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# PROBING FOR INCLUSIONS IN HEAT CONDUCTIVE BODIES 

Patricia Gaitan<br>Aix-Marseille Université, IUT Aix-en-Provence, France<br>Hiroshi Isozaki<br>University of Tsukuba, Japan<br>Olivier Poisson<br>Aix-Marseille Université, France<br>Samuli Siltanen<br>University of Helsinki, Finland<br>Janne Tamminen<br>Tampere University of Technology, Finland

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#### Abstract

This work deals with an inverse boundary value problem arising from the equation of heat conduction. Mathematical theory and algorithm is described in dimensions 1-3 for probing the discontinuous part of the conductivity from local temperature and heat flow measurements at the boundary. The approach is based on the use of complex spherical waves, and no knowledge is needed about the initial temperature distribution. In dimension two we show how conformal transformations can be used for probing deeper than is possible with discs. Results from numerical experiments in the one-dimensional case are reported, suggesting that the method is capable of recovering locations of discontinuities approximately from noisy data.


## 1. Introduction.

1.1. Inverse heat conductivity problem. Let $\Omega$ be a bounded open set in $\mathbb{R}^{d}$ with Lipschitz boundary $\partial \Omega$, and consider the following boundary value problem

$$
\left\{\begin{align*}
\partial_{t} v-\nabla \cdot(\gamma(x) \nabla v) & =0 \text { in }(0, T) \times \Omega  \tag{1}\\
v & =f \text { on }(0, T) \times \partial \Omega \\
\left.v\right|_{t=0} & =v_{0} \quad \text { in } \Omega
\end{align*}\right.
$$

where $\gamma(x) \in L^{\infty}(\Omega)$ such that $\gamma(x)>c$ for a constant $c>0$. Let $v^{f}$ be the unique solution to the above equation. The time-dependent Dirichlet-to-Neumann (DN) $\operatorname{map} \Lambda$ is then defined by

$$
\begin{equation*}
\Lambda:\left.f \rightarrow \gamma(x) \frac{\partial v^{f}}{\partial \nu}\right|_{\partial \Omega} \tag{2}
\end{equation*}
$$

where $\nu$ is the outer unit normal to $\partial \Omega$.

[^0]Physically, we consider a heat-conducting body modelled by the set $\bar{\Omega}$ and the strictly positive heat conductivity distribution $\gamma$ inside the body. The function $v_{0}$ is the initial temperature distribution in $\Omega$ over which we do not have control. We perform boundary measurements by applying the temperature $f(x, t)$ at the boundary $\partial \Omega$ during the time $0<t<T$ and measuring the resulting heat flux $\left.\gamma(x) \frac{\partial v^{f}}{\partial \nu}\right|_{\partial \Omega}$ through the boundary. The DN map $\Lambda$ defined in (2) is an ideal model of all possible infinite-precision measurements of the above type.

We study the ill-posed inverse problem of detecting conductivity inclusions inside $\Omega$ from the local knowledge of $\Lambda$. Namely, we assume that only a part $\Gamma \subset \partial \Omega$ is available for measurements and take as data the restrictions $\left.\gamma(x) \frac{\partial v^{f}}{\partial \nu}\right|_{\Gamma}$ of heat fluxes corresponding to functions $f$ that are supported in $\Gamma$. An inclusion is defined as a subdomain $\Omega_{1} \subset \Omega$ such that $\gamma(x)$ is perturbed on $\Omega_{1}$ from some known $\gamma_{0}(x)$. Our aim is to find the location of $\partial \Omega_{1}$ from the local measurements. The typical situation is drawn in the figure 1.


Figure 1.

The above inverse boundary value problem is related to nondestructive testing where one looks for anomalous materials inside a known material. One such example is monitoring a blast furnace used in ironmaking: the corroded thickness of the accreted refractory wall based on temperature and heat flux measurement on the accessible part of the furnace wall [11].

We remark that although $\Lambda$ depends on both $f$ and $v_{0}$, our method uses no information of the initial data $v_{0}$ to detect $\partial \Omega_{1}$.
1.2. Main theorems. Suppose we are given $\gamma_{0}(x) \in C^{\infty}(\bar{\Omega}), \gamma(x) \in L^{\infty}(\Omega)$ and an open subset $\Omega_{1} \subset \subset \Omega$, i.e. $\overline{\Omega_{1}} \subset \Omega$, such that for some constant $C_{0}>0$,

$$
C_{0}<\gamma_{0}(x)<C_{0}^{-1}, \quad C_{0}<\gamma(x)<C_{0}^{-1}, \quad \text { on } \quad \Omega,
$$

and

$$
\gamma(x)=\gamma_{0}(x), \quad \text { on } \quad \Omega_{0}=\Omega \backslash \overline{\Omega_{1}} .
$$

We put

$$
\gamma_{1}(x)=\gamma(x)-\gamma_{0}(x)
$$

Moreover, we shall assume that $\gamma_{1}(x)-\gamma_{0}(x)$ has a constant sign on $\Omega_{1}$. Our main purpose is to study discontinuous perturbations, however, we allow $\gamma(x)$ to be continuous. Hence, we impose the following assumption.
(A) $\inf _{x \in K}\left|\gamma_{1}(x)-\gamma_{0}(x)\right|>0$ for any compact set $K \subset \Omega_{1}$.

We define (formal) differential operators by

$$
\mathcal{A}_{0}=-\nabla \cdot\left(\gamma_{0} \nabla\right), \quad \mathcal{A}=-\nabla \cdot(\gamma \nabla) .
$$

We extend $\gamma_{0}(x)$ smoothly outside $\Omega$ so that $\gamma_{0}(x)=1$ for large $|x|$.
First, we consider the 1-dimensional problem. Namely, for the heat equation on $(a, b)$, we try to detect $\operatorname{dist}\left(a, \partial \Omega_{1}\right)$ from the measurement at $a$.

Theorem 1.1. Let $d=1, \Omega=(a, b)$. Suppose $\Omega_{1}=\left(a_{1}, b_{1}\right)$ with $a<a_{1}<b_{1}<b$. Take $x_{0} \in \mathbb{R}$ arbitrarily. Then there exists a real fucntion $\varphi_{\lambda}(x) \in C^{\infty}(\mathbb{R})$ depending on a large parameter $\lambda>0$ having the following properties.
(1) It satisfies

$$
\begin{equation*}
\left(\mathcal{A}_{0}+\lambda\right) \varphi_{\lambda}(x)=0, \quad x \in(a, b), \quad \lambda \gg 1 \tag{3}
\end{equation*}
$$

(2) For any compact sets $K_{-} \subset\left(-\infty, x_{0}\right)$, and $K_{+} \subset\left(x_{0}, \infty\right)$, there exist constants $C, \delta>0$ such that for $\lambda>C$

$$
\begin{aligned}
\left|\varphi_{\lambda}(x)\right| \geq C e^{\delta \sqrt{\lambda}}, & \left|\varphi_{\lambda}^{\prime}(x)\right| \geq C e^{\delta \sqrt{\lambda}}, \quad \forall x \in K_{-} \\
\left|\varphi_{\lambda}(x)\right| \leq C e^{-\delta \sqrt{\lambda}}, & \left|\varphi_{\lambda}^{\prime}(x)\right| \leq C e^{-\delta \sqrt{\lambda}}, \quad \forall x \in K_{+}
\end{aligned}
$$

(3) Take $0<T_{1} \leq T$ arbitrarily, and let

$$
\begin{gather*}
h_{\lambda}(x)=\left.\varphi_{\lambda}(x)\right|_{\partial \Omega}, \quad f_{\lambda}(t, x)=e^{\lambda t} h_{\lambda}(x) \\
I(\lambda)=e^{-\lambda T_{1}}\left(\Lambda f_{\lambda}\right)\left(T_{1}, a\right) h_{\lambda}(a)-\gamma_{0}(a) \frac{d h_{\lambda}}{d x}(a) h_{\lambda}(a) \tag{4}
\end{gather*}
$$

Then, if $\pm\left(\gamma_{1}-\gamma_{0}\right)>0$ on $\Omega_{1}$, we have

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty}(2 \sqrt{\lambda})^{-1} \log ( \pm I(\lambda))=-y\left(a_{1}\right) \tag{5}
\end{equation*}
$$

where $y(x)$ is defined by

$$
\begin{equation*}
y(x)=\int_{x_{0}}^{x} \frac{d t}{\sqrt{\gamma_{0}(t)}} \tag{6}
\end{equation*}
$$

For higher dimensions, we need the following notations. By dist $\left(x_{0}, A\right)$, we mean the distance between a point $x_{0} \in \mathbb{R}^{d}$ and a set $A \subset \mathbb{R}^{d}$. Let $B\left(x_{0}, R\right)=\{x \in$ $\left.\mathbb{R}^{d} ;\left|x-x_{0}\right|<R\right\}$. For dimensions $d \geq 2$, we take an open subset $\Gamma$ on the boundary $\partial \Omega$, and try to recover the location of $\partial \Omega_{1}$ from a measurement on $\Gamma$. Take $x_{0} \in \mathbb{R}^{d}$ arbitrarily. The roles of half-lines $\left(-\infty, x_{0}\right)$ and $\left(x_{0}, \infty\right)$ are played by $B\left(x_{0}, R\right)$ and $B\left(x_{0}, R\right)^{c}$, respectively.

