2\|_{h_1} \right\} \tag{5.1}$$

This gives the inequality

$$\|v\|_{h_1} \leq e^{\operatorname{dist}(h_1,h_2)} \|v\|_{h_2}$$

Given a metric h, define the distance between two spaces E, F to be

$$\operatorname{dist}_h(E,F) := \sup \left\{ \left\|f^\perp\right\|_h : e = f + f^\perp, \|e\|_h = 1, e \in E, f \in F, f^\perp \in F^\perp \right\} \bigcup$$
$$\bigcup \left\{ \left\|e^\perp\right\|_h : f = e + e^\perp, \left\|f\right\|_h = 1, f \in F, e \in E, e^\perp \in E^\perp \right\}.$$

Lemma 5.1. *Suppose we are given decompositions of $\mathbb{R}^d$ as*

$$\mathbb{R}^d = E \oplus E^\perp = F \oplus F^\perp$$

where perpendiculars are for a fixed metric h_0. Suppose given a sequence of metrics h_n^E, h_n^F satisfying

$$\operatorname{dist}(h_n^E, h_n^F) \leq \log(1 + \delta C_2 A^n) \tag{5.2}$$

for some constants C_2, A, δ. Assume further that we have constants $0 < \lambda < \mu$ and C such that $\forall f \in F, \forall f' \in F^\perp$

$$\left\|f\right\|_{h_n^F} \leq C\lambda^n \left\|f\right\|_{h_0} \tag{5.3}$$

$$C^{-1}\mu^n \left\|f'\right\|_{h_0} \leq \left\|f'\right\|_{h_n^F}. \tag{5.4}$$

Suppose that the above growth estimates hold analogously for E instead of F. Then we have the distance estimate

$$\operatorname{dist}_{h_0}(E,F) \leq C^2(2 + C_2 A)\delta^{\frac{\log\frac{\mu}{\lambda}}{\log A}}.$$

Remark 5.2. In fact, in the proof below we do not need the inequalities in the assumptions to hold for all n. We only need to know it for a value of n such that $c_3 \leq \delta A^n \leq c_4$, for some fixed constants c_3, c_4. Then the same conclusion holds (with different constants, but same δ and exponents).

Proof. Pick $e \in E$ with $\|e\|_{h_0} = 1$ and write it as $e = f + f^\perp$ where $f \in F, f^\perp \in F^\perp$. By the assumption on F we have

$$C^{-1}\mu^n \left\|f^\perp\right\|_{h_0} \leq \left\|f^\perp\right\|_{h_n^F}.$$

We also have the following chain of inequalities

$$\begin{aligned}\left\|f^{\perp}\right\|_{h_n^F} &= \left\|e-f\right\|_{h_n^F} \leq \|e\|_{h_n^F} + \left\|f\right\|_{h_n^F} \\ &\leq \|e\|_{h_n^E}(1+\delta C_2 A^n) + C\lambda^n \left\|f\right\|_{h_0} \quad \text{by (5.2) for } \|e\| \text{ and (5.3) for } \left\|f\right\| \\ &\leq \|e\|_{h_0} C\lambda^n(1+\delta C_2 A^n) + C\lambda^n \quad \text{by (5.3) for } \|e\| \text{ and since } \left\|f\right\|_{h_0} \leq \|e\|_{h_0} = 1.\end{aligned}$$

Combining the two inequalities, we find that

$$\begin{aligned}\left\|f^{\perp}\right\|_{h_0} &\leq 2C^2\left(\frac{\lambda}{\mu}\right)^n + C^2 C_2 \delta \left(\frac{\lambda A}{\mu}\right)^n \\ &\leq C^2\left(\frac{\lambda}{\mu}\right)^n (2+\delta C_2 A^n).\end{aligned}$$

Choose now n such that $1 \leq \delta A^n < A$, i.e. $0 \leq \log\delta + n\log A < \log A$. Then we have

$$\begin{aligned}\left\|f^{\perp}\right\|_{h_0} &\leq C^2(2+C_2 A)e^{\left(\log\frac{\lambda}{\mu}\right)\cdot\frac{-\log\delta}{\log A}} \\ &\leq C^2(2+C_2 A)\delta^{\frac{\log\frac{\mu}{\lambda}}{\log A}}\end{aligned}$$

