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ENTROPIES FOR A DIFFEOMORPHISM ON A NONCOMPACT MANIFOLD. APPLICATIONS TO RUELLE'S INEQUALITY AND PESIN'S FORMULA.

FELIPE RIQUELME

ABSTRACT. In this paper we study different notions of entropy of measure-preserving dynamical systems defined on noncompact spaces. We see that some classical results for compact spaces remain partially valid in this setting. We define a new kind of entropy for dynamical systems defined on noncompact Riemannian manifolds, which satisfies similar properties to the classical ones. As an application, we prove Ruelle's inequality for the geodesic flow in manifolds with pinched negative curvature. We discuss the case of equality and we give some corollaries.

1. INTRODUCTION

Let (X, \mathcal{B}, μ, T) be a measure-preserving dynamical system. The *measure theoretic entropy* of Kolmogorov-Sinai, denoted by $h_\mu(T)$, is an ergodic invariant measuring the exponential growth rate of the complexity of T with respect to μ . When (X, d) is a metric space and $T : X \rightarrow X$ is a continuous transformation, it is possible to measure the dynamical complexity from a metric point of view. For instance, when X is compact we can consider the exponential growth rate of the cardinality of a covering of X by dynamical balls, which is called *topological entropy*. We recall that for every $n \geq 0$ and $r > 0$, the (n, r) -dynamical ball at $x \in X$, denoted by $B_n(x, r)$, is the set of points $y \in X$ satisfying $d(T^i x, T^i y) < r$ for all $0 \leq i \leq n - 1$.

By imitating the definition of topological entropy, Katok considered in [Kat80] coverings by dynamical balls of sets with strictly positive measure. He showed that, when X is compact, the exponential growth rate of the cardinality of those coverings coincides with the measure theoretic entropy. More precisely, fix $0 < \delta < 1$ and denote by $N(n, r, \delta)$ the minimal cardinality of a covering by (n, r) -dynamical balls of a set of μ -measure $> 1 - \delta$. He proved that

$$h_\mu(T) = \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta) = \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta).$$

Another way to understand the complexity of a dynamical system is by considering the exponential decay of the measure of a generic dynamical ball. In [BK83] Brin and Katok showed that, when X is compact, then for μ -almost every $x \in X$, we have

$$(1.1) \quad h_\mu(T) = \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)) = \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)).$$

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In this paper we compare these different notions of entropy when the base-space is no longer compact. Moreover, we will study the complexity of a measure-preserving dynamical system from a geometric point of view.

Consider the particular case, where $T : M \rightarrow M$ is a continuous map defined on a Riemannian manifold. The Lebesgue measure \mathcal{L} induced by the Riemannian metric gives strictly positive measure to nonempty open sets. Hence, we can always compute the exponential decay of the Lebesgue measure of a dynamical ball (as in (1.1) for the Lebesgue measure). Naturally, since the Lebesgue measure is not necessarily T -invariant, the limits

$$\lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathcal{L}(B_n(x, r)) \quad \text{and} \quad \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mathcal{L}(B_n(x, r))$$

depend on $x \in M$. Since we are particularly interested in the measure theoretic entropy $h_\mu(T)$ of a T -invariant probability measure μ on M , we should consider the limits above for μ -generic dynamical balls. Having this in mind, we get

Theorem 1.1. *Let $T : M \rightarrow M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T -invariant probability measure μ . Then, we have*

$$(1.2) \quad h_\mu(T) \leq \sup_{K \text{ compact}} \operatorname{ess\,sup}_{x \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mathcal{L}(B_n(x, r)).$$

Observe that the measure μ appears in the right-hand term of the inequality (1.2) only when we consider the μ -essential supremum of the exponential decays of the volumes of dynamical balls. As in the case of local entropy, we can also take the limit in the right-hand term of (1.2) by considering the measure μ instead of \mathcal{L} . Following the strategy of the proof of Theorem 1.1, we obtain

Theorem 1.2. *Let $T : M \rightarrow M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T -invariant probability measure μ . Then, we have*

$$h_\mu(T) \leq \sup_{K \text{ compact}} \operatorname{ess\,sup}_{x \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mu(B_n(x, r)).$$

