ON SURFACES OF CLASS VII WITH NUMERICALLY ANTICANONICAL DIVISOR

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Abstract

We consider minimal compact complex surfaces S with Betti numbers $b_1 = 1$ and $n = b_2 > 0$. A theorem of Donaldson gives n exceptional line bundles. We prove that if in a deformation, these line bundles have sections, S is a degeneration of blown-up Hopf surfaces. Besides, if there exists an integer $m \geq 1$ and a flat line bundle F such that $H^0(S, -mK \otimes F) \neq 0$, then S contains a Global Spherical Shell. We apply this last result to complete classification of bihermitian surfaces.

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0 Introduction

A minimal compact complex surface S is said to be of the class VII_0 of Kodaira if the first Betti number satisfies $b_1(S) = 1$. A surface S is of class VII_0^+ if moreover $n := b_2(S) > 0$; these surfaces admit no nonconstant meromorphic functions. The major problem in classification of non-kählerian surfaces is to achieve the classification of surfaces S of class VII_0^+. All known surfaces of this

class contain Global Spherical Shells (GSS), i.e. admit a biholomorphic map $\varphi : U \to V$ from a neighbourhood $U \subset \mathbb{C}^2 \setminus \{0\}$ of the sphere $S^3 = \partial B^2$ onto an open set V such that $\Sigma = \varphi(S^3)$ does not disconnect S. Are there other surfaces ? In first section we investigate the general situation: A theorem of Donaldson [13] gives a $\mathbb{Z}$-base (E_i) of $H^2(S, \mathbb{Z})$, such that $E_iE_j = -\delta_{ij}$. These cohomology classes can be represented by line bundles L_i such that $K_SL_i = L_i^2 = -1$. Indeed, these line bundles generalize exceptional curves of the first kind, and since S is minimal, they have no section. Over the versal deformation $\mathcal{S} \to B$ of S these line bundles form families $\mathcal{L}_i$. We propose the following conjecture which can be easily checked for surfaces with GSS:
Conjecture 1: Let S be a surface in class VII_0^+ and $\mathcal{S} \to B$ be the versal deformation of $S \simeq S_0$ over the ball of dimension $h^1(S, \Theta)$. Then there exists $u \in \Delta$, $u \neq 0$, and flat line bundles F_i such that $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$ for $i = 0, \ldots, n-1$.
The main result of section 1 is (see theorem (13)),

Theorem 0. 1 *Let S be a surface in class VII_0^+ and $\mathcal{S} \to B$ its versal deformation. If there exists $u \in B$ and flat line bundles $F_i \in H^1(S, \mathbb{C}^\star)$ such that $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$ for $i = 0, \ldots, n-1$, then there is a non empty Zariski open set $U \subset B$ such that for all $u \in U$, S_u is a blown-up Hopf surface. In particular, S is a degeneration of blown-up Hopf surfaces.*

If a surface is a degeneration of blown-up Hopf surfaces, the fundamental group of a fiber is isomorphic to $\mathbb{Z} \times \mathbb{Z}_l$, hence taking a finite covering, once obtains a surface obtained by degeneration of blown-up primary Hopf surfaces. Notice that a finite quotient of a surface of class VII_0^+ containing a GSS still contains a GSS [8].

Conjecture 2: Let S be a surface of class VII_0^+. If S is a degeneration of blown-up primary Hopf surfaces, then S contains a cycle of rational curves.

A surface admitting a numerically anticanonical (NAC) divisor (see (20)), contains a cycle of rational curves. In section 2, we shall prove

Theorem 0. 2 *Let S be a surface of class VII_0^+. If S admits a NAC divisor, then S contains a GSS.*

It is a weak version of
Conjecture 3 (Nakamura [21]). Let S be a surface of class VII_0^+. If S contains a cycle of rational curves, S contains a GSS.
The proof is based on the fact that in $H^2(S, \mathbb{Z})$, a curve is equivalent to a class of the form $L_i - \sum_{j \in I} L_j$, with $I \neq \emptyset$. Intuitively L_i represents an exceptional curve of the first kind and C is then equivalent to an exceptional curve of the first kind blown-up several times (Card(I) times). It explains why curves have self-intersection ≤ -2. We recover a characterization of Inoue-Hirzebruch surfaces by Oeljeklaus, Toma & Zaffran [19]:

Theorem 0. 3 *Let S be a surface of class VII_0 with $b_2(S) > 0$. Then S is a Inoue-Hirzebruch surface if and only if there exists two flat line bundles F_1,*

F_2, two twisted vector fields $\theta_1 \in H^0(S, \Theta \otimes F_1)$, $\theta_2 \in H^0(S, \Theta \otimes F_2)$, such that $\theta_1 \wedge \theta_2(p) \neq 0$ at at least one point $p \in S$.

In section 3 we apply results of section 2 to complete the classification of bihermitian 4-manifolds M (see [1],[2] [22]), when $b_1(M) = 1$ and $b_2(M) > 0$: A bihermitian surface is a riemannian oriented connected 4-manifold (M, g) endowed with two integrable almost complex structures J_1, J_2 inducing the same orientation, orthogonal with respect to g and independent i.e. $J_1(x) \neq \pm J_2(x)$ for at least one point $x \in M$. This structure depends only on the conformal class c of g. A bihermitian surface is strongly bihermitian if $J_1(x) \neq \pm J_2(x)$ for every point $x \in M$. The key observation is that under these assumptions, (M, J_i), $i = 1, 2$ admit a numerically anticanonical divisor.

Theorem 0. 4 *Let (M, c, J_1, J_2) be a compact bihermitian surface with odd first Betti number.*
1) If (M, c, J_1, J_2) is strongly bihermitian (i.e $\mathcal{D} = \emptyset$), then the complex surfaces (M, J_i) are minimal and either a Hopf surface covered by a primary one associated to a contraction $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of the form

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a\alpha^{-1} z_2),$$

$$\text{with} \quad a, s \in \mathbb{C},\ 0 < |\alpha|^2 \leq a < |\alpha| < 1,\ (a^m - \alpha^{m+1})s = 0,$$

or else (M, J_i) are Inoue surfaces $S^+_{N,p,q,r;t}$, $S^-_{N,p,q,r}$.
2) If (M, c, J_1, J_2) is not strongly bihermitian, then $\mathcal{D}$ has at most two connected components, (M, J_i), $i = 1, 2$, contain GSS and the minimal models S_i of (M, J_i) are

- *Surfaces with GSS of intermediate type if $\mathcal{D}$ has one connected component*
- *Hopf surfaces of special type (see* [22] 2.2*), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if $\mathcal{D}$ has two connected components.*

Moreover, the blown-up points belong to the NAC divisors.

If moreover the metric g is anti-self-dual (ASD), we obtain

Corollary 0. 5 *Let (M, c, J_1, J_2) be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces (M, J_i), $i = 1, 2$, are*

- *Hopf surfaces of special type (see [22] 2.2),*
- *(parabolic) Inoue surfaces or*
- *even Inoue-Hirzebruch surfaces.*

Moreover, the blown-up points belong to the NAC divisors.

Remark 0. 6 Throughout the paper we shall use the following terminology:
1) A surface for which exists a nontrivial divisor D such that $D^2 = 0$ will be called a **Enoki surface**, they are obtained by holomorphic compactification of

an affine line bundle over an elliptic curve by a cycle D of rational curves [14]; otherwise they are associated to contracting holomorphic germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, tz_2)$$

[6] [9]. A Enoki surface with an elliptic curve will be called briefly a **Inoue surface (= parabolic Inoue surface)**: they are obtained by holomorphic compactification of a line bundle over an elliptic curve by a cycle of rational curves D; otherwise they are associated to the contracting germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n, tz_2).$$

For all these surfaces, the sum of opposite self-intersections of the $n = b_2(S)$ rational curves $D_0, \ldots, D_{n-1}$ is $\sigma_n(S) := -\sum_{i=0}^{n-1} D_i^2 = 2n$.
2) A surface with $2n < \sigma_n(S) < 3n$ will be called an **intermediate surface**;
3) A surface with $\sigma_n(S) = 3n$ a called a **Inoue-Hirzebruch (IH) surface** ([15] or [7] for a construction by contracting germs of mappings). An **even (=hyperbolic)** (resp. **odd (=half)**) **IH surfaces** has two (resp. one) cycle of rational curves.

We shall assume throughout the article that S admits no nonconstant meromorphic functions.

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1 Exceptional line bundles and degeneration of blown-up Hopf surfaces

Let S be a surface in class VII_0^+ with $n = b_2(S)$. Since S is not algebraic, $A^2 \leq 0$ for every divisor A. By adjunction formula it is easy to deduce that for every irreducible curve C, $KC \geq 0$, $C^2 \leq -2$ if C is a regular rational curve and $C^2 \leq 0$ if C is a rational curve with a double point or an elliptic curve. It is well known that S contains at most n rational curves, and at most one elliptic curve. By Hirzebruch index theorem, $b^- = b_2(S)$ whence the intersection form $Q : H^2(S, \mathbb{Z})/Torsion \to \mathbb{Z}$ is negative definite.

1.1 Exceptional line bundles

An irreducible curve which satisfies the two conditions $KC = C^2 = -1$ is an exceptional curve of the first kind; we generalize the notion to line bundles.

Definition 1. 7 *1) A line bundle $L \in H^1(S, \mathcal{O}^\star)$ is called an* **exceptional line bundle** *(of the first kind) if $KL = L^2 = -1$.*

2) An effective divisor E is called an exceptional divisor (of the first) kind if E is the inverse image $\Pi^\star C$ of an exceptional curve of the first kind C by a finite number of blowing-ups Π. Equivalentlly it is an effective reduced divisor which may be blown-down onto a regular point.

Using the fact that for a blowing-up $\Pi : S \to S'$, $K_S = \Pi^\star K_{S'} + C$ and the projection formula ([5] p11), it is easy to check that if E is an exceptional divisor, then $[E]$ is an exceptional line bundle. Moreover, the inverse image $\Pi^\star L$ of an exceptional line bundle L by a finite sequence of blowing-ups is still an exceptional line bundle.

The following theorem has been proved by I. Nakamura when S contains a cycle of rational curves [21] (1.7). It should be noticed that any surface with $b_1(S) = 1$ and $b_2(S) = 0$ is minimal and $H^1(S, \mathbb{C}^\star) \simeq H^1(S, \mathcal{O}^\star)$ [17] II, p699.

Theorem 1. 8 *Let S be a (not necessarily minimal) compact complex surface such that $b_1(S) = 1$, with second Betti number $n = b_2(S) > 0$. Then there exists n exceptional line bundles L_j, $j = 0, \dots, n-1$, unique up to torsion by a flat line bundle $F \in H^1(S, \mathbb{C}^\star)$ such that:*

- *$E_j = c_1(L_j)$, $0 \le j \le n-1$ is a $\mathbb{Z}$-basis of $H^2(S, \mathbb{Z})$,*
- *$K_S L_j = -1$ and $L_j L_k = -\delta_{jk}$,*
- *$K_S = L_0 + \cdots + L_{n-1}$ in $H^2(S, \mathbb{Z})$*
- *For every $i = 0, \dots, n-1$ and for every flat line bundle $F \in H^1(S, \mathbb{C}^\star)$,*

$$\mathcal{X}(L_i \otimes F) = 0.$$

- *If $h^0(S, L_i \otimes F) \neq 0$, there exists an exceptional divisor C_i and a (perhaps trivial) flat effective divisor P_i such that $L_i \otimes F = [C_i + P_i]$.*

Proof: 1) By Donaldson theorem [13],there exists a $\mathbb{Z}$-basis $(E_i)_i$, $0 \le i \le n-1$, of $H^2(S, \mathbb{Z})/Torsion$ such that $E_i E_j = -\delta_{i,j}$. Moreover, since $p_g = h^2(S, \mathcal{O}_S) = 0$, the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\star \to 0$ yields line bundles L_i such that $E_i = c_1(L_i)$. The line bundles L_i are unique up to tensor product by flat line bundles. For a surface of class VII, the group of flat line bundles is $H^1(S, \mathbb{C}^\star)$. Let $c = \sum n_i E_i \in H^2(S, \mathbb{Z})$. Then $c^2 = -\sum n_i^2$, therefore $c^2 = -1$ if and only if $c = \pm E_i$. Replacing if necessary L_i by L_i^{-1} we may suppose that $KL_i \le 0$ for $i = 0, \dots, n-1$. By Riemann-Roch formula

$$(*) \qquad \mathcal{X}(L_i \otimes F) = \mathcal{X}(\mathcal{O}_S) + \frac{1}{2}(L_i^2 - KL_i) = \frac{1}{2}(-1 - KL_i) \in \mathbb{Z},$$

therefore $KL_i \le -1$. Since (L_i) is a $\mathbb{Z}$-base of $H^2(S, \mathbb{Z})$, $K = \sum_i k_i L_i$ with $k_i = -KL_i \ge 1$. From $-n = K^2 = -\sum_{i=0}^{n-1} k_i^2$ we deduce that $k_i = 1$ for $i = 0, \dots, n-1$. From $(*)$ we obtain

$$\mathcal{X}(L_i \otimes F) = 0.$$

2) If $h^0(S, L_i \otimes F) > 0$, then $L_i \otimes F = [C_i]$ where C_i is an effective divisor. Let

$$C_i = n_1 G_1 + \cdots + n_p G_p$$

be a decomposition into irreducible components.

