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# ON SURFACES OF CLASS VII WITH NUMERICALLY ANTICANONICAL DIVISOR

Georges Dloussky

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# On surfaces of class $VII_0^+$ with numerically anticanonical divisor

Georges Dloussky

## Abstract

We consider minimal compact complex surfaces  $S$  with Betti numbers  $b_1 = 1$  and  $n = b_2 > 0$ . A theorem of Donaldson gives  $n$  exceptional line bundles. We prove that if in a deformation, these line bundles have sections,  $S$  is a degeneration of blown-up Hopf surfaces. Besides, if there exists an integer  $m \geq 1$  and a flat line bundle  $F$  such that  $H^0(S, -mK \otimes F) \neq 0$ , then  $S$  contains a Global Spherical Shell. We apply this last result to complete classification of bihermitian surfaces.

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## 0 Introduction

A minimal compact complex surface  $S$  is said to be of the class  $VII_0$  of Kodaira if the first Betti number satisfies  $b_1(S) = 1$ . A surface  $S$  is of class  $VII_0^+$  if moreover  $n := b_2(S) > 0$ ; these surfaces admit no nonconstant meromorphic functions. The major problem in classification of non-kählerian surfaces is to achieve the classification of surfaces  $S$  of class  $VII_0^+$ . All known surfaces of this

class contain Global Spherical Shells (GSS), i.e. admit a biholomorphic map  $\varphi : U \rightarrow V$  from a neighbourhood  $U \subset \mathbb{C}^2 \setminus \{0\}$  of the sphere  $S^3 = \partial B^2$  onto an open set  $V$  such that  $\Sigma = \varphi(S^3)$  does not disconnect  $S$ . Are there other surfaces? In first section we investigate the general situation: A theorem of Donaldson [13] gives a  $\mathbb{Z}$ -base  $(E_i)$  of  $H^2(S, \mathbb{Z})$ , such that  $E_i E_j = -\delta_{ij}$ . These cohomology classes can be represented by line bundles  $L_i$  such that  $K_S L_i = L_i^2 = -1$ . Indeed, these line bundles generalize exceptional curves of the first kind, and since  $S$  is minimal, they have no section. Over the versal deformation  $\mathcal{S} \rightarrow B$  of  $S$  these line bundles form families  $\mathcal{L}_i$ . We propose the following conjecture which can be easily checked for surfaces with GSS:

**Conjecture 1:** Let  $S$  be a surface in class  $VII_0^+$  and  $\mathcal{S} \rightarrow B$  be the versal deformation of  $S \simeq S_0$  over the ball of dimension  $h^1(S, \Theta)$ . Then there exists  $u \in \Delta$ ,  $u \neq 0$ , and flat line bundles  $F_i$  such that  $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$  for  $i = 0, \dots, n-1$ .

The main result of section 1 is (see theorem (13)),

**Theorem 0. 1** *Let  $S$  be a surface in class  $VII_0^+$  and  $\mathcal{S} \rightarrow B$  its versal deformation. If there exists  $u \in B$  and flat line bundles  $F_i \in H^1(S, \mathbb{C}^*)$  such that  $H^0(S_u, L_{i,u} \otimes F_i) \neq 0$  for  $i = 0, \dots, n-1$ , then there is a non empty Zariski open set  $U \subset B$  such that for all  $u \in U$ ,  $S_u$  is a blown-up Hopf surface. In particular,  $S$  is a degeneration of blown-up Hopf surfaces.*

If a surface is a degeneration of blown-up Hopf surfaces, the fundamental group of a fiber is isomorphic to  $\mathbb{Z} \times \mathbb{Z}_l$ , hence taking a finite covering, one obtains a surface obtained by degeneration of blown-up primary Hopf surfaces. Notice that a finite quotient of a surface of class  $VII_0^+$  containing a GSS still contains a GSS [8].

**Conjecture 2:** Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  is a degeneration of blown-up primary Hopf surfaces, then  $S$  contains a cycle of rational curves.

A surface admitting a numerically anticanonical (NAC) divisor (see (20)), contains a cycle of rational curves. In section 2, we shall prove

**Theorem 0. 2** *Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  admits a NAC divisor, then  $S$  contains a GSS.*

It is a weak version of

**Conjecture 3 (Nakamura [21]).** Let  $S$  be a surface of class  $VII_0^+$ . If  $S$  contains a cycle of rational curves,  $S$  contains a GSS.

The proof is based on the fact that in  $H^2(S, \mathbb{Z})$ , a curve is equivalent to a class of the form  $L_i - \sum_{j \in I} L_j$ , with  $I \neq \emptyset$ . Intuitively  $L_i$  represents an exceptional curve of the first kind and  $C$  is then equivalent to an exceptional curve of the first kind blown-up several times ( $\text{Card}(I)$  times). It explains why curves have self-intersection  $\leq -2$ . We recover a characterization of Inoue-Hirzebruch surfaces by Oeljeklaus, Toma & Zaffran [19]:

**Theorem 0. 3** *Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$ . Then  $S$  is a Inoue-Hirzebruch surface if and only if there exists two flat line bundles  $F_1$ ,*

$F_2$ , two twisted vector fields  $\theta_1 \in H^0(S, \Theta \otimes F_1)$ ,  $\theta_2 \in H^0(S, \Theta \otimes F_2)$ , such that  $\theta_1 \wedge \theta_2(p) \neq 0$  at at least one point  $p \in S$ .

In section 3 we apply results of section 2 to complete the classification of bihermitian 4-manifolds  $M$  (see [1],[2] [22]), when  $b_1(M) = 1$  and  $b_2(M) > 0$ : A bihermitian surface is a riemannian oriented connected 4-manifold  $(M, g)$  endowed with two integrable almost complex structures  $J_1, J_2$  inducing the same orientation, orthogonal with respect to  $g$  and independent i.e.  $J_1(x) \neq \pm J_2(x)$  for at least one point  $x \in M$ . This structure depends only on the conformal class  $c$  of  $g$ . A bihermitian surface is strongly bihermitian if  $J_1(x) \neq \pm J_2(x)$  for every point  $x \in M$ . The key observation is that under these assumptions,  $(M, J_i)$ ,  $i = 1, 2$  admit a numerically anticanonical divisor.

**Theorem 0. 4** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface with odd first Betti number.*

1) *If  $(M, c, J_1, J_2)$  is strongly bihermitian (i.e  $\mathcal{D} = \emptyset$ ), then the complex surfaces  $(M, J_i)$  are minimal and either a Hopf surface covered by a primary one associated to a contraction  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  of the form*

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a \alpha^{-1} z_2),$$

$$\text{with } a, s \in \mathbb{C}, 0 < |\alpha|^2 \leq a < |\alpha| < 1, (a^m - \alpha^{m+1})s = 0,$$

*or else  $(M, J_i)$  are Inoue surfaces  $S_{N,p,q,r;t}^+$ ,  $S_{N,p,q,r}^-$ .*

2) *If  $(M, c, J_1, J_2)$  is not strongly bihermitian, then  $\mathcal{D}$  has at most two connected components,  $(M, J_i)$ ,  $i = 1, 2$ , contain GSS and the minimal models  $S_i$  of  $(M, J_i)$  are*

- *Surfaces with GSS of intermediate type if  $\mathcal{D}$  has one connected component*
- *Hopf surfaces of special type (see [22] 2.2), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if  $\mathcal{D}$  has two connected components.*

*Moreover, the blown-up points belong to the NAC divisors.*

If moreover the metric  $g$  is anti-self-dual (ASD), we obtain

**Corollary 0. 5** *Let  $(M, c, J_1, J_2)$  be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces  $(M, J_i)$ ,  $i = 1, 2$ , are*

- *Hopf surfaces of special type (see [22] 2.2),*
- *(parabolic) Inoue surfaces or*
- *even Inoue-Hirzebruch surfaces.*

*Moreover, the blown-up points belong to the NAC divisors.*

**Remark 0. 6** Throughout the paper we shall use the following terminology:

1) A surface for which exists a nontrivial divisor  $D$  such that  $D^2 = 0$  will be called a **Enoki surface**, they are obtained by holomorphic compactification of

an affine line bundle over an elliptic curve by a cycle  $D$  of rational curves [14]; otherwise they are associated to contracting holomorphic germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, tz_2)$$

[6] [9]. A Enoki surface with an elliptic curve will be called briefly a **Inoue surface** (= **parabolic Inoue surface**): they are obtained by holomorphic compactification of a line bundle over an elliptic curve by a cycle of rational curves  $D$ ; otherwise they are associated to the contracting germs of maps

$$F(z_1, z_2) = (t^n z_1 z_2^n, tz_2).$$

For all these surfaces, the sum of opposite self-intersections of the  $n = b_2(S)$  rational curves  $D_0, \dots, D_{n-1}$  is  $\sigma_n(S) := -\sum_{i=0}^{n-1} D_i^2 = 2n$ .

2) A surface with  $2n < \sigma_n(S) < 3n$  will be called an **intermediate surface**;  
 3) A surface with  $\sigma_n(S) = 3n$  a called a **Inoue-Hirzebruch (IH) surface** ([15] or [7] for a construction by contracting germs of mappings). An **even** (=hyperbolic) (resp. **odd** (=half)) **IH surfaces** has two (resp. one) cycle of rational curves.

**We shall assume throughout the article that  $S$  admits no nonconstant meromorphic functions.**

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## 1 Exceptional line bundles and degeneration of blown-up Hopf surfaces

Let  $S$  be a surface in class  $\text{VII}_0^+$  with  $n = b_2(S)$ . Since  $S$  is not algebraic,  $A^2 \leq 0$  for every divisor  $A$ . By adjunction formula it is easy to deduce that for every irreducible curve  $C$ ,  $KC \geq 0$ ,  $C^2 \leq -2$  if  $C$  is a regular rational curve and  $C^2 \leq 0$  if  $C$  is a rational curve with a double point or an elliptic curve. It is well known that  $S$  contains at most  $n$  rational curves, and at most one elliptic curve. By Hirzebruch index theorem,  $b^- = b_2(S)$  whence the intersection form  $Q : H^2(S, \mathbb{Z})/\text{Torsion} \rightarrow \mathbb{Z}$  is negative definite.

### 1.1 Exceptional line bundles

An irreducible curve which satisfies the two conditions  $KC = C^2 = -1$  is an exceptional curve of the first kind; we generalize the notion to line bundles.

**Definition 1.7** 1) A line bundle  $L \in H^1(S, \mathcal{O}^*)$  is called an **exceptional line bundle** (of the first kind) if  $KL = L^2 = -1$ .

2) An effective divisor  $E$  is called an exceptional divisor (of the first) kind if  $E$  is the inverse image  $\Pi^*C$  of an exceptional curve of the first kind  $C$  by a finite number of blowing-ups  $\Pi$ . Equivalently it is an effective reduced divisor which may be blown-down onto a regular point.

Using the fact that for a blowing-up  $\Pi : S \rightarrow S'$ ,  $K_S = \Pi^*K_{S'} + C$  and the projection formula ([5] p11), it is easy to check that if  $E$  is an exceptional divisor, then  $[E]$  is an exceptional line bundle. Moreover, the inverse image  $\Pi^*L$  of an exceptional line bundle  $L$  by a finite sequence of blowing-ups is still an exceptional line bundle.

The following theorem has been proved by I. Nakamura when  $S$  contains a cycle of rational curves [21] (1.7). It should be noticed that any surface with  $b_1(S) = 1$  and  $b_2(S) = 0$  is minimal and  $H^1(S, \mathbb{C}^*) \simeq H^1(S, \mathcal{O}^*)$  [17] II, p699.

**Theorem 1. 8** *Let  $S$  be a (not necessarily minimal) compact complex surface such that  $b_1(S) = 1$ , with second Betti number  $n = b_2(S) > 0$ . Then there exists  $n$  exceptional line bundles  $L_j$ ,  $j = 0, \dots, n-1$ , unique up to torsion by a flat line bundle  $F \in H^1(S, \mathbb{C}^*)$  such that:*

- $E_j = c_1(L_j)$ ,  $0 \leq j \leq n-1$  is a  $\mathbb{Z}$ -basis of  $H^2(S, \mathbb{Z})$ ,
- $K_S L_j = -1$  and  $L_j L_k = -\delta_{jk}$ ,
- $K_S = L_0 + \dots + L_{n-1}$  in  $H^2(S, \mathbb{Z})$
- For every  $i = 0, \dots, n-1$  and for every flat line bundle  $F \in H^1(S, \mathbb{C}^*)$ ,

$$\mathcal{X}(L_i \otimes F) = 0.$$

- If  $h^0(S, L_i \otimes F) \neq 0$ , there exists an exceptional divisor  $C_i$  and a (perhaps trivial) flat effective divisor  $P_i$  such that  $L_i \otimes F = [C_i + P_i]$ .

