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MARCIN BILSKI, ADAM PARUSIŃSKI, AND GUILLAUME ROND

Abstract. T. Mostowski showed that every (real or complex) germ of an analytic set is homeomorphic to the germ of an algebraic set. In this paper we show that every (real or complex) analytic function germ, defined on a possibly singular analytic space, is topologically equivalent to a polynomial function germ defined on an affine algebraic variety.

1. Introduction and statement of results

The problem of approximation of analytic objects (sets or mappings) by algebraic ones has attracted many mathematicians, see e.g. [2] and the bibliography therein. Nevertheless there are very few positive results if one requires that the approximation gives a homeomorphism between the approximated object and the approximating one. In this paper we consider two cases of this problem: the local algebraicity of analytic sets and the local algebraicity of analytic functions. The problem can be considered over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

The local topological algebraicity of analytic sets has been established by Mostowski in [12]. More precisely, given an analytic set germ $(V, 0) \subset (\mathbb{K}^n, 0)$, Mostowski shows the existence of a local homeomorphism $\tilde{h} : (\mathbb{K}^{2n+1}, 0) \to (\mathbb{K}^{2n+1}, 0)$ such that, after the embedding $(V, 0) \subset (\mathbb{K}^n, 0) \subset (\mathbb{K}^{2n+1}, 0)$, the image $\tilde{h}(V)$ is algebraic. It is easy to see that Mostowski's proof together with Theorem 2 of [2] gives the following result.

Theorem 1.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be an analytic germ. Then there is a homeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $h(V)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

Mostowski's Theorem seems not to be widely known. Recently Fernández de Bobadilla showed, by a method different from that of Mostowski, the local topological algebraicity of complex hypersurfaces with one-dimensional singular locus, see [3]. We remark that in [12] Mostowski states his results only for $\mathbb{K} = \mathbb{R}$ but his proof works, word by word, for $\mathbb{K} = \mathbb{C}$.

The first purpose of this paper is to present a short proof of Theorem 1.1. We follow closely Mostowski's original approach that is based on two ideas, Płoski's version of Artin approximation, cf. [13], and Varchenko's theorem stating that the algebraic equisingularity of Zariski implies topological equisingularity. Our proof is shorter, but less elementary. We use a corollary of Neron Desingularization, that we call the Nested Artin-Płoski Approximation

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Theorem. This result is shown in Section 2 and the proof of Theorem 1.1 is given in Section 4. In Section 3 we recall Varchenko's results.

The second purpose of this paper is to show the local algebraicity of analytic functions.

Theorem 1.2. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ \sigma$ is the germ of a polynomial.*

The proof of Theorem 1.2, presented in Section 5, is based on the Nested Artin-Płoski Approximation Theorem and a refinement of Varchenko's method.

We end with the following generalization of Theorems 1.1 and 1.2.

Theorem 1.3. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $(V_i, 0) \subset (\mathbb{K}^n, 0)$ be a finite family of analytic set germs and let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there is a homeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ \sigma$ is the germ of a polynomial, and for each i, $\sigma^{-1}(V_i)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

Corollary 1.4. *Let $g : (V, p) \to (\mathbb{K}, 0)$ be an analytic function germ defined on the germ (V, p) of an analytic space. Then there exists an algebraic affine variety V_1, a point $p_1 \in V_1$, the germ of a polynomial function $g_1 : (V_1, p_1) \to (\mathbb{K}, 0)$ and a homeomorphism $\sigma : (V_1, p_1) \to (V, p)$ such that $g_1 = g \circ \sigma$.*

In Section 6 we present examples showing that Theorems 1.1, 1.2 and 1.3 are false if we replace "homeomorphism" by "diffeomorphism". We do not know whether these theorems hold true with "homeomorphism" replaced by "bi-lipschitz homeomorphism".

Remark 1.5. We often identify the germ at the origin of a $\mathbb{K}$-analytic function $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ with its Taylor series that is with a convergent power series. We say that a ($\mathbb{K}$-)analytic function or a germ is *Nash* if its graph is semi-algebraic. Thus $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ is the germ of a Nash function if and only if its Taylor series is an algebraic power series. A *Nash set* is the zero set of a finitely many Nash functions.

2. Nested Artin-Płoski Approximation Theorem

We set $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$. The ring of convergent power series in x_1,..., x_n is denoted by $\mathbb{K}\{x\}$. If A is a commutative ring then the ring of algebraic power series with coefficients in A is denoted by $A\langle x\rangle$.

The following result is a corollary of Theorem 11.4 [16] which itself is a corollary of Néron-Popescu desingularization (see [14], [16] or [17] for the proof of this desingularization theorem in whole generality or [15] for a proof in characteristic zero).

Theorem 2.1. *Let $f(x, y) \in \mathbb{K}\langle x\rangle[y]^p$ and let $y(x) \in \mathbb{K}\{x\}^m$ be a solution of $f(x, y) = 0$. Let us assume that $y_i(x)$ depends only on $(x_1, ..., x_{\sigma(i)})$ where $i \longmapsto \sigma(i)$ is an increasing function. Then there exist a new set of variables $z = (z_1, ..., z_s)$, an increasing function τ, convergent power series $z_i(x) \in \mathbb{K}\{x\}$ vanishing at 0 such that $z_1(x)$,..., $z_{\tau(i)}(x)$ depend only on $(x_1, ..., x_{\sigma(i)})$, and a vector of algebraic power series $y(x, z) \in \mathbb{K}\langle x, z\rangle^m$ solution of $f(x, y) = 0$ such that for every i, $y_i(x, z) \in \mathbb{K}\langle x_1, ..., x_{\sigma(i)}, z_1, ..., z_{\tau(i)}\rangle$, and $y(x) = y(x, z(x))$.*

Remark 2.2. Theorem 2.1 remains valid if we replace "convergent power series" by "formal power series".

For any i we set:

$$A_i = \mathbb{K}\langle x_1, ..., x_i\rangle,$$
$$B_i = \mathbb{K}\{x_1, ..., x_i\}.$$

We will need at several places the following two lemmas whose proofs are given later (for the definition and properties of an excellent ring see 7.8 [7] or [11]; a henselian local ring is a local ring satisfying the Implicit Function Theorem, see 18.5 [8]).

Lemma 2.3. *Let B be an excellent henselian local subring of $\mathbb{K}[[x_1, ..., x_{i-1}]]$ containing $\mathbb{K}\langle x_1, ..., x_{i-1}\rangle$ and whose maximal ideal is generated by x_1,..., x_{i-1}. Then the ring $A_i \otimes_{A_{i-1}} B$ is noetherian and its henselization is isomorphic to $B\langle x_i\rangle$.*

Lemma 2.4. *Let B be an excellent henselian local subring of $\mathbb{K}[[x_1, ..., x_{i-1}]]$ containing $\mathbb{K}\langle x_1, ..., x_{i-1}\rangle$ and whose maximal ideal is generated by x_1,..., x_{i-1}. Let I be an ideal of $B[x_i]$. Then the henselization of $\frac{B[x_i]_{(x_1,...,x_i)}}{I}$ is isomorphic to $\frac{B\langle x_i\rangle}{I}$.*

Proof of Theorem 2.1. By replacing $f(x, y)$ by $f(x, y(0) + y)$ we may assume that $y(0) = 0$.