Theorem 1.2. Suppose $d=2,3$. Take $B=B\left(x_{0}, R\right)$ satisfying
(C-1) $\quad \emptyset \neq \bar{B} \cap \partial \Omega \subset \Gamma$.
Take $x^{*} \in \partial B \backslash \bar{\Omega}$ arbitrarily. For an open set $\mathcal{O} \subset \mathbb{R}^{d}$, let

$$
\begin{equation*}
a_{B}(\mathcal{O})=\sup _{x \in \mathcal{O}} \frac{R^{2}-\left|x-x_{0}\right|^{2}}{\left|x-x^{*}\right|^{2}} \tag{7}
\end{equation*}
$$

and fix constants $T_{1}$ and $\mu$ such that

$$
\begin{equation*}
0<T_{1} \leq T, \quad 0<\mu<\frac{T_{1}}{\sqrt{2}\left(a_{B}(\Omega)-a_{B}\left(\Omega_{1}\right)\right)} \tag{8}
\end{equation*}
$$

Then there exists $\varphi_{\lambda}(x) \in C^{\infty}\left(\mathbb{R}^{d} \backslash\left\{x^{*}\right\}\right)$ depending on a large parameter $\lambda>0$ having the following properties.
(1) It satisfies

$$
\begin{equation*}
\left(\mathcal{A}_{0}+\lambda\right) \varphi_{\lambda}(x)=0, \quad 0<\left|x-x^{*}\right|<2 \sqrt{\lambda} R \tag{9}
\end{equation*}
$$

(2) For any compact sets $K_{-} \subset B\left(x_{0}, R\right) \backslash\left\{x^{*}\right\}$, and $K_{+} \subset{\overline{B\left(x_{0}, R\right)}}^{c}$, there exist constants $C, \delta>0$ such that for $\lambda>C$

$$
\begin{gathered}
\left|\varphi_{\lambda}(x)\right| \geq C e^{\delta \lambda}, \quad\left|\nabla_{x} \varphi_{\lambda}(x)\right| \geq C e^{\delta \lambda}, \quad \forall x \in K_{-} \\
\left|\varphi_{\lambda}(x)\right| \leq C e^{-\delta \lambda}, \quad\left|\nabla_{x} \varphi_{\lambda}(x)\right| \leq C e^{-\delta \lambda}, \quad \forall x \in K_{+}
\end{gathered}
$$

(3) We put

$$
\begin{gather*}
h_{\lambda}(x)=\left.\varphi_{\lambda}(x)\right|_{\partial \Omega}, \quad f_{\lambda}(t, x)=e^{\lambda t} h_{\lambda}(x) \\
I(\lambda)=e^{-\lambda T_{1}} \int_{\Gamma}\left(\Lambda f_{\lambda}\right)\left(T_{1}, x\right) \overline{h_{\lambda}(x)} d \sigma(x)-\int_{\Gamma} \gamma_{0}(x) \frac{\partial \varphi_{\lambda}(x)}{\partial \nu} \overline{h_{\lambda}(x)} d \sigma(x) \tag{10}
\end{gather*}
$$

$d \sigma(x)$ being the induced measure on $\partial \Omega$. Then we have

$$
\lim _{\lambda \rightarrow \infty}(\sqrt{2} \mu \lambda)^{-1} \log ( \pm I(\lambda))=a_{B}\left(\Omega_{1}\right)
$$

accordingly as $\pm\left(\gamma_{1}-\gamma_{0}\right)>0$.


Figure 2.
In Theorems 1.1 and $1.2, \varphi_{\lambda}(x)$ is constructed by using $\gamma_{0}(x)$ only. We put

$$
R_{1}=\operatorname{dist}\left(x_{0}, \partial \Omega_{1}\right)
$$

Let us remark that

$$
\begin{aligned}
& a_{B}\left(\Omega_{1}\right)>0 \Longleftrightarrow \\
& a_{B}\left(\Omega_{1}\right)<0 \Longleftrightarrow B\left(x_{0}, R\right) \cap \Omega_{1} \neq \emptyset \\
& B\left(x_{0}, R\right) \cap \overline{\Omega_{1}}=\emptyset \Longleftrightarrow \\
& R>R_{1} \\
& \\
& \hline
\end{aligned}
$$

Then by Theorem 1.2, if $\pm\left(\gamma_{1}-\gamma_{0}\right)>0$ on $\Omega_{1}$ and $R<R_{1}$, then $\pm I(\lambda)$ tends to 0 exponentially as $\lambda \rightarrow \infty$, and if $R>R_{1}$, then $\pm I(\lambda)$ tends to $\infty$ exponentially as $\lambda \rightarrow \infty$. This means, we can see whether $B\left(x_{0}, R\right)$ touches $\Omega_{1}$ or not using only the knowledge of $\gamma_{0}(x)$.

Our results also hold for $4 \leq d \leq 6$. However, to fix the idea, we shall consider the case $1 \leq d \leq 3$.
1.3. Detection algorithm. Suppose for the sake of simplicity that $\gamma_{0} \equiv 1$. The 1 -dimensional case is rather easy. We have only to use (5) to compute $a_{1}$. The detection algorithm for $d=2,3$ is as follows.

1. Take an open set $\Gamma \subset \partial \Omega$, and $x^{*}$ outside $\Omega$, but close to $\Gamma$.
2. Draw a straight line $l$ with end point $x^{*}$, take $x_{0} \in l$, and the ball $B=B\left(x_{0}, R\right)$ such that $(\bar{B} \cap \partial \Omega) \subset \Gamma$.
3. For large $\lambda>0$, compute $I(\lambda)$. The function

$$
\begin{gathered}
e^{\lambda t}\left|x-x^{*}\right|^{2-d} e^{-\zeta \cdot y(x)}, \quad y(x)=y^{*}+2 R \frac{x-x^{*}}{\left|x-x^{*}\right|^{2}}, \quad \zeta=\frac{\mu \lambda}{\sqrt{2}}\left(y^{*}+i y_{\perp}^{\prime}\right) \\
y^{*}=\frac{x^{*}-x_{0}}{\left|x^{*}-x_{0}\right|}, \quad y^{*} \cdot y_{\perp}^{*}=0, \quad\left|y_{\perp}^{*}\right|=1
\end{gathered}
$$

will give an approximation of the probing data.
4. If $I(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ we infer that $B$ does not intersect the inclusion.
5. If $I(\lambda) \rightarrow \infty$ (or $-\infty$ ) as $\lambda \rightarrow \infty$ we infer that $B$ intersects the inclusion and $\gamma_{1}(x)>\gamma(x)$ (or $\gamma_{1}(x)<\gamma(x)$ ) on it.
1.4. Transformation of exponentially growing solutions. The main idea of the proof (for $d=2,3$ ) consists in using the probing data of the form $e^{\lambda t} \varphi_{\lambda}(x)$, where $\varphi_{\lambda}(x)$ is exponentially growing as $\lambda \rightarrow \infty$ in the ball $B\left(x_{0}, R\right)$ and exponentially decaying outside $B\left(x_{0}, R\right)$. Such a method was found and used in [4] for the case of the elliptic problem: $\nabla \cdot(\gamma(x) \nabla u)=0$ by passing through the hyperbolic space. The essential feature of this idea is to use conformal transformation which maps a sphere to a plane. In this paper, we shall work entirely in the Euclidean space, and use inversion with respect to a sphere (this is also the case in [4]), which maps the ball $B\left(x_{0}, R\right)$ to the half-space $\left\{y \cdot y^{*}<0\right\}$. We are then led to consider the equation of the form

$$
\left(-\Delta+2 \zeta \cdot \nabla+q_{\lambda}\right) u=f
$$

where $\zeta \in \mathbb{C}^{d}$. Although in our case $q_{\lambda}(x)$ grows up linearly in $\lambda$, suitable change of variables enables us to reduce the construction of solutions to now standard method of Sylvester-Uhlmann [9]. Another important difference is that in [4], the point $x_{0}$, the center of the ball $B\left(x_{0}, R\right)$, is taken outside the covex hull of $\Omega$, while in our method it should satisfy (C-1) and that $x^{*} \in \partial B \backslash \bar{\Omega}$. This makes it possible to probe regions deeper than that of [4].