- If G is an elliptic curve or a rational curve with a double point then $KG = -G^2 \geq 0$.
- If G is a nonsingular rational curve, $KG = -2 - G^2 \geq -1$ and $KG = -1$ if and only if G is an exceptional curve of the first kind.

Therefore the condition

$$-1 = KC_i = \sum_i n_i K G_i$$

implies that there is an exceptional curve of the first kind, say G_p. Now we prove the result by induction on $n = b_2(S) \geq 1$.
If $n = 1$, there is only one exceptional line bundle L and if $h^0(S, L \otimes F) \neq 0$, S is not minimal, hence a blow-up of a surface S' with $b_2(S') = 0$. Then $L \otimes F = [C + P]$ where C is an exceptional curve of the first kind and $P = 0$ (if S' has no curve) or P is flat perhaps not trivial (if S' is a Hopf surface). Suppose that $n > 1$:

- If $G_p \sim E_i$, then $-F = [n_1 G_1 + \cdots + n_{p-1} G_{p-1} + (n_p - 1) G_p]$ is a flat line bundle and $L'_i = L_i \otimes F = [G_p]$.
- If $G_p \sim E_j$, $j \neq i$, there is a flat line bundle F such that $L'_j = L_j \otimes F = [G_p]$.

Therefore, replacing L_j by L'_j, and changing if necessary the numbering we may suppose that $L_{n-1} = [C_{n-1}]$ with C_{n-1} an exceptional curve of the first kind. Let $\Pi : S \to S'$ be the blowing-down of C_{n-1}. Since L_i, $i \neq n-1$ is trivial in a neighbourhood of C_{n-1}, $L'_i = \Pi_\star L_i$ is a line bundle such that $L_i = \Pi^\star \Pi_\star L_i$ and we check easily, by projection formula, that $(L'_i)_{0 \leq i \leq n-2}$ is a family of exceptional line bundles. If $h^0(L_i) \neq 0$, then $h^0(L'_i) \neq 0$, whence by induction hypothesis, $L'_i = [C'_i + P'_i]$ with C'_i an exceptional divisor and P'_i an effective flat divisor. Therefore $L_i = \Pi^\star L'_i = [\Pi^\star C'_i + \Pi^\star P'_i]$. $\square$

1.2 Families of exceptional or flat line bundles

If S is minimal then, for any $i = 0, \ldots, n-1$ and for any flat line bundle F, $h^0(S, L_i \otimes F) = 0$. In all known examples S has a deformation into a non minimal one, hence we consider now versal deformation of S.
Let $\Pi : \mathcal{S} \to B$ be the versal deformation of $S \simeq S_0$, where B is the unit ball of $\mathbb{C}^N$, $N = \dim \mathrm{H}^1(\mathrm{S}, \Theta)$. Standard arguments of spectral sequences yield

Proposition 1. 9 *For $i = 0, \dots, n-1$, there exist line bundles $\pi_i : \mathcal{L}_i \to \mathcal{S}$ such that for every $u \in B$, the restriction $\pi_{i,u} : L_{i,u} \to S_u$ is an exceptional line bundle,*
Moreover, if $\mathcal{K} \to \mathcal{S}$ is the relative canonical bundle, we have

$$\mathcal{K} \sim \bigotimes_{i=0}^{n-1} \mathcal{L}_i$$

in $H^2(\mathcal{S}, \mathbb{Z}) \simeq H^2(S_0, \mathbb{Z})$.

Since $\mathbb{C}^\star$ is commutative any representation $\rho : \pi_1(S) \to GL(1, \mathbb{C}) \simeq \mathbb{C}^\star$ factorize through $H_1(S, \mathbb{Z})$, therefore any representation (hence any flat line bundle), is defined by $\gamma \mapsto f$ with $f \in \mathbb{C}^\star$. We shall denote by $F^f \in H^1(S, \mathbb{C}^\star) \simeq \mathbb{C}^\star$ this line bundle and we have defined a group morphism

$$\begin{array}{cccc} \varphi : & \mathbb{C}^\star & \to & H^1(S, \mathbb{C}^\star) \\ & \lambda & \mapsto & F^\lambda \end{array}$$

Lemma 1. 10 *1) For any holomorphic function $f : B \to \mathbb{C}^\star$ there exists a unique flat line bundle $\mathcal{F}^f \to \mathcal{S}$ such that $(\mathcal{F}^f)_{|S_u} = F^{f(u)}$.*
2) There exists over $\mathcal{S} \times \mathbb{C}^\star$ a flat line bundle, called the **tautological flat line bundle** *$\mathcal{F}$ such that for any $(u, \lambda) \in B \times \mathbb{C}^\star$, $\mathcal{F}_{S_u \times \{\lambda\}} = F^\lambda$.*

Proof: Let $\omega : \tilde{\mathcal{S}} \to \mathcal{S}$ be the family of universal covering spaces of $\mathcal{S}$. Then the fundamental group $\pi_1(S)$ operates diagonally on $\tilde{\mathcal{S}} \times \mathbb{C}$ by $\gamma.(p, z) = (\gamma.p, f(\Pi\omega(p))z)$. The quotient manifold is $\mathcal{F}^f$.
A similar construction gives the tautological flat line bundle $\mathcal{F}$. $\square$

Examples 1. 11 1) Suppose that $S = S(F)$ is a primary Hopf surface defined by

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \qquad 0 < |\alpha_1| \le |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0.$$

If $s = 0$ (resp. $s \neq 0$), S contains at least two elliptic curves E_1, E_2 (resp. only one elliptic curve E_2), where

$$E_1 = \{z \neq 0 \mid z_1 = 0\}/\{\alpha_2^p | p \in \mathbb{Z}\}, \quad E_2 = \{z \neq 0 \mid z_2 = 0\}/\{\alpha_1^p | p \in \mathbb{Z}\}.$$

Then for $i = 1, 2$, $\varphi(\alpha_i) = [E_i]$. In fact, if S is a diagonal Hopf surface, the cocycle of the line bundle associated to $[E_i]$ is given by

$$(z_1, z_2, \lambda) \sim (\alpha_1 z_1, \alpha_2 z_2, \alpha_i \lambda).$$

2) Following [6], let $S = S(F)$ be the minimal surface containing a GSS with $b_2(S) > 0$ defined by

$$F(z) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2).$$

Then S contains a cycle of rational curves $\Gamma = D_0 + \cdots + D_{n-1}$ such that $D_i^2 = -2$, $i = 0, \ldots, n-1$ and $\Gamma^2 = 0$. Let $t = trDF(0) \neq 0$ be the trace of the surface, then $\varphi(t) = [\Gamma]$. In fact the equation of Γ is $z_2 = 0$.
If in the expression of F, there is at least one index i such that $a_i \neq 0$, S has no elliptic curve. If $M(S)$ is the intersection matrix of the rational curves then $\det M(S) = 0$, hence the curves do not generate $H^2(S, \mathbb{Z})$; for every $m \geq 1$, every F flat, $H^0(S, -mK + F) = 0$, therefore there is no NAC divisor.
If $F(z) = (t^n z_1 z_2^n, tz_2)$, i.e. S is a Inoue (parabolic) surface, S contains an elliptic curve E and $-K = [E + \Gamma]$.
3) Following [15], let $S = S_M = \mathbb{H} \times \mathbb{C}/G_M$ with $M \in SL(3, \mathbb{Z})$ a unimodular matrix with eigenvalues α, β, $\bar{\beta}$ such that $\alpha > 1$, $\beta \neq \bar{\beta}$. Denote by (a_1, a_2, a_3) a real eigenvector associated to α and (b_1, b_2, b_3) an eigenvector associated to β. It can be easily checked that (a_1, b_1), (a_2, b_2) and (a_3, b_3) are linearly independent over $\mathbb{R}$. Let G_M generated by

$$g_0 : (w, z) \mapsto (\alpha w, \beta z),$$

$$g_i : (w, z) \mapsto (w + a_i, z + b_i) \quad \text{for} \quad i = 1, 2, 3.$$

If G is generated by g_i, $i = 1, 2, 3$, $\omega = dw \wedge dz$ is invariant under G hence yields a 2-form on $\mathbb{H} \times \mathbb{C}/G$. Moreover, $g_0^\star \omega = \alpha\beta\omega$, hence yields a non-vanishing twisted 2-form over S_M and $K = F^{1/\alpha\beta}$. A line bundle has no section for there is no curve.

1.3 Degeneration of blown-up Hopf surfaces

All surfaces containing GSS are degeneration of blown-up primary Hopf surfaces as it can be easily checked using contracting germs of mappings. We show in this section that if a surface can be deformed into a non minimal one then over a Zariski open set in the base of the versal deformation, there are blown-up Hopf surfaces.
We need a lemma comparing the versal deformation of a surface S with the versal deformation of a blowing-up S' of S.

Lemma 1. 12 *Let S be a compact complex surface of the VII-class (not necessarily minimal), let $\Pi : S' \to S$ be the blowing-up of S at the point z_0 and*

$$e_{z_0} : H^0(S, \Theta) \to T_{z_0} S$$

be the evaluation of global vector fields at z_0. Then, if $\mathcal{V}$ is a covering of S such that $H^1(S, \Theta) = H^1(\mathcal{V}, \Theta)$,
1) $h^0(S', \Theta') = dim\ Ker\ e_{z_0}$;
2) There exists a covering $\mathcal{V}' = (V_i')_{i \geq 0}$ of S' such that $H^1(S', \Theta') = H^1(\mathcal{V}', \Theta')$ with the following properties:

- *i) V_0' is the inverse image by Π of a ball V_0 centered at z_0,*
- *ii) V_0' meets only one open subset of the covering, say V_1' along a spherical shell,*

iii) For all $i \geq 1$, the restriction of Π on V_i' is an isomorphism on its image V_i,

iv) the canonical mapping $\Pi^\star : H^1(\mathcal{V}, \Theta) \to H^1(\mathcal{V}', \Theta')$ is injective,

v) A base of $H^1(S', \Theta')$ may be obtained from a base of $H^1(\mathcal{V}, \Theta)$ by adding cocycles induced on V_{01}' by (at most two) non-vanishing vector fields Z^i on V_0 such that the vectors $Z^i(z_0)$ generate a suplementary subspace of $Im\ e_{z_0}$ in T_{z_0}.

In particular $h^1(S', \Theta') = h^1(S, \Theta) + codim\ Im\ e_{z_0}$.