For any i let $l(i)$ be the largest integer such that $y_1(x)$,..., $y_{l(i)}(x) \in \mathbb{K}\{x_1, ..., x_i\}$.

For any i let J_i be the kernel of the morphism

$$\varphi_i : \mathbb{K}\langle x_1, ..., x_i\rangle[y_1, ..., y_{l(i)}] \longrightarrow \mathbb{K}\{x_1, ..., x_i\} = B_i$$

defined by $\varphi_i(g(x, y)) = g(x, y(x))$. We define:

$$C_i = \frac{\mathbb{K}\langle x_1, ..., x_i\rangle[y_1, ..., y_{l(i)}]}{J_i}.$$

Then C_i is a finite type A_i-algebra and C_i is a sub-A_i-algebra of C_{i+1} since $J_i \subset J_{i+1}$. The morphism φ_i induces a morphism $C_i \longrightarrow B_i$ such that the following diagram is commutative:

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}$$

By Theorem 11.4 [16] (see also [18]) this diagram may be extended to a commutative diagram as follows

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D_1 & \longrightarrow & D_2 & \longrightarrow & \cdots & \longrightarrow & D_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}$$

where D_1 is a smooth A_1-algebra of finite type and D_i is a smooth $D_{i-1}\otimes_{A_{i-1}} A_i$-algebra of finite type for all $i>1$. We will denote by D'_{i-1} the ring $D_{i-1}\otimes_{A_{i-1}} A_i$ for all $i>1$ and set $D'_1=A_1$.

For any i let us write $D_i=\frac{D'_{i-1}[u_{i,1},...,u_{i,q_i}]}{I_i}$. We may make a change of coordinates (of the form $u_{i,j}\longmapsto u_{i,j}+c_{i,j}$ for some $c_{i,j}\in\mathbb{K}$) in such way that the image of $u_{i,j}$ is in the maximal ideal of B_i for any i and j. Thus $D_i\longrightarrow B_i$ factors through the localization morphism $D_i\longrightarrow (D_i)_{\mathfrak{m}_i}$ where $\mathfrak{m}_i=(x_1,...,x_i,u_{i,1},...,u_{i,q_i})$. Let D_i^h be the henselization of $(D_i)_{\mathfrak{m}_i}$. Since B_i is a henselian local ring, the morphism $D_i\longrightarrow B_i$ factors through D_i^h by the universal property of the henselization. Still by this universal property the composition of the morphisms $D_{i-1}\longrightarrow D_i\longrightarrow D_i^h$ factors through D_{i-1}^h. Thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_n\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots & \longrightarrow & C_n\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
D_1^h & \longrightarrow & D_2^h & \longrightarrow & \cdots & \longrightarrow & D_n^h\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_n
\end{array}$$

We will prove by induction that D_i^h is isomorphic to $\mathbb{K}\langle x_1,...,x_i,z_1,...,z_{\lambda(i)}\rangle$ where $i\longrightarrow\lambda(i)$ is an increasing function and the z_k are new indeterminates.

Since D_1^h is the henselisation of $(D_1)_{\mathfrak{m}_1}=\frac{\mathbb{K}\langle x_1\rangle[u_{1,1},...,u_{1,q_1}]}{I}_{(x_1,u_{1,1},...,u_{1,q_1})}$, D_1^h is isomorphic to $\frac{\mathbb{K}\langle x_1,u_{1,1},...,u_{1,q_1}\rangle}{I.\mathbb{K}\langle x_1,u_{1,1},...,u_{1,q_1}\rangle}$ by Lemma 2.4 and D_1 being smooth over $\mathbb{K}\langle x_1\rangle$ means that the matrix $\left(\frac{\partial f_j}{\partial u_k}(0,0)\right)_{i,j}$, where the f_j are generators of $I.\mathbb{K}\langle x_1,u_{1,1},...,u_{1,q_1}\rangle$, has maximal rank (by the jacobian criterion for smoothness, see Proposition 22.6.7 (iii) [6]). Thus by the Implicit Function Theorem the ring D_1^h is isomorphic to $\mathbb{K}\langle x_1,z_1,...,z_{\lambda(1)}\rangle$ for some new indeterminates z_1,..., $z_{\lambda(1)}$. This proves the induction propery for D_1^h.

Now let us assume that the induction property is true for D_{i-1}^h. By assumption D_i is smooth over $D_{i-1}\otimes_{A_{i-1}}A_i$. Thus D_i^h is smooth over the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$. By the universal property of the henselization the morphism from D_{i-1} to the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$ factors through D_{i-1}^h thus it factors through $D_{i-1}^h\otimes_{A_{i-1}}A_i$. Hence the henselization of $D_{i-1}\otimes_{A_{i-1}}A_i$ is isomorphic to the henselization of $D_{i-1}^h\otimes_{A_{i-1}}A_i$. But

$$D_{i-1}^h\otimes_{A_{i-1}}A_i=\mathbb{K}\langle x_1,...,x_{i-1},z_1,...,z_{\lambda(i-1)}\rangle\otimes_{\mathbb{K}\langle x_1,...,x_{i-1}\rangle}\mathbb{K}\langle x_1,...,x_i\rangle.$$

Its henselization is isomorphic to $\mathbb{K}\langle x_1,...,x_i,z_1,...,z_{\lambda(i-1)}\rangle$ by Lemma 2.3. This shows that D_i^h is smooth over $\mathbb{K}\langle x_1,...,x_i,z_1,...,z_{\lambda(i-1)}\rangle$ hence, by the Implicit Function Theorem as we did for D_1^h, D_i^h is isomorphic to $\mathbb{K}\langle x_1,...,x_i,z_1,...,z_{\lambda(i)}\rangle$ for some new indeterminates $z_{\lambda(i-1)+1}$,..., $z_{\lambda(i)}$.

Finally the morphisms $C_i \longrightarrow D_i^h$ define the $y_k(x,z)$ satisfying $f(x,y(x,z))=0$. The power series $z_j(x)$ are defined by the morphisms $D_i^h \longrightarrow B_i$ and the fact that $C_i \longrightarrow B_i$ factors through D_i^h yields $y(x)=y(x,z(x))$. □

Proof of Lemma 2.3. Let $\psi : A_i \otimes_{A_{i-1}} B \longrightarrow B\langle x_i\rangle$ be the morphism defined by $\psi(\sum_j a_j \otimes b_j) = \sum_j a_j b_j$ with $a_j \in A_i$ and $b_j \in B$ for any j. The morphism ψ is well defined since A_i and B are subrings of the ring $B\langle x_i\rangle$. The image of ψ is the subring of $B\langle x_i\rangle$ generated by A_i and B.
Let us prove that ψ is injective: Let $\sum_j a_j \otimes b_j \in \mathrm{Ker}(\psi)$ with $a_j \in A_i$ and $b_j \in B$ for any j. This means that $\sum_j a_j b_j = 0$. Let us write $a_j = \sum_{l\in\mathbb{N}} a_{j,l} x_i^l$ where $a_{j,l} \in A_{i-1}$ for any j and l. Thus we have

$$\sum_j a_{j,l} b_j = 0 \tag{2.1}$$

for any $l \in \mathbb{N}$ and this system of linear equations is equivalent to a finite system by noetherianity. The ring extension $A_{i-1} \longrightarrow B$ is flat since $A_{i-1} \longrightarrow \mathbb{K}[[x_1,...,x_{i-1}]]$ and $B \longrightarrow \mathbb{K}[[x_1,...,x_{i-1}]]$ are faithfully flat (they are completions of local noetherian rings, cf. [11] p. 46 and Theorem 8.14 p. 62). Thus the solution vector $(b_j)_j$ of (2.1) is a linear combination with coefficients in B of solution vectors in A_{i-1} (cf. [11] Theorem 7.6 p.49). Thus $(b_j)_j = \sum_k b'_k (a'_{j,k})_j$ where $b'_k \in B$ and, for any k, $(a'_{j,k})_j$ are vectors with entries in A_{i-1} which are solutions of (2.1). This means that

$$\sum_j a_j \otimes b_j = \sum_{j,k} a_j \otimes b'_k a'_{j,k} = \sum_k \sum_j a_j a'_{j,k} \otimes b'_k = \sum_k (\sum_l (\sum_j a_{j,l} a'_{j,k}) x_i^l) \otimes b'_k = 0.$$

Thus $\mathrm{Ker}(\psi) = (0)$.