It is interesting to find other conformal transformations mapping planes (lines) to some surfaces (curves) which are useful for the probing problem. Putting trivial transformations such as translation and orthogonal transformation aside, for $d=3$, the spherical inversion is essentially the unique conformal transformation for the Euclidean Laplacian. It can be seen by embedding the problem into the hyperbolic space, as was done in [4]. However, in 2-dimensions, the complex function theory provides us with lots of conformal transformations. We shall give in this paper one of such examples and use it for the inclusion detection problem in $\S 5$.
1.5. Literature review. There are lots of works on inverse problem of heat conductivity. In a recent article of [6], the following heat equation is considered for
domains $D \subset \subset \Omega$,

$$
\left\{\begin{array}{l}
u_{t}-\Delta u=0, \quad x \in \Omega \subset D, \quad 0<t<T \\
\frac{\partial u}{\partial \nu}+\rho u=0, \quad \text { on } \quad \partial D \\
u(x, 0)=0, \quad \text { in } \quad \Omega \backslash \bar{D}
\end{array}\right.
$$

$\nu$ being the unit normal on $\partial D$. Assuming that $D$ and $\rho$ are unknown, they recover the location and shape of $D$ from the knowledge of $u(x, t)$ and $\partial u(x, t) / \partial \nu$ on $\partial \Omega$. The proof is based on the exponentially growing solution of the form $\exp (\tau x \cdot(\omega+$ $\left.i \omega^{\perp}\right)$ ) with $\omega, \omega^{\perp} \in S^{2}, \omega \perp \omega^{\perp}$. This is an extreme case of the problem treated in this paper by taking the heat conducting coefficient $\gamma(x)=0$ on $\Omega_{1}$.

In [2] the interior boundary curve of an arbitrary-shaped annulus is reconstructed from overdetermined Cauchy-data on the exterior boundary curve, by assuming $\gamma=1$ and using the Newton method to a boundary integral equation approach. In [1] the shape of an inaccessible portion of the boundary is reconstructed by linearisation. In [3] the position of the boundary is reconstructed as a function of time, using the sideways heat equation. The proposed method of the present paper differs from these; it is a direct reconstruction method, and it finds inclusions in non-constant background conductivities. For other related results, see $[11,12,8]$ and the survey article [10].

The paper is organized as follows. In $\S 2$ and $\S 3$, we shall constuct the probing data $\varphi_{\lambda}(x)$. Theorems 1.1 and 1.2 are proved in $\S 4$. The use of conformal mapping in 2 -dimensions is explained in $\S 5$. In $\S 6$, we give some numerical results for the one dimensional case.
2. 1-dimensional trial function. Take $x_{0} \in \mathbb{R}$ arbitrarily, and make the change of variable by (6). Letting $\varphi_{\lambda}=\psi_{1}(y) e^{-\sqrt{\lambda} y}$, we transform the equation (3) into

$$
\psi_{1}^{\prime \prime}-(2 \sqrt{\lambda}-p(y)) \psi_{1}^{\prime}+\lambda p(y) \psi_{1}=0, \quad{ }^{\prime}=\frac{d}{d y}
$$

where $p(y)=\left(2 \sqrt{\gamma_{0}(x(y))}\right)^{-1} \frac{d \gamma_{0}}{d x}(x(y))=\frac{d}{d y} \log \left(\gamma_{0}(x(y))^{1 / 2}\right)$. Putting $P(y)=$ $\gamma_{0}(x(y))^{-1 / 4}$, we then have $2 \frac{d}{d y} P(y)=-P(y) p(y)$. Therefore, letting $\psi_{1}=P(y) \psi_{2}$, we have

$$
\begin{equation*}
\psi_{2}^{\prime \prime}-2 \sqrt{\lambda} \psi_{2}^{\prime}-Q(y) \psi_{2}=0 \tag{11}
\end{equation*}
$$

where $Q(y)=\frac{p^{\prime}(y)}{2}+\frac{p(y)^{2}}{4} \in C_{0}^{\infty}(\mathbb{R})$. Putting $\psi_{2}=1+\phi$, we then have

$$
\begin{equation*}
\phi^{\prime \prime}-2 \sqrt{\lambda} \phi^{\prime}-Q(y) \phi=Q \tag{12}
\end{equation*}
$$

We consider the integral operator

$$
(S u)(y)=\frac{1}{2 \sqrt{\lambda}} \int_{-\infty}^{y}\left(e^{-2 \sqrt{\lambda}(t-y)}-1\right) u(t) d t
$$

Take $A>0$ sufficiently large. If $u \in C^{2}((-\infty, A))$ satisfies $u, u^{\prime}, u^{\prime \prime} \in L^{1}((-\infty, A))$, we have

$$
\left(S u^{\prime \prime}\right)(y)=u(y)+2 \sqrt{\lambda}\left(S u^{\prime}\right)(y)
$$

Therefore, $\phi \in C^{2}((-\infty, A))$ is a solution to (12) satisfying $\phi, \phi^{\prime} \in L^{1}((-\infty, A))$ if and only if $\phi$ is a bounded solution to

$$
\phi=S Q \phi+S Q
$$

Since $Q(s)$ is compactly supported, we have

$$
\|S Q\|_{\mathbf{B}\left(L^{\infty}(I) ; L^{\infty}(I)\right)} \leq \frac{C}{\sqrt{\lambda}}, \quad I=(-\infty, A)
$$

which implies the existence of a solution $\psi_{2} \in C^{2}((-\infty, A))$ of (11) such that

$$
\left|\partial_{x}^{n}\left(\psi_{2}(y)-1\right)\right| \leq C(\sqrt{\lambda})^{-1+n}, \quad n=0,1,2
$$

Putting $\varphi_{\lambda}(x)=P(y) e^{\sqrt{\lambda} y} \psi_{2}(y)$, we obtain the following Theorem.
Theorem 2.1. There exists a solution $\varphi_{\lambda}(x)$ to the equation (3) such that

$$
\begin{equation*}
\varphi_{\lambda}(x)=\gamma_{0}(x)^{-1 / 4} e^{-\sqrt{\lambda} y(x)}\left(1+\phi_{\lambda}(x)\right) \tag{13}
\end{equation*}
$$

where $y(x)$ is defined by (6) and $\phi_{\lambda}(x)$ satisfies

$$
\left|\partial_{x}^{n} \phi_{\lambda}(x)\right| \leq C(\sqrt{\lambda})^{-1+n}, \quad n=0,1,2, \quad a \leq x \leq b
$$

3. Multi-dimensional trial function. The construction of $\varphi_{\lambda}(x)$ is more complicated in multi-dimensions than in the one-dimensional case. We shall prove the following theorem.

Theorem 3.1. Let $d=2,3$ and $x_{0}, x^{*}$ be as in Theorem 1.2. Take $y^{*}, y_{\perp}^{*} \in \mathbb{R}^{d}$ such that

$$
\begin{equation*}
y^{*}=\frac{x^{*}-x_{0}}{\left|x^{*}-x_{0}\right|}, \quad y^{*} \cdot y_{\perp}^{*}=0, \quad\left|y_{\perp}^{*}\right|=1 \tag{14}
\end{equation*}
$$

and let $\zeta \in \mathbf{C}^{d}$ be a complex vector such that

$$
\begin{equation*}
\zeta=\frac{\mu \lambda}{\sqrt{2}}\left(y^{*}+i y_{\perp}^{*}\right) \tag{15}
\end{equation*}
$$

where $\mu>0$ is a constant satisfying the condition (8). Let $y=y(x)$ be the inversion defined by

$$
\begin{equation*}
y=y^{*}+2 R \frac{x-x^{*}}{\left|x-x^{*}\right|^{2}} \tag{16}
\end{equation*}
$$

Then for large $\lambda>0$, there exists a solution $\varphi_{\lambda}(x)$ to the equation (9) in the region $0<\left|x-x^{*}\right|<2 \sqrt{\lambda} R$ having the following form:

$$
\begin{equation*}
\varphi_{\lambda}(x)=\left|x-x^{*}\right|^{2-d} \gamma_{0}(x)^{-1 / 2}\left(1+\phi_{\lambda}(x)\right) e^{-\zeta \cdot y(x)} \tag{17}
\end{equation*}
$$

where for any $0<\delta<1$, $\phi_{\lambda}(x)$ satisfies

$$
\begin{gather*}
\left\|\phi_{\lambda}(x)\right\|_{L^{\infty}(\Omega)} \leq C_{\delta} \mu^{-1} \lambda^{-\delta / 2}  \tag{18}\\
\left\|\nabla_{x} \phi_{\lambda}(x)\right\|_{L^{\infty}(\Omega)} \leq C_{\delta} \mu^{-1} \lambda^{(1-\delta) / 2}
\end{gather*}
$$

for a constant $C_{\delta}>0$, which also depends on $R$ and $x_{0}$ but is independent of $\mu$ and large $\lambda$. Moreover

$$
\begin{align*}
& \operatorname{Re} \zeta \cdot y(x)<0 \Longleftrightarrow\left|x-x_{0}\right|<R \\
& \operatorname{Re} \zeta \cdot y(x)=0 \Longleftrightarrow\left|x-x_{0}\right|=R,  \tag{19}\\
& \operatorname{Re} \zeta \cdot y(x)>0 \Longleftrightarrow\left|x-x_{0}\right|>R .
\end{align*}
$$