Proof: 1) is clear.
2) Let $\mathcal{U} = (U_i)_{i\geq 1}$ be a locally finite covering of S such that $H^1(S, \Theta) = H^1(\mathcal{U}, \Theta)$. It may be supposed that $z_0 \in U_i$ for $i = 1, \dots p$. Denote by $U_0' \subset\subset U_0$ balls centered at z_0 such that if $i > p$, then $U_0 \cap U_i = \emptyset$. Now, if $V_0 = U_0$, $V_1 = U_1 \backslash \overline{U_0'}$ and $V_i = U_i \backslash \overline{U_0}$, for $i > 1$, there are three coverings of S, $\mathcal{U}$, $\mathcal{U}_0 = (U_i)_{i\geq 0}$ and $\mathcal{V} = (V_i)_{i\geq 0}$ related by the relation

$$\mathcal{V} \prec \mathcal{U}_0 \prec \mathcal{U}.$$

The canonical mappings

$$H^1(\mathcal{U}, \Theta) \to H^1(\mathcal{U}_0, \Theta) \to H^1(\mathcal{V}, \Theta)$$

are isomorphism. We define a covering $\mathcal{V}' = (V_i')$ of S' by $V_i' = \Pi^{-1}(V_i)$. The canonical morphism $\Pi^\star$ is clearly injective and the evident mapping

$$s : H^1(\mathcal{V}', \Theta') \to H^1(\mathcal{V}, \Theta)$$

is clearly surjective. Let $\xi \notin Im\ e_{z_0}$ and θ a vector field on U_0 such that $\theta(z_0) = \xi$. Define $\eta = (\eta_{jk}) \in Z^1(\mathcal{V}, \Theta)$ by $\eta_{01} = \theta$ and $\eta_{jk} = 0$ if $\{j, k\} \neq \{0, 1\}$. If $\eta' = \Pi^\star(\eta) \in Z^1(\mathcal{V}', \Theta')$, η and η' are cocycles such that $s([\eta']) = [\eta] = 0$, but $[\eta'] \neq 0$. In fact if there exist vector fields X_0' on V_0' and X' on $S' \backslash \Pi^{-1}(U_0')$ such that $\theta' = X' - X_0'$ on V_{01}', we have $\theta = \Pi_\star X' - \Pi_\star X_0'$ on V_{01}. But since a vector field extends inside a ball, $\xi = \theta(z_0) = \Pi_\star X'(z_0)$ which is a contradiction. Therefore, $dim\ Ker\ s \geq codim\ Im\ e_{z_0}$ and it yields $h^1(\mathcal{V}', \Theta') \geq h^1(\mathcal{V}, \Theta) + codim\ Im\ e_{z_0}$. Now by Riemann-Roch-Hirzebruch-Atiyah-Singer theorem we have, since S and S' are VII-class surfaces,

$$h^1(S, \Theta) = h^0(S, \Theta) + 2b_2(S) \quad \text{and} \quad h^1(S', \Theta') = h^0(S', \Theta') + 2b_2(S').$$

Using $b_2(S') = b_2(S) + 1$, we obtain by a)

$$h^1(S', \Theta') = h^1(S, \Theta) + codim\ Im\ e_{z_0}$$

therefore

$$\begin{array}{rcl} h^1(\mathcal{V}, \Theta) + codim\ Im\ e_{z_0} & = & h^1(S, \Theta) + codim\ Im\ e_{z_0} \quad = \quad h^1(S', \Theta') \geq h^1(\mathcal{V}', \Theta') \\ & \geq & h^1(\mathcal{V}, \Theta) + codim\ Im\ e_{z_0} \end{array}$$

which completes the proof. □

Theorem 1. 13 *Let S be a surface of class VII_0^+, $n = b_2(S) \geq 1$ and $\mathcal{S} \to B$ its versal deformation. Assume that there is a point $v \in B$, $\lambda_i \in \mathbb{C}$, $i = 0, \ldots, n-1$ such that*

$$H^0(S_v, L_{i,v} \otimes F^{\lambda_i}) \neq 0.$$

Then there exists

- *Holomorphic functions $c_i : B \to \mathbb{C}$,*
- *Flat families of exceptional divisors $\mathcal{C}_i$ over $B \setminus H_i$, where $H_i = \{c_i = 0\}$,*

such that

- *For every $u \in B \setminus H_i$, $[C_{i,u}] = L_{i,u} \otimes F^{c_i(u)}$ and $F^{c_i(v)} = F^{\lambda_i}$,*
- *S_u is minimal if and only if $u \in M := \cap_{i=0}^{n-1} H_i$,*
- *S_u is a blown-up Hopf surface if and only if $u \in B \setminus \cup_{i=0}^{n-1} H_i$.*

Proof: 1) The surface S_v contains an exceptional curve of the first kind C. Changing if necessary the numbering we may suppose that $C = C_{n-1}$ and $L_{n-1,v} \otimes F^{\lambda_{n-1}} = [C_{n-1}]$. By stability theorem of Kodaira [16], there exists an open neighbourhood of $v \in B$ and a flat family $\mathcal{C}_{n-1}$ over V of exceptional curves of the first kind.
2) Let $pr_1 : \mathcal{S} \times \mathbb{C}^\star \to \mathcal{S}$ be the first projection and define the line bundle $\mathcal{M}_{n-1}$ over $\mathcal{S} \times \mathbb{C}^\star$ by

$$\mathcal{M}_{n-1} := pr_1^\star \mathcal{L}_{n-1} \otimes_{\mathcal{O}_{\mathcal{S}\times\mathbb{C}^\star}} \mathcal{F}$$

where $\mathcal{F}$ is the tautological flat line bundle of Lemma 1.10; by $p = \pi_{n-1} \times Id$, the sheaf $\mathcal{M}_{n-1}$ is flat over $B \times \mathbb{C}^\star$. For every $(u, \alpha) \in B \times \mathbb{C}^\star$, we have $(\mathcal{M}_{n-1})_{|S_u \times \{\alpha\}} \simeq (\mathcal{L}_{n-1})_{|S_u} \otimes F^\alpha$. By the semi-continuity theorem of Grauert, and because surfaces have no nonconstant meromorphic functions,

$$Z_{n-1} := \Big\{(u, \alpha) \in B \times \mathbb{C}^\star \mid h^0\big(S_u \times \{\alpha\}, (\mathcal{M}_{n-1})_{S_u \times \{\alpha\}}\big) = 1\Big\}$$

is an analytic subset of $B \times \mathbb{C}^\star$ and the dimension of the intersection $Z_{n-1} \cap V \times \mathbb{C}^\star$ is $N = \dim$ B by 1).
3) Let Z'_{n-1} be the irreducible component of Z_{n-1} such that $\mathcal{M}_{n-1|p^{-1}(Z'_{n-1} \cap V \times \mathbb{C}^\star)} = [\mathcal{C}_{n-1}]$. We have a flat family of curves

$$p : \mathcal{C}_{n-1} \to Z'_{n-1}$$

such that for $(u, \alpha) \in Z'_{n-1}$, $[C_{n-1,(u,\alpha)}] = L_{n-1,u} \otimes F^\alpha$. For $p = 0, 1$, the functions

$$\begin{array}{ccc} Z'_{n-1} & \to & \mathbb{N} \\ (u, \alpha) & \mapsto & h^p(C_{n-1,(u,\alpha)}, \mathcal{O}_{C_{n-1,(u,\alpha)}}) \end{array}$$

are constant (see [5] p 96). For $u \in V$ and α such that the section of $L_{n-1} \otimes F^\alpha$ vanishes on an exceptional curve of the first kind,

$$h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1, \quad h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$$

therefore $h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1$ and $h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$ everywhere, whence over each $(u,\alpha) \in Z'_{n-1}$ the analytic set is connected and does not contain any elliptic curve or cycle of rational curves.
Now, we show that the intersection $Z'_i \cap (V \times \mathbb{C}^\star)$ contains only one irreducible component: In fact, if $z = (u,\alpha)$ and $z' = (u,\alpha')$ are two points in Z'_{n-1} over $u \in V$, then

$$h^0(S_u, L_{n-1,u} \otimes F^\alpha) = h^0(S_u, L_{n-1,u} \otimes F^{\alpha'}) = 1,$$

whence $F^{\alpha/\alpha'}$ has a meromorphic section and by [20] (2.10), $F^{\alpha/\alpha'} = [D]$ with $D = mE + nF$, where E, F are elliptic curves or cycles of rational curves such that $E^2 = F^2 = 0$. It means that for $L_{n-1,u} \otimes F^\alpha = [C_{n-1,u}]$ and $L_{n-1,u} \otimes F^{\alpha'} = [C'_{n-1,u}]$ we have

$$[C_{n-1,u}] = L_{n-1,u} \otimes F^\alpha = L_{n-1,u} \otimes F^{\alpha'} \otimes F^{\alpha/\alpha'} = [C'_{n-1,u} + D]$$

and $C_{n-1,u}$ would not be connected, a contradiction.
As consequence Z'_i cannot accumulate on

$$B \times \{0\} \cap V \times \mathbb{P}^1(\mathbb{C}), \quad \text{or} \quad B \times \{\infty\} \cap V \times \mathbb{P}^1(\mathbb{C}).$$

By Remmert-Stein theorem the closure

$$G_{n-1} = \overline{Z'_{n-1}} \subset B \times \mathbb{P}^1(\mathbb{C})$$

is an irreducible analytic set of codimension one. The restriction

$$pr_1 : G_{n-1} \to B$$

is proper; therefore, $pr_1(G_{n-1})$ is an analytic subset of B. Since it contains the open set V, $pr_1(G_{n-1}) = B$. Now , (G_{n-1}, pr_1, B) is a ramified covering which has only one sheet over V, hence G_{n-1} is the graph of a holomorphic mapping $c_{n-1} : B \to \mathbb{P}^1(\mathbb{C})$. Define $H_{n-1} := c_{n-1}^{-1}(0) \cup c_{n-1}^{-1}(\infty)$. So far, we have considered the family of curves $\mathcal{C}_{n-1}$ over Z'_{n-1}. From now on, we shall consider it over $B_{n-1} := B \backslash H_{n-1}$.
4) If for $i \neq n-1$, $L_{i,v} \otimes F^{\lambda_i} = [C_{i,v}]$ with $C_{i,v}$ reducible, it means that $C_i = \pi^\star(C'_i)$ where π is a finite sequence of blowing-ups and C'_i is an exceptional curve of the first kind. By lemma (12), there is a deformation in which the blown-up points are moved outside C'_i. Therefore changing the point v in $V' \subset V$ we may suppose that $C_{i,v}$ is an exceptional curve of the first kind and apply 3).
5) We show now that for all $u \in B' := B \backslash \cup H_i$, S_u is a blown-up Hopf surface: By Iitaka theorem we may blow down the exceptional divisors $\mathcal{C}_{i|B'}$ over B'. Let $p : \mathcal{S} \to \mathcal{S}'$ be the canonical mapping and $\Pi' : \mathcal{S}' \to B'$ the induced family. Using classification of complex surfaces of class VII_0 with $b_2(S) = 0$ (see [18]), S'_u is a Hopf surface or a Inoue surface [15] of type S_M, $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$. We have to exclude the last three types: By [11] we may suppose that S has no non-trivial global vector field, therefore $N = h^1(S,\Theta) = 2n$ and restricting if necessary B we have $h^1(S_u,\Theta_u) = 2n$ for all $u \in B$. We denote

by $\mathcal{K}'$ the relative canonical bundle over $\mathcal{S}'$. For every $u \in B'$ there exists a unique $f(u) \in \mathbb{C}^\star$ such that $-K'_u = F^{f(u)}$; by Grauert theorem, $f : B' \to \mathbb{C}^\star$ is holomorphic. The relative canonical bundle $\mathcal{K}$ over $\mathcal{S}$ satisfies

$$\mathcal{K} = p^\star \mathcal{K}' \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i] = F^{f^{-1}} \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i].$$