Obviously $\mathrm{Im}(\psi)$ contains $B[x_i]$ whose henselization is $B\langle x_i\rangle$ by Lemma 2.4, thus ψ induces a surjective morphism between the henselization of $A_i \otimes_{A_{i-1}} B$ and $B\langle x_i\rangle$. This surjective morphism is also injective since ψ is injective and $A_i \otimes_{A_{i-1}} B$ is a domain (Indeed if $y \neq 0$ is in the henselization of $A_i \otimes_{A_{i-1}} B$, then y is a root of a non zero polynomial with coefficients in $A_i \otimes_{A_{i-1}} B$. Since $A_i \otimes_{A_{i-1}} B$ is a domain and $y \neq 0$ we may assume that this polynomial has a non zero constant term denoted by a. If the image of y in $B\langle x_i\rangle$ is zero then $\psi(a)=0$ which is a contradiction).
On the other hand $B\langle x_i\rangle$ is the henselization of $B[x_i]$ which is noetherian, thus $B\langle x_i\rangle$ is noetherian (cf. [8] Théorème 18.6.6.). This proves that the henselization of $A_i \otimes_{A_{i-1}} B$ is noetherian. Hence $A_i \otimes_{A_{i-1}} B$ is noetherian (cf. [8] Théorème 18.6.6.). □

Proof of Lemma 2.4. The elements of the henselization of a local ring A are algebraic over A by construction. Thus the henselization of $\frac{B[x_i]}{I}$ is a subring of $\frac{B\langle x_i\rangle}{I}$.
On the other hand let us prove first that $B\langle x_i\rangle$ is the henselization of $B[x_i]_{(x_1,...,x_i)}$. If $y \in B\langle x_i\rangle$, then y is a root of a polynomial $P(Y)$ with coefficients in $B[x_i]$. By Artin approximation Theorem (see Theorem 11.3 [16]), y may be approximated by elements which are in the henselization of $B[x_i]$. Since $P(Y)$ has only a finite number of roots, this means that y is in the henselization of $B[x_i]_{(x_1,...,x_i)}$. Thus $B\langle x_i\rangle$ is the henselization of $B[x_i]_{(x_1,...,x_i)}$.

Now the morphism $B[x_i] \longrightarrow \frac{B[x_i]}{I}$ induces a morphism $B\langle x_i\rangle \longrightarrow \left(\frac{B[x_i]}{I}_{(x_1,\ldots,x_i)}\right)^h$ of $B[x_i]$-algebras by the universal property of the henselization. It is clear that the kernel of this morphism is generated by I thus we get an injective morphism $\frac{B\langle x_i\rangle}{I} \longrightarrow \left(\frac{B[x_i]}{I}_{(x_1,\ldots,x_i)}\right)^h$ of $B[x_i]$-algebras. Since $\left(\frac{B[x_i]}{I}_{(x_1,\ldots,x_i)}\right)^h$ is a subring of $\frac{B\langle x_i\rangle}{I}$, this shows that the morphism $\frac{B\langle x_i\rangle}{I} \longrightarrow \left(\frac{B[x_i]}{I}_{(x_1,\ldots,x_i)}\right)^h$ is an isomorphism. $\square$

3. Algebraic Equisingularity of Zariski

Notation: Let $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$. Then we denote $x^i = (x_1, \ldots, x_i) \in \mathbb{C}^i$.

3.1. **Assumptions.** Let V be an analytic hypersurface of a neighborhood of the origin in $\mathbb{C}^l \times \mathbb{C}^n$ and let $W = V \cap (\mathbb{C}^l \times \{0\})$. Suppose there are given complex pseudopolynomials

$$F_i(t, x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t, x^{i-1}) x_i^{p_i - j}, \qquad i = 0, ..., n,$$

$t \in \mathbb{C}^l$, $x^i \in \mathbb{C}^i$, with complex analytic coefficients $a_{i-1,j}$, that satisfy

1. $V = F_n^{-1}(0)$.
2. $F_{i-1}(t, x^{i-1}) = 0$ if and only if $F_i(t, x^{i-1}, x_i) = 0$ considered as an equation on x_i with (t, x^{i-1}) fixed, has fewer roots than for generic (t, x^{i-1}).
3. $F_0 \equiv 1$.
4. There are positive reals $\delta_k > 0$, $k = 1, \ldots, l$, and $\varepsilon_j > 0$, $j = 1, \ldots, n$, such that F_i are defined on the polydiscs $U_i := \{|t_k| < \delta_k, |x_j| < \varepsilon_j, k = 1, \ldots, l, j = 1, \ldots, i\}$.
5. All roots of $F_i(t, x^{i-1}, x_i) = 0$, for $(t, x^{i-1}) \in U_{i-1}$, lie inside the circle of radius ε_i.
6. Either $F_i(t, 0) \equiv 0$ or $F_i \equiv 1$ (and in the latter case $F_k \equiv 1$ for all $k \leq i$).

We may take as F_{i-1} the Weierstrass polynomial associated to the reduced discriminant of F_i or a generalized discriminant (see the next section).

We shall denote $V_i = F_i^{-1}(0) \subset U_i$. For the parameter t fixed we write $V_t := V \cap (\{t\} \times \mathbb{C}^n)$, $V_{i,t} := V_i \cap (\{t\} \times \mathbb{C}^i)$, and $U_{i,t} = U_i \cap (\{t\} \times \mathbb{C}^i)$. We identify W and U_0.

Theorem 3.1. ([20] Theorem 1, [21] Theorem 1) *Under the above assumptions V is topologically equisingular along W with respect to the family of sections $V_t = V \cap (\{t\} \times \mathbb{C}^n)$. This means that for all $t \in W$ there is a homeomorphism $h_t : U_{n,0} \to U_{n,t}$ such that $h_t(V_0) = V_t$ and $h_t(0) = 0$.*

3.2. **Remarks on Varchenko's proof of Theorem 3.1.** As Varchenko states in Remark 1 of [20] a stronger result holds, the family V_t is topologically trivial, in the sense that the homeomorphisms h_t depend continuously on t. The details of the proof of Theorem 3.1 (with continuous dependence of h_t on t) are published in [19].