3.1. Schrödinger equation and spherical inversion. To prove Theorem 3.1, we need two changes of (in)dependent variables. For a solution $\varphi_{\lambda}$ to (9), we put $v=\sqrt{\gamma_{0}} \varphi_{\lambda}$. Then $v$ satisfies

$$
\begin{equation*}
-\Delta v+\left(\frac{\Delta \sqrt{\gamma_{0}}}{\sqrt{\gamma_{0}}}+\frac{\lambda}{\gamma_{0}}\right) v=0 \tag{20}
\end{equation*}
$$

Next we pass to the inversion (16). Letting $y^{\prime}=y-\left(y \cdot y^{*}\right) y^{*}$, and using $x^{*}=x_{0}+R y^{*}$, we have

$$
\begin{equation*}
y \cdot y^{*}=\frac{\left|x-x_{0}\right|^{2}-R^{2}}{\left|x-x^{*}\right|^{2}}, \quad y^{\prime}=\frac{2 R\left(x-x_{0}\right)^{\prime}}{\left|x-x^{*}\right|^{2}} \tag{21}
\end{equation*}
$$

The inverse map : $y \rightarrow x$ is given by

$$
\begin{equation*}
x=x^{*}+2 R \frac{y-y^{*}}{\left|y-y^{*}\right|^{2}} . \tag{22}
\end{equation*}
$$

Direct computation yields the following lemma.
Lemma 3.2. (1) The above transformation $x \rightarrow y$ maps
the ball $\left\{\left|x-x_{0}\right|<R\right\}$ to the half-space $\left\{y \cdot y^{*}<0\right\}$,
the sphere $\left\{\left|x-x_{0}\right|=R\right\}$ to the plane $\left\{y \cdot y^{*}=0\right\}$,
the outer region $\left\{\left|x-x_{0}\right|>R\right\}$ to the half-space $\left\{y \cdot y^{*}>0\right\}$,
where $y^{*}$ is defined by (14).
(2) For any $f(y) \in C^{\infty}\left(\mathbb{R}^{d}\right)$,

$$
4 R^{2} \Delta_{x} f(y(x))=\left.\left|y-y^{*}\right|^{2+d} \Delta_{y}\left(\left|y-y^{*}\right|^{2-d} f(y)\right)\right|_{y=y(x)}
$$

Here $\Delta_{x}$ means $\sum_{j=1}^{d}\left(\partial / \partial x_{j}\right)^{2}$.
For a solution $v(x)$ to (20), we put $w(y)=\left|y-y^{*}\right|^{2-d} v(x(y))$, where $x(y)$ is given by (22). Then by the above lemma, $w(y)$ satisfies

$$
\begin{gather*}
\left(-\Delta_{y}+q_{\lambda}(y)\right) w(y)=0  \tag{23}\\
q_{\lambda}(y)=\left.\frac{4 R^{2}}{\left|y-y^{*}\right|^{4}}\left(\frac{\Delta \sqrt{\gamma_{0}}}{\sqrt{\gamma_{0}}}+\frac{\lambda}{\gamma_{0}}\right)\right|_{x=x(y)}
\end{gather*}
$$

3.2. Proof of Theorem 3.1. Taking $\zeta$ from (15), we look for a solution $w(y)$ of (23) of the form

$$
w(y)=(1+\phi(y)) e^{-\zeta \cdot y}
$$

Since $\zeta^{2}=0, \phi$ satisfies

$$
\begin{equation*}
-\Delta \phi+2 \zeta \cdot \nabla \phi+q_{\lambda} \phi=-q_{\lambda} \tag{24}
\end{equation*}
$$

This is the equation treated in [9], with the difference that the potential $q_{\lambda}$ grows up linearly in $\lambda$. The remedy is to make the following change of variables and parameter :

$$
Y=\sqrt{\lambda}\left(y-y^{*}\right), \quad \eta=(\sqrt{\lambda})^{-1} \zeta
$$

Letting $\Phi(Y)=\phi(y)$, we then have

$$
\begin{gathered}
-\Delta_{Y} \Phi+2 \eta \cdot \nabla_{Y} \Phi+Q_{\lambda} \Phi=-Q_{\lambda} \\
Q_{\lambda}(Y)=\frac{q_{\lambda}(y)}{\lambda}=\left.\frac{4 R^{2}}{|Y|^{4}}\left(\frac{\Delta \sqrt{\gamma_{0}}}{\lambda \sqrt{\gamma_{0}}}+\frac{1}{\gamma_{0}}\right)\right|_{x=x(y)}
\end{gathered}
$$

The potential $Q_{\lambda}(Y)$ is now bounded in $\lambda$, however, it has a singularity at $Y=0$. We take $\chi_{\infty} \in C^{\infty}\left(\mathbb{R}^{d}\right)$ such that $\chi_{\infty}(Y)=1$ if $|Y| \geq 1 / 2$, $\chi_{\infty}(Y)=0$ if $|Y| \leq 1 / 4$, and define $V_{\lambda}(Y)$ by

$$
V_{\lambda}(Y)=\chi_{\infty}(Y) Q_{\lambda}(Y)
$$

We prove that if $|\eta|=\mu \sqrt{\lambda}$ is sufficiently large, the equation

$$
\begin{equation*}
\left(-\Delta_{Y}+2 \eta \cdot \nabla_{Y}+V_{\lambda}\right) \Psi=-V_{\lambda} \tag{25}
\end{equation*}
$$

admits a unique solution $\Psi \in L^{2, \sigma}$, where the weighted $L^{2}$-space $L^{2, \sigma}$ is defined by

$$
L^{2, \sigma}=\left\{u(Y) ;\left(1+|Y|^{2}\right)^{\sigma / 2} u \in L^{2}\left(\mathbb{R}^{d}\right)\right\}
$$

with obvious norm.
The following lemma can be proved easily, and the proof is omitted.
Lemma 3.3. If $\sigma<(6-d) / 2, V_{\lambda}$ satisfies the following inequalities:

$$
\begin{gathered}
\left\|(1+|Y|)^{4} V_{\lambda}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \leq C \\
\left\|V_{\lambda}\right\|_{L^{2, \sigma+1}}+\left\|\nabla_{Y} V_{\lambda}\right\|_{L^{2, \sigma+1}} \leq C \\
\left\|(1+|Y|)^{4} \nabla_{Y} V_{\lambda}\right\|_{L^{\infty}\left(\mathbb{R}^{d}\right)} \leq C
\end{gathered}
$$

where the constant $C$ does not depend on $\mu$ or on $\lambda \geq 1$.
Then thanks to [9, Theorem 2.3] and Lemma 3.3, for all $\sigma \in(-1,0),(25)$ admits a unique solution $\Psi \in L^{2, \sigma}$. Moreover, for $|\eta|>1$,

$$
\|\Psi\|_{L^{2, \sigma}} \leq C|\eta|^{-1}
$$

Letting ${ }^{\prime}=\partial / \partial Y_{j}$ and differentiating (25), we obtain

$$
\left(-\Delta_{Y}+2 \eta \cdot \nabla_{Y}+V_{\lambda}\right) \Psi^{\prime}=-V_{\lambda}^{\prime}(1+\Psi)
$$

Lemma 3.3 again implies that $\left\|V_{\lambda}^{\prime}(1+\Psi)\right\|_{L^{2, \sigma+1}} \leq C$, hence

$$
\left\|\nabla_{Y} \Psi\right\|_{L^{2, \sigma}} \leq C|\eta|^{-1}
$$

Let us recall the correspondence

$$
|Y|>1 \Longleftrightarrow\left|y-y^{*}\right|>\frac{1}{\sqrt{\lambda}} \Longleftrightarrow\left|x-x^{*}\right|<2 \sqrt{\lambda} R .
$$