If there exists a point $u \in B'$ such that S'_u is of type $S^{(+)}_{N,p,q,r;t}$ or $S^{(-)}_{N,p,q,r}$, then in a neighbourhood of u, all surfaces have the same type by [15] and by the theorem of Bombieri [15] p280, the function f is real valued hence constant, in particular globally defined. Therefore $\bigotimes_{i=0}^{n-1}[\mathcal{C}_i] = F^f \otimes \mathcal{K}$ extends over B, which is impossible. Remains the case S_M which is rigid: all the surfaces S'_u are isomorphic hence by (11) Example 3, $-K'_u = F^{\alpha\beta}$, therefore as before $\bigotimes_i [\mathcal{C}_i]$ should extend which is impossible.
6) The functions c_i are $\mathbb{C}$-valued: In fact, suppose that $c_i^{-1}(\infty) \neq \emptyset$; let A be an irreducible component of $c_i^{-1}(\infty)$. Since c has values in $\mathbb{C}$, there exists an index j such that c_j vanishes along A. Let μ (resp. ν) be the order of the pole (resp. zero) of c_i (resp. c_j) at a point $a \in A$. Then on a disc Δ containing a, $g := c_i^\nu c_j^\mu \in \mathcal{O}^\star(\Delta)$ and the family of positive divisors

$$[\nu \mathcal{C}_i + \mu \mathcal{C}_j] = \mathcal{L}_i^\nu \otimes \mathcal{L}_j^\mu \otimes F^g$$

extends on Δ. But it implies that $\mathcal{C}_i$ and $\mathcal{C}_j$ extend also, which yields a contradiction. □

Definition 1. 14 *Let S be a surface of class VII_0^+. We shall say that S is a degeneration of blown-up primary Hopf surfaces if there is a deformation $\mathcal{S} \to \Delta$ over the unit disc of $S \simeq S_0$, such that S_u is a blown-up primary Hopf surface for $u \neq 0$.*

If $\pi_1(S) = \mathbb{Z}$, a surface which can be deformed into a n times blown-up surface, is a degeneration of blown-up primary Hopf surfaces. The surface S_u is defined by a contraction

$$F_u(z) = (\alpha_1(u)z_1 + s(u)z_2^m, \alpha_2(u)z_2)$$
$$\text{with} \quad 0 < |\alpha_1(u)| \leq |\alpha_2(u)| < 1 \text{ and } s(u)(\alpha_2^m(u) - \alpha_1(u)) = 0$$

or

$$F_u(z) = (\alpha_1(u)z_1, \alpha_2(u)z_2 + s(u)z_1^m)$$
$$\text{with} \quad 0 < |\alpha_2(u)| \leq |\alpha_1(u)| < 1 \text{ and } s(u)(\alpha_1^m(u) - \alpha_2(u)) = 0.$$

In both cases there is at least one elliptic curve E_2 (resp. E_1) induced by $\{z_2 = 0\}$ (resp. $\{z_1 = 0\}$) and another E_1 (resp. E_2) induced by $\{z_1 = 0\}$ (resp. $\{z_2 = 0\}$) if $s = 0$. The trace

$$t(u) = tr(S'_u) = tr(S_u) = tr(DF_u(0)) = \alpha_1(u) + \alpha_2(u)$$

and the determinant

$$d(u) = \det DF_u(0) = \alpha_1(u)\alpha_2(u)$$

are bounded holomorphis functions on B' hence extend on B. They depend only on the conjugation class of F_u. We call $tr(u)$ the **trace of the surface** S_u.
By [17] II p696, we have the following description of the canonical bundle K'_u of S'_u: If S'_u is a diagonal Hopf surface then

$$K'_u = [-E_{1,u} - E_{2,u}] = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

If $s(u) \neq 0$, i.e. S is not diagonal, then since $\alpha_2^m(u) = \alpha_1(u)$,

$$K'_u = [-(m+1)E_{2,u}] = F^{\alpha_2(u)^{-(m+1)}} = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

By [5] I.9.1 (vii), the canonical bundle K_u of S_u over B' satisfies

$$K_u = p^\star K'_u \otimes \bigotimes_{0\leq i\leq n-1} [C_{i,u}].$$

If $\mathcal{K}$ denotes the relative canonical bundle and F denotes the tautological flat line bundle,

$$\mathcal{K} = F^{(\alpha_1\alpha_2)^{-1}} \otimes \bigotimes_i [\mathcal{C}_i] = F^f \otimes \bigotimes_i \mathcal{L}_i$$

where

$$f := \frac{\prod_i c_i}{\alpha_1\alpha_2}.$$

Since $\mathcal{K}$ and $\mathcal{L}_i$ are globally defined on B, F^f is globally defined on B and $f \in \mathcal{O}^\star(B)$. Twisting, for example, $\mathcal{L}_1$ by $F^{f^{-1}}$, we may suppose that $f = 1$ and then

$$c = \prod_{i=0}^{n-1} c_i = \alpha_1\alpha_2 = d \in \mathcal{O}^\star(B)$$

satisfies $\|c\|_\infty \leq 1$.

For contracting germs associated to surfaces containing GSS we refer to [6].

Proposition 1. 15 *Let S be a surface obtained by degeneration of blown-up minimal Hopf surfaces. If $tr(S) \neq 0$, then S contains a GSS and if $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a contracting germ of mappings associated to S, $tr(S) = trDF(0)$.*

Proof: One of the two functions α_i, $i = 1, 2$, say α_2, admit a limit $\alpha_2(0) \neq 0$, and $\alpha_1(0) = 0$. Therefore F^{α_2} is globally defined. Since $|\alpha_1(u)| < |\alpha_2(u)|$, the elliptic curve $E_{2,u}$ induced by $z_2 = 0$ exists for all $u \notin H$, hence $[E_{2,u}] = F^{\alpha_2}$. Since $u \mapsto h^0(S_u, F^{\alpha_2(u)})$ is upper-semicontinuous we obtain in S a flat cycle of rational curves $E_{2,0}$. By Enoki theorem, S contains a GSS. $\square$

2 Surfaces with a cycle of rational curves

Definition 2. 16 *Let S be a surface and C be an analytic subset of dimension one. We say that C is a r-cycle of rational curves if*

- *C is an elliptic curve when $r = 0$,*
- *C is a rational curve with a double point when $r = 1$, and*
- *$C = D_0 + \cdots + D_{r-1}$, is a reduced effective divisor such that D_i is a non-singular rational curve for $i = 0, \ldots, r-1$ and $D_0D_1 = \cdots = D_{r-2}D_{r-1} = D_{r-1}D_0 = 1$, $D_iD_j = 0$ in all other cases, when $r \geq 2$.*

We denote by $\sharp(C) = r \geq 0$ the number of rational curves of the cycle C.

With notations of theorem (8), we have

Theorem 2. 17 ([21] (1.7)) *Let S be a VII_0^+ surface with $n = b_2(S)$. Assume that S contains exactly one cycle C of rational curves such that $C^2 < 0$. Then*

- *$C \sim -(L_r + \cdots + L_{n-1})$ for some $1 \leq r \leq n-1$, if S is not an odd Inoue-Hirzebruch surface, or*
- *$C \sim -(L_0 + \cdots + L_{n-1}) + F_2$, with F_2 of order two, if S is an odd Inoue-Hirzebruch surface.*

Remark 2. 18 *If S is an even Inoue-Hirzebruch surface (=hyperbolic I-H surface), the two cycles C and C' satisfy*

$$C \sim -(L_r + \cdots + L_{n-1}), \quad C' \sim -(L_0 + \cdots + L_{r-1}).$$

The following lemma plays a crucial role in the computation of self-intersection of the cycle C.

Lemma 2. 19 ([21] (2.4)) *Let S be a surface of class VII_0^+ and without divisor D such that $D^2 = 0$. Let $L_I := \sum_{i\in I} L_i$, $L = L_I + F$, $F \in H^1(S, \mathcal{O}_S^\star)$ for a nonempty subset $I \subset [0, n-1]$. Then we have:*
i) If $I \neq [0, n-1]$, then $H^q(S, L) = 0$ for any q.
ii) If $L \otimes \mathcal{O}_C = \mathcal{O}_C$, then $I = [0, r-1]$, and $F = \mathcal{O}_S$, $K_S - L + C = \mathcal{O}_S$.
iii) If $LC_i = 0$ for any irreducible component C_i of C, then $I = [0, r-1]$.

2.1 Surfaces with numerically m-anticanonical divisor

Definition 2. 20 *Let S be a compact complex surface of the VII_0 class with $b_2(S) > 0$ and let $m \geq 0$ be an integer. We shall say that S admits a* **numerically m-anticanonical divisor** *if there exists a divisor D_m, and a flat line bundle F in $H^1(S, \mathbb{C}^\star)$ such that in $H^1(S, \mathcal{O}^\star)$,*

$$mK + F + [D_m] = 0.$$

We shall say that S admits a numerially anticanonical divisor, shortly a **NAC divisor**, *if there exists an integer m such that there exists a numerically m-anticanonical divisor.*

Remark 2. 21 Let S be a compact complex surface of the VII_0 class with $b_2(S) > 0$. Let $m \geq 1$ be an integer and $F \in H^1(S, \mathbb{C}^\star)$. Then $H^0(S, mK \otimes F) = 0$. In fact, a section of $mK \otimes F$ must vanish; let $[\Delta] = mK \otimes F$ be the associated divisor. Then

$$0 = (mK + F - \Delta)\Delta = mK\Delta - \Delta^2.$$

Since $mK\Delta \geq 0$ and $\Delta^2 \leq 0$, $\Delta^2 = 0$, therefore $b_2(S) = -K^2 = 0$.
This means that there is no numerically m-canonical divisors.

For the convenience of the reader we recall basic known facts with slightly different proofs (see [21] lemma (3.1))

Lemma 2. 22 *1) If D_m exists, it is a positive divisor, hence $H^0(S, -mK - F) \neq 0$.*
2) If an irreducible curve C meets the support $|D_m|$ of D_m then C is contained in $|D_m|$.

Proof: 1) We denote by D_i the irreducible components of D_m. Let $D_m = \sum_i k_i D_i = A - B$ where $A = \sum_{i|k_i>0} k_i D_i \geq 0$ and $B = \sum_{i|k_i<0}(-k_i) D_i \geq 0$ have no common component. If $B \neq 0$, $B^2 = \sum(-k_i)BD_i < 0$, there exists an index j such that $BD_j < 0$. Therefore,

$$0 \leq mKD_j = (-F - D_m)D_j = -D_m D_j = -AD_j + BD_j < 0$$

...a contradiction.
2) If C is an irreducible curve and meets $|D_m|$, $D_m C = -mKC \leq 0$, therefore C is contained in $|D_m|$. □

If there exists a non trivial divisor A such that $A^2 = 0$, then a numerically m-anticanonical divisor exists if and only if S is a Inoue surface. In this case, S contains an elliptic curve E, a cycle Γ of rational curves, $m = 1$ and $K + E + \Gamma = 0$. When there is no non-trivial flat divisor, the numerically m-anticanonical divisor D_m is clearly unique.

2.2 The reduction lemma

Definition 2. 23 *The least integer m such that there exists a numerically m-anticanonical divisor, is called the* **index** *of the surface S and will be denoted by $m(S)$.*
The index $m(S)$ of the surface S is the lcm of the denominators of the coefficients k_i of $D_1 = \sum_i k_i D_i$.