Proof: 1) By Donaldson theorem [13], there exists a  $\mathbb{Z}$ -basis  $(E_i)_i$ ,  $0 \leq i \leq n-1$ , of  $H^2(S, \mathbb{Z})/Torsion$  such that  $E_i E_j = -\delta_{i,j}$ . Moreover, since  $p_g = h^2(S, \mathcal{O}_S) = 0$ , the exponential exact sequence  $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$  yields line bundles  $L_i$  such that  $E_i = c_1(L_i)$ . The line bundles  $L_i$  are unique up to tensor product by flat line bundles. For a surface of class VII, the group of flat line bundles is  $H^1(S, \mathbb{C}^*)$ . Let  $c = \sum n_i E_i \in H^2(S, \mathbb{Z})$ . Then  $c^2 = -\sum n_i^2$ , therefore  $c^2 = -1$  if and only if  $c = \pm E_i$ . Replacing if necessary  $L_i$  by  $L_i^{-1}$  we may suppose that  $KL_i \leq 0$  for  $i = 0, \dots, n-1$ . By Riemann-Roch formula

$$(*) \quad \mathcal{X}(L_i \otimes F) = \mathcal{X}(\mathcal{O}_S) + \frac{1}{2}(L_i^2 - KL_i) = \frac{1}{2}(-1 - KL_i) \in \mathbb{Z},$$

therefore  $KL_i \leq -1$ . Since  $(L_i)$  is a  $\mathbb{Z}$ -base of  $H^2(S, \mathbb{Z})$ ,  $K = \sum_i k_i L_i$  with  $k_i = -KL_i \geq 1$ . From  $-n = K^2 = -\sum_{i=0}^{n-1} k_i^2$  we deduce that  $k_i = 1$  for  $i = 0, \dots, n-1$ . From (\*) we obtain

$$\mathcal{X}(L_i \otimes F) = 0.$$

2) If  $h^0(S, L_i \otimes F) > 0$ , then  $L_i \otimes F = [C_i]$  where  $C_i$  is an effective divisor. Let

$$C_i = n_1 G_1 + \cdots + n_p G_p$$

be a decomposition into irreducible components.

- If  $G$  is an elliptic curve or a rational curve with a double point then  $KG = -G^2 \geq 0$ .
- If  $G$  is a nonsingular rational curve,  $KG = -2 - G^2 \geq -1$  and  $KG = -1$  if and only if  $G$  is an exceptional curve of the first kind.

Therefore the condition

$$-1 = KC_i = \sum_i n_i KG_i$$

implies that there is an exceptional curve of the first kind, say  $G_p$ . Now we prove the result by induction on  $n = b_2(S) \geq 1$ .

If  $n = 1$ , there is only one exceptional line bundle  $L$  and if  $h^0(S, L \otimes F) \neq 0$ ,  $S$  is not minimal, hence a blow-up of a surface  $S'$  with  $b_2(S') = 0$ . Then  $L \otimes F = [C + P]$  where  $C$  is an exceptional curve of the first kind and  $P = 0$  (if  $S'$  has no curve) or  $P$  is flat perhaps not trivial (if  $S'$  is a Hopf surface). Suppose that  $n > 1$ :

- If  $G_p \sim E_i$ , then  $-F = [n_1 G_1 + \cdots + n_{p-1} G_{p-1} + (n_p - 1) G_p]$  is a flat line bundle and  $L'_i = L_i \otimes F = [G_p]$ .
- If  $G_p \sim E_j$ ,  $j \neq i$ , there is a flat line bundle  $F$  such that  $L'_j = L_j \otimes F = [G_p]$ .

Therefore, replacing  $L_j$  by  $L'_j$ , and changing if necessary the numbering we may suppose that  $L_{n-1} = [C_{n-1}]$  with  $C_{n-1}$  an exceptional curve of the first kind. Let  $\Pi : S \rightarrow S'$  be the blowing-down of  $C_{n-1}$ . Since  $L_i$ ,  $i \neq n-1$  is trivial in a neighbourhood of  $C_{n-1}$ ,  $L'_i = \Pi_* L_i$  is a line bundle such that  $L_i = \Pi^* \Pi_* L_i$  and we check easily, by projection formula, that  $(L'_i)_{0 \leq i \leq n-2}$  is a family of exceptional line bundles. If  $h^0(L_i) \neq 0$ , then  $h^0(L'_i) \neq 0$ , whence by induction hypothesis,  $L'_i = [C'_i + P'_i]$  with  $C'_i$  an exceptional divisor and  $P'_i$  an effective flat divisor. Therefore  $L_i = \Pi^* L'_i = [\Pi^* C'_i + \Pi^* P'_i]$ .  $\square$

## 1.2 Families of exceptional or flat line bundles

If  $S$  is minimal then, for any  $i = 0, \dots, n-1$  and for any flat line bundle  $F$ ,  $h^0(S, L_i \otimes F) = 0$ . In all known examples  $S$  has a deformation into a non minimal one, hence we consider now versal deformation of  $S$ .

Let  $\Pi : \mathcal{S} \rightarrow B$  be the versal deformation of  $S \simeq S_0$ , where  $B$  is the unit ball of  $\mathbb{C}^N$ ,  $N = \dim H^1(S, \Theta)$ . Standard arguments of spectral sequences yield

**Proposition 1. 9** For  $i = 0, \dots, n-1$ , there exist line bundles  $\pi_i : \mathcal{L}_i \rightarrow \mathcal{S}$  such that for every  $u \in B$ , the restriction  $\pi_{i,u} : L_{i,u} \rightarrow S_u$  is an exceptional line bundle,

Moreover, if  $\mathcal{K} \rightarrow \mathcal{S}$  is the relative canonical bundle, we have

$$\mathcal{K} \sim \bigotimes_{i=0}^{n-1} \mathcal{L}_i$$

in  $H^2(\mathcal{S}, \mathbb{Z}) \simeq H^2(S_0, \mathbb{Z})$ .

Since  $\mathbb{C}^*$  is commutative any representation  $\rho : \pi_1(S) \rightarrow GL(1, \mathbb{C}) \simeq \mathbb{C}^*$  factorize through  $H_1(S, \mathbb{Z})$ , therefore any representation (hence any flat line bundle), is defined by  $\gamma \mapsto f$  with  $f \in \mathbb{C}^*$ . We shall denote by  $F^f \in H^1(S, \mathbb{C}^*) \simeq \mathbb{C}^*$  this line bundle and we have defined a group morphism

$$\begin{aligned} \varphi : \mathbb{C}^* &\rightarrow H^1(S, \mathbb{C}^*) \\ \lambda &\mapsto F^\lambda \end{aligned}$$

**Lemma 1. 10** 1) For any holomorphic function  $f : B \rightarrow \mathbb{C}^*$  there exists a unique flat line bundle  $\mathcal{F}^f \rightarrow \mathcal{S}$  such that  $(\mathcal{F}^f)|_{S_u} = F^{f(u)}$ .

2) There exists over  $\mathcal{S} \times \mathbb{C}^*$  a flat line bundle, called the **tautological flat line bundle**  $\mathcal{F}$  such that for any  $(u, \lambda) \in B \times \mathbb{C}^*$ ,  $\mathcal{F}_{S_u \times \{\lambda\}} = F^\lambda$ .

Proof: Let  $\omega : \tilde{\mathcal{S}} \rightarrow \mathcal{S}$  be the family of universal covering spaces of  $\mathcal{S}$ . Then the fundamental group  $\pi_1(S)$  operates diagonally on  $\tilde{\mathcal{S}} \times \mathbb{C}$  by  $\gamma \cdot (p, z) = (\gamma \cdot p, f(\Pi\omega(p))z)$ . The quotient manifold is  $\mathcal{F}^f$ .

A similar construction gives the tautological flat line bundle  $\mathcal{F}$ .  $\square$

**Examples 1. 11** 1) Suppose that  $S = S(F)$  is a primary Hopf surface defined by

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0.$$

If  $s = 0$  (resp.  $s \neq 0$ ),  $S$  contains at least two elliptic curves  $E_1, E_2$  (resp. only one elliptic curve  $E_2$ ), where

$$E_1 = \{z \neq 0 \mid z_1 = 0\} / \{\alpha_2^p \mid p \in \mathbb{Z}\}, \quad E_2 = \{z \neq 0 \mid z_2 = 0\} / \{\alpha_1^p \mid p \in \mathbb{Z}\}.$$

Then for  $i = 1, 2$ ,  $\varphi(\alpha_i) = [E_i]$ . In fact, if  $S$  is a diagonal Hopf surface, the cocycle of the line bundle associated to  $[E_i]$  is given by

$$(z_1, z_2, \lambda) \sim (\alpha_1 z_1, \alpha_2 z_2, \alpha_i \lambda).$$

2) Following [6], let  $S = S(F)$  be the minimal surface containing a GSS with  $b_2(S) > 0$  defined by

$$F(z) = (t^n z_1 z_2^n + \sum_{i=0}^{n-1} a_i t^{i+1} z_2^{i+1}, t z_2).$$



Then  $S$  contains a cycle of rational curves  $\Gamma = D_0 + \dots + D_{n-1}$  such that  $D_i^2 = -2$ ,  $i = 0, \dots, n-1$  and  $\Gamma^2 = 0$ . Let  $t = \text{tr}DF(0) \neq 0$  be the trace of the surface, then  $\varphi(t) = [\Gamma]$ . In fact the equation of  $\Gamma$  is  $z_2 = 0$ .

If in the expression of  $F$ , there is at least one index  $i$  such that  $a_i \neq 0$ ,  $S$  has no elliptic curve. If  $M(S)$  is the intersection matrix of the rational curves then  $\det M(S) = 0$ , hence the curves do not generate  $H^2(S, \mathbb{Z})$ ; for every  $m \geq 1$ , every  $F$  flat,  $H^0(S, -mK + F) = 0$ , therefore there is no NAC divisor.

If  $F(z) = (t^n z_1 z_2^n, t z_2)$ , i.e.  $S$  is a Inoue (parabolic) surface,  $S$  contains an elliptic curve  $E$  and  $-K = [E + \Gamma]$ .

3) Following [15], let  $S = S_M = \mathbb{H} \times \mathbb{C}/G_M$  with  $M \in SL(3, \mathbb{Z})$  a unimodular matrix with eigenvalues  $\alpha, \beta, \bar{\beta}$  such that  $\alpha > 1$ ,  $\beta \neq \bar{\beta}$ . Denote by  $(a_1, a_2, a_3)$  a real eigenvector associated to  $\alpha$  and  $(b_1, b_2, b_3)$  an eigenvector associated to  $\beta$ . It can be easily checked that  $(a_1, b_1)$ ,  $(a_2, b_2)$  and  $(a_3, b_3)$  are linearly independent over  $\mathbb{R}$ . Let  $G_M$  generated by

$$g_0 : (w, z) \mapsto (\alpha w, \beta z),$$

$$g_i : (w, z) \mapsto (w + a_i, z + b_i) \quad \text{for } i = 1, 2, 3.$$

If  $G$  is generated by  $g_i$ ,  $i = 1, 2, 3$ ,  $\omega = dw \wedge dz$  is invariant under  $G$  hence yields a 2-form on  $\mathbb{H} \times \mathbb{C}/G$ . Moreover,  $g_0^* \omega = \alpha \beta \omega$ , hence yields a non-vanishing twisted 2-form over  $S_M$  and  $K = F^{1/\alpha\beta}$ . A line bundle has no section for there is no curve.

### 1.3 Degeneration of blown-up Hopf surfaces

All surfaces containing GSS are degeneration of blown-up primary Hopf surfaces as it can be easily checked using contracting germs of mappings. We show in this section that if a surface can be deformed into a non minimal one then over a Zariski open set in the base of the versal deformation, there are blown-up Hopf surfaces.

We need a lemma comparing the versal deformation of a surface  $S$  with the versal deformation of a blowing-up  $S'$  of  $S$ .

**Lemma 1. 12** *Let  $S$  be a compact complex surface of the VII-class (not necessarily minimal), let  $\Pi : S' \rightarrow S$  be the blowing-up of  $S$  at the point  $z_0$  and*

$$e_{z_0} : H^0(S, \Theta) \rightarrow T_{z_0} S$$

*be the evaluation of global vector fields at  $z_0$ . Then, if  $\mathcal{V}$  is a covering of  $S$  such that  $H^1(S, \Theta) = H^1(\mathcal{V}, \Theta)$ ,*

*1)  $h^0(S', \Theta') = \dim \text{Ker } e_{z_0}$ ;*

*2) There exists a covering  $\mathcal{V}' = (V'_i)_{i \geq 0}$  of  $S'$  such that  $H^1(S', \Theta') = H^1(\mathcal{V}', \Theta')$  with the following properties:*

*i)  $V'_0$  is the inverse image by  $\Pi$  of a ball  $V_0$  centered at  $z_0$ ,*

*ii)  $V'_0$  meets only one open subset of the covering, say  $V'_1$  along a spherical shell,*

iii) For all  $i \geq 1$ , the restriction of  $\Pi$  on  $V'_i$  is an isomorphism on its image  $V_i$ ,

iv) the canonical mapping  $\Pi^* : H^1(\mathcal{V}, \Theta) \rightarrow H^1(\mathcal{V}', \Theta')$  is injective,

v) A base of  $H^1(S', \Theta')$  may be obtained from a base of  $H^1(\mathcal{V}, \Theta)$  by adding cocycles induced on  $V'_{01}$  by (at most two) non-vanishing vector fields  $Z^i$  on  $V_0$  such that the vectors  $Z^i(z_0)$  generate a supplementary subspace of  $\text{Im } e_{z_0}$  in  $T_{z_0}$ .