Consider now a C^1 -diffeomorphism $f : M \rightarrow M$ of a Riemannian manifold and μ an f -invariant probability measure on M . The set of (Lyapunov-Perron) regular points in M , denoted by Λ_f , is the set of points where the asymptotic eigenvalues of the linearized dynamics of df are (somehow) well defined. The logarithm of these eigenvalues are called the Lyapunov exponents of f (see definition in Section 3). When M is compact, Ruelle's inequality says that the measure theoretic entropy of f , with respect to μ , is bounded from above by the sum of positive Lyapunov exponents (see [Rue78]). More precisely,

$$h_\mu(f) \leq \int \sum_{\lambda_j(x) > 0} \lambda_j(x) \dim(E_j(x)) d\mu(x).$$

In [Riq15], we showed that Ruelle's inequality is not always satisfied when M is noncompact. However, under reasonable assumptions on the manifold, Ruelle's inequality is satisfied for the geodesic flow in negative curvature.

Theorem 1.3. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature. Assume that the derivatives of the sectional*

curvature are uniformly bounded. Then, for all (g^t) -invariant probability measures μ on T^1X , we have

$$h_\mu((g^t)) \leq \int \sum_{\lambda_j(v) > 0} \lambda_j(v) \dim(E_j(v)) d\mu(v).$$

We discuss under which conditions we have equality in Ruelle's inequality for the geodesic flow. Recall that this has been widely studied in [Pes77, LS82, Led84a, LY85] for generic compact manifolds and $C^{1+\alpha}$ -diffeomorphisms. Following the strategies in [LS82], [Led84a] and [OP04], we show

Theorem 1.4. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ be a (g^t) -invariant probability measure on T^1X . Then μ has absolutely continuous conditional measures on unstable manifolds if and only if*

$$h_\mu((g^t)) = \int \sum_{\lambda_j(v) > 0} \lambda_j(v) \dim(E_j(v)) d\mu(v).$$

In section 2 we give some general definitions of entropy and we establish, in the noncompact setting, comparison results that are classical in the compact case. In section 3 we prove Theorems 1.1 and 1.2. In section 4 we recall some properties of the geodesic flow in negative curvature and we give the proofs of Theorems 1.3 and 1.4. Some corollaries are also stated at the end of that section.

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2. PRELIMINARIES

In this section we consider a probability space (X, \mathcal{B}, μ) and a measurable transformation $T : X \rightarrow X$ preserving the measure μ . Recall that μ is *ergodic* if every T -invariant measurable set $A \subset X$ satisfies $\mu(A) \in \{0, 1\}$.

2.1. Measure theoretic entropy. Let \mathcal{P} be a countable measurable partition of X . The entropy of \mathcal{P} with respect to μ , denoted by $H_\mu(\mathcal{P})$, is defined as

$$H_\mu(\mathcal{P}) = - \sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

For all $n \geq 0$, define the partition \mathcal{P}^n as the measurable partition consisting of all possible intersections of elements of $T^{-i}\mathcal{P}$, for all $i = 0, \dots, n-1$. The entropy of T with respect to the partition \mathcal{P} is then defined as the limit

$$h_\mu(T, \mathcal{P}) = \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu(\mathcal{P}^n).$$

The *measure theoretic entropy of T , with respect to μ* , is the supremum of the entropies $h_\mu(T, \mathcal{P})$ over all measurable finite partitions \mathcal{P} of X , i.e.

$$h_\mu(T) = \sup_{\mathcal{P} \text{ finite}} h_\mu(T, \mathcal{P}).$$

2.2. Katok δ -entropies. Suppose now that (X, d) is a metric space, $T : X \rightarrow X$ is a continuous transformation and μ is a Borel T -invariant probability measure on X . For every $n \geq 0$ the dynamical distance d_n is defined by

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y), \quad \text{for all } x, y \in X.$$

The (n, r) -dynamical ball centered at x , denoted by $B_n(x, r)$, is the r -ball centered at x for the dynamical distance d_n . Note that since T is continuous, the (n, r) -dynamical balls are open subsets of X . Let A be a subset of X . A (n, r) -covering of A is a covering of A by (n, r) -dynamical balls. A (n, r) -separated set in A is a subset E of A such that for every $x, y \in E$, if $x \neq y$ then $d_n(x, y) \geq r$.

Definition 2.1. Let $K \subset X$ be a compact set. Denote by

- (1) $N(n, r, K)$ the minimal cardinality of a (n, r) -covering of K , i.e. a covering of K by (n, r) -balls, and
- (2) $S(n, r, K)$ the maximal cardinality of a (n, r) -separated set in K .