An analogous argument gives the estimate for vectors in F, thus proving the claim. □

Remark 5.3. Suppose given two sequences of matrices A_n, B_n with

$$\|A_n - B_n\| \leq \delta L^n$$

Then we have the bound $\|A_n v\| \leq \|B_n v\| + \delta L^n \|v\|$. If we also had the Oseledets-type estimate

$$\|v\| \leq C\mu^{-n}\|B_n v\|$$

we could conclude $\|A_n v\| \leq \|B_n v\|(1+\delta C L^n \mu^{-n})$. If we defined a sequence of metrics by

$$\begin{aligned}\|v\|_{h_n^A} &:= \|A_n v\| \\ \|v\|_{h_n^B} &:= \|B_n v\|\end{aligned}$$

then we could apply the above lemma (with $A = \frac{L}{\mu}$) and get a conclusion similar to that of Lemma 5.3.3 from [BP07]. The advantage of using the metric formulation is that the metrics defined above can come from linear operators with the same source, but different targets.

The next result allows us to intersect subspaces in the past and future Oseledets filtrations.

Lemma 5.4. *Let K be a compact metric space and let F_+, F_- be two Hölder continuous maps to Grassmanians*

$$F_\pm : K \to Gr(k_\pm, \mathbb{R}^d)$$

with Hölder exponents ν_+, ν_-. Suppose that for all $x \in K$, we have that $F_+(x)$ and $F_-(x)$ are in general position, i.e. intersect in a space of dimension $r := k_+ + k_- - d$.

Then the map

$$I : K \to Gr(r, \mathbb{R}^d)$$

assigning to x the intersection of $F_+(x)$ *and* $F_-(x)$ *is Hölder of exponent* $\min(\nu_+, \nu_-)$.

Proof. We omit the $\mathbb{R}^d$ from the notation for Grassmanians. Inside $Gr(k_+) \times Gr(k_-) \times Gr(r)$ define the closed algebraic set

$$\Gamma := \{(V_+, V_-, I) : I \subseteq V_+ \cap V_-\}.$$

We have projections

$$p_{12} : \Gamma \to Gr(k_+) \times Gr(k_-)$$
$$p_3 : \Gamma \to Gr(r).$$

Over the Zariski-open set $U \subset Gr(k_+) \times Gr(k_-)$ of planes in general position we know that the map

$$p_{12} : \left(\Gamma \cap p_{12}^{-1}(U)\right) \to U$$

is a smooth (even algebraic) bijection. Denote its inverse by $p_{12,U}^{-1}$ assumed to take values in $Gr(k_+) \times Gr(k_-) \times Gr(r)$.

By assumption, the map $F_+ \times F_- : K \to U$ lands inside the open set, and has compact image. In particular, the first derivative of $p_{12,U}^{-1}$ is uniformly bounded on this image.

The function I giving the intersection of F_+ and F_- over K can now be written as the composition $p_3 \circ p_{12,U}^{-1} \circ (F_+ \times F_-)$. It is therefore Hölder, with exponent $\min(\nu_+, \nu_-)$. □

We next explain how cocycles on vector bundles can be reduced to cocycles over matrices.

Lemma 5.5. *Suppose* $V \to X$ *is a vector bundle with metric over a connected manifold* X. *Then we can find another vector bundle with metric* $W \to X$ *and an isomorphism* $V \oplus W \cong \underline{\mathbb{R}}^N$.

Here $\underline{\mathbb{R}}^N$ *denotes the trivial vector bundle over* X, *with the standard metric. The regularity of* W *and the isomorphism can be taken as good as that of* V *and* X.

Proof. First, we show that V can be trivialized using finitely many open sets (not necessarily connected). Then we find the isometric embedding.