In particular  $h^1(S', \Theta') = h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0}$ .

Proof: 1) is clear.

2) Let  $\mathcal{U} = (U_i)_{i \geq 1}$  be a locally finite covering of  $S$  such that  $H^1(S, \Theta) = H^1(\mathcal{U}, \Theta)$ . It may be supposed that  $z_0 \in U_i$  for  $i = 1, \dots, p$ . Denote by  $U'_0 \subset\subset U_0$  balls centered at  $z_0$  such that if  $i > p$ , then  $U_0 \cap U_i = \emptyset$ . Now, if  $V_0 = U_0$ ,  $V_1 = U_1 \setminus \overline{U'_0}$  and  $V_i = U_i \setminus \overline{U_0}$ , for  $i > 1$ , there are three coverings of  $S$ ,  $\mathcal{U}$ ,  $\mathcal{U}_0 = (U_i)_{i \geq 0}$  and  $\mathcal{V} = (V_i)_{i \geq 0}$  related by the relation

$$\mathcal{V} \prec \mathcal{U}_0 \prec \mathcal{U}.$$

The canonical mappings

$$H^1(\mathcal{U}, \Theta) \rightarrow H^1(\mathcal{U}_0, \Theta) \rightarrow H^1(\mathcal{V}, \Theta)$$

are isomorphism. We define a covering  $\mathcal{V}' = (V'_i)$  of  $S'$  by  $V'_i = \Pi^{-1}(V_i)$ . The canonical morphism  $\Pi^*$  is clearly injective and the evident mapping

$$s : H^1(\mathcal{V}', \Theta') \rightarrow H^1(\mathcal{V}, \Theta)$$

is clearly surjective. Let  $\xi \notin \text{Im } e_{z_0}$  and  $\theta$  a vector field on  $U_0$  such that  $\theta(z_0) = \xi$ . Define  $\eta = (\eta_{jk}) \in Z^1(\mathcal{V}, \Theta)$  by  $\eta_{01} = \theta$  and  $\eta_{jk} = 0$  if  $\{j, k\} \neq \{0, 1\}$ . If  $\eta' = \Pi^*(\eta) \in Z^1(\mathcal{V}', \Theta')$ ,  $\eta$  and  $\eta'$  are cocycles such that  $s([\eta']) = [\eta] = 0$ , but  $[\eta'] \neq 0$ . In fact if there exist vector fields  $X'_0$  on  $V'_0$  and  $X'$  on  $S' \setminus \Pi^{-1}(U'_0)$  such that  $\theta' = X' - X'_0$  on  $V'_{01}$ , we have  $\theta = \Pi_* X' - \Pi_* X'_0$  on  $V_{01}$ . But since a vector field extends inside a ball,  $\xi = \theta(z_0) = \Pi_* X'(z_0)$  which is a contradiction. Therefore,  $\dim \text{Ker } s \geq \text{codim } \text{Im } e_{z_0}$  and it yields  $h^1(\mathcal{V}', \Theta') \geq h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0}$ . Now by Riemann-Roch-Hirzebruch-Atiyah-Singer theorem we have, since  $S$  and  $S'$  are VII-class surfaces,

$$h^1(S, \Theta) = h^0(S, \Theta) + 2b_2(S) \quad \text{and} \quad h^1(S', \Theta') = h^0(S', \Theta') + 2b_2(S').$$

Using  $b_2(S') = b_2(S) + 1$ , we obtain by a)

$$h^1(S', \Theta') = h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0}$$

therefore

$$\begin{aligned} h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0} &= h^1(S, \Theta) + \text{codim } \text{Im } e_{z_0} = h^1(S', \Theta') \geq h^1(\mathcal{V}', \Theta') \\ &\geq h^1(\mathcal{V}, \Theta) + \text{codim } \text{Im } e_{z_0} \end{aligned}$$

which completes the proof.  $\square$

**Theorem 1. 13** *Let  $S$  be a surface of class  $VII_0^+$ ,  $n = b_2(S) \geq 1$  and  $\mathcal{S} \rightarrow B$  its versal deformation. Assume that there is a point  $v \in B$ ,  $\lambda_i \in \mathbb{C}$ ,  $i = 0, \dots, n-1$  such that*

$$H^0(S_v, L_{i,v} \otimes F^{\lambda_i}) \neq 0.$$

*Then there exists*

- *Holomorphic functions  $c_i : B \rightarrow \mathbb{C}$ ,*
- *Flat families of exceptional divisors  $\mathcal{C}_i$  over  $B \setminus H_i$ , where  $H_i = \{c_i = 0\}$ ,*

*such that*

- *For every  $u \in B \setminus H_i$ ,  $[C_{i,u}] = L_{i,u} \otimes F^{c_i(u)}$  and  $F^{c_i(v)} = F^{\lambda_i}$ ,*
- *$S_u$  is minimal if and only if  $u \in M := \bigcap_{i=0}^{n-1} H_i$ ,*
- *$S_u$  is a blown-up Hopf surface if and only if  $u \in B \setminus \bigcup_{i=0}^{n-1} H_i$ .*

Proof: 1) The surface  $S_v$  contains an exceptional curve of the first kind  $C$ . Changing if necessary the numbering we may suppose that  $C = C_{n-1}$  and  $L_{n-1,v} \otimes F^{\lambda_{n-1}} = [C_{n-1}]$ . By stability theorem of Kodaira [16], there exists an open neighbourhood of  $v \in B$  and a flat family  $\mathcal{C}_{n-1}$  over  $V$  of exceptional curves of the first kind.

2) Let  $pr_1 : \mathcal{S} \times \mathbb{C}^* \rightarrow \mathcal{S}$  be the first projection and define the line bundle  $\mathcal{M}_{n-1}$  over  $\mathcal{S} \times \mathbb{C}^*$  by

$$\mathcal{M}_{n-1} := pr_1^* \mathcal{L}_{n-1} \otimes_{\mathcal{O}_{\mathcal{S} \times \mathbb{C}^*}} \mathcal{F}$$

where  $\mathcal{F}$  is the tautological flat line bundle of Lemma 1.10; by  $p = \pi_{n-1} \times Id$ , the sheaf  $\mathcal{M}_{n-1}$  is flat over  $B \times \mathbb{C}^*$ . For every  $(u, \alpha) \in B \times \mathbb{C}^*$ , we have  $(\mathcal{M}_{n-1})|_{S_u \times \{\alpha\}} \simeq (\mathcal{L}_{n-1})|_{S_u} \otimes F^\alpha$ . By the semi-continuity theorem of Grauert, and because surfaces have no nonconstant meromorphic functions,

$$Z_{n-1} := \left\{ (u, \alpha) \in B \times \mathbb{C}^* \mid h^0(S_u \times \{\alpha\}, (\mathcal{M}_{n-1})_{S_u \times \{\alpha\}}) = 1 \right\}$$

is an analytic subset of  $B \times \mathbb{C}^*$  and the dimension of the intersection  $Z_{n-1} \cap V \times \mathbb{C}^*$  is  $N = \dim B$  by 1).

3) Let  $Z'_{n-1}$  be the irreducible component of  $Z_{n-1}$  such that  $\mathcal{M}_{n-1}|_{p^{-1}(Z'_{n-1} \cap V \times \mathbb{C}^*)} = [C_{n-1}]$ . We have a flat family of curves

$$p : \mathcal{C}_{n-1} \rightarrow Z'_{n-1}$$

such that for  $(u, \alpha) \in Z'_{n-1}$ ,  $[C_{n-1,(u,\alpha)}] = L_{n-1,u} \otimes F^\alpha$ . For  $p = 0, 1$ , the functions

$$\begin{aligned} Z'_{n-1} &\rightarrow \mathbb{N} \\ (u, \alpha) &\mapsto h^p(C_{n-1,(u,\alpha)}, \mathcal{O}_{C_{n-1,(u,\alpha)}}) \end{aligned}$$

are constant (see [5] p 96). For  $u \in V$  and  $\alpha$  such that the section of  $L_{n-1} \otimes F^\alpha$  vanishes on an exceptional curve of the first kind,

$$h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1, \quad h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$$

therefore  $h^0(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 1$  and  $h^1(C_{n-1,u}, \mathcal{O}_{C_{n-1,u}}) = 0$  everywhere, whence over each  $(u, \alpha) \in Z'_{n-1}$  the analytic set is connected and does not contain any elliptic curve or cycle of rational curves.

Now, we show that the intersection  $Z'_i \cap (V \times \mathbb{C}^*)$  contains only one irreducible component: In fact, if  $z = (u, \alpha)$  and  $z' = (u, \alpha')$  are two points in  $Z'_{n-1}$  over  $u \in V$ , then

$$h^0(S_u, L_{n-1,u} \otimes F^\alpha) = h^0(S_u, L_{n-1,u} \otimes F^{\alpha'}) = 1,$$

whence  $F^{\alpha/\alpha'}$  has a meromorphic section and by [20] (2.10),  $F^{\alpha/\alpha'} = [D]$  with  $D = mE + nF$ , where  $E, F$  are elliptic curves or cycles of rational curves such that  $E^2 = F^2 = 0$ . It means that for  $L_{n-1,u} \otimes F^\alpha = [C_{n-1,u}]$  and  $L_{n-1,u} \otimes F^{\alpha'} = [C'_{n-1,u}]$  we have

$$[C_{n-1,u}] = L_{n-1,u} \otimes F^\alpha = L_{n-1,u} \otimes F^{\alpha'} \otimes F^{\alpha/\alpha'} = [C'_{n-1,u} + D]$$

and  $C_{n-1,u}$  would not be connected, a contradiction.

As consequence  $Z'_i$  cannot accumulate on

$$B \times \{0\} \cap V \times \mathbb{P}^1(\mathbb{C}), \quad \text{or} \quad B \times \{\infty\} \cap V \times \mathbb{P}^1(\mathbb{C}).$$

By Remmert-Stein theorem the closure

$$G_{n-1} = \overline{Z'_{n-1}} \subset B \times \mathbb{P}^1(\mathbb{C})$$

is an irreducible analytic set of codimension one. The restriction

$$pr_1 : G_{n-1} \rightarrow B$$

is proper; therefore,  $pr_1(G_{n-1})$  is an analytic subset of  $B$ . Since it contains the open set  $V$ ,  $pr_1(G_{n-1}) = B$ . Now,  $(G_{n-1}, pr_1, B)$  is a ramified covering which has only one sheet over  $V$ , hence  $G_{n-1}$  is the graph of a holomorphic mapping  $c_{n-1} : B \rightarrow \mathbb{P}^1(\mathbb{C})$ . Define  $H_{n-1} := c_{n-1}^{-1}(0) \cup c_{n-1}^{-1}(\infty)$ . So far, we have considered the family of curves  $\mathcal{C}_{n-1}$  over  $Z'_{n-1}$ . From now on, we shall consider it over  $B_{n-1} := B \setminus H_{n-1}$ .

4) If for  $i \neq n-1$ ,  $L_{i,v} \otimes F^{\lambda_i} = [C_{i,v}]$  with  $C_{i,v}$  reducible, it means that  $C_i = \pi^*(C'_i)$  where  $\pi$  is a finite sequence of blowing-ups and  $C'_i$  is an exceptional curve of the first kind. By lemma (12), there is a deformation in which the blown-up points are moved outside  $C'_i$ . Therefore changing the point  $v$  in  $V' \subset V$  we may suppose that  $C_{i,v}$  is an exceptional curve of the first kind and apply 3).