Therefore, for $\sigma \in(-1,0)$, there exists a unique solution $\phi(y) \in L^{2, \sigma}$ of (24) in the region $\left|y-y^{*}\right|>1 / \sqrt{\lambda}$. Letting $\Omega^{\prime}=\{y(x) ; x \in \Omega\}$, and noticing that $|\eta|=\mu \sqrt{\lambda}$, we then have

$$
\begin{aligned}
\|\phi\|_{L^{2}\left(\Omega^{\prime}\right)} & \leq C_{\sigma}(1+\lambda)^{|\sigma| / 2}\left\|\left(1+\lambda\left|y-y^{*}\right|^{2}\right)^{\sigma / 2} \phi\right\|_{L^{2}\left(\Omega^{\prime}\right)} \\
& \leq C_{\sigma}(1+\lambda)^{|\sigma| / 2} \lambda^{-d / 2}\|\Psi(Y)\|_{L^{2, \sigma}} \\
& \leq C_{\sigma} \mu^{-1} \lambda^{(|\sigma|-d-1) / 2}
\end{aligned}
$$

Similarly, since $\nabla_{y} \phi(y)=\sqrt{\lambda} \nabla_{Y} \Psi(Y)$, we obtain

$$
\left\|\nabla_{y} \phi\right\|_{L^{2}\left(\Omega^{\prime}\right)} \leq C_{\sigma} \mu^{-1} \lambda^{(|\sigma|-d) / 2}
$$

and then, generally,

$$
\|\phi\|_{H^{s}\left(\Omega^{\prime}\right)} \leq C_{\sigma, s} \mu^{-1} \lambda^{(|\sigma|-d+s-1) / 2}, \quad s>0
$$

Thanks to Sobolev's embeddings in $\mathbb{R}^{d}, d=2,3$, we obtain if $s>d / 2$ :

$$
\begin{gathered}
\|\phi\|_{L^{\infty}\left(\Omega^{\prime}\right)} \leq C_{\sigma, s} \mu^{-1} \lambda^{(|\sigma|-d+s-1) / 2} \\
\|\nabla \phi\|_{L^{\infty}\left(\Omega^{\prime}\right)} \leq C_{\sigma, s} \mu^{-1} \lambda^{(|\sigma|-d+s) / 2}
\end{gathered}
$$

Hence letting $\delta=1-|\sigma|$, if $\mu \sqrt{\lambda}$ is sufficiently large we obtain

$$
\begin{aligned}
\|\phi\|_{L^{\infty}(\Omega)} & \leq C_{\delta} \mu^{-1} \lambda^{-\delta / 2} \\
\|\nabla \phi\|_{L^{\infty}(\Omega)} & \leq C_{\delta} \mu^{-1} \lambda^{(1-\delta) / 2}
\end{aligned}
$$

The estimates (19) follow from Lemma 3.2.

## 4. Detection of inclusions.

4.1. Stationary D-N map. We define the stationary Dirichlet-Neumann maps $\Lambda_{S}^{0}$ and $\Lambda_{S} \in B\left(H^{1 / 2}(\partial \Omega) ; H^{-1 / 2}(\partial \Omega)\right)$ as follows:

$$
\Lambda_{S}^{0}(h)=\gamma_{0} \frac{\partial u_{0}}{\partial \nu}, \quad \Lambda_{S}(h)=\gamma \frac{\partial u}{\partial \nu}
$$

where $u_{0}$ and $u$ are solutions to

$$
\begin{align*}
& \left\{\begin{aligned}
\left(\mathcal{A}_{0}+\lambda\right) u_{0} & =0
\end{aligned} \begin{array}{rl} 
& \text { on } \\
u_{0} & =
\end{array} \text { on } \partial \Omega,\right. \\
& \left\{\begin{array}{rlll}
(\mathcal{A}+\lambda) u & = & \text { on } \Omega, \\
u & & \text { on } & \partial \Omega .
\end{array}\right. \tag{26}
\end{align*}
$$

Let $\langle,\rangle_{\partial \Omega}$ be the inner product on $L^{2}(\partial \Omega)$ and put

$$
\begin{equation*}
J_{S}(h)=\left\langle\left(\Lambda_{S}-\lambda_{S}^{0}\right) h, h\right\rangle_{\partial \Omega} \tag{27}
\end{equation*}
$$

The following lemma was proven in [7] in the case $\lambda=0$.
Lemma 4.1. The following identities hold:

$$
\begin{gather*}
J_{S}(h)=\int_{\Omega}\left\{\left(\gamma-\gamma_{0}\right)\left|\nabla u_{0}\right|^{2}-\gamma\left|\nabla\left(u-u_{0}\right)\right|^{2}-\lambda\left|u-u_{0}\right|^{2}\right\} d x  \tag{28}\\
J_{S}(h)=\int_{\Omega}\left\{\frac{\gamma_{0}\left(\gamma-\gamma_{0}\right)}{\gamma}\left|\nabla u_{0}\right|^{2}+\frac{1}{\gamma}\left|\gamma \nabla u-\gamma_{0} \nabla u_{0}\right|^{2}+\lambda\left|u-u_{0}\right|^{2}\right\} d x \tag{29}
\end{gather*}
$$

Proof. By Green's formula, we have for $g \in H^{1}(\Omega)$

$$
\begin{align*}
\left\langle\Lambda_{S}^{0} h, g\right\rangle_{\partial \Omega} & =\int_{\Omega}\left(\gamma_{0} \nabla u_{0} \nabla \bar{g}+\lambda u_{0} \bar{g}\right) d x  \tag{30}\\
\left\langle\Lambda_{S} h, g\right\rangle_{\partial \Omega} & =\int_{\Omega}(\gamma \nabla u \nabla \bar{g}+\lambda u \bar{g}) d x \tag{31}
\end{align*}
$$

Letting $g=u_{0}$ in (30) and $g=u$ in (31), we have

$$
\begin{align*}
\left\langle\Lambda_{S}^{0} h, h\right\rangle_{\partial \Omega} & =\int_{\Omega}\left(\gamma_{0}\left|\nabla u_{0}\right|^{2}+\lambda\left|u_{0}\right|^{2}\right) d x  \tag{32}\\
\left\langle\Lambda_{S} h, h\right\rangle_{\partial \Omega} & =\int_{\Omega}\left(\gamma|\nabla u|^{2}+\lambda|u|^{2}\right) d x \tag{33}
\end{align*}
$$

Relation (30) with $g=u-u_{0}$ gives:

$$
\begin{equation*}
0=\int_{\Omega}\left\{\gamma_{0} \nabla u_{0} \cdot \nabla\left(\bar{u}-\overline{u_{0}}\right)+\lambda u_{0}\left(\bar{u}-\overline{u_{0}}\right)\right\} d x \tag{34}
\end{equation*}
$$

From (33) - (32) we have

$$
\begin{aligned}
J_{S}(h)= & \int_{\Omega}\left\{\left(\gamma-\gamma_{0}\right)|\nabla u|^{2}+\gamma_{0}\left(|\nabla u|^{2}-\left|\nabla u_{0}\right|^{2}\right)+\lambda\left(|u|^{2}-\left|u_{0}\right|^{2}\right)\right\} d x \\
= & \int_{\Omega}\left\{\left(\gamma-\gamma_{0}\right)|\nabla u|^{2}+\gamma_{0}\left|\nabla\left(u-u_{0}\right)\right|^{2}\right. \\
& +\gamma_{0}\left(\nabla\left(u-u_{0}\right) \cdot \nabla \overline{u_{0}}+\nabla u_{0} \cdot \nabla\left(\bar{u}-\overline{u_{0}}\right)\right) \\
& \left.+\lambda\left(\left|u-u_{0}\right|^{2}+\left(u-u_{0}\right) \overline{u_{0}}+u_{0}\left(\bar{u}-\overline{u_{0}}\right)\right)\right\} d x
\end{aligned}
$$

Thanks to(34) we obtain

$$
\begin{equation*}
J_{S}(h)=\int_{\Omega}\left\{\left(\gamma-\gamma_{0}\right)|\nabla u|^{2}+\gamma_{0}\left|\nabla\left(u-u_{0}\right)\right|^{2}+\lambda\left|u-u_{0}\right|^{2}\right\} d x \tag{35}
\end{equation*}
$$

Now letting $\psi=\gamma \nabla u-\gamma_{0} \nabla u_{0}$, we take notice of the following equality

$$
\begin{aligned}
& \left(\gamma-\gamma_{0}\right)|\nabla u|^{2}+\gamma_{0}\left|\nabla\left(u-u_{0}\right)\right|^{2} \\
= & \gamma^{-2}\left\{\left(\gamma-\gamma_{0}\right)\left|\gamma_{0} \nabla u_{0}+\psi\right|^{2}+\gamma_{0}\left|\left(\gamma_{0}-\gamma\right) \nabla u_{0}+\psi\right|^{2}\right\} \\
= & \gamma^{-1} \gamma_{0}\left(\gamma-\gamma_{0}\right)\left|\nabla u_{0}\right|^{2}+\gamma^{-1}|\psi|^{2} .
\end{aligned}
$$

This and (35) imply (29). By exchanging ( $\left.\gamma, u, u-u_{0}\right)$ and $\left(\gamma_{0}, u_{0}, u_{0}-u\right)$ in (35), we can prove (28).