Lemma 2.2 below is classical (see for instance [Wal82, page 169]). It says that the cardinalities $N(n, r, K)$ and $S(n, r, K)$ are comparable.

Lemma 2.2. *For all compact subsets $K \subset X$, we have*

$$N(n, r, K) \leq S(n, r, K) \leq N(n, r/2, K).$$

Recall that Bowen's definition of topological entropy, for continuous transformations on compact metric spaces, is the following:

$$\begin{aligned} h_{top}(T) &= \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, X) \\ &= \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log S(n, r, X). \end{aligned}$$

From the measure-point of view, Katok proposed the following definition of entropy in [Kat80]. For every $0 < \delta < 1$, denote by $N(n, r, \delta)$ the minimal cardinality of a (n, r) -covering of a set of μ -measure greater than $1 - \delta$. Observe that this number is finite since every compact subset of measure greater than $1 - \delta$ admits a finite (n, r) -covering.

Definition 2.3. Let $0 < \delta < 1$. The lower and upper δ -entropies of Katok, denoted respectively by $\underline{h}_\mu^\delta(T)$ and $\overline{h}_\mu^\delta(T)$, are defined as

$$\underline{h}_\mu^\delta(T) = \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta)$$

and

$$\overline{h}_\mu^\delta(T) = \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta).$$

Proposition 2.4. *Let $0 < \delta_2 \leq \delta_1 < 1$, then*

$$\underline{h}_\mu^{\delta_1}(T) \leq \underline{h}_\mu^{\delta_2}(T) \quad \text{and} \quad \overline{h}_\mu^{\delta_1}(T) \leq \overline{h}_\mu^{\delta_2}(T).$$

Proof. We define \mathcal{B}_i , for $i = 1, 2$, by $\mathcal{B}_i = \{B : \mu(B) > 1 - \delta_i\}$. Since $\mathcal{B}_2 \subset \mathcal{B}_1$, we obtain

$$\begin{aligned} \underline{h}_\mu^{\delta_1}(T) &= \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \min\{N(n, r, B) : B \in \mathcal{B}_1\} \\ &\leq \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log \min\{N(n, r, B) : B \in \mathcal{B}_2\} \\ &= \underline{h}_\mu^{\delta_2}(T) \end{aligned}$$

The other inequality can be proved similarly. \square

Suppose that X is a compact metric space. Katok proved in [Kat80] that the lower and upper δ -entropies are equal and coincide with the measure theoretic entropy. In his proof the assumption of compactness for X is only used to show that $\overline{h}_\mu^\delta(T) \leq h_\mu(T)$. The other inequality is only based on the fact that $h_\mu(T)$ can be approximate by entropies $h_\mu(T, \mathcal{P})$, with respect to partitions satisfying $\mu(\partial\mathcal{P}) = 0$, where $\partial\mathcal{P}$ is the union of the boundaries of the elements of \mathcal{P} . The validity of this inequality in the noncompact case has been also remarked in [GK01] to compute the topological entropy of the geodesic flow in the modular surface.

Theorem 2.5 (Katok). *Let X be a complete metric space and let $T : X \rightarrow X$ be a continuous transformation. If μ is an ergodic T -invariant probability measure, then for all $0 < \delta < 1$, we have*

$$h_\mu(T) \leq \underline{h}_\mu^\delta(T).$$

2.3. Local entropies of Brin-Katok. The aim of this subsection is to understand some relations between previous notions of entropy and local entropy. The notion of local entropy was introduced by Brin and Katok in [BK83].

Definition 2.6. The lower and upper *local entropies* of T , denoted respectively by $\underline{h}_\mu^{loc}(T)$ and $\overline{h}_\mu^{loc}(T)$, are defined as

$$\underline{h}_\mu^{loc}(T) = \operatorname{ess\,inf}_{x \in X} \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r))$$

and

$$\overline{h}_\mu^{loc}(T) = \operatorname{ess\,sup}_{x \in X} \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)).$$

Suppose that μ is an ergodic measure. The monotonicity of the functions $r \mapsto \liminf_n -\frac{1}{n} \log \mu(B_n(x, r))$ and $r \mapsto \limsup_n -\frac{1}{n} \log \mu(B_n(x, r))$, together with Birkhoff's Ergodic Theorem, imply the following:

Lemma 2.7. *Let X be a complete metric space and let $T : X \rightarrow X$ be a continuous transformation. If μ is an ergodic T -invariant probability measure on X , then*

$$\underline{h}_\mu^{loc}(T) = \int \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x)$$

and

$$\overline{h}_\mu^{loc}(T) = \int \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x).$$

As said above, when X is a compact metric space, the lower and upper local entropies coincide and are equal to the measure theoretic entropy. As in the case of δ -entropies, only one inequality in the proof requires the compactness assumption on X . Thus, in the general case following exactly the proof in [BK83], we obtain

Theorem 2.8 (Brin-Katok). *Let X be a complete metric space and let $T : X \rightarrow X$ be a continuous transformation. If μ is an ergodic T -invariant probability measure on X , then*

$$(2.1) \quad h_\mu(T) \leq \underline{h}_\mu^{loc}(T).$$

Following Ledrappier, we give sufficient conditions in order to get an equality in (2.1) for a large class of transformations defined on Riemannian manifolds.

Theorem 2.9. *Let $T : M \rightarrow M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T -invariant probability measure. Then*

$$h_\mu(T) = \underline{h}_\mu^{loc}(T).$$

Proof. As consequence of Theorem 2.8, we only need to prove that $h_\mu(T) \geq \underline{h}_\mu^{loc}(T)$. This follows from Proposition 2.10 below.

Proposition 2.10 (Ledrappier [Led13], Proposition 6.3). *Let $T : M \rightarrow M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T -invariant probability measure. Then, for every compact set $K \subset M$ such that $\mu(K) > 0$ and all $0 < r < 1$, there exists a partition \mathcal{P} of K with finite entropy such that, if $\mathcal{P} = \widehat{\mathcal{P}} \cup \{M \setminus K\}$, then for μ -almost every $x \in K$, there is a strictly increasing sequence of positive integers $(n_k)_{k \geq 0}$ such that*

$$\mathcal{P}^{n_k}(x) \subset B_{n_k}(x, r),$$

for all $k \geq 0$.

Let \mathcal{P} as in Proposition 2.10. Then, the partition \mathcal{P} has finite entropy and for μ -a.e. $x \in M$, we have

$$(2.2) \quad \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(x)) \geq \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)),$$

Observe that the left-hand side of inequality (2.2) is μ -almost everywhere a limit because of Shannon-McMillan-Breiman Theorem. By Lemma 2.7, for every $\varepsilon > 0$ there exists $r_0 > 0$ such that for all $0 < r < r_0$, we have

$$\int \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x) \geq \underline{h}_\mu^{loc}(T) - \varepsilon.$$

Using inequality (2.2), together with Shannon-McMillan-Breiman Theorem, we obtain

$$\begin{aligned} h_\mu(T) &\geq h_\mu(T, \mathcal{P}) = \int \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu(\mathcal{P}^n(x)) d\mu(x) \\ &\geq \int \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mu(B_n(x, r)) d\mu(x) \geq \underline{h}_\mu^{loc}(T) - \varepsilon. \end{aligned}$$

Hence, the desired inequality follows when $\varepsilon \rightarrow 0$. □

Remark 2.11. *The sequence $(n_k)_k$ of positive integers given in Proposition 2.10 is by construction the strictly increasing sequence of return times in K , which is μ -a.e. infinite as a result of Poincaré Recurrence Theorem.*

Lemma 2.12. *Let $T : M \rightarrow M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T -invariant probability measure. Then*

$$\sup_K \operatorname{ess\,sup}_{x \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mu(B_n(x, r)) \leq h_\mu(T).$$

Proof. Since T is Lipschitz, Proposition 2.10, Remark 2.11 and Shannon-McMillan-Breiman Theorem imply that, for every compact set $K \subset M$ such that $\mu(K) > 0$, and every $0 < r < 1$, we have

$$\limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mu(B_n(x, r)) \leq h_\mu(T),$$

for μ -a.e. $x \in K$. This implies the desired inequality. \square

To end this subsection we give a relation between the upper δ -entropy and the upper local entropy in a general setting.