To begin, since X is a manifold it has finite Lebesgue covering dimension, say n. Therefore, there exist a cover of X by open sets U_α such that at most $n+1$ intersect at any point, and we have trivializations $U_\alpha \times \mathbb{R}^d \cong V|_{U_\alpha}$.

Take a partition of unity ϕ_α subordinated to the covering, and for any set $I = \{\alpha_1, \dots, \alpha_i\}$ define the open sets

$$W_I := \{x \in X : \phi_\alpha(x) < \phi_{\alpha_j}(x), \forall \alpha \notin I, \alpha_j \in I\}$$

The sets W_I are open by the local finiteness of the covering, and V can be trivialized on each of them. Furthermore, if $I \neq I'$ and of the same cardinality, then $W_I \cap W_{I'} = \emptyset$. We can thus define the $n+1$ open sets

$$W_i := \bigcup_{\#I=i} W_I \qquad i = 1, \dots, n+1$$

These sets are open, give a cover of X and V is trivialized on each of them.

Next, take a partition of unity ρ_i adapted to W_i. Let $v_1^{(i)}, \dots, v_d^{(i)}$ be the global sections of V obtained from the coordinate sections in each trivialization, multiplied by the ρ_i to have support in W_i.

We can now define a surjection

$$\begin{aligned} X \times \mathbb{R}^{d(n+1)} &\twoheadrightarrow V \\ \overrightarrow{e}_{(i-1)d+j} &\mapsto v_j^{(i)} \qquad i = 1, \dots, n+1, \qquad j = 1, \dots, d \end{aligned}$$

Here $\overrightarrow{e}_l$ denotes the l-th coordinate vector in $\mathbb{R}^{d(n+1)}$.

Let W be the kernel bundle of the surjection just defined, and let V' be its orthogonal complement. The surjection induces an isomorphism $V' \to V$, which we can invert to an injection $V \to \underline{\mathbb{R}}^N$.

Letting h be the initial metric on V, the embedding just described induces a different metric h'. The space of metrics on a fiber of the bundle is $\mathrm{SL}_d(\mathbb{R})/SO_d(\mathbb{R})$, which is contractible. Moreover, we have unique geodesics given as exponentials of symmetric operators (symmetric for the underlying metric). See for example [BH99, pg. 324] for a discussion of these properties.

This means that precomposing the injection $V \to \underline{\mathbb{R}}^N$ with uniquely defined linear operators on the fibers of V, we can arrange the map to be isometric. We have thus obtained the desired isometric decomposition $V \oplus W = \underline{\mathbb{R}}^N$ □

5.2. **Applications to the Kontsevich-Zorich cocycle.** We refer to [Zor06] for a general introduction to flat surfaces and the Kontsevich-Zorich (KZ) cocycle. The results of Eskin, Mirzakhani, and Mohammadi [EM13, EMM13] lead us to consider affine invariant manifolds, since those support the $\mathrm{SL}_2\,\mathbb{R}$-invariant measures.

In this section, we prove two types of results in that context. The first one is that the main theorem C applies to the KZ cocycle. The second one is similar to the results of Chaika-Eskin [CE15].

Theorem 5.6. *Let $\mathcal{M}$ be an affine invariant manifold and let μ be the corresponding ergodic $\mathrm{SL}_2\,\mathbb{R}$-invariant probability measure (see [EM13]). Let E be the Kontsevich-Zorich cocycle (or any of its tensor powers) and let $\{\lambda_i\}$ be its Lyapunov exponents, with Oseledets subspaces E^i.*

Then there exists $\nu_i > 0$ such that for any $\varepsilon > 0$ there exists a compact set K_ε with $\mu(K_\varepsilon) > 1 - \varepsilon$ and such that the spaces E^i vary ν_i-Hölder continuously on K_ε.

The AGY norm. Before the start the proof, we recall some properties of a norm defined by Avila-Gouëzel-Yoccoz (see [AGY06, Sect. 2.2.2] and [AG13, Sect. 5]). For the definitions below, we potentially need to pass to a finite cover of $\mathcal{M}$ to avoid orbifold issues.