The above lemma implies

$$
\int_{\Omega} \frac{\gamma_{0}\left(\gamma-\gamma_{0}\right)}{\gamma}\left|\nabla u_{0}\right|^{2} d x \leq J_{S}(h) \leq \int_{\Omega}\left(\gamma-\gamma_{0}\right)\left|\nabla u_{0}\right|^{2} d x
$$

which yields the following corollary.
Corollary 1. If $\gamma(x)-\gamma_{0}(x)$ has a constant sign, then $J_{S}(f)$ has the same sign and there exists a constant $C>0$ such that

$$
\begin{equation*}
C \int_{\Omega}\left|\gamma-\gamma_{0}\right|\left|\nabla u_{0}\right|^{2} d x \leq\left|J_{S}(h)\right| \leq C^{-1} \int_{\Omega}\left|\gamma-\gamma_{0}\right|\left|\nabla u_{0}\right|^{2} d x \tag{36}
\end{equation*}
$$

### 4.2. Uniform estimate for the parabolic equation.

Lemma 4.2. Let $u_{0}$ be a solution to the equation $\left(\mathcal{A}_{0}+\lambda\right) u_{0}=0$ in $\Omega$. Construct solutions $u$ to (26) with $h=u_{0}$, and $v(t)$ to (1) with $f=e^{\lambda t} u_{0}$. Then for any $0<t \leq T$ and $m \geq 0$, there exists a constant $C_{t m}>0$ independent of $\lambda$ such that

$$
\left\|v(t, \cdot)-e^{\lambda t} u(\cdot)\right\|_{H^{m}\left(\Omega \backslash \Omega_{1}\right)} \leq C_{t m}\left(\left\|v_{0}\right\|_{L^{2}(\Omega)}+\left\|u_{0}\right\|_{H^{1}(\Omega)}\right)
$$

Proof. Letting $w=v(t)-e^{\lambda t} u$, we have

$$
\left\{\begin{array}{cl}
\partial_{t} w+\mathcal{A} w & =0 \text { in }(0, T) \times \Omega \\
w(t) & =0 \text { on }(0, T) \times \partial \Omega \\
w(0) & =v_{0}-u \quad \text { on } \Omega
\end{array}\right.
$$

Thus, $w(t)=e^{-t A_{D}}\left(v_{0}-u\right)$, where $A_{D}$ denotes the self-adjoint operator $\mathcal{A}$ with the homogeneous Dirichlet boundary condition. Therefore, for any $t>0$ and $m \geq 0$, $w(t) \in D\left(\left(A_{D}\right)^{m}\right)$. By the regularity theorem for elliptic operators, we then have

$$
\|w(t)\|_{H^{2 m}\left(\Omega \backslash \Omega_{1}\right)} \leq C_{m}\left(\|w(t)\|_{L^{2}(\Omega)}+\left\|\left(A_{D}\right)^{m} w(t)\right\|_{L^{2}(\Omega)}\right) \leq C_{t m}\left\|v_{0}-u\right\|_{L^{2}(\Omega)}
$$

Let $\phi$ be the solution of

$$
\left\{\begin{array}{cllll}
\mathcal{A} \phi & = & 0 & \text { on } & \Omega \\
\phi & = & u_{0} & \text { on } & \partial \Omega
\end{array}\right.
$$

Then $\|\phi\|_{H^{1}(\Omega)} \leq C\left\|u_{0}\right\|_{H^{1 / 2}(\partial \Omega)}$ and $u-\phi$ satisfies

$$
\left\{\begin{array}{clc}
(A+\lambda)(u-\phi) & =-\lambda \phi \text { on } \Omega \\
u-\phi & = & 0 \quad \text { on } \partial \Omega
\end{array}\right.
$$

Since $A_{D}$ is positive definite, $\|u-\phi\|_{L^{2}(\Omega)} \leq C \lambda\left\|\left(A_{D}+\lambda\right)^{-1} \phi\right\|_{L^{2}(\Omega)} \leq C\|\phi\|_{L^{2}(\Omega)} \leq$ $C\left\|u_{0}\right\|_{H^{1 / 2}(\partial \Omega)}$, where $C$ is independent of $\lambda>0$. Hence we obtain for $0<t \leq T$ and $m \geq 0$

$$
\begin{aligned}
\|w(t, \cdot)\|_{H^{m}\left(\Omega \backslash \Omega_{1}\right)} & \leq C_{t}\left\|v_{0}-u\right\|_{L^{2}(\Omega)} \leq C_{t}\left(\left\|v_{0}\right\|_{L^{2}(\Omega)}+\left\|u_{0}\right\|_{H^{1 / 2}(\partial \Omega)}\right) \\
& \leq C_{t}\left(\left\|v_{0}\right\|_{L^{2}(\Omega)}+\left\|u_{0}\right\|_{H^{1}(\Omega)}\right) .
\end{aligned}
$$

4.3. Proof of Theorems $\mathbf{1 . 1}$ and 1.2. For the sake of simplicity, we restrict the proof to the case $\gamma_{1}(x)>\gamma_{0}(x)$ on $\Omega_{1}$. Let $y(x)$ be defined by (6) for $d=1$, and by (16) for $d=2,3$. Letting $y^{*}$ be defined by (14), we put

$$
y_{1}(x)=\left\{\begin{array}{cc}
y(x) & (d=1) \\
y(x) \cdot y^{*} & (d=2,3)
\end{array}\right.
$$

and for a bounded open set $\mathcal{O} \subset \mathbb{R}^{d}$

$$
a(\mathcal{O})=\sup \left\{-y_{1}(x) ; x \in \mathcal{O}\right\}
$$

Note that $a(\mathcal{O})=a_{B}(\mathcal{O})$ for $d=2,3$ by (7) and (21). We put

$$
\kappa=\left\{\begin{array}{l}
2 \sqrt{\lambda}, \quad \text { if } \quad d=1 \\
\sqrt{2} \mu \lambda, \quad \text { if } \quad d=2,3
\end{array}\right.
$$

Let $\varphi_{\lambda}(x)$ be as in Theorem 2.1 or Theorem 3.1. Thanks to (13) and (17), we have

$$
\begin{equation*}
C \kappa^{2} e^{-\kappa y_{1}(x)} \leq\left|\nabla \varphi_{\lambda}(x)\right|^{2} \leq C^{-1} \kappa^{2} e^{-\kappa y_{1}(x)} \tag{37}
\end{equation*}
$$

We then put

$$
\begin{gathered}
J(\mathcal{O}, \kappa)=\int_{\mathcal{O}} e^{-\kappa y_{1}(x)} d x \\
J_{\gamma}(\kappa)=\int_{\Omega}\left(\gamma-\gamma_{0}\right)\left|\nabla \varphi_{\lambda}(x)\right|^{2} d x
\end{gathered}
$$

Lemma 4.3. (1) For a non-empty bounded open set $\mathcal{O} \subset \mathbb{R}^{d}$, we have

$$
\begin{equation*}
\lim _{\kappa \rightarrow \infty} \kappa^{-1} \log J(\mathcal{O}, \kappa)=a(\mathcal{O}) \tag{38}
\end{equation*}
$$

(2) We have

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \kappa^{-1} \log J_{\gamma}(\kappa)=a\left(\Omega_{1}\right) \tag{39}
\end{equation*}
$$

Proof. Letting $\mathcal{O}_{\epsilon}=\left\{x \in \mathcal{O} ; a(\Omega)-\epsilon \leq-y_{1}(x) \leq a(\mathcal{O})\right\}$, we have

$$
\int_{\mathcal{O}_{\epsilon}} e^{\kappa(a(\mathcal{O})-\epsilon)} d x \leq \int_{\mathcal{O}} e^{-\kappa y_{1}(x)} d x \leq \int_{\mathcal{O}} e^{\kappa a(\mathcal{O})} d x
$$

Taking the logarithm, and letting $\kappa \rightarrow \infty$, we get (38). Let $\mathcal{O}$ be a bounded open set such that $\overline{\mathcal{O}} \subset \Omega_{1}$. Then in view of (37), we have

$$
C \kappa^{2} J(\mathcal{O}, \kappa) \leq J_{\gamma}(\kappa) \leq C^{-1} \kappa^{2} J\left(\Omega_{1}, \kappa\right)
$$

which implies by (38)

$$
a(\mathcal{O}) \leq \liminf _{\kappa \rightarrow \infty} \kappa^{-1} \log J_{\gamma}(\kappa) \leq \limsup _{\kappa \rightarrow \infty} \kappa^{-1} \log J_{\gamma}(\kappa) \leq a\left(\Omega_{1}\right)
$$

Since $\mathcal{O} \subset \subset \Omega_{1}$ can be taken arbitrarily, we get (39).