Theorem 2.13. *Let $T : X \rightarrow X$ be a continuous transformation of a metric space (X, d) and μ an ergodic T -invariant probability measure. Then for all $0 < \delta < 1$, we have*

$$\overline{h}_\mu^\delta(T) \leq \overline{h}_\mu^{\operatorname{loc}}(T).$$

Proof. Fix $\varepsilon > 0$ and $0 < r < 1$. Define the set $X(\varepsilon, r, n') \subset X$, for $n' \geq 1$, by

$$X(\varepsilon, r, n') = \left\{ x \in X : -\frac{1}{n} \log \mu(B_n(x, r)) \leq \overline{h}_\mu^{\operatorname{loc}}(T) + \varepsilon, \text{ for all } n \geq n' \right\}.$$

Note that $\mu(X(\varepsilon, r, n'))$ goes to 1 when $n' \rightarrow \infty$ and $r \rightarrow 0$. Take $r > 0$ small enough and $n'_0 = n'_0(r) > 0$ large enough such that $\mu(X(\varepsilon, r, n')) > 1 - \delta$ for every $n' \geq n'_0$. Let $K \subset X(\varepsilon, r, n'_0)$ be a compact set such that $\mu(K) > 1 - \delta$. We are going to find an upper bound of $S(n, r, K)$ (the maximal cardinality of a (n, r) -separated subset of K), for every $n \geq n'_0$. Let E be a maximal (n, r) -separated set in K . Since $(n, r/2)$ -balls centered at E are disjoint, we have

$$\sum_{x \in E} \mu(B_n(x, r)) = \mu \left(\bigcup_{x \in E} B_n(x, r/2) \right) \leq 1.$$

As the (n, r) -balls with center in K satisfy $\mu(B_n(x, r)) \geq \exp \left(-n \left(\overline{h}_\mu^{\operatorname{loc}}(T) + \varepsilon \right) \right)$, it follows that

$$\#E = S(n, r, K) \leq \exp \left(n \left(\overline{h}_\mu^{\operatorname{loc}}(T) + \varepsilon \right) \right).$$

Therefore, by Lemma 2.2, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, K) &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log S(n, r, K) \\ &\leq \overline{h}_\mu^{\operatorname{loc}}(T) + \varepsilon. \end{aligned}$$

Hence, for every $r > 0$ small enough we can find a compact $K \subset X$ such that $\mu(K) > 1 - \delta$ and

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, K) \leq \overline{h}_\mu^{\operatorname{loc}}(T) + \varepsilon.$$

In particular,

$$\begin{aligned} \overline{h}_\mu^\delta(T) &= \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta) \\ &\leq \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, K) \\ &\leq \overline{h}_\mu^{\text{loc}}(T) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the conclusion follows. \square

3. EXPONENTIAL DECAY OF THE VOLUME OF DYNAMICAL BALLS

This section is divided in two parts. In the first part we define the Riemannian local entropy $h_\mu^{\mathcal{L}}(T)$ and we compare it with the classical measure entropy. In the second part, inspired by this property, we will show that the sum of positive Lyapunov exponents equals the exponential decay of the volume of dynamical balls for the dynamic of df on TM .

3.1. A Riemannian local entropy for Riemannian manifolds. Our goal is to define a local entropy of a measure μ measuring the Riemannian measure of a μ -typical dynamical ball. Moreover, we want to be able to compare it with the classical entropy. It turns out that the essential supremum of the exponential decay for the Riemannian measure of μ -typical dynamical balls is an interesting quantity to consider.

Definition 3.1. Let $T : M \rightarrow M$ be a continuous transformation of a Riemannian manifold, preserving an ergodic T -invariant probability measure μ . For every compact set $K \subset M$, we define the *local Riemannian entropy of T with respect to μ over K* , denoted by $h_\mu^{\mathcal{L}}(T, K)$, as

$$h_\mu^{\mathcal{L}}(T, K) = \text{ess sup}_{x \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mathcal{L}(B_n(x, r)),$$

where the essential supremum is with respect to the measure μ . We define the *local Riemannian entropy of T with respect to μ* , denoted by $h_\mu^{\mathcal{L}}(T)$, as

$$h_\mu^{\mathcal{L}}(T) = \sup_K h_\mu^{\mathcal{L}}(T, K),$$

where the supremum is taken over all the compact subsets of M .

The following theorem shows that the lower δ -entropy of Katok is bounded from above by the local Riemannian entropy. We stress the fact that the measure μ only appears in the definition of local Riemannian entropy when considering μ -typical dynamical balls. We don't know related results in the literature.