Denote by H^1_{rel} the real bundle of relative cohomology. We denote by ω both a point in the affine manifold $\mathcal{M}$ and the cohomology class in H^1_{rel} that it represents. The norm

on H^1_{rel} is then defined by

$$\|\alpha\|_\omega := \sup_{\text{saddle } \gamma} \frac{|\alpha(\gamma)|}{|\omega(\gamma)|} \tag{5.5}$$

Recall that γ is *a saddle* if on the flat surface represented by ω, the class γ can be realized as a straight line connecting two singular points of the flat metric.

Identifying the relative cohomology with the tangent space to the stratum, the expression (5.5) gives a complete metric ([AGY06, Cor. 2.13]). Given $x \in \mathcal{M}$, denote by $W^u(x)$ the unstable leaf through x, and by $E^u(x) \subseteq H^1_{rel}(x)$ the tangent space to the unstable leaf. We have the exponential map (linear in period coordinates)

$$\begin{aligned} \Psi_x : H^1_{rel}(x) &\to W^u(x) \\ \alpha &\mapsto x + \alpha \end{aligned}$$

The main properties of the metric relative to the Teichmüller flow are summarized below. We denote by Dg_t the induced cocycle on the tangent bundle, and by g_t the Kontsevich-Zorich cocycle (i.e. the flat Gauss-Manin connection).

(1) [AGY06, ineq. 2.13] For $x \in \mathcal{M}$, $v \in H^1_{rel}(x)$ we have the growth bounds

$$e^{-2t} \|v\|_x \leq \left\| Dg_t v \right\|_{g_t x} \leq e^{2t} \|v\|_x \tag{5.6}$$

(2) [AG13, Prop. 5.3] If we denote by $B(0,r)_x$ the ball of radius r in $H^1_{rel}(x)$ then Ψ_x is well-defined on $B(0,1/2)$ for all $x \in \mathcal{M}$, and for all $v \in B(0,1/2)$ we have

$$d_{W^u(x)}(x, \Psi_x(v)) \leq 2 \|v\|_x \tag{5.7}$$

(3) [AG13, Prop. 5.3] For all $w \in E^u(x)$ and $v \in B(0,1/2)$ we have

$$\frac{1}{2} \|w\|_{\Psi_x(v)} \leq \|w\|_x \leq 2 \|w\|_{\Psi_x(v)} \tag{5.8}$$

In the above inequality, the vector w is transported to $\Psi_x(v)$ using the flat connection along the exponential map.

In order to invoke Lemma 5.1, we note that its proof works in the same way for Finsler metrics. The distance between Finsler metrics is defined by equation (5.1).

Alternatively, to a Finsler metric one can associate canonically the Riemannian metric coming from the John ellipsoid construction (see [Bal97, Thm. 3.1]), and each controls the other by absolute constants (depending on dimension only). Recall that given a bounded convex body $K \subset \mathbb{R}^n$, there is a canonically associated ellipsoid $E \subseteq K$ (the John ellipsoid) with the following two properties. First, E is the ellipsoid of maximal volume and satisfying $E \subseteq K$, and moreover $K \subseteq \sqrt{n}E$, where $\sqrt{n}E$ denotes the rescaling by $\sqrt{n}$ based at the center of E. Interpreting a convex body as the unit ball of a norm, this means that we can replace a Finsler metric with a canonical inner product, such that each is within a bounded factor of the other.

Proof of Theorem 5.6. We prove Hölder continuity for the forward Oseledets filtration, the proof for the backward one being similar. Combining them using Lemma 5.4, the desired claim follows.

Define the forward Oseledets filtration by

$$F^i(\theta)_x := \oplus_{i\geq j} E^j(\theta)_x$$

Fix $\varepsilon_1 > 0$ arbitrarily small. Then for C sufficiently large, there exists a compact set $K_\varepsilon \subseteq \mathcal{M}$ of measure at least $1 - \varepsilon/10$ on which the Oseledets theorem holds uniformly:

$$C^{-1} e^{(\lambda_i - \varepsilon_1)t} \|v_i\| \leq \left\| g_t v_i \right\| \leq C e^{(\lambda_i + \varepsilon_1)t} \|v_i\| \qquad \forall i, \forall v_i \in E^i_x, \forall x \in K_\varepsilon$$

By passing to a further compact subset, we can assume the angle between Oseledets subspaces is uniformly bounded away from zero.