The proof of [12] (1.3) works under the following relaxed hypothesis:

Lemma 2. 24 (Reduction lemma) *Let S be a surface of class VII_0 with $b_2(S) > 0$ and index $m = m(S) > 1$. Then there exists a diagram*

$$\begin{tikzcd}
& S' \arrow[dr, "\pi'"] & \\
T \arrow[ur, "c"] \arrow[d, "\rho"'] & & Z' \\
Z \arrow[dr, "\pi"'] & & T' \arrow[u, "\rho'"'] \arrow[dl, "c'"] \\
& S &
\end{tikzcd}$$

where

i) (Z, π, S) *is a m-fold ramified covering space of* S*, branched over* D_m*, endowed with an automorphism group isomorphic to* $\mathbb{U}_m$ *which acts transitively on the fibers,*

ii) (T, ρ, Z) *is the minimal desingularization of* Z*,*

iii) (T, c, S') *is the contraction of the (possible) exceptional curves of the first kind,*

iv) S' *is a surface of class* VII_0*, with* $b_2(S) > 0$*, with action of* $\mathbb{U}_m$*, with index* $m(S') = 1$*,*

v) (S', π', Z') *is the quotient space of* S' *by* $\mathbb{U}_m$*,*

vi) (T', ρ', Z') *the minimal desingularization of* Z'*,*

vii) (T', c', S) *is the contraction of the (possible) exceptional curves of the first kind,*

such that the restriction over $S \setminus D$ *is commutative, i.e.*

$$\theta := \pi \circ \rho \circ c^{-1} = c' \circ {\rho'}^{-1} \circ \pi' : S' \setminus D' \to S \setminus D$$

and $(S' \setminus D', S \setminus D)$ *is a m-fold non ramified covering. Moreover*

- *S contains a GSS if and only if S' contains a GSS, S and*
- *The maximal divisors D and D' of S and S' respectively have the same number of cycles and branches.*

Corollary 2. 25 *If* S *admits a NAC divisor* D_m*, then the support of* D_m *contains a cycle.*

Proof: We may suppose that there is no divisor such that $D^2 = 0$. By reduction lemma it is sufficient to prove that the support of D_{-K} contains a cycle. Suppose that there exists a divisor D_{-K} and a flat line bundle F such that $K + D_{-K} + F = 0$. By Cartan-Serre duality

$$h^2(S, O_S(-D_{-K})) = h^2(S, K + F) = h^0(S, -F) = \left\{ \begin{array}{ll} 0 & \text{if F} \neq 0 \\ 1 & \text{if F} = 0 \end{array} \right.$$

By Riemann-Roch formula,

$$h^0(S,-F) - h^1(S,-F) = h^0(S,-F) - h^1(S,-F) + h^0(S,K+F) = \chi(F) = 0,$$

hence

$$h^1(S,-F) = \begin{cases} 0 & \text{if F} \neq 0 \\ 1 & \text{if F} = 0 \end{cases}$$

We have

$$0 \to \mathcal{O}_S(-D_{-K}) \to \mathcal{O}_S \to \mathcal{O}_{D_{-K}} \to 0$$

- If $F \neq 0$, the long exact sequence yields

  $$h^0(S,\mathcal{O}_{D_{-K}}) = h^1(S,\mathcal{O}_{D_{-K}}) = 1.$$

  The support of D_{-K} is connected and by [20] (2.7), $h^1(S,\mathcal{O}_{(D_{-K})_{red}}) \geq 1$ hence $(D_{-K})_{red}$ contains a cycle of rational curves.

- If $F = 0$, the associated long exact sequence and [20] (2.2.1) imply

  $$1 \leq h^0(S,\mathcal{O}_{D_{-K}}) = h^1(S,\mathcal{O}_{D_{-K}}) \leq 2.$$

  As before $h^1(S,\mathcal{O}_{(D_{-K})_{red}}) \geq 1$ and there is at least one cycle.
  If $h^1(S,\mathcal{O}_{(D_{-K})_{red}}) = 2$, then by the already quoted result (2.2.1) there are two cycles of rational curves.

$\square$

2.3 Characterization of Inoue-Hirzebruch surfaces

Lemma 2. 26 *Let S be a surface with a NAC divisor $D_{-K} = \sum k_i D_i$. We suppose that the maximal divisor contains a cycle of rational curves $C = D_0 + \cdots + D_{s-1}$ with $s \geq 1$ irreducible curves. If there exists $j \leq s-1$ such that $k_j = 1$, then $k_i = 1$ for all $i = 1, \ldots, s-1$ and C has no branch.*

Proof: **Case $s = 1$**: Since D_0 is a rational curve with a double point, the adjunction formula yields

$$D_0^2 = -KD_0 = D_{-K}D_0 = \sum k_i D_i D_0 = D_0^2 + \sum_{i>0} k_i D_i D_0.$$

hence $\sum_{i>0} k_i D_i D_0 = 0$ and C has no branch.
Case $s \geq 2$: By adjunction formula,

$$2 + D_j^2 = -KD_j = \sum_i k_i D_i D_j = D_j^2 + \sum_{i \neq j} k_i D_i D_j,$$

whence $2 = \sum_{i \neq j} k_i D_i D_j$. Since $\sum_{i \neq j} D_i D_j \geq 2$, D_j meets at most two curves D_{j-1} and D_{j+1} (one if the cycle contains two curves) and $k_{j-1} = k_{j+1} = 1$. By connectivity we conclude. $\square$

Proposition 2. 27 *Let S be a surface with a NAC divisor $D_{-K} = \sum k_i D_i$ and let $C = D_0 + \cdots + D_{s-1}$ be a cycle contained in the support of D_{-K}. One of the following conditions holds:*

i) There exists an index $0 \leq j \leq s-1$ such that $k_j = 1$, then S is a Inoue surface or a Inoue-Hirzebruch surface.

ii) For every $0 \leq j \leq s-1$, $k_j \geq 2$, then C has at least one branch and the support $D = |D_{-K}|$ of D_{-K} is connected. More precisely, if $k = \max\{k_i \mid 0 \leq i \leq s-1\}$, there exists a curve D_j and a branch $H_j > 0$ such that $k_j = k$ and $D_j H_j > 0$.

In particular, each connected component of $|D_K|$ contains a cycle.

Definition 2. 28 *With preceeding notations, a curve D_j in the cycle such that $D_j H > 0$ will be called the root of the branch H.*

Proof of (27): Taking if necessary a double covering it may be supposed, by [20](2.14), that the cycle has at least two curves, hence all the curves are regular. Let

$$D_{-K} = A + B = \sum_{i=0}^{p-1} k_i D + \sum_{i=p}^{p+q} k_i D_i$$

where the support of $A = \sum_{i=0}^{p-1} k_i D_i$ is the connected component of the cycle $C = \sum_{i=0}^{s-1} D_i$.
If B contains another cycle resp. an elliptic curve), then S is a Inoue-Hirzebruch surface [20] (8.1) (resp. a Inoue surface [20] (10.2)); suppose therefore that the support $|B|$ of B is simply-connected. We have to show that $|D_{-K}|$ is connected: In fact, let B_0 be a connected component of $|B|$. There is a proper mapping $p : S \to S'$ onto a normal surface S' with normal singularities $a = p(|A|)$ and $b = p(B_0)$. Since B is simply connected, F is trivial on a strictly pseudo-convex neighbourhood U of B and thus a holomorphic section of $-K - F$ yields a non vanishing holomorphic 2-form on $U \setminus B$, i.e. (S', b) is Gorenstein. If (S', b) would be an elliptic singularity, a two-fold covering T of S should contain three exceptional connected divisors such that their contractions $q : T \to T'$ would fullfil $h^0(T', R^1 q_\star \mathcal{O}_T) = 3$. However, by Leray spectral sequence, there is an exact sequence
(†)
$0 \to H^1(T', \mathcal{O}_{T'}) \to H^1(T, \mathcal{O}_T) \to H^0(T', R^1 q_\star \mathcal{O}_T) \to H^2(T', \mathcal{O}_{T'}) \to H^2(T, \mathcal{O}_S)$

where

$$p_g = h^2(T, \mathcal{O}_T) = 0, \quad q = h^1(T, \mathcal{O}_T) = 1 \quad \text{and} \quad h^2(T', \mathcal{O}_{T'}) = h^0(T', \omega_{T'}) \leq 1,$$

by Serre-Grothendieck duality and because T' has no non-constant meromorphic functions. By (†) we obtain a contradiction, hence (S', b) is a Gorenstein rational singularity, hence a Du Val singularity. However, such a singularity has a trivial canonical divisor, therefore $B = 0$ and $D_{-K} = A$.

i) Suppose that there is an index $j \leq s-1$ such that $k_j = 1$. By lemma (26), C has no branch and $s = p$. By adjunction formula, we have for every $0 \leq i \leq s-1$,

$$D_i^2 + 2 = -KD_i = D_{-K} D_i = k_{i-1} + k_{i+1} + k_i D_i^2,$$

therefore

$$(\star) \qquad (k_{i-1}-1)+(k_{i+1}-1)=(k_i-1)(-D_i^2).$$

By $(*)$, $k_i=1$ for all $0\leq i\leq p-1$.

- If for every $0\leq i\leq s-1$, $D_i^2=-2$, S is a Enoki surface by [14]. Moreover, by hypothesis, S admits a NAC divisor, hence is a Inoue surface or
- There is at least one index $k\leq s-1$ such that $D_k^2\leq -3$ whence $C^2<0$. If there is no other cycle, we have shown that $D_{-K}=C$, hence

$$-b_2(S)=K^2=C^2,$$

  S is a Inoue-Hirzebruch surface by [20] (9.2).

In case ii), if the maximum k is reached at a curve D_i, $i\leq s-1$, which is not a root, then applying again $(\star)$, we obtain $k_{i-1}=k_{i+1}=k_i=k$. By connexity, we reach a root. □

Proposition 2. 29 *Let S be a surface of class VII_0^+ admitting a NAC divisor D_m. Then one of the following conditions is fulfilled:*
i) The maximal divisor D is connected and contains a cycle with at least one branch;
ii) S is a Inoue surface or a (even or odd) Inoue-Hirzebruch surface.

Proof: By (25) S contains at least one cycle By (24) S' admits a NAC divisor D_{-K}. Then by (27), and (24) S' and S are of the same type. □

2.4 Surfaces with singular rational curve

Theorem 2. 30 *Let S be a surface with $n=b_2(S)$ containing a singular rational curve D_0 with a double point. Then*
1) $D_0\sim -(L_1+\cdots+L_{n-1})$ and the connected component containing D_0 is $D_0+D_1+\cdots+D_p$ for $1\leq p\leq n-1$, $D_i\sim L_i-L_{i-1}$, $1\leq i\leq p$. In particular $D_0^2=-(n-1)$, $D_1^2=\cdots=D_p^2=-2$.
2) The following conditions are equivalent:
i) There exists an integer $m\geq 1$ such that there exists a numerically m-anticanonical divisor,
ii) S contains a GSS.
When these conditions are fulfilled then S is either a Inoue-Hirzebruch surface and $m(S)=1$ or D_0 has a branch and its index satisfies $m(S)=n-1$.

Proof: 1) By (17), there exists $r\geq 1$ such that $D_0=-(L_r+\cdots+L_{n-1})$. We have $L_0D_0=0$, hence $r=1$ by (19) iii). We show by induction on $i\geq 1$ that if there is a non singular rational curve D_i such that $(D_0+\cdots+D_{i-1})D_i\neq 0$ then D_i is unique and $D_i=L_i-L_{i-1}$. By [20] (2.2.4), and unicity $D_0D_i=\cdots=D_{i-2}D_i=0$, $D_{i-1}D_i=1$ if $i\geq 2$.
Suppose that there is a curve, say D_1 such that $D_0D_1=1$. By [21] (2.6) (see lemma 32 below) we set $D_1\sim L_1-L_{I_1}$, then

$$1=D_0D_1=-(L_1+\cdots+L_{n-1})(L_1-L_{I_1})=1+(L_1+\cdots+L_{n-1})L_{I_1},$$

hence $I_1 = \{0\}$ and $D_1 \sim L_1 - L_0$. If there were another curve, say D_2 meeting D_0, the same argument shows that $D_2 = L_2 - L_0$, but in this case $D_1 D_2 = -1$ and it is impossible. It shows the unicity. Suppose that for $i > 1$, D_i exists and $D_i \sim L_i - L_{I_i}$. We have the equations

$$\begin{array}{ccccccc}
0 & = & D_0 D_i & = & -(L_1 + \cdots + L_{n-1})(L_i - L_{I_i}) & = & 1 + (L_1 + \cdots + L_{n-1}) L_{I_i} \\
0 & = & D_1 D_i & = & -(L_1 - L_0)(L_i - L_{I_i}) & = & -L_1 L_{I_i} + L_0 L_{I_i} \\
\vdots & & & & & \vdots & \\
0 & = & D_{i-2} D_i & = & -(L_{i-2} - L_{i-3})(L_i - L_{I_i}) & = & -L_{i-2} L_{I_i} + L_{i-3} L_{I_i} \\
1 & = & D_{i-1} D_i & = & -(L_{i-1} - L_{i-2})(L_i - L_{I_i}) & = & -L_{i-1} L_{I_i} + L_{i-2} L_{I_i}
\end{array}$$

which yield from the last to the second equation

$$i - 1 \in I_i,\ i - 2 \notin I_i,\ \ldots,\ 0 \notin I_i.$$

The first one implies that I_i contains exactly one index, hence

$$D_i \sim L_i - L_{i-1}.$$

We prove unicity as for $i = 1$.