Theorem 3.2. *Let $T : M \rightarrow M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T -invariant probability measure μ . If $K \subset M$ is a compact set of strictly positive μ -measure, then for all $1 - \mu(K)^2 < \delta < 1$, we have*

$$\underline{h}_\mu^\delta(T) \leq h_\mu^{\mathcal{L}}(T, K).$$

Proof. If $h_\mu^{\mathcal{L}}(T, K) = \infty$ there is nothing to prove. Suppose that $h_\mu^{\mathcal{L}}(T, K) < \infty$. For $\varepsilon > 0$, $r > 0$ and $n' \geq 1$, we define the set $K_{\varepsilon, r, n'}$ as

$$K_{\varepsilon, r, n'} = \{x \in K : \mathcal{L}(B_n(x, r)) \geq \exp(-n(h_\mu^{\mathcal{L}}(T, K) + \varepsilon)), \text{ for every } n \geq n' \\ \text{such that } T^n x \in K\}.$$

Note that the measure $\mu(K_{\varepsilon, r, n'})$ goes to $\mu(K)$ when $n' \rightarrow \infty$ and $r \rightarrow 0$. For all $0 < \eta < \mu(K)/2$ there exist $r > 0$ and $n'_0 \geq 1$ (depending on r) such that $\mu(K_{\varepsilon, r, n'_0}) > \mu(K) - \eta/2$. Let $K_0 \subset K_{\varepsilon, r, n'_0}$ be a compact set with measure $\mu(K_0) > \mu(K) - \eta$. We are going to estimate the cardinality of a minimal (n, r) -covering of K_0 for $n \geq n'_0$. The problem is that in general, if $x, x' \in K$ are different, the first time of return in K is also different for these two points. The ergodicity assumption for the dynamical system will help us to erase this problem.

Birkhoff's Ergodic Theorem implies that $\frac{1}{n} \sum_{i=0}^{n-1} \mu(K_0 \cap T^{-i} K_0)$ converge to $\mu(K_0)^2$. In particular, there is a sequence $(\phi(n))_n$ strictly increasing of integers such that $\mu(K_0 \cap T^{-\phi(n)} K_0)$ converge to $L(K_0) \geq \mu(K_0)^2$. Let $0 < \lambda < L(K_0)/2$. Then, there is an integer $n_1 \geq 1$ such that $\mu(K_0 \cap T^{-\phi(n)} K_0) > L(K_0) - \lambda$ for all $n \geq n_1$. Let $\delta(K_0, \lambda) = 1 - (\mu(K_0)^2 - \lambda)$ and set $K_{\phi(n)} = K_0 \cap T^{-\phi(n)} K_0$. The μ -measure of $K_{\phi(n)}$ satisfies, for all $n \geq n_1$

$$\mu(K_{\phi(n)}) > L(K_0) - \lambda \geq \mu(K_0)^2 - \lambda = 1 - \delta(K_0, \lambda).$$

Let E be a maximal set $(\phi(n), r)$ -separated in $K_{\phi(n)}$, for $n \geq \max\{n_0, n_1\}$. Then

$$\begin{aligned} \mathcal{L}(V_r(K)) &\geq \mathcal{L}\left(\bigcup_{x \in E} B_{\phi(n)}(x, r/2)\right) \\ &\geq \sum_{x \in E} \mathcal{L}(B_{\phi(n)}(x, r/2)) \\ &\geq \#E \exp(-n(h_\mu^{\mathcal{L}}(T, K) + \varepsilon)). \end{aligned}$$

Therefore, the cardinality of E is bounded from above by

$$\#E \leq \mathcal{L}(V_r(K)) \exp(n(h_\mu^{\mathcal{L}}(T, K) + \varepsilon)).$$

Hence, using Lemma 2.2 and the estimation from above of the cardinality of a maximal set $(\phi(n), r)$ -separated in $K_{\phi(n)}$, we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta(K_0, \lambda)) &\leq \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log N(\phi(n), r, \delta(K_0, \lambda)) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log N(\phi(n), r, K_{\phi(n)}) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{\phi(n)} \log S(\phi(n), r, K_{\phi(n)}) \\ &\leq h_\mu^{\mathcal{L}}(T, K) + \varepsilon. \end{aligned}$$

In particular, we have shown that for every $r > 0$ small enough and every n large enough (depending on r), there exists a compact set $K_{\phi(n)}$ of μ -measure $\mu(K_{\phi(n)}) \geq \delta(K_0, \lambda)$. Therefore, the sequence of inequalities above implies that

$$\begin{aligned} \underline{h}_\mu^{\delta(K_0, \lambda)}(T) &= \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log N(n, r, \delta(K_0, \lambda)) \\ &\leq h_\mu^{\mathcal{L}}(T, K) + \varepsilon. \end{aligned}$$