5) We show now that for all  $u \in B' := B \setminus H_i$ ,  $S_u$  is a blown-up Hopf surface: By Itaka theorem we may blow down the exceptional divisors  $\mathcal{C}_{i|B'}$  over  $B'$ . Let  $p : \mathcal{S} \rightarrow \mathcal{S}'$  be the canonical mapping and  $\Pi' : \mathcal{S}' \rightarrow B'$  the induced family. Using classification of complex surfaces of class VII<sub>0</sub> with  $b_2(\mathcal{S}) = 0$  (see [18]),  $S'_u$  is a Hopf surface or a Inoue surface [15] of type  $S_M$ ,  $S_{N,p,q,r;t}^{(+)}$  or  $S_{N,p,q,r}^{(-)}$ . We have to exclude the last three types: By [11] we may suppose that  $\mathcal{S}$  has no non-trivial global vector field, therefore  $N = h^1(\mathcal{S}, \Theta) = 2n$  and restricting if necessary  $B$  we have  $h^1(S_u, \Theta_u) = 2n$  for all  $u \in B$ . We denote

by  $\mathcal{K}'$  the relative canonical bundle over  $\mathcal{S}'$ . For every  $u \in B'$  there exists a unique  $f(u) \in \mathbb{C}^*$  such that  $-K'_u = F^{f(u)}$ ; by Grauert theorem,  $f : B' \rightarrow \mathbb{C}^*$  is holomorphic. The relative canonical bundle  $\mathcal{K}$  over  $\mathcal{S}$  satisfies

$$\mathcal{K} = p^* \mathcal{K}' \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i] = F^{f^{-1}} \otimes \bigotimes_{i=0}^{n-1} [\mathcal{C}_i].$$

If there exists a point  $u \in B'$  such that  $S'_u$  is of type  $S_{N,p,q,r;t}^{(+)}$  or  $S_{N,p,q,r}^{(-)}$ , then in a neighbourhood of  $u$ , all surfaces have the same type by [15] and by the theorem of Bombieri [15] p280, the function  $f$  is real valued hence constant, in particular globally defined. Therefore  $\bigotimes_{i=0}^{n-1} [\mathcal{C}_i] = F^f \otimes \mathcal{K}$  extends over  $B$ , which is impossible. Remains the case  $S_M$  which is rigid: all the surfaces  $S'_u$  are isomorphic hence by (11) Example 3,  $-K'_u = F^{\alpha\beta}$ , therefore as before  $\bigotimes_i [\mathcal{C}_i]$  should extend which is impossible.

6) The functions  $c_i$  are  $\mathbb{C}$ -valued: In fact, suppose that  $c_i^{-1}(\infty) \neq \emptyset$ ; let  $A$  be an irreducible component of  $c_i^{-1}(\infty)$ . Since  $c$  has values in  $\mathbb{C}$ , there exists an index  $j$  such that  $c_j$  vanishes along  $A$ . Let  $\mu$  (resp.  $\nu$ ) be the order of the pole (resp. zero) of  $c_i$  (resp.  $c_j$ ) at a point  $a \in A$ . Then on a disc  $\Delta$  containing  $a$ ,  $g := c_i^\nu c_j^\mu \in \mathcal{O}^*(\Delta)$  and the family of positive divisors

$$[\nu \mathcal{C}_i + \mu \mathcal{C}_j] = \mathcal{L}_i^\nu \otimes \mathcal{L}_j^\mu \otimes F^g$$

extends on  $\Delta$ . But it implies that  $\mathcal{C}_i$  and  $\mathcal{C}_j$  extend also, which yields a contradiction.  $\square$

**Definition 1. 14** *Let  $S$  be a surface of class  $VII_0^+$ . We shall say that  $S$  is a degeneration of blown-up primary Hopf surfaces if there is a deformation  $\mathcal{S} \rightarrow \Delta$  over the unit disc of  $S \simeq S_0$ , such that  $S_u$  is a blown-up primary Hopf surface for  $u \neq 0$ .*

If  $\pi_1(S) = \mathbb{Z}$ , a surface which can be deformed into a  $n$  times blown-up surface, is a degeneration of blown-up primary Hopf surfaces. The surface  $S_u$  is defined by a contraction

$$F_u(z) = (\alpha_1(u)z_1 + s(u)z_2^m, \alpha_2(u)z_2)$$

$$\text{with } 0 < |\alpha_1(u)| \leq |\alpha_2(u)| < 1 \text{ and } s(u)(\alpha_2^m(u) - \alpha_1(u)) = 0$$

or

$$F_u(z) = (\alpha_1(u)z_1, \alpha_2(u)z_2 + s(u)z_1^m)$$

$$\text{with } 0 < |\alpha_2(u)| \leq |\alpha_1(u)| < 1 \text{ and } s(u)(\alpha_1^m(u) - \alpha_2(u)) = 0.$$

In both cases there is at least one elliptic curve  $E_2$  (resp.  $E_1$ ) induced by  $\{z_2 = 0\}$  (resp.  $\{z_1 = 0\}$ ) and another  $E_1$  (resp.  $E_2$ ) induced by  $\{z_1 = 0\}$  (resp.  $\{z_2 = 0\}$ ) if  $s = 0$ . The trace

$$t(u) = \text{tr}(S'_u) = \text{tr}(S_u) = \text{tr}(DF_u(0)) = \alpha_1(u) + \alpha_2(u)$$

and the determinant

$$d(u) = \det DF_u(0) = \alpha_1(u)\alpha_2(u)$$

are bounded holomorphic functions on  $B'$  hence extend on  $B$ . They depend only on the conjugation class of  $F_u$ . We call  $tr(u)$  the **trace of the surface**  $S_u$ .

By [17] II p696, we have the following description of the canonical bundle  $K'_u$  of  $S'_u$ : If  $S'_u$  is a diagonal Hopf surface then

$$K'_u = [-E_{1,u} - E_{2,u}] = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

If  $s(u) \neq 0$ , i.e.  $S$  is not diagonal, then since  $\alpha_2^m(u) = \alpha_1(u)$ ,

$$K'_u = [-(m+1)E_{2,u}] = F^{\alpha_2(u)^{-(m+1)}} = F^{(\alpha_1(u)\alpha_2(u))^{-1}}.$$

By [5] I.9.1 (vii), the canonical bundle  $K_u$  of  $S_u$  over  $B'$  satisfies

$$K_u = p^* K'_u \otimes \bigotimes_{0 \leq i \leq n-1} [C_{i,u}].$$

If  $\mathcal{K}$  denotes the relative canonical bundle and  $F$  denotes the tautological flat line bundle,

$$\mathcal{K} = F^{(\alpha_1\alpha_2)^{-1}} \otimes \bigotimes_i [C_i] = F^f \otimes \bigotimes_i \mathcal{L}_i$$

where

$$f := \frac{\prod_i c_i}{\alpha_1\alpha_2}.$$

Since  $\mathcal{K}$  and  $\mathcal{L}_i$  are globally defined on  $B$ ,  $F^f$  is globally defined on  $B$  and  $f \in \mathcal{O}^*(B)$ . Twisting, for example,  $\mathcal{L}_1$  by  $F^{f^{-1}}$ , we may suppose that  $f = 1$  and then

$$c = \prod_{i=0}^{n-1} c_i = \alpha_1\alpha_2 = d \in \mathcal{O}^*(B)$$

satisfies  $\|c\|_\infty \leq 1$ .

For contracting germs associated to surfaces containing GSS we refer to [6].

**Proposition 1. 15** *Let  $S$  be a surface obtained by degeneration of blown-up minimal Hopf surfaces. If  $tr(S) \neq 0$ , then  $S$  contains a GSS and if  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  is a contracting germ of mappings associated to  $S$ ,  $tr(S) = trDF(0)$ .*

Proof: One of the two functions  $\alpha_i$ ,  $i = 1, 2$ , say  $\alpha_2$ , admit a limit  $\alpha_2(0) \neq 0$ , and  $\alpha_1(0) = 0$ . Therefore  $F^{\alpha_2}$  is globally defined. Since  $|\alpha_1(u)| < |\alpha_2(u)|$ , the elliptic curve  $E_{2,u}$  induced by  $z_2 = 0$  exists for all  $u \notin H$ , hence  $[E_{2,u}] = F^{\alpha_2}$ . Since  $u \mapsto h^0(S_u, F^{\alpha_2(u)})$  is upper-semicontinuous we obtain in  $S$  a flat cycle of rational curves  $E_{2,0}$ . By Enoki theorem,  $S$  contains a GSS.  $\square$

## 2 Surfaces with a cycle of rational curves

**Definition 2. 16** Let  $S$  be a surface and  $C$  be an analytic subset of dimension one. We say that  $C$  is a  $r$ -cycle of rational curves if

- $C$  is an elliptic curve when  $r = 0$ ,
- $C$  is a rational curve with a double point when  $r = 1$ , and
- $C = D_0 + \cdots + D_{r-1}$ , is a reduced effective divisor such that  $D_i$  is a non-singular rational curve for  $i = 0, \dots, r-1$  and  $D_0D_1 = \cdots = D_{r-2}D_{r-1} = D_{r-1}D_0 = 1$ ,  $D_iD_j = 0$  in all other cases, when  $r \geq 2$ .

We denote by  $\sharp(C) = r \geq 0$  the number of rational curves of the cycle  $C$ .

With notations of theorem (8), we have

**Theorem 2. 17 ([21] (1.7))** Let  $S$  be a  $VII_0^+$  surface with  $n = b_2(S)$ . Assume that  $S$  contains exactly one cycle  $C$  of rational curves such that  $C^2 < 0$ . Then

- $C \sim -(L_r + \cdots + L_{n-1})$  for some  $1 \leq r \leq n-1$ , if  $S$  is not an odd Inoue-Hirzebruch surface, or
- $C \sim -(L_0 + \cdots + L_{n-1}) + F_2$ , with  $F_2$  of order two, if  $S$  is an odd Inoue-Hirzebruch surface.

**Remark 2. 18** If  $S$  is an even Inoue-Hirzebruch surface (=hyperbolic I-H surface), the two cycles  $C$  and  $C'$  satisfy

$$C \sim -(L_r + \cdots + L_{n-1}), \quad C' \sim -(L_0 + \cdots + L_{r-1}).$$

The following lemma plays a crucial role in the computation of self-intersection of the cycle  $C$ .

**Lemma 2. 19 ([21] (2.4))** Let  $S$  be a surface of class  $VII_0^+$  and without divisor  $D$  such that  $D^2 = 0$ . Let  $L_I := \sum_{i \in I} L_i$ ,  $L = L_I + F$ ,  $F \in H^1(S, \mathcal{O}_S^*)$  for a nonempty subset  $I \subset [0, n-1]$ . Then we have:

- If  $I \neq [0, n-1]$ , then  $H^q(S, L) = 0$  for any  $q$ .
- If  $L \otimes \mathcal{O}_C = \mathcal{O}_C$ , then  $I = [0, r-1]$ , and  $F = \mathcal{O}_S$ ,  $K_S - L + C = \mathcal{O}_S$ .
- If  $LC_i = 0$  for any irreducible component  $C_i$  of  $C$ , then  $I = [0, r-1]$ .

### 2.1 Surfaces with numerically $m$ -anticanonical divisor

**Definition 2. 20** Let  $S$  be a compact complex surface of the  $VII_0$  class with  $b_2(S) > 0$  and let  $m \geq 0$  be an integer. We shall say that  $S$  admits a **numerically  $m$ -anticanonical divisor** if there exists a divisor  $D_m$ , and a flat line bundle  $F$  in  $H^1(S, \mathbb{C}^*)$  such that in  $H^1(S, \mathcal{O}^*)$ ,

$$mK + F + [D_m] = 0.$$

We shall say that  $S$  admits a numerically anticanonical divisor, shortly a **NAC divisor**, if there exists an integer  $m$  such that there exists a numerically  $m$ -anticanonical divisor.

**Remark 2. 21** Let  $S$  be a compact complex surface of the  $VII_0$  class with  $b_2(S) > 0$ . Let  $m \geq 1$  be an integer and  $F \in H^1(S, \mathbb{C}^*)$ . Then  $H^0(S, mK \otimes F) = 0$ . In fact, a section of  $mK \otimes F$  must vanish; let  $[\Delta] = mK \otimes F$  be the associated divisor. Then

$$0 = (mK + F - \Delta)\Delta = mK\Delta - \Delta^2.$$

Since  $mK\Delta \geq 0$  and  $\Delta^2 \leq 0$ ,  $\Delta^2 = 0$ , therefore  $b_2(S) = -K^2 = 0$ . This means that there is no numerically  $m$ -canonical divisors.

For the convenience of the reader we recall basic known facts with slightly different proofs (see [21] lemma (3.1))

**Lemma 2. 22** 1) If  $D_m$  exists, it is a positive divisor, hence  $H^0(S, -mK - F) \neq 0$ .

2) If an irreducible curve  $C$  meets the support  $|D_m|$  of  $D_m$  then  $C$  is contained in  $|D_m|$ .

Proof: 1) We denote by  $D_i$  the irreducible components of  $D_m$ . Let  $D_m = \sum_i k_i D_i = A - B$  where  $A = \sum_{i|k_i > 0} k_i D_i \geq 0$  and  $B = \sum_{i|k_i < 0} (-k_i) D_i \geq 0$  have no common component. If  $B \neq 0$ ,  $B^2 = \sum (-k_i) B D_i < 0$ , there exists an index  $j$  such that  $B D_j < 0$ . Therefore,

$$0 \leq mK D_j = (-F - D_m) D_j = -D_m D_j = -A D_j + B D_j < 0$$

... a contradiction.

2) If  $C$  is an irreducible curve and meets  $|D_m|$ ,  $D_m C = -mK C \leq 0$ , therefore  $C$  is contained in  $|D_m|$ .  $\square$

If there exists a non trivial divisor  $A$  such that  $A^2 = 0$ , then a numerically  $m$ -anticanonical divisor exists if and only if  $S$  is a Inoue surface. In this case,  $S$  contains an elliptic curve  $E$ , a cycle  $\Gamma$  of rational curves,  $m = 1$  and  $K + E + \Gamma = 0$ . When there is no non-trivial flat divisor, the numerically  $m$ -anticanonical divisor  $D_m$  is clearly unique.