The homeomorphisms h_t are constructed in [19] inductively by lifting step by step the homeomorphisms

$$h_{i,t} : U_{i,0} \to U_{i,t},$$

so that $h_{i,t}(x^{i-1}, x_i) = (h_{i-1,t}(x^{i-1}), h_{i,t,i}(x^i))$, $h_{i,t}(V_{i,0}) = V_{i,t}$, $h_{i,t}(0) = 0$. If $h_{i-1,t}$ depends continuously on t, then the number of roots of $F_i(h_{i-1,t}(x^{i-1}), x_i) = 0$ is independent of t.

Therefore, if $F_n = G_1 \cdots G_k$, then the number of roots of each $G_j(h_{n-1,t}(x^{n-1}), x_n) = 0$ is independent of t, see Lemma 2.2 of [19]. In particular h_t preserves not only $V = F_n^{-1}(0)$ but also each of $G_j^{-1}(0)$. Thus [19] implies the following.

Theorem 3.2. *The homeomorphisms h_t of Theorem 3.1 can be chosen continuous in t. If $F_n = G_1 \cdots G_k$ then for each $s = 1, \dots, k$, $h_t(G_s^{-1}(0) \cap (\{0\} \times \mathbb{C}^n)) = G_s^{-1}(0) \cap (\{t\} \times \mathbb{C}^n)$.*

4. Mostowski's Theorem.

In this section we show Theorem 1.1

4.1. **Generalized discriminants.** Let $f(T) = T^p + \sum_{j=1}^p a_i T^{p-i} = \prod_{j=1}^p (T - T_i)$. Then the expressions

$$\sum_{r_1,\dots r_{j-1}} \prod_{k<l, k,l \neq r_1,\dots r_{j-1}} (T_k - T_l)^2$$

are symmetric in $T_1, \dots, T_p$ and hence polynomials in $a = (a_1, \dots, a_p)$. We denote these polynomials by $\Delta_j(a)$. Thus Δ_1 is the standard discriminant and f has exactly $p-j$ distinct roots if and only if $\Delta_1 = \cdots = \Delta_j = 0$ and $\Delta_{j+1} \neq 0$.

4.2. **Construction of a normal system of equations.** Let be given a finite set of pseudopolynomials $g_1, \dots, g_k \in \mathbb{C}\{x\}$:

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}) x_n^{r_s - j}.$$

The coefficients $a_{n-1,s,j}$ can be arranged in a row vector $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ where $p_n := \sum_s r_s$. Let f_n be the product of the g_s's. The generalized discriminants $\Delta_{n,i}$ of f_n are polynomials in a_{n-1}. Let j_n be a positive integer such that

$$\Delta_{n,i}(a_{n-1}) \equiv 0 \qquad i < j_n, \tag{4.1}$$

and $\Delta_{n,j_n}(a_{n-1}) \not\equiv 0$. Then, after a linear change of coordinates x^{n-1}, we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1})(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2}) x_{n-1}^{p_{n-1}-j}).$$

where $u_{n-1}(0) \neq 0$ and for all j, $a_{n-2,j}(0) = 0$. We denote

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2}) x_{n-1}^{p_{n-1}-j}$$

and the vector of its coefficients $a_{n-2,j}$ by $a_{n-2} \in \mathbb{C}\{x^{n-2}\}^{p_{n-1}}$. Let j_{n-1} be the positive integer such that the first $j_{n-1} - 1$ generalized discriminants $\Delta_{n-1,i}$ of f_{n-1} are identically zero and $\Delta_{n-1,j_{n-1}}$ is not. Then again we define $f_{n-2}(x^{n-2})$ as the Weierstrass polynomial associated to $\Delta_{n-1,j_{n-1}}$.

We continue this construction and define a sequence of pseudopolynomials $f_i(x^i)$, $i = 1, \dots, n-1$, such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}) x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first non identically zero generalized discriminant $\Delta_{i+1,j_{i+1}}(a_i)$ of f_{i+1}, where we denote in general $a_i = (a_{i,1}, \dots, a_{i,p_{i+1}})$,

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i)(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}) x_i^{p_i-j}), \quad i = 0, ..., n-1. \tag{4.2}$$

Thus the vector of functions a_i satisfies

$$\Delta_{i+1,k}(a_i) \equiv 0 \qquad k < j_{i+1}, \quad i = 0, ..., n-1. \tag{4.3}$$

This means in particular that

$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \text{ and } \Delta_{1,j_1}(a_0) \equiv u_0,$$

where u_0 is a non-zero constant.

4.3. **Approximation by Nash functions.** Consider (4.2) and (4.3) as a system of polynomial equations on $a_i(x^i)$, $u_i(x^i)$. By construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables $z = (z_1, ..., z_s)$, an increasing function τ, and convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0 such that $z_1(x)$,..., $z_{\tau(i)}(x)$ depend only on $(x_1, ..., x_i)$, algebraic power series $u_i(x^i, z) \in \mathbb{C}\langle x^i, z_1, ..., z_{\tau(i)}\rangle$ and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{C}\langle x^{(i)}, z_1, ..., z_{\tau(i)}\rangle^{p_i}$, such that $a_i(x^i, z)$, $u_i(x^i, z)$ are solutions of (4.2), (4.3) and $a_i(x^i) = a_i(x^i, z(x^i))$, $u_i(x^i) = u_i(x^i, z(x^i))$.

For $t \in \mathbb{C}$ we define

$$F_n(t, x) = \prod_s G_s(t, x), \quad G_s(t, x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1}, tz(x^{n-1})) x_n^{r_s-j}$$

$$F_i(t, x) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}, tz(x^{i-1}) x_i^{p_i-j}, \quad i = 0, ..., n-1.$$

Finally we set $F_0 \equiv 1$. Because $u_i(0, 0) = u_i(0, z(0)) \neq 0$, the family $F_i(t, x)$ satisfies the assumptions of Theorem 3.1 with $|t| < R$ for any $R < \infty$.

Corollary 4.1. *Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be an analytic germ defined by $g_1 = ... = g_k = 0$ with $g_s \in \mathbb{K}\{x\}$. Then there are algebraic power series $\hat{g}_s \in \mathbb{K}\langle x\rangle$ and a homeomorphism germ $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $h(g_s^{-1}(0)) = \hat{g}_s^{-1}(0)$ for $s = 1, ..., k$. In particular, $h(V)$ is the Nash set germ $\{\hat{g}_1 = ... = \hat{g}_k = 0\}$.*

Proof. For $\mathbb{K} = \mathbb{C}$ we set $\hat{g}_i(x) = G_i(0, x)$ and then the corollary follows from Theorem 3.2. The real case follows from the complex one because if the pseudopolynomials F_i of subsection 3.1 have real coefficients then the homeomorphisms h_t constructed in [19] are conjugation invariant, cf. §6 of. [19]. □

Now Theorem 1.1 follows from Corollary 4.1 and the following result.

Theorem 4.2. ([2] Theorem 2.) *Let $(V, 0) \subset (\mathbb{K}^n, 0)$ be a Nash set germ. Then there is a local Nash diffeomorphism $\sigma : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $\sigma(V)$ is the germ of an algebraic subset of $\mathbb{K}^n$.*

5. Topological equivalence between analytic and algebraic function germs

In this section we show Theorem 1.2 and Theorem 1.3.

5.1. A variant of Varchenko's method.

We replace the assumptions (2) and (3) of Subsection 3.1 by

(2') There are $q_i \in \mathbb{N}$ such that $x_1^{q_i} F_{i-1}(t, x^{i-1}) = 0$ if and only if the equation $F_i(t, x^{i-1}, x_i) = 0$, has fewer roots than for generic (t, x^{i-1}).