Since $\lambda > 0$ is arbitrary and from Proposition 2.4, we have $\underline{h}_\mu^\delta(T) \leq h_\mu^\mathcal{L}(T, K) + \varepsilon$ for all $1 - \mu(K_0)^2 < \delta < 1$. Since $\eta > 0$ is arbitrary, we have $\underline{h}_\mu^\delta(T) \leq h_\mu^\mathcal{L}(T, K) + \varepsilon$, for all $1 - \mu(K)^2 < \delta < 1$. Since $\varepsilon > 0$ is arbitrary, the conclusion of the theorem follows. \square

A direct consequence of Theorem 3.2, by choosing a sequence of compact sets $(K_n)_n$ such that $\mu(K_n) \rightarrow 1$ when $n \rightarrow 1$, is the following corollary.

Corollary 3.3. *Let $T : M \rightarrow M$ be a continuous transformation of a complete Riemannian manifold, preserving an ergodic T -invariant probability measure μ . Then, for all $0 < \delta < 1$, we have*

$$\underline{h}_\mu^\delta(T) \leq h_\mu^\mathcal{L}(T).$$

Finally we can prove Theorem 1.1.

Proof of Theorem 1.1. This is a consequence of Theorem 2.5 and Corollary 3.3. \square

Remark 3.4. *We consider the Riemannian measure in the definition of local Riemannian entropy because it will be useful in the proof of Theorem 1.3. However, we might also have considered the measure μ as in the case of local entropies. Hence, for all $0 < \delta < 1$, we have*

$$(3.1) \quad \underline{h}_\mu^\delta(T) \leq \sup_K \operatorname{ess\,sup}_{x \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ T^n x \in K}} -\frac{1}{n} \log \mu(B_n(x, r)).$$

Proof of Theorem 1.2. This is a consequence of Theorem 2.5 and inequality (3.1). \square

In Theorem 2.9 we established that the measure theoretic entropy and the lower local entropy, of a Lipschitz transformation defined on complete Riemannian manifolds, coincide. We prove now the analogue result for the measure theoretic entropy and the lower δ -entropies.

Theorem 3.5. *Let $T : M \rightarrow M$ be a Lipschitz transformation of a complete Riemannian manifold and μ an ergodic T -invariant probability measure. Then, for all $0 < \delta < 1$, we have*

$$h_\mu(T) = \underline{h}_\mu^\delta(T).$$

Proof. On the one hand, Theorem 2.5 implies that $h_\mu(T) \leq \underline{h}_\mu^\delta(T)$. On the other hand, Inequality (3.1) and Lemma 2.12 imply that $\underline{h}_\mu^\delta(T) \leq h_\mu(T)$. \square

3.2. Lyapunov exponents and exponential decay of linearized dynamical balls. Now we will study the dynamics of a smooth transformation of a noncompact Riemannian manifold. Let (M, g) be a Riemannian manifold and $f : M \rightarrow M$ a C^1 -diffeomorphism. For $x \in M$, let $\|\cdot\|_x$ denote the Riemannian norm induced by g on $T_x M$. The point x is said to be (Lyapunov-Perron) *regular* if there exist numbers $\{\lambda_i(x)\}_{i=1}^{l(x)}$, called *Lyapunov exponents*, and a decomposition of the tangent space at x into $T_x M = \bigoplus_{i=1}^{l(x)} E_i(x)$ such that for every tangent vector $v \in E_i(x) \setminus \{0\}$, we have

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|d_x f^n v\|_{f^n x} = \lambda_i(x).$$

Let Λ be the set of regular points. By Oseledec's Theorem ([Ose68], [Led84b]), if μ is an f -invariant probability measure on M such that $\log \|df^{\pm 1}\|$ is μ -integrable,

then the set Λ has full μ -measure. Moreover, the functions $x \mapsto \lambda_j(x)$ and $x \mapsto \dim(E_j(x))$ are μ -measurable and f -invariant. In particular, if μ is ergodic, they are μ -almost everywhere constant. In that case, we denote by $\{\lambda_j\}_{j=1}^l$ the Lyapunov exponents. Note that for μ -a.e. $x \in M$, for every $v \in T_x M \setminus \{0\}$ the limit $\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\|_{f^n x}$ exists. More precisely, if $v = \sum_j v_j$ is the Oseledec's decomposition of v , then

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|d_x f^n(v)\|_{f^n x} =: \lambda(x, v)$$

is the largest Lyapunov exponent associated to a vector $v_j \neq 0$.