2) By 1) the intersection matrix of the connected component of the cycle is the $(p+1, p+1)$ matrix, $p \geq 0$,

$$M = \begin{pmatrix}
-(n-1) & 1 & 0 & \cdots & \cdots & 0 \\
1 & -2 & 1 & 0 & & \vdots \\
0 & 1 & -2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 & 0 \\
\vdots & & \ddots & 1 & -2 & 1 \\
0 & \cdots & \cdots & 0 & 1 & -2
\end{pmatrix}$$

and it is easy to check that

$$\det M = (-1)^{p+1} \big[(n-1)(p+1) - p\big].$$

A numerically m-anticanonical divisor $D_m = \sum_{i=0}^{p} k_i D_i$ supported by the connected component containing the cycle satisfies the linear system

$$M \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_p \end{pmatrix} = \begin{pmatrix} -m(n-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore

$$(*) \qquad k_i = \frac{m(n-1)(p+1-i)}{(n-1)(p+1) - p}, \quad \text{for every } 0 \leq i \leq p.$$

Notice that

$$k_i = (p+1-i) k_p \quad \text{for every } 0 \leq i \leq p,$$

therefore all k_i are integers if and only if k_p is an integer. By assumption

$$D_m^2 = (-mK)^2 = -m^2 n,$$

and we have $D_m D_i = -mKD_i = 0$ for $i = 1, \ldots, p$. We have to consider two cases:
First case: $p \geq 1$ then

$$-m^2 n = \sum_{i=0}^{p} k_i D_i D_m = k_0 D_0 D_m = k_0^2 D_0^2 + k_0 k_1 = -\frac{m^2(n-1)^2(p+1)}{(n-1)(p+1)-p}$$

and this condition is equivalent to

$$p = n - 1.$$

Hence S has a numerically m-anticanonical divisor if and only if S contains n rational curves if and only if S contains a GSS by [12].
Replacing in $(*)$ we obtain

$$k_i = \frac{m(n-i)}{n-1}, \quad \text{for every } 0 \leq i \leq n-1.$$

Hence we conclude

$$m(S) = n - 1.$$

Second case: $p = 0$, then

$$-m^2 n = k_0^2 D_0^2 = -k_0^2(n-1)$$

However, this equation has no solution in integers, therefore the support of D_m is not connected. We conclude by (29).

□

Corollary 2. 31 *If S in the VII_0^+ class has a NAC divisor and contains a rational curve with a double point, then S contains a GSS.*

2.5 Cycles with at least two rational curves

Lemma 2. 32 ([21] (2.5), (2.6), (2.7)) *1) Let D be a nonsingular rational curve. If $D \sim a_0 L_0 + \cdots + a_{n-1} L_{n-1}$, then there exists a unique $i \in [0, n-1]$ such that $a_i = 1$ or -2, and $a_j = 0$ or -1 for $j \neq i$.*
2) If D is not contained in the cycle then $D \sim L_i - L_I$ for some $i \in [0, n-1]$ and $I \subset [0, n-1]$, $i \notin I$.
3) Let D_1 and D_2 two distinct nonsingular rational curves such that $D_1 \sim L_{i_1} - L_{I_1}$, $D_2 \sim L_{i_2} - L_{I_2}$, then $i_1 \neq i_2$.

Proposition 2. 33 *Let S be a surface containing a cycle $C = D_0 + \cdots + D_{s-1}$ of $s \geq 2$ nonsingular rational curves. Suppose that there exists a rational curve E such that $E.C = 1$, and let $D = \sum_i D_i$ be the maximal connected divisor containing C. Then every curve D_i is of the type $D_i \sim L_i - L_I$ for $i \notin I$.*

Proof: By (32), every curve D_i is of type $D_i \sim L_i - L_I$ (type **a**) or $D_i \sim -2L_i - L_I$ (type **b**). Suppose that C contains at least one curve of type **b**. Since $(-2L_i - L_I)(-2L_j - L_J) \leq 0$ two such curves cannot meet, in particular if $s = 2$, C contains at most one curve of type **b**.
Suppose $s \geq 3$. If D_i of type **b**, D_i meets curves D_j of type **a**. We have

$$1 = (-2L_i - L_I)(L_j - L_J) = -2L_iL_j - L_jL_I + 2L_iL_J + L_IL_J$$

and

- either $j \in I$, $i \neq j$, $i \notin J$, $I \cap J = \emptyset$, whence

$$D_i + D_j \sim -2L_i - L_I + L_j - L_J = -2L_i - L_{I\cup J\setminus\{j\}}.$$

- or $j \notin I$, $i = j$, $i \notin J$ and $I \cap J$ contains one element, say k. Setting $I' = I \setminus \{k\}$ and $J' = J \setminus \{k\}$ we obtain

$$D_i + D_j \sim -2L_k - L_{\{i\}\cup I'\cup J'}.$$

By ([21] (1.7)), there is a proper smooth family of compact surfaces $\pi : \mathcal{S} \to \Delta$ over the unit disc, a flat divisor $\mathcal{C}$ such that $C_0 = C$ and for $u \neq 0$, $C_u \sim D' + (C - D_i - D_j)$, with $D' \sim D_i + D_j$ of type **b**, in particular $\sharp(C_u) = \sharp(C) - 1$. Therefore, repeting if necessary such a deformation we obtain a contradiction if C contains two curves of type **b**. Now if C contains a curve of type **b**, we take a double covering $p : S' \to S$ of S. The surface S' of type VII_0 satisfies $b_2(S') = 2b_2(S)$ and by [20] (2.14) contains a cycle of $2s$ rational curves. Applying the same arguments to S', $p^\star D_i \sim -2p^\star L_i - p^\star L_I$ is the union of two rational curves of type **b**, hence a contradiction. Finally, there is no curve of type **b**. $\square$

Lemma 2. 34 *Let S be a surface containing a cycle $C = D_0 + \cdots + D_{s-1}$ of $s \geq 2$ nonsingular rational curves and $D = \sum_{i=0}^{p} D_i$ the maximal connected divisor containing C.*
1) Let D_j, D_k and D_l three distinct nonsingular rational curves, $D_j \sim L_{i_j} - L_{I_j}$, $D_k \sim L_{i_k} - L_{I_k}$, $D_l \sim L_{i_l} - L_{I_l}$. Then $I_j \cap I_k \cap I_l = \emptyset$.
2) If D_j, D_k are two distinct nonsingular rational curves, then $\mathrm{Card}(I_j \cap I_k) \leq 1$.

Proof: The surface S can be deformed into a blown-up Hopf surface $\pi : \mathcal{S} \to \Delta$ with a flat family $\mathcal{C}$, where $C_0 = C = D_0 + \ldots + D_{s-1} \sim -(L_r + \cdots + L_{n-1})$, and C_u, $u \neq 0$, is an elliptic curve blown-up $n - r$ times. We denote by $\Pi = \Pi_{0,u} \cdots \Pi_{n-1,u} : S_u \to S'_u$ the composition of blowing-ups and $(\Pi_{i+1} \cdots \Pi_{n-1})^\star(E_{i,u}) \sim L_i$. We have $D_j \sim L_{i_j} - L_{I_j}$, $j = 0, \ldots, s-1$, hence D_j is homologous to an exceptional rational curve of the first kind E_{i_j} blown-up $\mathrm{Card}(I_j)$ times. Since an exceptional curve cannot blow-up three rational curves we have the first assertion. Moreover two distinct exceptional rational curve of the kind cannot be blown-up two times by the same curves, whence the second assertion. $\square$

Proposition 2. 35 *Let S be a minimal surface containing a cycle $C = D_0 + \cdots + D_{s-1} \sim -(L_r + \cdots + L_{n-1})$ of $s \geq 1$ nonsingular rational curves. Then, $r = s$ and numbering properly the line bundles L_i, for $i = 0, \ldots, r-1$, we have $D_i \sim L_i - L_{I_i}$. Moreover $\cup_{i=0}^{s-1} I_i = [0, n-1]$ and*

$$\sharp(C) - C^2 = b_2(S).$$

Proof: 1) The case $s = 1$ has been proved in [21] (see lemma (30)).
2) **If $s = 2$**, $D_0 \sim L_{i_0} - L_{I_0}$, $D_1 \sim L_{i_1} - L_{I_1}$. We have

$$2 = D_0 D_1 = -L_{i_0} L_{I_1} - L_{i_1} L_{I_0} + L_{I_0} L_{I_1},$$

whence $i_1 \in I_0$, $i_0 \in I_1$, $I_0 \cap I_1 = \emptyset$. Setting $I'_0 = I_0 \setminus \{i_1\}$ and $I'_1 = I_1 \setminus \{i_0\}$, we obtain

$$D_0 \sim L_{i_0} - L_{i_1} - L_{I'_0}, \quad D_1 \sim L_{i_1} - L_{i_0} - L_{I'_1}, \quad \text{with } I'_0 \cap I'_1 = \emptyset,\ \{i_0, i_1\} \cap (I'_0 \cup I'_1) = \emptyset.$$

Therefore

$$-(L_r + \cdots + L_{n-1}) \sim C = D_0 + D_1 \sim -(L_{I'_0} + L_{I'_1})$$

i.e. $I'_0 \cup I'_1 = [r, n-1]$. Let $I = \{i_0, i_1\} \cup I'_0 \cup I'_1$ and $I' = [0, n-1] \setminus I$. Of course, $L_{I'}.D_0 = L_{I'}.D_1 = 0$, whence by (19) 3), $I' = [0, r-1]$ which is impossible. Therefore $I' = \emptyset$ and $\{i_0, i_1\} \cup I'_0 \cup I'_1 = [0, n-1]$, i.e. $r = 2$ and

$$\sharp(C) - C^2 = 2 + (n-2) = b_2(S).$$

3) **If $s \geq 3$**, $D_j \sim L_{i_j} - L_{I_j}$, $j = 0, \ldots, s-1$ and we may suppose that

$$D_0 D_1 = \ldots = D_{s-2} D_{s-1} = D_{s-1} D_0 = 1.$$

Since there is a deformation of S in which $D_j + D_{j+1} \sim L_{i_j} + L_{i_{j+1}} - L_{I_j} - L_{I_{j+1}}$ is deformed into a nonsingular rational curve D'_u in a (perhaps non minimal) surface, contained in a cycle C'_u, hence $D_j + D_{j+1}$ must be of the homological type **a**, and either $i_j \in I_{j+1}$, $i_{j+1} \notin I_j$ or $i_j \notin I_{j+1}$, $i_{j+1} \in I_j$. Moreover the equality

$$1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j} + L_{I_j} L_{I_{j+1}}$$

implies that $I_j \cap I_{j+1} = \emptyset$, whence

$$\forall j, \quad 1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j}.$$

4) Now, **we show by induction on $s \geq 2$ that for all j, $0 \leq j \leq s-1$, there is a unique index, denoted $\sigma(j)$, such that $i_j \in I_{\sigma(j)}$.** This assertion is evident if $s = 2$, therefore we suppose that $s \geq 3$. By 3) it is possible to choose the numbering in such a way that $i_{s-1} \in I_{s-2}$. We choose a deformation $\Pi : \mathcal{S} \to \Delta$ of S over the disc endowed with a flat family $\mathcal{C}$ of curves such that $C_0 = C$, and for $u \neq 0$,

$$D'_u \sim D_{s-2} + D_{s-1} \sim L_{i_{s-2}} + L_{i_{s-1}} - (L_{I_{s-2}} + L_{I_{s-1}}) = L_{i_{s-2}} - (L_{I'_{s-2}} + L_{I_{s-1}}),$$

with $I'_{s-2} = I_{s-2} \setminus \{i_{s-1}\}$. Consider the cycle $C_u = D_{0,u} + \cdots + D_{s-3,u} + D'_u$ of $s-1$ rational curves in the (perhaps non minimal) surface S_u; by the induction hypothesis, since $i_{s-1} \in I_{s-2}$ and $I'_{s-2} \cap I_{s-1} = \emptyset$, for all $0 \leq j \leq s-2$, there is a unique index $\sigma(j)$, $0 \leq \sigma(j) \leq s-1$ such that $i_j \in I_\sigma(j)$. Repeting this argument with $D_{s-3} + D_{s-2}$ we obtain the result. Therefore we have a well-defined mapping $\sigma : \{0, \dots, s-1\} \to \{0, \dots, s-1\}$ such that for all j, $0 \leq j \leq s-1$, $i_j \in I_{\sigma(j)}$.
5) Setting $I'_j = I_j \setminus \sigma^{-1}(j)$ we have

$$\begin{aligned} -(L_r + \cdots + L_{n-1}) &\sim C = D_0 + \cdots + D_{s-1} \sim L_{i_0} + \cdots + L_{i_{s-1}} - (L_{I_0} + \cdots + L_{I_{s-1}}) \\ &\sim -(L_{I'_0} + \cdots + L_{I'_{s-1}}) \end{aligned}$$

therefore

$$[r, n-1] = I'_0 \cup \cdots \cup I'_{s-1} \quad \text{with} \quad \{i_0, \dots, i_{s-1}\} \cap \left(I'_0 \cup \cdots \cup I'_{s-1}\right) = \emptyset$$