(3') $F_1 \equiv 1$.

Then Varchenko's method gives the following result.

Theorem 5.1. *Under the above assumptions, V is topologically equisingular along W with respect to the family of sections $V_t = V \cap (\{t\} \times \mathbb{C}^n)$. Moreover all the sections $V \cap \{x_1 = const\}$ are also equisingular. This means that for all $t \in W$ there is a homeomorphism $h_t : U_{n,0} \to U_{n,t}$ such that $h_t(V_0) = V_t$, $h_t(0) = 0$, and h_t preserves the levels of x_1*

$$h_t(x_1, ..., x_n) = (x_1, \hat{h}_t(x_1, ..., x_n)). \tag{5.1}$$

Indeed, recall that the homeomorphisms h_t are constructed inductively by lifting step by step the homeomorphisms $h_{i,t} : U_{i,0} \to U_{i,t}$, so that $h_{i,t}(x^{i-1}, x_i) = (h_{i-1,t}(x^{i-1}), h_{i,t,i}(x^i))$. At each stage such lifts $h_{i,t}$ exist and preserve the zero set of F_i if $h_{i-1,t}$ depends continuously on t and preserves the discriminant set of F_i, see [19] sections 2 and 3.

Because $F_1 \equiv 1$, by (2'), the discriminant set of F_2 is either empty or given by $x_1 = 0$. Therefore we may take $h_{1,t}(x_1) = x_1$. Then we show by induction on i that each $h_{i,t}$ can be lifted so that the lift $h_{i+1,t}$ preserves the zero set of F_{i+1} and the values of x_1. The former condition follows by inductive assumption, $h_{i,t}$ preserves the discriminant set of F_{i+1}. The latter condition is satisfied trivially since $h_{i+1,t}$ is a lift of $h_{i,t}$.

5.2. Equisingularity of functions.

We apply Theorem 5.1 to study the equisingularity of analytic function germs as follows. Let $G(t, y) : (\mathbb{C}^l \times \mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ be analytic, $y = (y_1, ..., y_{n-1})$. We associate to G its graph $V = \{(t, x_1, x_2, ..., x_n); x_1 = G_t(x_2, ..., x_n)\}$, thus fixing the following notation

$$x = (x_1, x_2, ..., x_n) = (x_1, y) \tag{5.2}$$

We consider G as an analytic family of analytic function germs $G_t : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$ parametrized by $t \in W$, where W is a neighborhood of the origin in $\mathbb{C}^l$.

Theorem 5.2. *Suppose that V and W satisfy the assumptions of Theorem 5.1. Then the family of analytic function germs G_t is topologically equisingular. This means that there is a family of local homeomorphisms $\sigma_t : (\mathbb{C}^{n-1}, 0) \to \mathbb{C}^{n-1}, 0)$ such that*

$$G_0 = G_t \circ \sigma_t.$$

Proof. It follows from (5.1) by setting $\sigma_t(y) = \hat{h}_t(G_0(y), y)$. Indeed, since h_t preserves V we have

$$h_t(G_0(y), y) = (G_t(\hat{h}_t(G_0(y), y)), \hat{h}_t(G_0(y), y)),$$

and since it preserves the levels of x_1

$$G_0(y) = G_t(\hat{h}_t(G_0(y), y)).$$

□

5.3. **Construction of a normal system of equations for a finite family of function germs.** Let $g_m : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$, $m = 1, ..., p$, be a finite family of analytic function germs that we assume not identically equal to zero. After a linear change of coordinates $(x_2, ..., x_n)$, $\prod_{m=1}^{p}(x_1 - g_m(x_2, ..., x_n))$ is equivalent to a pseudopolynomial, that is we may write

$$\prod_{m=1}^{p}(x_1 - g_m(x_2, ..., x_n)) = u_n(x)(x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j}(x^{n-1})x_n^{p_n-j}),$$

where $u_n(0) \neq 0$ and $a_{n-1,j}(0) = 0$. We denote

$$f_n(x) = x_n^{p_n} + \sum_{j=1}^{p_n} a_{n-1,j}(x^{n-1})x_n^{p_n-j}$$

so that

$$u_n(x)f_n(x) = \prod_{m=1}^{p}(x_1 - \sum_{k=2}^{n} x_k b_{m,k}(x_2, ..., x_n)) \tag{5.3}$$

with $g_m = \sum_{k=2}^{n} x_k b_{m,k}$. We denote by $b \in \mathbb{C}\{x\}^{p(n-1)}$ the vector of the coefficients $b_{m,k}$ and by $a_{n-1} \in \mathbb{C}\{x^{n-1}\}^{p_n}$ the one of the coefficients $a_{n-1,j}$.

The generalized discriminants $\Delta_{n,i}$ of f_n are polynomials in a_{n-1}. Let j_n be a positive integer such that

$$\Delta_{n,i}(a_{n-1}) \equiv 0 \qquad i < j_n,$$

and $\Delta_{n,j_n}(a_{n-1}) \not\equiv 0$. After a change of coordinates $(x_2, ..., x_{n-1})$ we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1})x_1^{q_{n-1}}(x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}),$$

where $u_{n-1}(0) \neq 0$ and $a_{n-2,j}(0) = 0$. We denote $f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}$ and the vector of its coefficients $a_{n-2,j}$ by $a_{n-2} \in \mathbb{C}\{x^{n-2}\}^{p_{n-1}}$. Let j_{n-1} be the positive integer such that the first $j_{n-1} - 1$ generalized discriminants $\Delta_{n-1,i}$ of f_{n-1} are identically zero and $\Delta_{n-1,j_{n-1}}$ is not. Then again we divide $\Delta_{n-1,j_{n-1}}$ by the maximal power of x_1 and, after a change of coordinates $(x_2, ..., x_{n-2})$, denote the associated Weierstrass polynomial by $f_{n-2}(x^{n-2})$.

We continue this construction and define a sequence of pseudopolynomials $f_i(x^i)$, $i = 1, \ldots, n-1$, such that $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$ is the Weierstrass polynomial associated to the first non identically zero generalized discriminant $\Delta_{i,j_i}(a_{i+1})$ of f_{i+1}, divided by the maximal power of x_1, where we denote in general $a_i = (a_{i,1}, \ldots, a_{i,p_i})$,

$$\Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i)x_1^{q_i}(x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}), \quad i = 0, ..., n-1. \tag{5.4}$$

Thus the vector of functions a_i satisfies

$$\Delta_{i+1,k}(a_{i-1}) \equiv 0 \qquad k < j_{i+1}, \quad i = 0, ..., n-1. \tag{5.5}$$

These equations mean in particular that

$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \text{ and } \Delta_{1,j_1}(a_0) \equiv u_0 x_1^{q_0}. \tag{5.6}$$

where u_0 is a non-zero constant. Hence $f_1 \equiv 1$.