Since the forward Oseledets filtration only depends on the position of the point on unstable leaves, given x and $y \in W^u(x)$, it suffices to estimate $\mathrm{dist}(F^i(x), F^i(y))$ in terms of $d_{W^u(x)}(x, y)$. Assuming $d_{W^u(x)}(x, y) \leq 1/4$, using the properties of the AGY norm we have $y = \Psi_x(v)$ for some $v \in H^1_{rel}(x)$.

Denote by GM_v the Gauss-Manin connection from x to $x + v$ along the exponential map. If $w \in E_x$ is a vector, we define new norms by flowing forward:

$$\|w\|_{x,t} := \left\| g_t w \right\|_{g_t x}$$

A similar norm is defined on E_y and we wish to compare $\|w\|_{x,t}$ and $\|GM_v(w)\|_{y,t}$.

Set $\delta = \|v\|_x$ and $t = \frac{-1}{2} \log \delta - 2$. Note that with this choice we have $1/100 < \delta e^{2t} < 1/4$. We would like to apply Lemma 5.1 (see Remark 5.2 for why it suffices to check assumptions at a well-chosen time). The Oseledets-type assumptions of Lemma 5.1 (i.e. (5.3) and (5.4)) on the filtrations hold by the choice of compact sets. We need to estimate the distance between norms, i.e. check assumption (5.2) in Lemma 5.1.

Thus, given $w \in E_x$, we need to estimate the ratio of $\left\| g_t w \right\|_{g_t x}$ and $\left\| GM_{Dg_t v}(g_t w) \right\|_{g_t y}$. By our choice of t and using property (5.6) on the AGY norm, we have $\left\| Dg_t v \right\|_{g_t x} \leq 1/2$ Therefore, using property (5.8) of the AGY norm we find

$$\frac{1}{2} \left\| GM_{Dg_t v}(g_t w) \right\|_{g_t y} \leq \left\| g_t w \right\|_{g_t x} \leq 2 \left\| GM_{Dg_t v}(g_t w) \right\|_{g_t y}$$

Therefore, the assumptions of Lemma 5.1 are satisfied, with $A = e^2$ (coming from property (5.6) of the AGY norm). We can now conclude the forward Oseledets filtration F^i varies Hölder-continuously with exponent $\frac{1}{2} \log \frac{\lambda_i}{\lambda_{i+1}} - \varepsilon'$. Combining the forward and backward filtrations using Lemma 5.4, we find individual Oseledets subspaces E^i are Hölder with exponent $\nu_i := \frac{1}{2} \min\left(\log \frac{\lambda_i}{\lambda_{i+1}}, \log \frac{\lambda_{i-1}}{\lambda_i}\right) - \varepsilon'$. □

Following Chaika-Eskin [CE15], the next result refines the above one when restricting to a single $\mathrm{SL}_2\,\mathbb{R}$-orbit. Recall the notation for 1-parameter subgroups of $\mathrm{SL}_2\,\mathbb{R}$:

$$g_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \quad r_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad g_t^\theta := r_\theta^{-1} g_t r_\theta$$

Theorem 5.7. *Let $x \in \mathcal{H}(\kappa)$ be a flat surface in a stratum and let E be the Kontsevich-Zorich cocycle. If E is some tensor power of the KZ cocycle, let $k \geq 1$ be the smallest order of the tensor product which contains it.*

For $g \in \mathrm{SL}_2\,\mathbb{R}$ denote by $A(g,x) : E_x \to E_{gx}$ the matrix of the cocycle E. By Theorem 1.2 in [CE15] *there exist $\lambda_1 > \cdots > \lambda_k$ such that for a.e. $\theta \in S^1$ we have*

(1) *There exists a decomposition $E_x = \oplus_i E^i(\theta)_x$ with $E^i(\theta)_x$ measurably varying in θ*
(2) *We have*

$$\lim_{t\to\pm\infty} \frac{1}{t} \log \left\| A(g_t^\theta, x) v_i \right\| = \lambda_i \quad \forall v_i \in E^i(\theta)_x$$

Let $\nu_i := \frac{1}{2k} \min\left(\log \frac{\lambda_i}{\lambda_{i+1}}, \log \frac{\lambda_{i-1}}{\lambda_i}\right) - \varepsilon'$, for arbitrarily small ε'.