If $I = \{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1}$ and $I' = [0, n-1] \setminus I$, we have $L_{I'} D_j = 0$ for all $j = 0, \dots, s-1$, then if $I' \neq \emptyset$, lemma (19) would imply that $I' = [0, r-1]$ and this is impossible, hence $I' = \emptyset$ and $\{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1} = [0, n-1]$, i.e. $r = s$ and $\{i_0, \dots, i_{r-1}\} = [0, r-1]$. Finally

$$\sharp(C) - C^2 = s + (n - r) = n = b_2(S).$$

□

Corollary 2. 36 *Let S be a minimal surface with $b_2(S) \geq 1$. If E is a k-cycle, $k = \sharp(E) \geq 0$, such that $i_\star H_1(E, \mathbb{Z}) = H_1(S, \mathbb{Z})$, then $\sharp(E) - E^2 = b_2(S)$.*

Remark 2. 37 If $[H_1(S, \mathbb{Z}) : i_\star H_1(E, \mathbb{Z})] = 2$, S is an odd Inoue-Hirzebruch surface with only one cycle C. Then we have $\sharp(C) - C^2 = 2b_2(S)$ (see [20] (2.13)).

For surfaces containing a divisor D such that $D^2 = 0$, i.e. Enoki surfaces, the situation is well understood: They all contain GSS, D is a cycle of $b_2(S)$ rational curves $D = D_0 + \cdots + D_{n-1}$ such that $D_0^2 = \cdots = D_{n-1}^2 = -2$ and S admits a numerically anticanonical divisor if and only if S is a Inoue surface, in which case $h^0(S, -K) = 1$. For surfaces such that $D^2 < 0$ for any divisor we have the following theorem:

Theorem 2. 38 *Let S be a compact complex surface of class VII_0^+. We suppose that there is no divisor such that $D^2 = 0$. Then the following properties are equivalent:*

i) S contains a GSS,

ii) S admits a NAC divisor.

iii) S contains $b_2(S)$ rational curves,

Proof: $i) \Leftrightarrow iii)$ by [12].
$i) \Rightarrow ii)$ The intersection matrix $M(S)$ is negative definite and the rational curves give a $\mathbb{Q}$-base of $H^2(S,\mathbb{Q})$, whence there is an integer $m \geq 1$ and a divisor D_m such that in $H^2(S,\mathbb{Z})$, $mK + D_m = 0$. By lemma (22), D_m is effective.
$ii) \Rightarrow i)$ By lemma (24) we may suppose that the index satisfies $m(S) = 1$, i.e. S has a NAC divisor D_{-K}. By theorem (30) we may suppose that there is no singular curve. By lemma (27) there are two cases: S is a Inoue-Hirzebruch surface, in particular contains a GSS, or S contains a cycle with at least one branch. Therefore we have to prove the result in the second case. Let $D = \sum_{i=0}^p D_i$ be the maximal divisor, where $C = D_0 + \cdots + D_{s-1}$ is the cycle. By (33), we have in $H^2(S,\mathbb{Z})$,

$$-(L_0 + \cdots + L_{n-1}) = -K = \sum_{i=0}^{p} k_i D_i = \sum_{i=0}^{p} k_i (L_i - L_{I_i}),$$

where $k_i \geq 1$. If $p < n-1$, the curve D_{n-1} is missing, hence there is exactly one index j such that $n-1 \in I_j$ and $k_j = 1$. Moreover by Proposition (35), $\cup_{i=0}^{s-1} I_i = [0, n-1]$, therefore $j \leq s-1$. By lemma (26), S would be a Inoue-Hirzebruch surface, which is a contradiction. □

Corollary 2. 39 ([19]) *Let S be a surface of class VII_0 with $b_2(S) > 0$. Then S is a Inoue-Hirzebruch surface if and only if there exists two twisted vector fields $\theta_1 \in H^0(S, \Theta \otimes F_1)$, $\theta_2 \in H^0(S, \Theta \otimes F_2)$, where F_1, F_2 are flat line bundles, such that $\theta_1 \wedge \theta_2(p) \neq 0$ at at least one point $p \in S$.*

Proof: $\theta_1 \wedge \theta_2$ is a non trivial section of $-K \otimes F$ whence S contains a GSS. We conclude by [10] th. 5.5. □

3 On classification of bihermitian surfaces

3.1 Conformal and complex structures

We consider connected oriented conformal 4-manifolds (M^4, c) with two complex (i.e. integrable almost-complex) structures J_1, J_2 which induce the same orientation. Given a riemannian metric g in c, (M^4, g, J_1, J_2) is called a **bihermitian surface** relatively to the conformal class c if

- J_i are orthogonal with respect to the metric, i.e. $g(J_iX, J_iY) = g(X, Y)$, $i = 1, 2$,
- J_1 and J_2 are independent, i.e. there is a point $x \in M$ such that $J_1(x) \neq \pm J_2(x)$.

The triple (c, J_1, J_2) is called a bihermitian structure on M^4. If moreover $J_1(x) \neq \pm J_2(x)$ everywhere, the bihermitian structure (c, J_1, J_2) is called **strongly bihermitian**.

Given two such almost-complex structures J_1 and J_2, denote by f the smooth function, called the angle function

$$f = \frac{1}{4}\mathrm{tr}({}^{\mathrm{t}}\mathrm{J}_1\mathrm{J}_2) = -\frac{1}{4}\mathrm{tr}(\mathrm{J}_1\mathrm{J}_2).$$

By Cauchy-Schwarz inequality, $|f| \leq 1$ and $f(x) = \pm 1$ if and only if $J_1(x) = \pm J_2(x)$. Since $J_1 J_2 \in SO(4)$ and $J_2 J_1$ is the inverse of $J_1 J_2$, it is easy to check that

$$J_1 J_2 + J_2 J_1 = -2f\, Id.$$

Moreover $J_1(x)$ and $J_2(x)$ anticommute if and only if $f(x) = 0$.

Another conformal structure is provided by the Weyl curvature tensor: The riemannien curvature tensor R of type (3,1) has a classical decomposition, under the orthogonal group $O(4)$, into three parts given by the scalar curvature, the Ricci curvature tensor without trace and the Weyl curvature tensor W of type (3,1) which is a conformal invariant [4].
Let $* : \bigwedge^2 T^\star M \to \bigwedge^2 T^\star M$ be the star-Hodge operator, with $\star^2 = Id$ and two eigenvalues ± 1. We denote by $\bigwedge^2_+$ (resp. $\bigwedge^2_-$) the eigenspace associated to $+1$ (resp. -1). Since $*$ depends only on c, the splitting $\bigwedge^2 T^\star M = \bigwedge^2_+ \oplus \bigwedge^2_-$ is a conformal invariant. Let $W(g) : \bigwedge^2 T^\star M \to \bigwedge^2 T^\star M$ be the Weyl curvature tensor of type (2,2), with restrictions $W_\pm \in End(\bigwedge^2_\pm)$ over $\bigwedge^2_\pm$. The riemannian conformal class of (M^4, g) is called **anti-self-dual (ASD)** if $*W(g) = -W(g)$, or equivalently $W_+ = 0$.
The geometric meaning of ASD condition stems from Atiyah-Hitchin-Singer theorem [3]: Let

$$Z = \{J \in SO(TM) \mid J^2 = -Id\} = SO(TM)/U(2) \to M$$

be the twistor space, i.e. the space of all orthogonal almost-complex structures over (M, g), inducing the orientation of M. The fiber is isomorphic to the Riemann sphere S^2. Since there is only one complex structure on S^2, Z is a differentiable fiber bundle with complex fiber $\mathbb{P}^1(\mathbb{C})$. The complex structure on the fiber and on the base yield a canonical almost-complex structure $\mathbb{J}$ of the twistor space Z, which is not integrable in general. The theorem of Atiyah-Hitchin-Singer asserts that $\mathbb{J}$ is integrable if and only if the metric g is ASD, hence any compatible complex structure at a point $x \in M$ extends into a compatible complex structure over a neighbourhood of x. However, it does not extends to the whole manifold and perhaps (M, c) admits no complex structure (for instance S^4). **The aim is to classify compact 4-manifolds with several compatible complex structures, or at least to give necessary conditions for their existence**. When there are more than two compatible almost structures, we have

Proposition 3. 40 ([22]) *If an oriented riemannian 4-manifold (M, g) admits three independent compatible complex structures then the metric g is anti-self-dual.*

By Pontecorvo's classification of ASD bihermitian surfaces [22] Prop. 3.7, these surfaces are hyperhermitian. Following C.P. Boyer [4] we define **hyperhermitian complex surfaces** as oriented compact conformal 4-manifolds $(M, c, \mathcal{F})$ with a 2-sphere $\mathcal{F}$ of compatible complex structures generated by two anti-commuting ones. A hyperhermitian 4-manifold $(M, c, \mathcal{F})$ must be one of the following

- A flat complex torus
- A K3 surface with Ricci-flat Kähler metric, or
- A special Hopf surface

in particular they all have $b_2(M) = 0$. We refer to [22] and [4] for details.
Now we focus on the case where first Betti number $b_1(M)$ is odd; for the even case see [2].

3.2 Numerically anticanonical divisor of a bihermitian surface

For the convenience of the reader we recall the results used in the sequel (see [1, 2, 22]): Denote by

$$F_i^g(.\ ,\ .) = g(J_i\ .\ ,\ .), \quad i = 1, 2$$

the Kähler forms of (g, J_1) and (g, J_2) respectively, θ_1^g, θ_2^g their Lee forms, i.e.

$$dF_i^g = \theta_i^g \wedge F_i^g, \quad i = 1, 2$$

We furthermore denote by $[J_1, J_2] = J_1 J_2 - J_2 J_1$ the commutator of J_1 and J_2 and we consider the real J_i-anti-invariant 2-form

$$\Phi^g(.\ ,\ .) = \frac{1}{2} g([J_1, J_2].\ ,\ .),$$

and the corresponding complex $(0, 2)$-forms

$$\sigma_i^g(.\ ,\ .) = \Phi^g(.\ ,\ .) + i\Phi^g(J_i.\ ,\ .).$$

Then σ_i^g, $i = 1, 2$ are smooth sections of the anti-canonical bundle $K_{J_i}^{-1} \simeq \bigwedge_{J_i}^{(0,2)}(M)$ of (M, J_i) and $\sigma_i^g(x) = 0$ if and only if $\Phi^g(x) = 0$ if and only if $J_1(x) = \pm J_2(x)$. Therefore the common zero set of σ_i is exactly $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$, where

$$\mathcal{D}^+ = f^{-1}(1) = \{x \in M \mid J_1(x) = J_2(x)\},$$
$$\mathcal{D}^- = f^{-1}(-1) = \{x \in M \mid J_1(x) = -J_2(x)\}.$$

Lemma 3. 41 ([1, 2]) *Let (M, c, J_1, J_2) be a bihermitian surface. Then, for any metric g in the conformal class c, the 1-forms θ_1^g, θ_2^g and σ_1^g satisfy the following properties:*
i) If M is compact, then $d(\theta_1^g + \theta_2^g) = 0$,
ii) $\bar{\partial}_{J_1} \sigma_1^g = -\frac{1}{2}(\theta_1^g + \theta_2^g)^{(0,1)} \otimes \sigma_1^g$,
where $(\ .\ )^{(0,1)}$ denotes the $(0, 1)$ part and $\bar{\partial}_{J_1}$ is the Cauchy-Riemann operator relatively to J_1.