5.4. **Approximation by Nash functions.** Consider (5.3), (5.4), (5.5), as a system of polynomial equations on $a_i(x^i)$, $u_i(x^i)$, and $b(x)$. By construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables $z = (z_1, ..., z_s)$, an increasing function τ, and convergent power series $z_i(x) \in \mathbb{C}\{x\}$ vanishing at 0 such that $z_1(x)$,..., $z_{\tau(i)}(x)$ depend only on $(x_1, ..., x_i)$, algebraic power series $u_i(x^i, z) \in \mathbb{C}\langle x^i, z_1, ..., z_{\tau(i)}\rangle$, and vectors of algebraic power series $a_i(x^i, z) \in \mathbb{C}\langle x^{(i)}, z_1, ..., z_{\tau(i)}\rangle^{p_i}$, $b(x, z) \in \mathbb{C}\langle x, z\rangle^{n-1}$, such that $a_i(x^i, z)$, $u_i(x^i, z)$, $b(x, z)$, are solutions of (5.3), (5.4), (5.5) and $a_i(x^i) = a_i(x^i, z(x^i))$, $u_i(x^i) = u_i(x^i, z(x^i))$, $b(x) = b(x, z(x))$.

For $t \in \mathbb{C}$ we define

$$F_i(t, x) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1}, tz(x^{i-1}))x_i^{p_i-j}.$$

In particular, by (5.6), $F_1 \equiv 1$. Since

$$u_n(x, tz(x))F_n(t, x) = \prod_{m=1}^{p}(x_1 - \sum_{k=2}^{n} x_k b_{m,k}(x, tz(x))),$$

by the Implicit Function Theorem there are algebraic power series $G_m \in \mathbb{C}\langle t, y_1...., y_{n-1}\rangle$ such that

$$F_n^{-1}(0) = \bigcup_m \{(t, x); x_1 = G_m(t, x_2, ..., x_n)\}$$

as germs at the origin. Then $g_m(y) = G_m(1, y)$ and $G_m(0, y) \in \mathbb{C}\langle y\rangle$. We denote $\hat{g}_m(y) = G_m(0, y)$.

Because $u_i(0, 0) = u_i(0, z(0)) \neq 0$, the family $F_i(t, x)$ satisfies the assumptions of Theorem 5.1 with $|t| < R$ for arbitrary $R < \infty$. By Theorem 5.1 there is a continuous family of homomorphism germs $h_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $h_t(x) = (x_1, \hat{h}_t(x_1, x_2, ..., x_n))$, such that

$$h_t(g_m(y), y) = (G_m(t, \hat{h}_t(g_m(y), y)), \hat{h}_t(g_m(y), y)),$$

Fix one m, for instance $m = 1$, and set

$$\sigma_t(y) = \hat{h}_t(g_1(y), y)$$

as in the proof of Theorem 5.2 (we use here the notation (5.2)). Then $g_1(y) = G_1(t, \sigma_t(y))$ and in particular

$$g_1(y) = \hat{g}_1(\sigma_0(y)). \tag{5.7}$$

It is not true in general that $g_m(y) = \hat{g}_m(\sigma_0(y))$ since the homeomorphism σ_t is defined by restricting h_t to the graph of G_1. If we define

$$\sigma_{m,t}(y) = \hat{h}_t(g_m(y), y)$$

then we have

$$g_m(y) = \hat{g}_m(\sigma_{m,0}(y))$$

Both homeomorphisms coincide on $X_m = \{y \in (\mathbb{C}^{n-1}, 0); (g_m - g_1)(y) = 0\}$. Therefore if we define $\hat{X}_m = \{y \in (\mathbb{C}^{n-1}, 0); (\hat{g}_m - \hat{g}_1)(y) = 0\}$ then

$$\sigma_0(\hat{X}_m) = X_m. \tag{5.8}$$

Therefore we have the following result.

Proposition 5.3. *Let $(V_i, 0) \subset (\mathbb{K}^n, 0)$ be a finite family of analytic set germs and let $g : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$ be an analytic function germ. Then there are Nash set germs $(\hat{V}_i, 0) \subset (\mathbb{K}^n, 0)$, an algebraic power series $\hat{g} \in \mathbb{K}\langle x\rangle$, and a homeomorphism germ $\hat{\sigma} : (\mathbb{K}^{n-1}, 0) \to (\mathbb{K}^{n-1}, 0)$ such that $\sigma(\hat{V}_i) = V_i$ and $g \circ \hat{\sigma} = \hat{g}$.*

Proof. Let $\mathbb{K} = \mathbb{C}$. Choose a finite family $g_m : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}, 0)$, $m = 1, ..., p$, of analytic function such that $g_1 = g$ and for every i, the ideal of V_i is generated by some of the differences $g_m - g_1$. We apply to the family g_m the procedure of subsections 5.3 and 5.4 and set $\hat{\sigma} = \sigma_0$. The claim now follows from (5.7) and (5.8).

The real case follows from the complex one because if the pseudopolynomials F_i of subsection 3.1 have real coefficients then the homeomorphisms h_t constructed in [19] are conjugation invariant, cf. §6 of. [19]. □

5.5. **Proof of Theorem 1.2 and Theorem 1.3.** It suffices to show Theorem 1.3. It will follow from Proposition 5.3 and the next two results.

Theorem 5.4. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $f_i : (\mathbb{K}^n, 0) \to (\mathbb{K}, 0)$, be a finite family of Nash function germs. Then there is a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and analytic (even Nash) units $u_i : (\mathbb{K}^n, 0) \to \mathbb{K}$, $u_i(0) \neq 0$, such that for all i, $u_i(x)f_i(h(x))$ are germs of polynomials.*

Proof. For $\mathbb{K} = \mathbb{C}$ Theorem 5.4 follows from Theorem 5 of [2]. Indeed, (i) implies (ii) of this theorem gives :
If $(V, 0) \subset (\mathbb{K}^n, 0)$ is a Nash set germ then there is a Nash diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that for any analytic irreducible component W of $(V, 0)$ the ideal of functions vanishing on $h(W)$ is generated by polynomials.

Now, if $\mathbb{K} = \mathbb{C}$, it suffices to apply the above result to $(V, 0)$ defined as the zero set of the product of f_i's.

If $\mathbb{K} = \mathbb{R}$ such a set theoretic statement is not sufficent but in this case Theorem 5.4 follows from the proof of Theorem 5 of [2]. We sketch this argument below.

First we consider $\mathbb{K} = \mathbb{C}$. Choose representatives $f_i : U \to \mathbb{C}$ of the germs f_i, $i = 1, ..., m$, and let $f = (f_1, ..., f_m) : U \to \mathbb{C}^m$. By Artin-Mazur Theorem, [1] Theorem 8.4.4, [2] Proposition 2, there is an algebraic set $X \subset \mathbb{C}^n \times \mathbb{C}^N$ of dimension n, a polynomial map $\Phi : \mathbb{C}^n \times \mathbb{C}^N \to \mathbb{C}^m$, and a Nash map $s : U \to X$ such that $f = \Phi \circ s$, $s : U \to s(U)$ is a Nash diffeomorphism and $s(U) \cap Sing(X) = \emptyset$. ($X$ is the normalization of the Zariski closure of the graph of f.) We may assume that $p = s(0)$ is the origin in $\mathbb{C}^n \times \mathbb{C}^N$.

Let $\pi : X \to \mathbb{C}^n$ be a generic linear projection. Then the germ h of $(\pi \circ s)^{-1}$ satisfies the claim. Indeed, denote $X_i = X \cap \Phi_i^{-1}(0)$. Then for each $i = 1, ..., m$, $Z_i = \pi(X_i)$ is an algebraic subset of $\mathbb{C}^n$ and moreover, π induces a local isomorphism $(X_i, 0) \to (Z_i, 0)$. We fix a reduced polynomial P_i that defines Z_i. Then $f_i \circ h$, as a germ at the origin, vanishes exactly on Z_i and hence equals a power of P_i times an analytic unit.