## 2.2 The reduction lemma

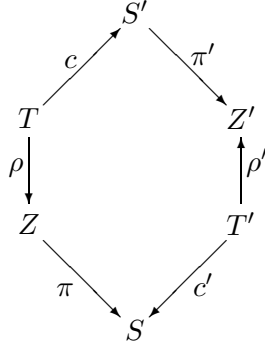
**Definition 2. 23** The least integer  $m$  such that there exists a numerically  $m$ -anticanonical divisor, is called the **index** of the surface  $S$  and will be denoted by  $m(S)$ .

The index  $m(S)$  of the surface  $S$  is the lcm of the denominators of the coefficients  $k_i$  of  $D_1 = \sum_i k_i D_i$ .

The proof of [12] (1.3) works under the following relaxed hypothesis:

**Lemma 2. 24 (Reduction lemma)** Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$  and index  $m = m(S) > 1$ . Then there exists a diagram





where

- i)  $(Z, \pi, S)$  is a  $m$ -fold ramified covering space of  $S$ , branched over  $D_m$ , endowed with an automorphism group isomorphic to  $\mathbb{U}_m$  which acts transitively on the fibers,
- ii)  $(T, \rho, Z)$  is the minimal desingularization of  $Z$ ,
- iii)  $(T, c, S')$  is the contraction of the (possible) exceptional curves of the first kind,
- iv)  $S'$  is a surface of class  $VII_0$ , with  $b_2(S') > 0$ , with action of  $\mathbb{U}_m$ , with index  $m(S') = 1$ ,
- v)  $(S', \pi', Z')$  is the quotient space of  $S'$  by  $\mathbb{U}_m$ ,
- vi)  $(T', \rho', Z')$  the minimal desingularization of  $Z'$ ,
- vii)  $(T', c', S)$  is the contraction of the (possible) exceptional curves of the first kind,

such that the restriction over  $S \setminus D$  is commutative, i.e.

$$\theta := \pi \circ \rho \circ c^{-1} = c' \circ \rho'^{-1} \circ \pi' : S' \setminus D' \rightarrow S \setminus D$$

and  $(S' \setminus D', S \setminus D)$  is a  $m$ -fold non ramified covering. Moreover

- $S$  contains a GSS if and only if  $S'$  contains a GSS,  $S$  and
- The maximal divisors  $D$  and  $D'$  of  $S$  and  $S'$  respectively have the same number of cycles and branches.

**Corollary 2. 25** *If  $S$  admits a NAC divisor  $D_m$ , then the support of  $D_m$  contains a cycle.*

Proof: We may suppose that there is no divisor such that  $D^2 = 0$ . By reduction lemma it is sufficient to prove that the support of  $D_{-K}$  contains a cycle. Suppose that there exists a divisor  $D_{-K}$  and a flat line bundle  $F$  such that  $K + D_{-K} + F = 0$ . By Cartan-Serre duality

$$h^2(S, \mathcal{O}_S(-D_{-K})) = h^2(S, K + F) = h^0(S, -F) = \begin{cases} 0 & \text{if } F \neq 0 \\ 1 & \text{if } F = 0 \end{cases}$$

By Riemann-Roch formula,

$$h^0(S, -F) - h^1(S, -F) = h^0(S, -F) - h^1(S, -F) + h^0(S, K + F) = \chi(F) = 0,$$

hence

$$h^1(S, -F) = \begin{cases} 0 & \text{if } F \neq 0 \\ 1 & \text{if } F = 0 \end{cases}$$

We have

$$0 \rightarrow \mathcal{O}_S(-D_{-K}) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_{D_{-K}} \rightarrow 0$$

- If  $F \neq 0$ , the long exact sequence yields

$$h^0(S, \mathcal{O}_{D_{-K}}) = h^1(S, \mathcal{O}_{D_{-K}}) = 1.$$

The support of  $D_{-K}$  is connected and by [20] (2.7),  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) \geq 1$  hence  $(D_{-K})_{red}$  contains a cycle of rational curves.

- If  $F = 0$ , the associated long exact sequence and [20] (2.2.1) imply

$$1 \leq h^0(S, \mathcal{O}_{D_{-K}}) = h^1(S, \mathcal{O}_{D_{-K}}) \leq 2.$$

As before  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) \geq 1$  and there is at least one cycle.

If  $h^1(S, \mathcal{O}_{(D_{-K})_{red}}) = 2$ , then by the already quoted result (2.2.1) there are two cycles of rational curves.

□

### 2.3 Characterization of Inoue-Hirzebruch surfaces

**Lemma 2. 26** *Let  $S$  be a surface with a NAC divisor  $D_{-K} = \sum k_i D_i$ . We suppose that the maximal divisor contains a cycle of rational curves  $C = D_0 + \dots + D_{s-1}$  with  $s \geq 1$  irreducible curves. If there exists  $j \leq s-1$  such that  $k_j = 1$ , then  $k_i = 1$  for all  $i = 1, \dots, s-1$  and  $C$  has no branch.*

**Proof:** **Case  $s = 1$ :** Since  $D_0$  is a rational curve with a double point, the adjunction formula yields

$$D_0^2 = -KD_0 = D_{-K}D_0 = \sum k_i D_i D_0 = D_0^2 + \sum_{i>0} k_i D_i D_0.$$

hence  $\sum_{i>0} k_i D_i D_0 = 0$  and  $C$  has no branch.

**Case  $s \geq 2$ :** By adjunction formula,

$$2 + D_j^2 = -KD_j = \sum_i k_i D_i D_j = D_j^2 + \sum_{i \neq j} k_i D_i D_j,$$

whence  $2 = \sum_{i \neq j} k_i D_i D_j$ . Since  $\sum_{i \neq j} D_i D_j \geq 2$ ,  $D_j$  meets at most two curves  $D_{j-1}$  and  $D_{j+1}$  (one if the cycle contains two curves) and  $k_{j-1} = k_{j+1} = 1$ . By connectivity we conclude. □

**Proposition 2. 27** *Let  $S$  be a surface with a NAC divisor  $D_{-K} = \sum k_i D_i$  and let  $C = D_0 + \dots + D_{s-1}$  be a cycle contained in the support of  $D_{-K}$ . One of the following conditions holds:*

i) There exists an index  $0 \leq j \leq s - 1$  such that  $k_j = 1$ , then  $S$  is a Inoue surface or a Inoue-Hirzebruch surface.

ii) For every  $0 \leq j \leq s - 1$ ,  $k_j \geq 2$ , then  $C$  has at least one branch and the support  $D = |D_{-K}|$  of  $D_{-K}$  is connected. More precisely, if  $k = \max\{k_i \mid 0 \leq i \leq s - 1\}$ , there exists a curve  $D_j$  and a branch  $H_j > 0$  such that  $k_j = k$  and  $D_j H_j > 0$ .

In particular, each connected component of  $|D_K|$  contains a cycle.

**Definition 2. 28** With preceding notations, a curve  $D_j$  in the cycle such that  $D_j H > 0$  will be called the root of the branch  $H$ .

Proof of (27): Taking if necessary a double covering it may be supposed, by [20](2.14), that the cycle has at least two curves, hence all the curves are regular. Let

$$D_{-K} = A + B = \sum_{i=0}^{p-1} k_i D + \sum_{i=p}^{p+q} k_i D_i$$

where the support of  $A = \sum_{i=0}^{p-1} k_i D_i$  is the connected component of the cycle  $C = \sum_{i=0}^{s-1} D_i$ .

If  $B$  contains another cycle resp. an elliptic curve), then  $S$  is a Inoue-Hirzebruch surface [20] (8.1) (resp. a Inoue surface [20] (10.2)); suppose therefore that the support  $|B|$  of  $B$  is simply-connected. We have to show that  $|D_{-K}|$  is connected: In fact, let  $B_0$  be a connected component of  $|B|$ . There is a proper mapping  $p : S \rightarrow S'$  onto a normal surface  $S'$  with normal singularities  $a = p(|A|)$  and  $b = p(B_0)$ . Since  $B$  is simply connected,  $F$  is trivial on a strictly pseudo-convex neighbourhood  $U$  of  $B$  and thus a holomorphic section of  $-K - F$  yields a non vanishing holomorphic 2-form on  $U \setminus B$ , i.e.  $(S', b)$  is Gorenstein. If  $(S', b)$  would be an elliptic singularity, a two-fold covering  $T$  of  $S$  should contain three exceptional connected divisors such that their contractions  $q : T \rightarrow T'$  would fullfil  $h^0(T', R^1 q_* \mathcal{O}_T) = 3$ . However, by Leray spectral sequence, there is an exact sequence

$$(\dagger) \quad 0 \rightarrow H^1(T', \mathcal{O}_{T'}) \rightarrow H^1(T, \mathcal{O}_T) \rightarrow H^0(T', R^1 q_* \mathcal{O}_T) \rightarrow H^2(T', \mathcal{O}_{T'}) \rightarrow H^2(T, \mathcal{O}_S)$$

where

$$p_g = h^2(T, \mathcal{O}_T) = 0, \quad q = h^1(T, \mathcal{O}_T) = 1 \quad \text{and} \quad h^2(T', \mathcal{O}_{T'}) = h^0(T', \omega_{T'}) \leq 1,$$

by Serre-Grothendieck duality and because  $T'$  has no non-constant meromorphic functions. By  $(\dagger)$  we obtain a contradiction, hence  $(S', b)$  is a Gorenstein rational singularity, hence a Du Val singularity. However, such a singularity has a trivial canonical divisor, therefore  $B = 0$  and  $D_{-K} = A$ .

i) Suppose that there is an index  $j \leq s - 1$  such that  $k_j = 1$ . By lemma (26),  $C$  has no branch and  $s = p$ . By adjunction formula, we have for every  $0 \leq i \leq s - 1$ ,

$$D_i^2 + 2 = -K D_i = D_{-K} D_i = k_{i-1} + k_{i+1} + k_i D_i^2,$$

therefore

$$(\star) \quad (k_{i-1} - 1) + (k_{i+1} - 1) = (k_i - 1)(-D_i^2).$$

By  $(\star)$ ,  $k_i = 1$  for all  $0 \leq i \leq p - 1$ .

- If for every  $0 \leq i \leq s - 1$ ,  $D_i^2 = -2$ ,  $S$  is a Enoki surface by [14]. Moreover, by hypothesis,  $S$  admits a NAC divisor, hence is a Inoue surface or
- There is at least one index  $k \leq s - 1$  such that  $D_k^2 \leq -3$  whence  $C^2 < 0$ . If there is no other cycle, we have shown that  $D_{-K} = C$ , hence

$$-b_2(S) = K^2 = C^2,$$

$S$  is a Inoue-Hirzebruch surface by [20] (9.2).

In case ii), if the maximum  $k$  is reached at a curve  $D_i$ ,  $i \leq s - 1$ , which is not a root, then applying again  $(\star)$ , we obtain  $k_{i-1} = k_{i+1} = k_i = k$ . By connexity, we reach a root.  $\square$

**Proposition 2. 29** *Let  $S$  be a surface of class  $VII_0^+$  admitting a NAC divisor  $D_m$ . Then one of the following conditions is fulfilled:*

- i) *The maximal divisor  $D$  is connected and contains a cycle with at least one branch;*
- ii)  *$S$  is a Inoue surface or a (even or odd) Inoue-Hirzebruch surface.*

Proof: By (25)  $S$  contains at least one cycle By (24)  $S'$  admits a NAC divisor  $D_{-K}$ . Then by (27), and (24)  $S'$  and  $S$  are of the same type.  $\square$

## 2.4 Surfaces with singular rational curve

**Theorem 2. 30** *Let  $S$  be a surface with  $n = b_2(S)$  containing a singular rational curve  $D_0$  with a double point. Then*

1)  $D_0 \sim -(L_1 + \dots + L_{n-1})$  and the connected component containing  $D_0$  is  $D_0 + D_1 + \dots + D_p$  for  $1 \leq p \leq n - 1$ ,  $D_i \sim L_i - L_{i-1}$ ,  $1 \leq i \leq p$ . In particular  $D_0^2 = -(n - 1)$ ,  $D_1^2 = \dots = D_p^2 = -2$ .

2) *The following conditions are equivalent:*

- i) *There exists an integer  $m \geq 1$  such that there exists a numerically  $m$ -anticanonical divisor,*
- ii)  *$S$  contains a GSS.*

*When these conditions are fulfilled then  $S$  is either a Inoue-Hirzebruch surface and  $m(S) = 1$  or  $D_0$  has a branch and its index satisfies  $m(S) = n - 1$ .*

Proof: 1) By (17), there exists  $r \geq 1$  such that  $D_0 = -(L_r + \dots + L_{n-1})$ . We have  $L_0 D_0 = 0$ , hence  $r = 1$  by (19) iii). We show by induction on  $i \geq 1$  that if there is a non singular rational curve  $D_i$  such that  $(D_0 + \dots + D_{i-1})D_i \neq 0$  then  $D_i$  is unique and  $D_i = L_i - L_{i-1}$ . By [20] (2.2.4), and unicity  $D_0 D_i = \dots = D_{i-2} D_i = 0$ ,  $D_{i-1} D_i = 1$  if  $i \geq 2$ .