Proof: i) By [2], $\left(d(\theta_1^g+\theta_2^g)\right)_+=0$. Let $\delta=-\star d\star$ be the adjoint of d. Setting $\varphi=\theta_1^g+\theta_2^g$, $\Delta d\varphi=d\delta d\varphi=-d\star d\star d\varphi=0$, for by assumption $*d\varphi=-d\varphi$. Hence $\delta d\varphi=0$, and $|d\varphi|^2=(d\varphi,d\varphi)=(\varphi,\delta d\varphi)=0$.
ii) The proof in [2] Lemma 3 is local. $\square$

Proposition 3. 42 *Let (M,c,J_1,J_2) be a compact bihermitian surface. Then there exists a topologically trivial line bundle $L\in H^1(M,\mathbb{R}_+^\star)$ such that σ_i is a non-trivial holomorphic section of $K_{J_i}^{-1}\otimes L$, in particular*

$$H^0(M,K_{J_i}^{-1}\otimes L)\neq 0,\quad i=1,2,$$

and $\mathcal{D}^+$, $\mathcal{D}^-$ are empty or complex curves for both (M,J_1) and (M,J_2). Moreover, if (c,J_1,J_2) is strongly bihermitian, then $K_{J_i}=L$, in particular, $b_2(M)=0$.

Proof: By (41), $\theta_1^g+\theta_2^g$ is closed, hence there is an open covering $(U_j)_{j\in I}$ and $\mathcal{C}^\infty$ functions $\Phi_j:U_j\to\mathbb{R}$, such that for the local metric $g_j=\exp(\Phi_j)g$, the local Lee forms satisfy $\theta_1^{g_j}+\theta_2^{g_j}=0$, i.e. $(\theta_1^g+\theta_2^g)_{|U_j}=-2d\Phi_j$. Setting $c_{jk}=\exp(\Phi_j-\Phi_k)\in\mathbb{R}_+^\star$, $\sigma_i^j=\sigma_i^{g_j}$, we obtain a topologically trivial line bundle $L=[(c_{jk})]\in H^1(M,\mathbb{R}_+^\star)$, and a holomorphic section $\sigma_i=(\sigma_i^j)$ of $K_{J_i}^{-1}\otimes L$. If (c,J_1,J_2) is strongly bihermitian $\mathcal{D}=\emptyset$, therefore σ_i is a non-vanishing holomorphic section of $K_{J_i}^{-1}\otimes L$. $\square$

3.3 Bihermitian surfaces with odd first Betti number

The Kodaira dimension of a compact bihermitian surface with odd first Betti number is $\kappa=-\infty$ ([1] thm 1), hence the minimal models of (M,J_1) and of (M,J_2) are in class VII_0. In this section, we shall complete (and simplify) the classification theorem of V. Apostolov [1]. We need first

Lemma 3. 43 *Let S be a complex surface and let $\Pi:S'\to S$ be the blowing-up of $x\in S$, and $E=\Pi^{-1}(x)$ its exceptional curve.*
1) If there exists a flat line bundle L' on S' such that $H^0(S,K_{S'}^{-1}\otimes L')\neq 0$, then for $L=\Pi_\star L'$, $H^0(S,K_S^{-1}\otimes L)\neq 0$.
2) If there exists a flat line bundle L on S such that $H^0(S,K_S^{-1}\otimes L)\neq 0$, and if Π blows-up a point on the effective twisted anticanonical divisor, then for $L'=\Pi^\star L$, $H^0(S',K_{S'}^{-1}\otimes L')\neq 0$.

Proof: 1) The coherent sheaf $L=\Pi_\star L'$ is locally trivial since on a simply connected neighbourhood U of the exceptional curve E, $L_{|U}$ is trivial. The line bundle $K_S^{-1}\otimes L$ has a section on $S\setminus\Pi(E)$ which extends by Hartogs theorem.
2) We have $K_{S'}^{-1}\otimes L'=\pi^\star(K_S^{-1}\otimes L)-E$ hence a section σ of $K_S^{-1}\otimes L$ yields a section of $K_{S'}^{-1}\otimes L'$ if and only if x belongs to the zero set of σ. $\square$

Theorem 3. 44 *Let (M,c,J_1,J_2) be a compact bihermitian surface with odd first Betti number.*

1) If (M, c, J_1, J_2) is strongly bihermitian (i.e $\mathcal{D} = \emptyset$), then the complex surfaces (M, J_i) are minimal and either a Hopf surface covered by a primary one associated to a contraction $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ of the form

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a\alpha^{-1} z_2),$$

$$\text{with} \quad a, s \in \mathbb{C},\ 0 < |\alpha|^2 \leq a < |\alpha| < 1,\ (a^m - \alpha^{m+1})s = 0,$$

or else (M, J_i) are Inoue surfaces $S^+_{N,p,q,r;t}$, $S^-_{N,p,q,r}$.
2) If (M, c, J_1, J_2) is not strongly bihermitian, then $\mathcal{D}$ has at most two connected components, (M, J_i), $i = 1, 2$, contain GSS and the minimal models S_i of (M, J_i) are

- *Surfaces with GSS of intermediate type if $\mathcal{D}$ has one connected component*
- *Hopf surfaces of special type (see* [22] 2.2*), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if $\mathcal{D}$ has two connected components.*

Moreover, the blown-up points belong to the NAC divisors.

Proof: 0) Since the fundamental group of a surface with GSS is isomorphic to $\mathbb{Z}$, since $\pi_1(S^{(+)}_{N,p,q,r;t}) \neq \mathbb{Z}$, $\pi_1(S^{(-)}_{N,p,q,r}) \neq \mathbb{Z}$, and $S^{(-)}_{N,p,q,r}$ is a quotient of a surface $S^{(+)}_{N,p,q,r;t}$ ([15] p 276), (M, J_1) and (M, J_2) must be of the same type.
1) If (M, c, J_1, J_2) is strongly bihermitian, (M, J_i) are minimal using (42). Applying the classification of minimal surfaces of class VII_0 with $b_2(S) = 0$ [18], we have to consider Hopf surfaces and Inoue surfaces. We derive from Proposition (42) that $-K_{J_i} \in H^1(M, \mathbb{R}^\star_+)$. On one hand the anticanonical line bundle of a surface S_M is not real (11), hence we may exclude it. On second hand, a finite covering of a bihermitian surface is bihermitian and if a primary Hopf surface is associated to the contraction

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \qquad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0,$$

$-K = L^{\alpha_1\alpha_2}$, therefore $a = \alpha_1\alpha_2 \in \mathbb{R}^\star_+$ and it is easy to check that the requested conditions are fulfilled.
2) If (M, c, J_1, J_2) is not strongly bihermitian, (M, J_i) admits a non-trivial effective NAC divisor D_{-K_i} whose support is $\mathcal{D}$, and as noticed in [1], 3.3, it is the same for the minimal model S_i of (M, J_i). Using theorem (38), S_i contains GSS and the support of the NAC divisor has at most two components by (29). If $\Pi = \Pi_m \cdots \Pi_0 : (M, J_i) \to S_i$ is the blowing-down of the exceptional curves, the blown-up points belong to the successive NAC divisors by (43), hence $\mathcal{D}$ has at most two connected components. $\square$

Corollary 3. 45 *Let (M, c, J_1, J_2) be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces (M, J_i), $i = 1, 2$, are*

- *Hopf surfaces of special type (see* [22] 2.2*),*
- *(parabolic) Inoue surfaces or*

- *even Inoue-Hirzebruch surfaces.*

Moreover, the blown-up points belong to the NAC divisors.

Proof: By [22] 3.11, (M, J_i) has a minimal model S_i in the class VII_0 and

$$H^0(M, -K_{J_i}) \neq 0,$$

therefore S_i is not a Inoue surface $S^{(+)}_{N,p,q,r;t}$, $S^{(-)}_{N,p,q,r}$. If $b_2(S_i) = 0$, S_i is a Hopf surface, and if $b_2(S_i) > 0$, S_i contains a GSS by (38). The existence of a (non-twisted) global section of $-K$ is equivalent to the existence of a metric $g \in c$ such that $\theta^g_1 + \theta^g_2 = 0$. In this situation, [2] Prop.4 asserts that $\mathcal{D}_+$ and $\mathcal{D}_-$ are both non-empty. This is possible only when S_i is parabolic Inoue or an even Inoue-Hirzebruch surface. □

References

[1] V. Apostolov: Bihermitian surfaces with odd first Betti number. *Math. Z. 238 (2001), 555-568*

[2] V. Apostolov, P. Gauduchon, G. Grantcharov : Bihermitian structures on complex surfaces. *Proc. London Math. Soc. 79, 414-428 (1999)*

[3] M.F Atiyah, N.J. Hitchin, & I.M. Singer : Self-duality in 4-dimensional Riemannian Geometry. *Proc. Roy. Soc. London, Ser. A 362 (1978) 425-461.*

[4] C.P. Boyer : A note on hyperhermitian manifolds. *Proc. Amer. Math. Soc. 102, 157-164 (1988)*

[5] W. Barth, C. Peters, A. Van de Ven : Compact Complex Surfaces. *Springer (1984).*

[6] G. Dloussky : Structure des surfaces de Kato. *Mémoire de la S.M.F 112.*n°14 *(1984).*

[7] G. Dloussky : Une construction élémentaire des surfaces d'Inoue-Hirzebruch. *Math. Ann. 280, (1988), 663-682.*

[8] G. Dloussky : Complex surfaces with Betti Numbers $b_1 = 1$, $b_2 > 0$ and finite quotients. *Contemp. Math. (AMS) 288, (2001), 305-309.*

[9] G. Dloussky & F. Kohler : Classification of singular germs of mappings and deformations of compact surfaces of class VII_0. *Ann. Polonici Math. LXX (1998), 49-83*

[10] G. Dloussky, K. Oeljeklaus : Vector fields and foliations on compact surfaces of class VII_0. *Ann. Inst. Fourier 49, 5 (1999), 1503-1545*

[11] G. Dloussky, K. Oeljeklaus, M. Toma : Surfaces de la classe VII_0 admettant un champ de vecteurs. *Comment. Math. Helvet. 76 (2001), 640-664*

[12] G. Dloussky, K. Oeljeklaus, M. Toma : Class VII_0 surfaces with b_2 curves. *Tohoku Math. J. 55 (2003), 283-309*

[13] S.K. Donaldson : The orientation of Yang-Mills moduli space and 4-manifolds topology. *J. Differential Geometry 26 (1987) 397-428.*

[14] I. Enoki : Surfaces of class VII_0 with curves. *Tôhoku Math. J. 33, (1981), 453-492.*

[15] M. Inoue On surfaces of Class VII_0, *Inventiones math. 24 (1974) 269-310.*

[16] K. Kodaira On stability of compact submanifolds of complex manifolds. *Ann.Math. 85 (1963), 79-94.*

[17] K. Kodaira On the structure of compact complex analytic surfaces I, II. *Am. J. of Math. vol.86, 751-798 (1964); vol.88, 682-721 (1966).*

[18] M. Lübke , A. Teleman: The Kobayashi-Hitchin Correspondance. *World Scientific (1995).*

[19] K. Oeljeklaus, M. Toma & D. Zafran: Une caractérisation des surfaces d'Inoue-Hirzebruch. *Ann. Inst. Fourier 51, 5 (2001), 1243-1257.*

[20] I. Nakamura : On surfaces of class VII_0 with curves. *Invent. Math. 78,(1984), 393-443.*

[21] I. Nakamura On surfaces of class VII_0 with curves II.*Tohoku Math. J. 42 (1990), 475-516.*

[22] M. Pontecorvo : Complex structures on Riemannian four-manifolds. *Math. Ann. 309, 159-177 (1997)*

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