If $\mathbb{K}=\mathbb{R}$ then we apply the complex case to the complexifications of the f_i's keeping the construction conjugation invariant. In particular the linear projection can be chosen real (that is conjugation invariant). Indeed, this projection is from the Zariski open dense subset $\mathcal{U}$ of the set of linear projections $L(\mathbb{C}^n\times\mathbb{C}^N,\mathbb{C}^n)$. Then $\mathcal{U}\cap L(\mathbb{R}^n\times\mathbb{R}^N,\mathbb{R}^n)$ is non-empty. (A complex polynomial of M variables that vanishes on $\mathbb{R}^M$ is identically equal to zero). This ends the proof. □

Theorem 5.5. *Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Let $f:(\mathbb{K}^n,0)\to(\mathbb{K},0)$ be an analytic function germ and let $u:(\mathbb{K}^n,0)\to\mathbb{K}$ be an analytic unit, $u(0)\neq 0$ ($u(0)>0$ if $\mathbb{K}=\mathbb{R}$). Let $(V_i,0)\subset(\mathbb{K}^n,0)$ be a finite family of analytic set germs. Then there is a homeomorphism germ $\sigma:(\mathbb{K}^n,0)\to(\mathbb{K}^n,0)$ such that $(\sigma(V_i),0)=(V_i,0)$ for each i and $uf=f\circ\sigma$.*

Proof. If $\mathbb{K}=\mathbb{C}$ we suppose additionally that the segment that joins $u(0)$ and 1 does not contain 0. The general case can be reduced to this one.

Fix a small neighborhood U of the origin in $\mathbb{K}^n$ so that the representatives $V_i\subset U$, $f:U\to\mathbb{K}$, and $u:U\to\mathbb{K}$ are well-defined. In the proof we often shrink U when necessary. Let I denote a small neighborhood of $[0,1]$ in $\mathbb{R}$. We construct a Thom stratification of the deformation $\Psi(x,t)=(F(x,t),t):U\times I\to\mathbb{K}\times I$, where

$$F(x,t)=f(x)(1-t+tu(x))=f(x)(1+t(u(x)-1)),\quad (x,t)\in U\times I,$$

that connects $f(x)=F(x,0)$ and $u(x)f(x)=F(x,1)$. Then we conclude by the second Thom-Mather Isotopy Lemma. For the Thom stratification we refer the reader to [4], Ch. 1, and for the Thom-Mather Isotopy Lemmas to [4], Ch. 2.

Fix a Thom stratification $\mathcal{S}'=\{S'_j\}$ of $f:U\to\mathbb{K}$ such that each V_i is a union of strata. That means that $\mathcal{S}'$ is a Whitney stratification of U, compatible with $f^{-1}(0)$ and $f^{-1}(\mathbb{K}\setminus\{0\})$, that satisfies Thom's a_f condition. (It is well-known that such a stratification exists, the existence of a_f regular stratifications was first proved in the complex analytic case by H. Hironaka in [5], using resolution of singularities, under the assumption "sans éclatement" which is always satisfied for functions. In the real subanalytic case it was first shown in [9])

First we show that $\mathcal{S}=\{S_j=S'_j\times I\}$ as a stratification of $U\times I$ satisfies a_F condition

(a_F) *for every stratum $S\subset F^{-1}(\mathbb{K}\setminus\{0\})$ and every sequence of points $p_i=(x_i,t_i)\in S$ that converges to a point $p_0=(x_0,t_0)\in S_0\subset F^{-1}(0)$, such that $\ker d_{p_i}F|_{T_{p_i}S}\to T$, we have $T\supset T_{p_0}S_0$.*

By the curve selection lemma it suffices to check this condition on every real analytic curve $p(s)=(x(s),t(s)):[0,\varepsilon)\to S\cup S_0$, $p(0)\in S_0$ and $p(s)\in S$ for $s>0$. Since $\mathcal{S}'$ satisfies a_f, the condition a_F for $\mathcal{S}$ follows from the following lemma.

Lemma 5.6. *Let $S=S'\times I\subset F^{-1}(\mathbb{K}\setminus\{0\})$ and $S_0\subset F^{-1}(0)$ be two strata of $\mathcal{S}$ and let $p(s)=(x(s),t(s)):[0,\varepsilon)\to U\times I$ be a real analytic curve such that $p_0=p(0)\in S_0$ and $p(s)\in S$ for $s>0$. Then for $s>0$ and small, $\operatorname{grad}_x F|_S(p(s))\neq 0$ and*

$$\lim_{s\to 0}\frac{\operatorname{grad}F|_S(p(s))}{\|\operatorname{grad}F|_S(p(s))\|}=\lim_{s\to 0}\frac{(\operatorname{grad}f|_{S'}(x(s)),0)}{\|\operatorname{grad}f|_{S'}(x(s))\|} \tag{5.9}$$

Proof. By assumption $f(x(s))=\sum_{i=i_0}^{\infty}a_is^i$, with $i_0>0$ and $a_{i_0}\neq 0$. By differenting we obtain

$$|\frac{df}{ds}|=|\langle\operatorname{grad}f|_{S'},x'(s)\rangle|\leq\|\operatorname{grad}f|_{S'}\|\|x'(s)\|.$$

Hence there exists $C > 0$ such that for small $s > 0$

$$|f(x(s))| \le sC\| \operatorname{grad} f|_{S'}\|. \tag{5.10}$$

Moreover

$$\operatorname{grad} F|_S(p(s)) = (\operatorname{grad} f|_{S'}(x(s)), 0)(1 + t(s)(u(x(s)) - 1) \tag{5.11}$$

$$+ f(x(s))(t(s) \operatorname{grad} u|_{S'}(x(s)), u(x) - 1). \tag{5.12}$$

Now (5.9) follows easily from (5.10) and (5.11). □

Finally $\mathcal{S}$ as a stratification of $U \times I$ together with $((\mathbb{K}\setminus\{0\}) \times I, \{0\} \times I)$ as a stratification of $\mathbb{K} \times I$ is a Thom stratification of Ψ. Indeed, $\mathcal{S}$ is a Whitney stratification as the product of a Whitney stratification of U times I. Secondly, for any pair of strata $S = S' \times I \subset F^{-1}(\mathbb{K}\setminus\{0\})$ and $S_0 = S'_0 \times I \subset F^{-1}(0)$ it satisfies a_F and hence also a_Ψ condition. Therefore Theorem 5.5 follows from the second Thom-Mather Isotopy Lemma, [4], Ch. 2 (5.8). □

Now we may conclude the proof of Theorem 1.3. By Proposition 5.3 we may assume that g is a Nash function germ and the V_i's are Nash sets germs. Moreover by Theorem 5.4, after composing with the Nash diffeomorphism h, we may assume that g equals a polynomial times an analytic unit and that the ideal of analytic function germs defining each V_i is generated by polynomials. In particular each V_i is algebraic. Finally we apply Theorem 5.5 to show that, after composing with a homeomorphism preserving each V_i, g becomes a polynomial.