We put

$$
h_{\lambda}(x)=\left.\varphi_{\lambda}(x)\right|_{\partial \Omega}
$$

By Corollary 4.2 and (39), we have
Lemma 4.4. Let $J_{S}(f)$ be defined by (27). Then we have

$$
\lim _{\kappa \rightarrow \infty} \kappa^{-1} \log J_{S}\left(h_{\lambda}\right)=a\left(\Omega_{1}\right)
$$

Before entering into the probing problem, we need to study geometric properties of the map : $x \rightarrow y_{1}(x)$. The next lemma follows from a direct computation using $R=\left|x_{0}-x^{*}\right|$ and (21).

Lemma 4.5. Let $d \geq 2$. For $c \neq 1$, let

$$
x_{c}=\frac{x_{0}-c x^{*}}{1-c}, \quad R_{c}=\frac{R}{|1-c|}
$$

Then we have

$$
\left\{x \in \mathbb{R}^{d} ;-y_{1}(x)=c\right\}=B\left(x_{c}, R_{c}\right)
$$

Note that $B\left(x_{c}, R_{c}\right)$ is the sphere passing through $x^{*}$ with center on the line $l=\left\{x^{*}+t\left(x_{0}-x^{*}\right) ; t \in \mathbb{R}\right\}$. When $c$ grows up, so does its radius, and if $c=0$, it is the sphere $B\left(x_{0}, R\right)$.

Now, by definition, there exists $\bar{x} \in \overline{\Omega_{1}}$ such that $-y_{1}(\bar{x})=a\left(\Omega_{1}\right)$. Then $\Omega_{1}$ lies outside $B\left(x_{c}, R_{c}\right)$ with $c=a\left(\Omega_{1}\right)$. Assume that the open set $\Gamma \subset \partial \Omega$ satisfies the condition (C-1). Then by a suitable choice of $c<a\left(\Omega_{1}\right)$, we see that $\Omega_{2}=$ $\Omega \backslash \overline{B\left(x_{c}, R_{c}\right)}$ satisfies
(C-2) $\quad \partial \Omega \backslash \Gamma \subset \partial \Omega_{2}, \quad \bar{\Omega} \cap \overline{B(x, R)}=\emptyset, \quad a\left(\Omega_{2}\right)<a\left(\Omega_{1}\right)$.
The following lemma gives a relation between the time-dependent measurement $I(\lambda)$ defined by (4) or (10) and the stationary one $J_{S}\left(h_{\lambda}\right)$.

Lemma 4.6. Assume the condition $(C-1)$ for $d=2,3$. Then there exists $\delta>0$ such that the following estimate holds:

$$
\left|\frac{I(\lambda)-J_{S}\left(h_{\lambda}\right)}{J_{S}\left(h_{\lambda}\right)}\right| \leq e^{-\delta \kappa}, \quad \forall \lambda>1 / \delta
$$

Proof. Let $f_{\lambda}(t, x)=e^{\lambda t} \varphi_{\lambda}(x)$, and $v(t, x)$ the solution of (1) with $f=f_{\lambda}$. Let $u_{\lambda}$ be the solution of (26) with $h=h_{\lambda}$. Letting $w=v-e^{\lambda t} u_{\lambda}$, we then have

$$
\begin{aligned}
I(\lambda) & =J_{S}\left(h_{\lambda}\right)+R_{\Gamma}(\lambda)+R_{S}(\lambda) \\
R_{\Gamma}(\lambda) & =e^{-\lambda T_{1}} \int_{\Gamma} \gamma \frac{\partial w\left(T_{1}, \cdot\right)}{\partial \nu} \overline{h_{\lambda}} d \sigma \\
R_{S}(\lambda) & =-\int_{\partial \Omega \backslash \Gamma}\left(\Lambda_{S}-\Lambda_{S}^{0}\right) h_{\lambda} \overline{h_{\lambda}} d \sigma
\end{aligned}
$$

Then, thanks to lemma 4.2, we obtain:

$$
\begin{aligned}
\left|R_{\Gamma}\right| & \leq C e^{-\lambda T_{1}}\left\|\frac{\partial w\left(T_{1}, \cdot\right)}{\partial \nu}\right\|_{H^{1 / 2}(\partial \Omega)}\left\|h_{\lambda}\right\|_{H^{1 / 2}(\partial \Omega)} \\
& \leq C e^{-\lambda T_{1}}\|w\|_{H^{2}\left(\Omega \backslash \Omega_{1}\right)}\left\|\varphi_{\lambda}\right\|_{H^{1}(\Omega)} \\
& \leq C e^{-\lambda T_{1}}\left(\left\|v_{0}\right\|^{2}+\left\|\varphi_{\lambda}\right\|_{H^{1}(\Omega)}^{2}\right)
\end{aligned}
$$

By (37), we have $\left\|\varphi_{\lambda}\right\|_{H^{1}(\Omega)}^{2} \leq C \kappa^{2} e^{\kappa a(\Omega)}$. Hence

$$
\begin{equation*}
\left|R_{\Gamma}\right| \leq C \kappa^{2} e^{-\lambda T_{1}+\kappa a(\Omega)} \tag{40}
\end{equation*}
$$

Let us estimate $R_{S}$. If $d=1$ we set $\Omega_{2}=\left(a_{2}, b\right) \subset \Omega$ with $a_{1}<a_{2}<b$. Recall that $\Omega_{1}=\left(a_{1}, b_{1}\right)$. If $d=2,3$, we take $\Omega_{2}$ satisfying the condition $(C-2)$. Then we have

$$
\begin{align*}
\left|R_{S}\right| & \leq C\left(\left\|u_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}+\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}\right)\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}  \tag{41}\\
& \leq C\left(\|\tilde{w}\|_{H^{1}\left(\Omega_{2}\right)}+2\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}\right)\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}
\end{align*}
$$

with $\tilde{w}=u_{\lambda}-\varphi_{\lambda}$. Using (35) we have $\|\tilde{w}\|_{H^{1}(\Omega)} \leq \sqrt{J_{S}\left(h_{\lambda}\right)}$. Hence, by (36), we obtain

$$
\left|R_{S}\right| \leq C\left(\sqrt{J_{S}\left(h_{\lambda}\right)}+\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)}\right)\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)} .
$$

By (37), (41) and

$$
\left\|\varphi_{\lambda}\right\|_{H^{1}\left(\Omega_{2}\right)} \leq C \kappa e^{\kappa a\left(\Omega_{2}\right) / 2}
$$

Lemma 4.5 implies that for any $\epsilon>0$, there exists $\kappa_{0}>0$ such that

$$
\begin{equation*}
e^{\kappa\left(a\left(\Omega_{1}\right)-\epsilon\right)} \leq J_{S}\left(h_{\lambda}\right) \leq e^{\kappa\left(a\left(\Omega_{1}\right)+\epsilon\right)}, \quad \forall \kappa>\kappa_{0} \tag{42}
\end{equation*}
$$

In view of (40), (41) and (42), we have

$$
\begin{align*}
\left|\frac{I(\lambda)-J_{S}\left(h_{\lambda}\right)}{J_{S}\left(h_{\lambda}\right)}\right| & \leq \frac{\left|R_{\Gamma}(\lambda)\right|}{J_{S}\left(h_{\lambda}\right)}+\frac{\left|R_{S}(\lambda)\right|}{J_{S}\left(h_{\lambda}\right)}  \tag{43}\\
& \leq C \kappa^{2}\left(e^{-\lambda T_{1}+\kappa\left(a(\Omega)-a\left(\Omega_{1}\right)+\epsilon\right)}+e^{\kappa\left(a\left(\Omega_{2}\right)-a\left(\Omega_{1}\right)+2 \epsilon\right)}\right)
\end{align*}
$$

Now we use the geometric configuration of $\Omega_{1}$ and $\Omega_{2}$. In the case $d=1, y(x)$ is monotone increasing. Hence

$$
a\left(\Omega_{2}\right)=\sup _{x \in \Omega_{2}}\left(-y_{1}(x)\right)=-y\left(a_{2}\right)<-y\left(a_{1}\right)=\sup _{x \in \Omega_{1}}\left(-y_{1}(x)\right)=a\left(\Omega_{1}\right)
$$

In the case $d>1$, we have $a\left(\Omega_{2}\right)<a\left(\Omega_{1}\right)$ by ( $C-2$ ). By the condition (8), we have $-\lambda T_{1}+\kappa\left(a(\Omega)-a\left(\Omega_{1}\right)\right)<-\delta \kappa$ for some $\delta>0$. Therefore choosing $\epsilon$ small enough in (43), we prove the lemma.