Then for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq S^1$ of Lebesgue measure at least $1-\varepsilon$ such that the subspaces $E_i(\theta)$ vary ν_i-Hölder continuously for $\theta \in K_\varepsilon$.

Proof. First we show that the forward (resp. backward) Oseledets filtrations are Hölder. Then we combine the results to conclude individual subspaces are Hölder.

Define the forward Oseledets filtration by

$$F^i(\theta)_x := \oplus_{i \geq j} E^j(\theta)_x$$

Define also norms on E_x by

$$\|v\|_{g_t^\theta} := \left\| A(g_t^\theta, x) v \right\|$$

Fix $\varepsilon_1 > 0$ arbitrarily small. Then for C sufficiently large, there exists a compact set $K_1 \subseteq S^1$ of measure at least $1 - \varepsilon/10$ on which the Oseledets theorem holds uniformly:

$$C^{-1} e^{(\lambda_i - \varepsilon_1)t} \|v_i\| \leq \|v_i\|_{g_t^\theta} \leq C e^{(\lambda_i + \varepsilon_1)t} \|v_i\| \quad \forall i, \forall v_i \in E^i(\theta)_x, \forall \theta \in K_1$$

By Lusin's theorem, we can further restrict to a compact subset $K_2 \subseteq K_1$ of measure at least $1-\varepsilon$ such that $E_i(\theta)$ vary continuously on it. In particular, the distance between $E^i(\theta)$ and $E^j(\theta)$ for $i \neq j$ is uniformly bounded away from zero on K_2.

Consider now the decomposition $E = F^i(\theta) \oplus F^i(\theta)^\perp$. From the uniform boundedness of the angle between Oseledets subspaces, there exists $C_1 > 0$ such that

$$\|v\|_{g_t^\theta} \leq C_1 e^{(\lambda_i + \varepsilon_1)t} \|v\| \qquad \forall v \in F^i(\theta)$$
$$C_1^{-1} e^{(\lambda_{i-1} - \varepsilon_1)t} \leq \|v\|_{g_t^\theta} \qquad \forall v \in F^i(\theta)^\perp$$

We can now apply Lemma 5.1 to conclude the Hölder continuity of $F^i(\theta)$, provided we control the divergence of metrics. We need to find a A, C_2 such that

$$\mathrm{dist}\left(\|-\|_{g_t^{\theta_1}}, \|-\|_{g_t^{\theta_2}} \right) \leq A|\theta_1 - \theta_2| e^{tC_2}$$

By results of Forni (see [For02]) the norm of the Kontsevich-Zorich cocycle on a Teichmüller disk is bounded by the hyperbolic distance between points. Namely, if we E is contained in a k-th tensor product of the KZ cocycle, then

$$\left\|A(x, gx)\right\| \leq e^{k\cdot \mathrm{dist}(x,gx)}$$

Next, from hyperbolic geometry we have for t bounded away from zero

$$e^{\frac{1}{2}\,\mathrm{dist}\left(g_t^{\theta_1}x, g_t^{\theta_2}x\right)} \leq C_3|\theta_1 - \theta_2|e^t$$

Plugging in these estimates into Lemma 5.1, we get Hölder continuity of $F^i(\theta)$ with exponent $\nu_i = \frac{1}{2k}\log\frac{\lambda_{i-1}}{\lambda_i}$.

An analogous result holds for the backward Oseledets filtration. Therefore, combining the filtrations using Lemma 5.4 we obtain the desired result. □

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