Suppose that there is a curve, say  $D_1$  such that  $D_0 D_1 = 1$ . By [21] (2.6) (see lemma 32 below) we set  $D_1 \sim L_1 - L_{I_1}$ , then

$$1 = D_0 D_1 = -(L_1 + \dots + L_{n-1})(L_1 - L_{I_1}) = 1 + (L_1 + \dots + L_{n-1})L_{I_1},$$

hence  $I_1 = \{0\}$  and  $D_1 \sim L_1 - L_0$ . If there were another curve, say  $D_2$  meeting  $D_0$ , the same argument shows that  $D_2 = L_2 - L_0$ , but in this case  $D_1 D_2 = -1$  and it is impossible. It shows the unicity. Suppose that for  $i > 1$ ,  $D_i$  exists and  $D_i \sim L_i - L_{I_i}$ . We have the equations

$$\begin{aligned} 0 &= D_0 D_i &= -(L_1 + \cdots + L_{n-1})(L_i - L_{I_i}) &= 1 + (L_1 + \cdots + L_{n-1})L_{I_i} \\ 0 &= D_1 D_i &= -(L_1 - L_0)(L_i - L_{I_i}) &= -L_1 L_{I_i} + L_0 L_{I_i} \\ &\vdots &&\vdots \\ 0 &= D_{i-2} D_i &= -(L_{i-2} - L_{i-3})(L_i - L_{I_i}) &= -L_{i-2} L_{I_i} + L_{i-3} L_{I_i} \\ 1 &= D_{i-1} D_i &= -(L_{i-1} - L_{i-2})(L_i - L_{I_i}) &= -L_{i-1} L_{I_i} + L_{i-2} L_{I_i} \end{aligned}$$

which yield from the last to the second equation

$$i - 1 \in I_i, \quad i - 2 \notin I_i, \quad \dots, \quad 0 \notin I_i.$$

The first one implies that  $I_i$  contains exactly one index, hence

$$D_i \sim L_i - L_{i-1}.$$

We prove unicity as for  $i = 1$ .

2) By 1) the intersection matrix of the connected component of the cycle is the  $(p+1, p+1)$  matrix,  $p \geq 0$ ,

$$M = \begin{pmatrix} -(n-1) & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & & \vdots \\ 0 & 1 & -2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & -2 & 1 \\ 0 & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}$$

and it is easy to check that

$$\det M = (-1)^{p+1} [(n-1)(p+1) - p].$$

A numerically  $m$ -anticanonical divisor  $D_m = \sum_{i=0}^p k_i D_i$  supported by the connected component containing the cycle satisfies the linear system

$$M \begin{pmatrix} k_0 \\ k_1 \\ \vdots \\ k_p \end{pmatrix} = \begin{pmatrix} -m(n-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore

$$(*) \quad k_i = \frac{m(n-1)(p+1-i)}{(n-1)(p+1)-p}, \quad \text{for every } 0 \leq i \leq p.$$

Notice that

$$k_i = (p+1-i)k_p \quad \text{for every } 0 \leq i \leq p,$$

therefore all  $k_i$  are integers if and only if  $k_p$  is an integer. By assumption

$$D_m^2 = (-mK)^2 = -m^2n,$$

and we have  $D_m D_i = -mK D_i = 0$  for  $i = 1, \dots, p$ . We have to consider two cases:

**First case:**  $p \geq 1$  then

$$-m^2n = \sum_{i=0}^p k_i D_i D_m = k_0 D_0 D_m = k_0^2 D_0^2 + k_0 k_1 = -\frac{m^2(n-1)^2(p+1)}{(n-1)(p+1)-p}$$

and this condition is equivalent to

$$p = n - 1.$$

Hence  $S$  has a numerically  $m$ -anticanonical divisor if and only if  $S$  contains  $n$  rational curves if and only if  $S$  contains a GSS by [12].

Replacing in (\*) we obtain

$$k_i = \frac{m(n-i)}{n-1}, \quad \text{for every } 0 \leq i \leq n-1.$$

Hence we conclude

$$m(S) = n - 1.$$

**Second case:**  $p = 0$ , then

$$-m^2n = k_0^2 D_0^2 = -k_0^2(n-1)$$

However, this equation has no solution in integers, therefore the support of  $D_m$  is not connected. We conclude by (29). □

**Corollary 2. 31** *If  $S$  in the  $VII_0^+$  class has a NAC divisor and contains a rational curve with a double point, then  $S$  contains a GSS.*

## 2.5 Cycles with at least two rational curves

**Lemma 2. 32** ([21] (2.5), (2.6), (2.7)) *1) Let  $D$  be a nonsingular rational curve. If  $D \sim a_0 L_0 + \dots + a_{n-1} L_{n-1}$ , then there exists a unique  $i \in [0, n-1]$  such that  $a_i = 1$  or  $-2$ , and  $a_j = 0$  or  $-1$  for  $j \neq i$ .*

*2) If  $D$  is not contained in the cycle then  $D \sim L_i - L_I$  for some  $i \in [0, n-1]$  and  $I \subset [0, n-1]$ ,  $i \notin I$ .*

*3) Let  $D_1$  and  $D_2$  two distinct nonsingular rational curves such that  $D_1 \sim L_{i_1} - L_{I_1}$ ,  $D_2 \sim L_{i_2} - L_{I_2}$ , then  $i_1 \neq i_2$ .*

**Proposition 2. 33** *Let  $S$  be a surface containing a cycle  $C = D_0 + \dots + D_{s-1}$  of  $s \geq 2$  nonsingular rational curves. Suppose that there exists a rational curve  $E$  such that  $E.C = 1$ , and let  $D = \sum_i D_i$  be the maximal connected divisor containing  $C$ . Then every curve  $D_i$  is of the type  $D_i \sim L_i - L_I$  for  $i \notin I$ .*

Proof: By (32), every curve  $D_i$  is of type  $D_i \sim L_i - L_I$  (type **a**) or  $D_i \sim -2L_i - L_I$  (type **b**). Suppose that  $C$  contains at least one curve of type **b**. Since  $(-2L_i - L_I)(-2L_j - L_J) \leq 0$  two such curves cannot meet, in particular if  $s = 2$ ,  $C$  contains at most one curve of type **b**.

Suppose  $s \geq 3$ . If  $D_i$  of type **b**,  $D_i$  meets curves  $D_j$  of type **a**. We have

$$1 = (-2L_i - L_I)(L_j - L_J) = -2L_iL_j - L_jL_I + 2L_iL_J + L_IL_J$$

and

- either  $j \in I$ ,  $i \neq j$ ,  $i \notin J$ ,  $I \cap J = \emptyset$ , whence

$$D_i + D_j \sim -2L_i - L_I + L_j - L_J = -2L_i - L_{I \cup J \setminus \{j\}}.$$

- or  $j \notin I$ ,  $i = j$ ,  $i \notin J$  and  $I \cap J$  contains one element, say  $k$ . Setting  $I' = I \setminus \{k\}$  and  $J' = J \setminus \{k\}$  we obtain

$$D_i + D_j \sim -2L_k - L_{\{i\} \cup I' \cup J'}.$$

By ([21] (1.7)), there is a proper smooth family of compact surfaces  $\pi : \mathcal{S} \rightarrow \Delta$  over the unit disc, a flat divisor  $\mathcal{C}$  such that  $C_0 = C$  and for  $u \neq 0$ ,  $C_u \sim D' + (C - D_i - D_j)$ , with  $D' \sim D_i + D_j$  of type **b**, in particular  $\sharp(C_u) = \sharp(C) - 1$ . Therefore, repeating if necessary such a deformation we obtain a contradiction if  $C$  contains two curves of type **b**. Now if  $C$  contains a curve of type **b**, we take a double covering  $p : S' \rightarrow S$  of  $S$ . The surface  $S'$  of type VII<sub>0</sub> satisfies  $b_2(S') = 2b_2(S)$  and by [20] (2.14) contains a cycle of  $2s$  rational curves. Applying the same arguments to  $S'$ ,  $p^*D_i \sim -2p^*L_i - p^*L_I$  is the union of two rational curves of type **b**, hence a contradiction. Finally, there is no curve of type **b**.  $\square$

**Lemma 2. 34** *Let  $S$  be a surface containing a cycle  $C = D_0 + \dots + D_{s-1}$  of  $s \geq 2$  nonsingular rational curves and  $D = \sum_{i=0}^p D_i$  the maximal connected divisor containing  $C$ .*

1) *Let  $D_j, D_k$  and  $D_l$  three distinct nonsingular rational curves,  $D_j \sim L_{i_j} - L_{I_j}$ ,  $D_k \sim L_{i_k} - L_{I_k}$ ,  $D_l \sim L_{i_l} - L_{I_l}$ . Then  $I_j \cap I_k \cap I_l = \emptyset$ .*

2) *If  $D_j, D_k$  are two distinct nonsingular rational curves, then  $\text{Card}(I_j \cap I_k) \leq 1$ .*

Proof: The surface  $S$  can be deformed into a blown-up Hopf surface  $\pi : \mathcal{S} \rightarrow \Delta$  with a flat family  $\mathcal{C}$ , where  $C_0 = C = D_0 + \dots + D_{s-1} \sim -(L_r + \dots + L_{n-1})$ , and  $C_u$ ,  $u \neq 0$ , is an elliptic curve blown-up  $n - r$  times. We denote by  $\Pi = \Pi_{0,u} \dots \Pi_{n-1,u} : S_u \rightarrow S'_u$  the composition of blowing-ups and  $(\Pi_{i+1} \dots \Pi_{n-1})^*(E_{i,u}) \sim L_i$ . We have  $D_j \sim L_{i_j} - L_{I_j}$ ,  $j = 0, \dots, s - 1$ , hence  $D_j$  is homologous to an exceptional rational curve of the first kind  $E_{i_j}$  blown-up  $\text{Card}(I_j)$  times. Since an exceptional curve cannot blow-up three rational curves we have the first assertion. Moreover two distinct exceptional rational curve of the kind cannot be blown-up two times by the same curves, whence the second assertion.  $\square$

**Proposition 2. 35** *Let  $S$  be a minimal surface containing a cycle  $C = D_0 + \dots + D_{s-1} \sim -(L_r + \dots + L_{n-1})$  of  $s \geq 1$  nonsingular rational curves. Then,  $r = s$  and numbering properly the line bundles  $L_i$ , for  $i = 0, \dots, r-1$ , we have  $D_i \sim L_i - L_{I_i}$ . Moreover  $\cup_{i=0}^{s-1} I_i = [0, n-1]$  and*

$$\sharp(C) - C^2 = b_2(S).$$

Proof: 1) The case  $s = 1$  has been proved in [21] (see lemma (30)).