Now Theorems 1.1 and 1.2 follow from Lemma 4.7 and (42).
5. 2-dimensional conformal map. In this section, we identify $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$ with a complex number $z=x_{1}+i x_{2} \in \mathbb{C}$. Consider a univalent holomorphic function $F(z)$ defined in a neighborhood of $\Omega$, and the conformal map : $z \rightarrow w=y_{1}+i y_{2}=$ $F(z)$. As in $\S 3$, we look for a solution of (9) in the form $\varphi_{\lambda}(x)=(1+\phi(y)) e^{-\zeta \cdot F(x)}$, where $\phi$ satisfies (24). Note that $\zeta \cdot F(z)$ is identified with $\frac{\mu \lambda}{\sqrt{2}} F(z)$. Assume that $0 \notin \bar{\Omega}$ and let us choose $F(z)=z^{-n}-\rho^{-n}$ with $\rho>0$ and $n=1,2, \ldots$ The case $n=1$ corresponds to the inversion we have studied in $\S 3$, where $R$ corresponds to $2 / \rho$.

Consider the curve $\operatorname{Re} F(z)=0$ that separates the set of $x$ where $\varphi_{\lambda}(x)$ has an exponential growth from the set of $x$ where $\varphi_{\lambda}(x)$ has an exponential decay as $\lambda \rightarrow \infty$. Writing $z=r e^{i \alpha}$, we have that $\operatorname{Re} F(z)=0 \Longleftrightarrow r=\rho(\cos (n \alpha))^{1 / n}$. It appears that if $\Omega$ is convex, then $n=3$ is a better choice than $n=1,2$ since the curve $\operatorname{Re} F(z)=0$ goes more deeply inside $\Omega$ (cf. figure 3 ).


Figure 3.
6. Numerical tests in the one-dimensional case. We take $\gamma_{0}=1$ which implies

$$
\begin{align*}
h_{\lambda}(x) & =e^{-\sqrt{\lambda}\left(x-x_{0}\right)} \\
f_{\lambda}(t, x) & =e^{\lambda t} h_{\lambda}(x) \\
I\left(T_{1}, \lambda\right) & =\left[e^{-\lambda T_{1}}\left(\Lambda f_{\lambda}\right)\left(T_{1}, a\right)-\frac{\partial h_{\lambda}}{\partial x}(a)\right] h_{\lambda}(a)  \tag{44}\\
\lim _{\lambda \rightarrow \infty} \frac{\log \left(I\left(T_{1}, \lambda\right)\right)}{2 \sqrt{\lambda}} & =x_{0}-a_{1} \tag{45}
\end{align*}
$$

The equation (45) is calculated with a finite $\lambda$, so we have the approximative reconstruction equation

$$
\begin{equation*}
a_{1} \approx x_{0}-\frac{\log \left(I\left(T_{1}, \lambda\right)\right)}{2 \sqrt{\lambda}} \tag{46}
\end{equation*}
$$

which is the more accurate the larger the parameter $\lambda$ is. In practice larger values here implies very large values of $f_{\lambda}$ on the boundary, so numerical errors will be a problem. However for initial testing purposes we proceed by using $\lambda=1.0,1.5,2.0, \ldots, 20.0$. Also we choose $a=0, b=1, x_{0}=0$ and $T=1$.

Four test cases of the following heat conductivity are considered:

$$
\begin{aligned}
& \gamma=\gamma_{1}, \quad a_{1} \leq x \leq b_{1}, \\
& \gamma=\gamma_{0}, \quad \text { otherwise }
\end{aligned}
$$

where the values of $a_{1}, b_{1}$ and $\gamma_{1}$ are given in table 6 . These test cases are pictured in figure 4. The solutions $v$ in (1) are calculated with the finite element method (FEM) so that there are $N_{x}=200$ points of $x \in[0,1]$ and $N_{t}=100$ points of $t \in[0,1]$. We consider three different initial temperature distributions:

$$
\begin{aligned}
v(x, 0) & =1 \\
v(x, 0) & =2 \sin (\pi x) \\
v(x, 0) & =4|\sin (4 \pi x)|
\end{aligned}
$$

In figure 5 we display the solution function $v(x, t), v(x, 0)=1, \lambda=2$ calculated by FEM for the test case 1 , smaller numbers of $N_{x}$ and $N_{t}$ are only used for printing purposes.

|  | $a_{1}$ | $b_{1}$ | $\gamma_{1}$ |
| :---: | :---: | :---: | :---: |
| Case 1 | 0.1 | 0.3 | 0.5 |
| Case 2 | 0.1 | 0.3 | 5 |
| Case 3 | 0.4 | 0.6 | 0.5 |
| Case 4 | 0.4 | 0.6 | 5 |

Table 1. Parameters $a_{1}, b_{1}$ and $\gamma_{1}$ for the test cases

To simulate measurement noise we define the vector

$$
\begin{equation*}
L_{n}=L \cdot(1+c \cdot r), \tag{47}
\end{equation*}
$$

where $L$ is the vector containing the values of $\Lambda f_{\lambda}\left(T_{1}, 0\right)=\left.\gamma(x) \frac{\partial v^{f}}{\partial \nu}\right|_{x=0}$ at the FEM mesh points and $r$ is a normally distributed random vector (white noise) of the same size and the parameter $c$ is adjusted separately for each $\lambda$ so that the relative noise is approximately two percent:

$$
\begin{equation*}
\frac{\left\|L_{n}-L\right\|}{\|L\|} \approx 0.02 \tag{48}
\end{equation*}
$$

The logarithm of the indicator function (44) is calculated for each $t \in[0,1]$ and $\lambda=1.0,1.5,2.0, \ldots, 20.0$. This information can be presented in several ways. In figure 6 one can see how the logarithm of the indicator function, noisy and nonnoisy, behaves as a function of $T_{1}$, for $\lambda=1$ and $\lambda=10$, in test case 1 . We conclude that the value $\lambda=1$ is too small to get accurate indicator function values, and that $\lambda=10$ provides good values throughout the time interval $[T / 2, T]$. In figure 7 we have a similar graph for $\lambda=10$, but for each initial temperature distribution. We conclude that the initial distribution $v(x, 0)$ does not change the indicator function values in the interval $[T / 2, T]$ and thus it will not affect the reconstructions. In figure 8 one can see how the logarithm of the indicator function, noisy and nonnoisy, behaves as a function of $\lambda$, for several choices of $T_{1}$, in test case 1. Notice that in these graphs the analytical value based on (46) is also shown.

Based on the observation that the values of the indicator function are good between the time interval $[T / 2, T]$ we use the following indicator function value:

$$
\begin{equation*}
I^{\star}(\lambda)=\frac{1}{N_{t}^{\star}} \sum_{T_{1}=T / 2}^{T} I\left(T_{1}, \lambda\right), \tag{49}
\end{equation*}
$$

where $I\left(T_{1}, \lambda\right)$ is defined by (44) and $N_{t}^{\star}$ is the number of FEM mesh points $t \in[0,1]$ between the time values $T / 2$ and $T$. The value $I^{\star}(\lambda)$ is then inserted into (46), from which the reconstruction $a_{1}$ is calculated with $\lambda=1.0,1.5,2.0, \ldots, 20.0$. In figure 9 the reconstructed $a_{1}$, noisy and non-noisy, is shown as a function of $\lambda$, for each test case.

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E-mail address: patricia.gaitan@univmed.fr
E-mail address: isozakih@math.tsukuba.ac.jp
E-mail address: poisson@cmi.univ-mrs.fr
E-mail address: samuli.siltanen@tut.fi
E-mail address: janne.p.tamminen@tut.fi


Figure 4. The test case heat conductivities

Case 1, $\lambda=2, N x=100, N t=50$


Figure 5. The solution function $v(x, t), \lambda=2$ calculated by FEM for test case $1, v(x, 0)=1$.

Case 1, no noise


Case 1, noisy


Figure 6. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of $T_{1}$ for $\lambda=1$ and $\lambda=10$ in test case 1 .

Case $1, \lambda=10$, no noise


Case $1, \lambda=10$, noisy


Figure 7. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of $T_{1}$ for $\lambda=10$ in test case 1, with three different initial data $v(x, 0)=1, v(x, 0)=$ $2 \sin (\pi x)$ and $v(x, 0)=4|\sin (4 \pi x)|$. Note that the effect of the initial data vanishes as $T_{1}$ grows.

Case 1, no noise


Case 1, noisy


Figure 8. The logarithm of the indicator function, non-noisy on the left and noisy on the right, as a function of $\lambda$ for several choices of $T_{1}$ in test case 1.

## Case 1



Case 2



Figure 9. The reconstructed $a_{1}$, noisy and non-noisy, as a function of $\lambda$.


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