6. Examples

Example 6.1. We give here an example showing that the $\mathcal{C}^1$ analog of Theorems 1.1 or 1.2 is false in the real case (this example is well-known, see [2] for example). The germ $(V, 0) \subset (\mathbb{R}^3, 0)$, defined by the vanishing of

$$f(t, x, y) = xy(y - x)(y - (3 + t)x)(y - \gamma(t)x)$$

where $\gamma(t) \in \mathbb{R}\{t\}$ is transcendental and $\gamma(0) = 4$, is not C^1-diffeomorphic to the germ of an algebraic set as follows from the argument of Whitney, cf. Section 14 of [22]. Indeed V is the union of five smooth surfaces intersecting along the t-axis and its tangent cone at the point $(t, 0, 0)$ is the union of five planes intersecting along a line. The cross-ratio of the first four planes is $3 + t$ and the cross-ratio of the first three and the last plane is $\gamma(t)$. Since the cross-ratio is preserved by linear maps, these two cross-ratios are preserved by C^1-diffeomorphisms. But these two cross-ratios are algebraically independent thus the image of V under a C^1-diffeomorphisms cannot be algebraic.

Example 6.2. The previous example also shows that the C^1 analogs of Theorems 1.1 or 1.2 are false in the complex case. Define V in a neighborhood of 0 in $\mathbb{C}^3$ by the vanishing of the polynomial of Example 6.1. Modifying Whitney's argument (cf. [22], pp 240, 241) we will show that the germ of V at 0 is not C^1-equivalent to any Nash germ in $\mathbb{C}^3$.

For any $(t, 0, 0) \in \mathbb{C}^3$ with $|t|$ small, the tangent cone to V at $(t, 0, 0)$ is the union of five two-dimensional $\mathbb{C}$-linear spaces $L_{1,t}, \ldots, L_{5,t}$, where $L_{j,t}$ corresponds to the j'th factor of f. Suppose that there is a C^1-diffeomorphism $\Phi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ such that $\Phi(V)$ is a germ of a Nash set in $\mathbb{C}^3$. Then the tangent cone to $\Phi(V)$ at $\Phi(t, 0, 0)$ is the union of $d_{(t,0,0)}\Phi(L_{j,t})$, for $j = 1,..., 5$. In particular, every $d_{(t,0,0)}\Phi(L_{j,t})$ is a $\mathbb{C}$-linear subspace of $\mathbb{C}^3$ of dimension 2.

Let us check that for every $t \in \mathbb{R}$ with $|t|$ small and for every pairwise distinct $k_1, \dots, k_4 \in \{1, \dots, 5\}$, the cross-ratio of $L_{k_1,t}$,..., $L_{k_4,t}$ is equal to the cross-ratio of $d_{(t,0,0)}\Phi(L_{k_1,t})$,..., $d_{(t,0,0)}\Phi(L_{k_4,t})$. By a real line in $\mathbb{C}^3$ we mean a set of the form $\{a + tb : t \in \mathbb{R}\}$ where $a, b \in \mathbb{C}^3$. For $t \in \mathbb{R}$, $L_{k_1,t}, \dots, L_{k_4,t}$ are defined by real equations so there is a real line $l_t \subset \mathbb{C}^3$ intersecting $L_{k_1,t} \cup \dots \cup L_{k_4,t}$ at exactly four points, say $a_{1,t}$,..., $a_{4,t}$. Then the cross-ratio of $L_{k_1,t}, \dots, L_{k_4,t}$ equals the cross-ratio of $a_{1,t}$,..., $a_{4,t}$. Moreover, $d_{(t,0,0)}\Phi(l_t)$ is also a real line and it intersects $d_{(t,0,0)}\Phi(L_{k_1,t}) \cup \dots \cup d_{(t,0,0)}\Phi(L_{k_4,t})$ at $d_{(t,0,0)}\Phi(a_{1,t})$,..., $d_{(t,0,0)}\Phi(a_{4,t})$. The cross-ratio of the last four points equals that of $a_{1,t}$,..., $a_{4,t}$ because $d_{(t,0,0)}\Phi$ is $\mathbb{R}$-linear and $a_{1,t}, \dots, a_{4,t} \in l_t$. Since the cross-ratio of $d_{(t,0,0)}\Phi(a_{1,t})$,..., $d_{(t,0,0)}\Phi(a_{4,t})$ equals the cross-ratio of $d_{(t,0,0)}\Phi(L_{k_1,t})$,..., $d_{(t,0,0)}\Phi(L_{k_4,t})$, we obtain our claim.

Now observe that the complex t-axis is the singular locus of V and its image S by Φ is the singular locus of $\Phi(V)$. Clearly, S is a smooth complex Nash curve. Moreover, the cross-ratios h_1, h_2 of $d_{(t,0,0)}\Phi(L_{1,t}), \dots, d_{(t,0,0)}\Phi(L_{4,t})$ and $d_{(t,0,0)}\Phi(L_{1,t})$, $d_{(t,0,0)}\Phi(L_{2,t})$, $d_{(t,0,0)}\Phi(L_{3,t})$, $d_{(t,0,0)}\Phi(L_{5,t})$, respectively, depend algebraically on $s = \Phi(t, 0, 0) \in S$ (cf. [22], p 241), i.e. $h_1, h_2 : S \to \mathbb{C}$ are complex Nash functions. On the other hand, the cross-ratios of $L_{1,t}, \dots, L_{4,t}$ and of $L_{1,t}, L_{2,t}, L_{3,t}, L_{5,t}$ equal $3 + t$ and $\gamma(t)$, respectively.

The last two paragraphs imply that for $t \in \mathbb{R}$ with $|t|$ small, we have $h_1(\Phi(t, 0, 0)) = 3 + t$ and $h_2(\Phi(t, 0, 0)) = \gamma(t)$. Since S is a smooth complex Nash curve, we may assume that h_1, h_2 are defined in some neighborhood of $0 \in \mathbb{C}$ and that $\Psi(t) = \Phi(t, 0, 0)$ is a map into $\mathbb{C}$. We have $\Psi(t) = \Psi_1(t) + i\Psi_2(t)$ where Ψ_1, Ψ_2 are real valued continuous functions and $h_1(s) = u_1(s) + iv_1(s)$, where u_1, v_1 are real Nash functions, and $u_1(\Psi_1(t), \Psi_2(t)) = 3 + t$, and $v_1(\Psi_1(t), \Psi_2(t)) = 0$ for $t \in \mathbb{R}$ with $|t|$ small. Since u_1, v_1 satisfy the Cauchy-Riemann equations and h_1 is not constant, neither of u_1, v_1 is constant. Consequently, $\Psi_1|_{\mathbb{R}}$, $\Psi_2|_{\mathbb{R}}$ are semi-algebraic functions, which contradicts the fact that $h_2(\Psi(t)) = \gamma(t)$ for real t.

Example 6.3. Theorem 1.2 cannot be extended to many functions or to maps to $\mathbb{K}^m$, $m > 1$. For example the one variable analytic germs x and $e^x - 1$ cannot be made polynomial (or Nash) simultaneously by composing with the same homeomorphism.

Example 6.4. The key point in the previous examples is the fact that two one variable functions which are algebraically independent remain algebraically independent after composition with a homeomorphism. Theorem 1.2 also cannot be extended to many functions, even if we assume them algebraically depended. For instance, one variable Nash germs x and $y(x) = \sqrt{\varphi(x)} - 2$, with $\varphi(x) = (x - 1)(x + 2)(x - 2)$, cannot be made polynomial simultaneously since the cubic $y^2 = \varphi(x)$ is not rational.

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