2) **If  $s = 2$** ,  $D_0 \sim L_{i_0} - L_{I_0}$ ,  $D_1 \sim L_{i_1} - L_{I_1}$ . We have

$$2 = D_0 D_1 = -L_{i_0} L_{I_1} - L_{i_1} L_{I_0} + L_{I_0} L_{I_1},$$

whence  $i_1 \in I_0$ ,  $i_0 \in I_1$ ,  $I_0 \cap I_1 = \emptyset$ . Setting  $I'_0 = I_0 \setminus \{i_1\}$  and  $I'_1 = I_1 \setminus \{i_0\}$ , we obtain

$$D_0 \sim L_{i_0} - L_{i_1} - L_{I'_0}, \quad D_1 \sim L_{i_1} - L_{i_0} - L_{I'_1}, \quad \text{with } I'_0 \cap I'_1 = \emptyset, \{i_0, i_1\} \cap (I'_0 \cup I'_1) = \emptyset.$$

Therefore

$$-(L_r + \dots + L_{n-1}) \sim C = D_0 + D_1 \sim -(L_{I'_0} + L_{I'_1})$$

i.e.  $I'_0 \cup I'_1 = [r, n-1]$ . Let  $I = \{i_0, i_1\} \cup I'_0 \cup I'_1$  and  $I' = [0, n-1] \setminus I$ . Of course,  $L_{I'} \cdot D_0 = L_{I'} \cdot D_1 = 0$ , whence by (19) 3),  $I' = [0, r-1]$  which is impossible. Therefore  $I' = \emptyset$  and  $\{i_0, i_1\} \cup I'_0 \cup I'_1 = [0, n-1]$ , i.e.  $r = 2$  and

$$\sharp(C) - C^2 = 2 + (n-2) = b_2(S).$$

3) **If  $s \geq 3$** ,  $D_j \sim L_{i_j} - L_{I_j}$ ,  $j = 0, \dots, s-1$  and we may suppose that

$$D_0 D_1 = \dots = D_{s-2} D_{s-1} = D_{s-1} D_0 = 1.$$

Since there is a deformation of  $S$  in which  $D_j + D_{j+1} \sim L_{i_j} + L_{i_{j+1}} - L_{I_j} - L_{I_{j+1}}$  is deformed into a nonsingular rational curve  $D'_u$  in a (perhaps non minimal) surface, contained in a cycle  $C'_u$ , hence  $D_j + D_{j+1}$  must be of the homological type **a**, and either  $i_j \in I_{j+1}$ ,  $i_{j+1} \notin I_j$  or  $i_j \notin I_{j+1}$ ,  $i_{j+1} \in I_j$ . Moreover the equality

$$1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j} + L_{I_j} L_{I_{j+1}}$$

implies that  $I_j \cap I_{j+1} = \emptyset$ , whence

$$\forall j, \quad 1 = D_j D_{j+1} = -L_{i_j} L_{I_{j+1}} - L_{i_{j+1}} L_{I_j}.$$

4) Now, we show by induction on  $s \geq 2$  that for all  $j$ ,  $0 \leq j \leq s-1$ , there is a unique index, denoted  $\sigma(j)$ , such that  $i_j \in I_{\sigma(j)}$ . This assertion is evident if  $s = 2$ , therefore we suppose that  $s \geq 3$ . By 3) it is possible to choose the numbering in such a way that  $i_{s-1} \in I_{s-2}$ . We choose a deformation  $\Pi : \mathcal{S} \rightarrow \Delta$  of  $S$  over the disc endowed with a flat family  $\mathcal{C}$  of curves such that  $C_0 = C$ , and for  $u \neq 0$ ,

$$D'_u \sim D_{s-2} + D_{s-1} \sim L_{i_{s-2}} + L_{i_{s-1}} - (L_{I_{s-2}} + L_{I_{s-1}}) = L_{i_{s-2}} - (L_{I'_{s-2}} + L_{I_{s-1}}),$$



with  $I'_{s-2} = I_{s-2} \setminus \{i_{s-1}\}$ . Consider the cycle  $C_u = D_{0,u} + \cdots + D_{s-3,u} + D'_u$  of  $s-1$  rational curves in the (perhaps non minimal) surface  $S_u$ ; by the induction hypothesis, since  $i_{s-1} \in I_{s-2}$  and  $I'_{s-2} \cap I_{s-1} = \emptyset$ , for all  $0 \leq j \leq s-2$ , there is a unique index  $\sigma(j)$ ,  $0 \leq \sigma(j) \leq s-1$  such that  $i_j \in I_{\sigma(j)}$ . Repeating this argument with  $D_{s-3} + D_{s-2}$  we obtain the result. Therefore we have a well-defined mapping  $\sigma : \{0, \dots, s-1\} \rightarrow \{0, \dots, s-1\}$  such that for all  $j$ ,  $0 \leq j \leq s-1$ ,  $i_j \in I_{\sigma(j)}$ .

5) Setting  $I'_j = I_j \setminus \sigma^{-1}(j)$  we have

$$\begin{aligned} -(L_r + \cdots + L_{n-1}) &\sim C = D_0 + \cdots + D_{s-1} \sim L_{i_0} + \cdots + L_{i_{s-1}} - (L_{I_0} + \cdots + L_{I_{s-1}}) \\ &\sim -(L_{I'_0} + \cdots + L_{I'_{s-1}}) \end{aligned}$$

therefore

$$[r, n-1] = I'_0 \cup \cdots \cup I'_{s-1} \quad \text{with} \quad \{i_0, \dots, i_{s-1}\} \cap (I'_0 \cup \cdots \cup I'_{s-1}) = \emptyset$$

If  $I = \{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1}$  and  $I' = [0, n-1] \setminus I$ , we have  $L_{I'} D_j = 0$  for all  $j = 0, \dots, s-1$ , then if  $I' \neq \emptyset$ , lemma (19) would imply that  $I' = [0, r-1]$  and this is impossible, hence  $I' = \emptyset$  and  $\{i_0, \dots, i_{s-1}\} \cup I'_0 \cup \cdots \cup I'_{s-1} = [0, n-1]$ , i.e.  $r = s$  and  $\{i_0, \dots, i_{r-1}\} = [0, r-1]$ . Finally

$$\sharp(C) - C^2 = s + (n-r) = n = b_2(S).$$

□

**Corollary 2. 36** *Let  $S$  be a minimal surface with  $b_2(S) \geq 1$ . If  $E$  is a  $k$ -cycle,  $k = \sharp(E) \geq 0$ , such that  $i_* H_1(E, \mathbb{Z}) = H_1(S, \mathbb{Z})$ , then  $\sharp(E) - E^2 = b_2(S)$ .*

**Remark 2. 37** If  $[H_1(S, \mathbb{Z}) : i_* H_1(E, \mathbb{Z})] = 2$ ,  $S$  is an odd Inoue-Hirzebruch surface with only one cycle  $C$ . Then we have  $\sharp(C) - C^2 = 2b_2(S)$  (see [20] (2.13)).

For surfaces containing a divisor  $D$  such that  $D^2 = 0$ , i.e. Enoki surfaces, the situation is well understood: They all contain GSS,  $D$  is a cycle of  $b_2(S)$  rational curves  $D = D_0 + \cdots + D_{n-1}$  such that  $D_0^2 = \cdots = D_{n-1}^2 = -2$  and  $S$  admits a numerically anticanonical divisor if and only if  $S$  is a Inoue surface, in which case  $h^0(S, -K) = 1$ . For surfaces such that  $D^2 < 0$  for any divisor we have the following theorem:

**Theorem 2. 38** *Let  $S$  be a compact complex surface of class  $VII_0^+$ . We suppose that there is no divisor such that  $D^2 = 0$ . Then the following properties are equivalent:*

- i)  $S$  contains a GSS,
- ii)  $S$  admits a NAC divisor.
- iii)  $S$  contains  $b_2(S)$  rational curves,

Proof:  $i) \Leftrightarrow iii)$  by [12].

$i) \Rightarrow ii)$  The intersection matrix  $M(S)$  is negative definite and the rational curves give a  $\mathbb{Q}$ -base of  $H^2(S, \mathbb{Q})$ , whence there is an integer  $m \geq 1$  and a divisor  $D_m$  such that in  $H^2(S, \mathbb{Z})$ ,  $mK + D_m = 0$ . By lemma (22),  $D_m$  is effective.

$ii) \Rightarrow i)$  By lemma (24) we may suppose that the index satisfies  $m(S) = 1$ , i.e.  $S$  has a NAC divisor  $D_{-K}$ . By theorem (30) we may suppose that there is no singular curve. By lemma (27) there are two cases:  $S$  is a Inoue-Hirzebruch surface, in particular contains a GSS, or  $S$  contains a cycle with at least one branch. Therefore we have to prove the result in the second case. Let  $D = \sum_{i=0}^p D_i$  be the maximal divisor, where  $C = D_0 + \dots + D_{s-1}$  is the cycle. By (33), we have in  $H^2(S, \mathbb{Z})$ ,

$$-(L_0 + \dots + L_{n-1}) = -K = \sum_{i=0}^p k_i D_i = \sum_{i=0}^p k_i (L_i - L_{I_i}),$$

where  $k_i \geq 1$ . If  $p < n - 1$ , the curve  $D_{n-1}$  is missing, hence there is exactly one index  $j$  such that  $n - 1 \in I_j$  and  $k_j = 1$ . Moreover by Proposition (35),  $\cup_{i=0}^{s-1} I_i = [0, n - 1]$ , therefore  $j \leq s - 1$ . By lemma (26),  $S$  would be a Inoue-Hirzebruch surface, which is a contradiction.  $\square$

**Corollary 2. 39 ([19])** *Let  $S$  be a surface of class  $VII_0$  with  $b_2(S) > 0$ . Then  $S$  is a Inoue-Hirzebruch surface if and only if there exists two twisted vector fields  $\theta_1 \in H^0(S, \Theta \otimes F_1)$ ,  $\theta_2 \in H^0(S, \Theta \otimes F_2)$ , where  $F_1, F_2$  are flat line bundles, such that  $\theta_1 \wedge \theta_2(p) \neq 0$  at at least one point  $p \in S$ .*

Proof:  $\theta_1 \wedge \theta_2$  is a non trivial section of  $-K \otimes F$  whence  $S$  contains a GSS. We conclude by [10] th. 5.5.  $\square$

## 3 On classification of bihermitian surfaces

### 3.1 Conformal and complex structures

We consider connected oriented conformal 4-manifolds  $(M^4, c)$  with two complex (i.e. integrable almost-complex) structures  $J_1, J_2$  which induce the same orientation. Given a riemannian metric  $g$  in  $c$ ,  $(M^4, g, J_1, J_2)$  is called a **bihermitian surface** relatively to the conformal class  $c$  if

- $J_i$  are orthogonal with respect to the metric, i.e.  $g(J_i X, J_i Y) = g(X, Y)$ ,  $i = 1, 2$ ,
- $J_1$  and  $J_2$  are independent, i.e. there is a point  $x \in M$  such that  $J_1(x) \neq \pm J_2(x)$ .

The triple  $(c, J_1, J_2)$  is called a bihermitian structure on  $M^4$ . If moreover  $J_1(x) \neq \pm J_2(x)$  everywhere, the bihermitian structure  $(c, J_1, J_2)$  is called **strongly bihermitian**.

Given two such almost-complex structures  $J_1$  and  $J_2$ , denote by  $f$  the smooth function, called the angle function

$$f = \frac{1}{4}\text{tr}({}^t J_1 J_2) = -\frac{1}{4}\text{tr}(J_1 J_2).$$

By Cauchy-Schwarz inequality,  $|f| \leq 1$  and  $f(x) = \pm 1$  if and only if  $J_1(x) = \pm J_2(x)$ . Since  $J_1 J_2 \in SO(4)$  and  $J_2 J_1$  is the inverse of  $J_1 J_2$ , it is easy to check that

$$J_1 J_2 + J_2 J_1 = -2f \text{Id}.$$

Moreover  $J_1(x)$  and  $J_2(x)$  anticommute if and only if  $f(x) = 0$ .

Another conformal structure is provided by the Weyl curvature tensor: The riemannian curvature tensor  $R$  of type (3,1) has a classical decomposition, under the orthogonal group  $O(4)$ , into three parts given by the scalar curvature, the Ricci curvature tensor without trace and the Weyl curvature tensor  $W$  of type (3,1) which is a conformal invariant [4].

Let  $*$  :  $\wedge^2 T^*M \rightarrow \wedge^2 T^*M$  be the star-Hodge operator, with  $\star^2 = \text{Id}$  and two eigenvalues  $\pm 1$ . We denote by  $\wedge_+^2$  (resp.  $\wedge_-^2$ ) the eigenspace associated to  $+1$  (resp.  $-1$ ). Since  $*$  depends only on  $c$ , the splitting  $\wedge^2 T^*M = \wedge_+^2 \oplus \wedge_-^2$  is a conformal invariant. Let  $W(g) : \wedge^2 T^*M \rightarrow \wedge^2 T^*M$  be the Weyl curvature tensor of type (2,2), with restrictions  $W_\pm \in \text{End}(\wedge_\pm^2)$  over  $\wedge_\pm^2$ . The riemannian conformal class of  $(M^4, g)$  is called **anti-self-dual (ASD)** if  $*W(g) = -W(g)$ , or equivalently  $W_+ = 0$ .

The geometric meaning of ASD condition stems from Atiyah-Hitchin-Singer theorem [3]: Let

$$Z = \{J \in SO(TM) \mid J^2 = -\text{Id}\} = SO(TM)/U(2) \rightarrow M$$

be the twistor space, i.e. the space of all orthogonal almost-complex structures over  $(M, g)$ , inducing the orientation of  $M$ . The fiber is isomorphic to the Riemann sphere  $S^2$ . Since there is only one complex structure on  $S^2$ ,  $Z$  is a differentiable fiber bundle with complex fiber  $\mathbb{P}^1(\mathbb{C})$ . The complex structure on the fiber and on the base yield a canonical almost-complex structure  $\mathbb{J}$  of the twistor space  $Z$ , which is not integrable in general. The theorem of Atiyah-Hitchin-Singer asserts that  $\mathbb{J}$  is integrable if and only if the metric  $g$  is ASD, hence any compatible complex structure at a point  $x \in M$  extends into a compatible complex structure over a neighbourhood of  $x$ . However, it does not extend to the whole manifold and perhaps  $(M, c)$  admits no complex structure (for instance  $S^4$ ). **The aim is to classify compact 4-manifolds with several compatible complex structures, or at least to give necessary conditions for their existence.** When there are more than two compatible almost structures, we have

**Proposition 3. 40 ([22])** *If an oriented riemannian 4-manifold  $(M, g)$  admits three independent compatible complex structures then the metric  $g$  is anti-self-dual.*

By Pontecorvo's classification of ASD bihermitian surfaces [22] Prop. 3.7, these surfaces are hyperhermitian. Following C.P. Boyer [4] we define **hyperhermitian complex surfaces** as oriented compact conformal 4-manifolds  $(M, c, \mathcal{F})$  with a 2-sphere  $\mathcal{F}$  of compatible complex structures generated by two anti-commuting ones. A hyperhermitian 4-manifold  $(M, c, \mathcal{F})$  must be one of the following

- A flat complex torus
- A K3 surface with Ricci-flat Kähler metric, or
- A special Hopf surface

in particular they all have  $b_2(M) = 0$ . We refer to [22] and [4] for details. Now we focus on the case where first Betti number  $b_1(M)$  is odd; for the even case see [2].

### 3.2 Numerically anticanonical divisor of a bihermitian surface

For the convenience of the reader we recall the results used in the sequel (see [1, 2, 22]): Denote by

$$F_i^g(\cdot, \cdot) = g(J_i \cdot, \cdot), \quad i = 1, 2$$

the Kähler forms of  $(g, J_1)$  and  $(g, J_2)$  respectively,  $\theta_1^g, \theta_2^g$  their Lee forms, i.e.

$$dF_i^g = \theta_i^g \wedge F_i^g, \quad i = 1, 2$$

We furthermore denote by  $[J_1, J_2] = J_1 J_2 - J_2 J_1$  the commutator of  $J_1$  and  $J_2$  and we consider the real  $J_i$ -anti-invariant 2-form

$$\Phi^g(\cdot, \cdot) = \frac{1}{2}g([J_1, J_2] \cdot, \cdot),$$

and the corresponding complex  $(0, 2)$ -forms

$$\sigma_i^g(\cdot, \cdot) = \Phi^g(\cdot, \cdot) + i\Phi^g(J_i \cdot, \cdot).$$

Then  $\sigma_i^g$ ,  $i = 1, 2$  are smooth sections of the anti-canonical bundle  $K_{J_i}^{-1} \simeq \wedge_{J_i}^{(0,2)}(M)$  of  $(M, J_i)$  and  $\sigma_i^g(x) = 0$  if and only if  $\Phi^g(x) = 0$  if and only if  $J_1(x) = \pm J_2(x)$ . Therefore the common zero set of  $\sigma_i$  is exactly  $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$ , where

$$\begin{aligned} \mathcal{D}^+ &= f^{-1}(1) = \{x \in M \mid J_1(x) = J_2(x)\}, \\ \mathcal{D}^- &= f^{-1}(-1) = \{x \in M \mid J_1(x) = -J_2(x)\}. \end{aligned}$$

**Lemma 3. 41** ([1, 2]) *Let  $(M, c, J_1, J_2)$  be a bihermitian surface. Then, for any metric  $g$  in the conformal class  $c$ , the 1-forms  $\theta_1^g, \theta_2^g$  and  $\sigma_1^g$  satisfy the following properties:*

i) *If  $M$  is compact, then  $d(\theta_1^g + \theta_2^g) = 0$ ,*

ii)  *$\bar{\partial}_{J_1} \sigma_1^g = -\frac{1}{2}(\theta_1^g + \theta_2^g)^{(0,1)} \otimes \sigma_1^g$ ,*

*where  $(\cdot)^{(0,1)}$  denotes the  $(0, 1)$  part and  $\bar{\partial}_{J_1}$  is the Cauchy-Riemann operator relatively to  $J_1$ .*

Proof: i) By [2],  $(d(\theta_1^g + \theta_2^g))_+ = 0$ . Let  $\delta = -\star d\star$  be the adjoint of  $d$ . Setting  $\varphi = \theta_1^g + \theta_2^g$ ,  $\Delta d\varphi = d\delta d\varphi = -d\star d\star d\varphi = 0$ , for by assumption  $\star d\varphi = -d\varphi$ . Hence  $\delta d\varphi = 0$ , and  $|d\varphi|^2 = (d\varphi, d\varphi) = (\varphi, \delta d\varphi) = 0$ .

ii) The proof in [2] Lemma 3 is local.  $\square$

**Proposition 3. 42** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface. Then there exists a topologically trivial line bundle  $L \in H^1(M, \mathbb{R}_+^*)$  such that  $\sigma_i$  is a non-trivial holomorphic section of  $K_{J_i}^{-1} \otimes L$ , in particular*

$$H^0(M, K_{J_i}^{-1} \otimes L) \neq 0, \quad i = 1, 2,$$

and  $\mathcal{D}^+, \mathcal{D}^-$  are empty or complex curves for both  $(M, J_1)$  and  $(M, J_2)$ .

Moreover, if  $(c, J_1, J_2)$  is strongly bihermitian, then  $K_{J_i} = L$ , in particular,  $b_2(M) = 0$ .

Proof: By (41),  $\theta_1^g + \theta_2^g$  is closed, hence there is an open covering  $(U_j)_{j \in I}$  and  $C^\infty$  functions  $\Phi_j : U_j \rightarrow \mathbb{R}$ , such that for the local metric  $g_j = \exp(\Phi_j)g$ , the local Lee forms satisfy  $\theta_1^{g_j} + \theta_2^{g_j} = 0$ , i.e.  $(\theta_1^g + \theta_2^g)|_{U_j} = -2d\Phi_j$ . Setting  $c_{jk} = \exp(\Phi_j - \Phi_k) \in \mathbb{R}_+^*$ ,  $\sigma_i^j = \sigma_i^{g_j}$ , we obtain a topologically trivial line bundle  $L = [(c_{jk})] \in H^1(M, \mathbb{R}_+^*)$ , and a holomorphic section  $\sigma_i = (\sigma_i^j)$  of  $K_{J_i}^{-1} \otimes L$ . If  $(c, J_1, J_2)$  is strongly bihermitian  $\mathcal{D} = \emptyset$ , therefore  $\sigma_i$  is a non-vanishing holomorphic section of  $K_{J_i}^{-1} \otimes L$ .  $\square$

### 3.3 Bihermitian surfaces with odd first Betti number

The Kodaira dimension of a compact bihermitian surface with odd first Betti number is  $\kappa = -\infty$  ([1] thm 1), hence the minimal models of  $(M, J_1)$  and of  $(M, J_2)$  are in class VII<sub>0</sub>. In this section, we shall complete (and simplify) the classification theorem of V. Apostolov [1]. We need first

**Lemma 3. 43** *Let  $S$  be a complex surface and let  $\Pi : S' \rightarrow S$  be the blowing-up of  $x \in S$ , and  $E = \Pi^{-1}(x)$  its exceptional curve.*

1) *If there exists a flat line bundle  $L'$  on  $S'$  such that  $H^0(S, K_{S'}^{-1} \otimes L') \neq 0$ , then for  $L = \Pi_\star L'$ ,  $H^0(S, K_S^{-1} \otimes L) \neq 0$ .*

2) *If there exists a flat line bundle  $L$  on  $S$  such that  $H^0(S, K_S^{-1} \otimes L) \neq 0$ , and if  $\Pi$  blows-up a point on the effective twisted anticanonical divisor, then for  $L' = \Pi^* L$ ,  $H^0(S', K_{S'}^{-1} \otimes L') \neq 0$ .*

Proof: 1) The coherent sheaf  $L = \Pi_\star L'$  is locally trivial since on a simply connected neighbourhood  $U$  of the exceptional curve  $E$ ,  $L|_U$  is trivial. The line bundle  $K_S^{-1} \otimes L$  has a section on  $S \setminus \Pi(E)$  which extends by Hartogs theorem. 2) We have  $K_{S'}^{-1} \otimes L' = \pi^*(K_S^{-1} \otimes L) - E$  hence a section  $\sigma$  of  $K_S^{-1} \otimes L$  yields a section of  $K_{S'}^{-1} \otimes L'$  if and only if  $x$  belongs to the zero set of  $\sigma$ .  $\square$

**Theorem 3. 44** *Let  $(M, c, J_1, J_2)$  be a compact bihermitian surface with odd first Betti number.*

1) If  $(M, c, J_1, J_2)$  is strongly bihermitian (i.e  $\mathcal{D} = \emptyset$ ), then the complex surfaces  $(M, J_i)$  are minimal and either a Hopf surface covered by a primary one associated to a contraction  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)$  of the form

$$F(z_1, z_2) = (\alpha z_1 + s z_2^m, a \alpha^{-1} z_2),$$

$$\text{with } a, s \in \mathbb{C}, 0 < |\alpha|^2 \leq a < |\alpha| < 1, (a^m - \alpha^{m+1})s = 0,$$

or else  $(M, J_i)$  are Inoue surfaces  $S_{N,p,q,r;t}^+$ ,  $S_{N,p,q,r}^-$ .

2) If  $(M, c, J_1, J_2)$  is not strongly bihermitian, then  $\mathcal{D}$  has at most two connected components,  $(M, J_i)$ ,  $i = 1, 2$ , contain GSS and the minimal models  $S_i$  of  $(M, J_i)$  are

- Surfaces with GSS of intermediate type if  $\mathcal{D}$  has one connected component
- Hopf surfaces of special type (see [22] 2.2), Inoue (parabolic) surfaces or Inoue-Hirzebruch surfaces if  $\mathcal{D}$  has two connected components.

Moreover, the blown-up points belong to the NAC divisors.

Proof: 0) Since the fundamental group of a surface with GSS is isomorphic to  $\mathbb{Z}$ , since  $\pi_1(S_{N,p,q,r;t}^{(+)}) \neq \mathbb{Z}$ ,  $\pi_1(S_{N,p,q,r}^{(-)}) \neq \mathbb{Z}$ , and  $S_{N,p,q,r}^{(-)}$  is a quotient of a surface  $S_{N,p,q,r;t}^{(+)}$  ([15] p 276),  $(M, J_1)$  and  $(M, J_2)$  must be of the same type.

1) If  $(M, c, J_1, J_2)$  is strongly bihermitian,  $(M, J_i)$  are minimal using (42). Applying the classification of minimal surfaces of class VII<sub>0</sub> with  $b_2(S) = 0$  [18], we have to consider Hopf surfaces and Inoue surfaces. We derive from Proposition (42) that  $-K_{J_i} \in H^1(M, \mathbb{R}_+^*)$ . On one hand the anticanonical line bundle of a surface  $S_M$  is not real (11), hence we may exclude it. On second hand, a finite covering of a bihermitian surface is bihermitian and if a primary Hopf surface is associated to the contraction

$$F(z) = (\alpha_1 z_1 + s z_2^m, \alpha_2 z_2), \quad 0 < |\alpha_1| \leq |\alpha_2| < 1, \quad (\alpha_2^m - \alpha_1)s = 0,$$

$-K = L^{\alpha_1 \alpha_2}$ , therefore  $a = \alpha_1 \alpha_2 \in \mathbb{R}_+^*$  and it is easy to check that the requested conditions are fulfilled.

2) If  $(M, c, J_1, J_2)$  is not strongly bihermitian,  $(M, J_i)$  admits a non-trivial effective NAC divisor  $D_{-K_i}$  whose support is  $\mathcal{D}$ , and as noticed in [1], 3.3, it is the same for the minimal model  $S_i$  of  $(M, J_i)$ . Using theorem (38),  $S_i$  contains GSS and the support of the NAC divisor has at most two components by (29). If  $\Pi = \Pi_m \cdots \Pi_0 : (M, J_i) \rightarrow S_i$  is the blowing-down of the exceptional curves, the blown-up points belong to the successive NAC divisors by (43), hence  $\mathcal{D}$  has at most two connected components.  $\square$

**Corollary 3. 45** *Let  $(M, c, J_1, J_2)$  be a compact ASD bihermitian surface with odd first Betti number. Then the minimal models of the complex surfaces  $(M, J_i)$ ,  $i = 1, 2$ , are*

- Hopf surfaces of special type (see [22] 2.2),
- (parabolic) Inoue surfaces or

- *even Inoue-Hirzebruch surfaces.*

Moreover, the blown-up points belong to the NAC divisors.

Proof: By [22] 3.11,  $(M, J_i)$  has a minimal model  $S_i$  in the class  $\text{VII}_0$  and

$$H^0(M, -K_{J_i}) \neq 0,$$

therefore  $S_i$  is not a Inoue surface  $S_{N,p,q,r;t}^{(+)}$ ,  $S_{N,p,q,r}^{(-)}$ . If  $b_2(S_i) = 0$ ,  $S_i$  is a Hopf surface, and if  $b_2(S_i) > 0$ ,  $S_i$  contains a GSS by (38). The existence of a (non-twisted) global section of  $-K$  is equivalent to the existence of a metric  $g \in c$  such that  $\theta_1^g + \theta_2^g = 0$ . In this situation, [2] Prop.4 asserts that  $\mathcal{D}_+$  and  $\mathcal{D}_-$  are both non-empty. This is possible only when  $S_i$  is parabolic Inoue or an even Inoue-Hirzebruch surface.  $\square$

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