Let $x \in \Lambda$. Define $E^{su}(x)$ as

$$E^{su}(x) = \bigoplus_{\lambda_j(x) > 0} E_j(x).$$

Consider the function $\chi^+ : M \rightarrow \mathbb{R}$ defined as $\chi^+(x) = \sum_{\lambda_j(x) > 0} \lambda_j(x) \dim(E_j(x))$ if $x \in \Lambda$ and $\chi^+(x) = 0$ otherwise. If μ is an ergodic f -invariant probability measure on M , we denote by $\chi^+(\mu)$ (or simply χ^+ when there is no ambiguity) the essential value of the function $\chi^+(x)$ with respect to μ .

Denote by $B_x(0, r)$ the r -ball centered at 0 in $(T_x M, g_x)$.

Definition 3.6. We define the *tangent (n, r) -dynamical ball* $\mathcal{C}(x, n, r)$ on $T_x M$ as

$$\mathcal{C}(x, n, r) = \bigcap_{i=0}^{n-1} (d_x f^i)^{-1}(B_{f^i x}(0, r)) \subset T_x M.$$

Let vol_x be the Euclidean volume on $T_x M$ induced by g_x . Theorem 3.7 below says that the Lyapunov exponents describe the exponential decay of the vol_x -volume of a tangent dynamical ball, as follows

Theorem 3.7. *Let (M, g) be a Riemannian manifold and $f : M \rightarrow M$ a C^1 -diffeomorphism. Suppose that μ is an f -invariant probability measure such that $\log \|df^{\pm 1}\| \in L^1(\mu)$. Then, for μ -a.e. $x \in M$, we have*

$$\begin{aligned} \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \text{vol}_x(\mathcal{C}(x, n, r)) &= \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \text{vol}_x(\mathcal{C}(x, n, r)) \\ &= \chi^+(x). \end{aligned}$$

Before giving a proof of Theorem 3.7, we need the following technical lemma.

Lemma 3.8. *Let (M, g) be a Riemannian manifold of dimension d and $f : M \rightarrow M$ a C^1 -diffeomorphism. Suppose that μ is an ergodic f -invariant probability measure such that $\log \|df^{\pm 1}\| \in L^1(\mu)$. Then, for every $\varepsilon > 0$, there exists a compact set $K \subset M$ such that for every $x \in K$ there is a sequence $(t_n)_n$ of strictly positive real numbers such that*

- (1) the μ -measure of K is greater than $1 - \varepsilon$,
- (2) the exponential decay of t_n is smaller than 2ε , i.e.

$$\limsup -\frac{1}{n} \log t_n \leq 2\varepsilon,$$

(3) for every $r > 0$ there exist constants $C, C' > 0$ (depending on K, r and l) such that, for every integer $n \geq 0$ and every $x \in K$, we have

$$(3.2) \quad \text{vol}_x(\mathcal{C}(x, n, r)) \geq Ct_n^d \prod_{\lambda_j > 0} \exp(-n \dim(E_j(x))(\lambda_j + \varepsilon))$$

and

$$(3.3) \quad \text{vol}_x(\mathcal{C}(x, n, r)) \leq C' \prod_{\lambda_j > 0} \exp(-n \dim(E_j(x))(\lambda_j - \varepsilon)).$$

Proof. Let $\varepsilon > 0$ be such that $\varepsilon \leq \min\{|\lambda_j| : \lambda_j \neq 0\}/100$. For every integer $k \geq 1$, define the set $M_{\varepsilon, k}$ as

$$M_{\varepsilon, k} = \{x \in \Lambda : \forall v \in T_x^1 M, \forall |i| \geq k, \exp(i(\lambda(x, v) - \varepsilon)) \leq \|d_x f^i v\| \leq \exp(i(\lambda(x, v) + \varepsilon))\}.$$

Oseledec's Theorem implies that $\mu(M \setminus M_{\varepsilon, k})$ goes to 0 when k goes to infinity. In particular, there exists $k_0 \geq 1$ such that $\mu(M \setminus M_{\varepsilon, k_0}) \leq \varepsilon/2$. Since μ is a Borel measure, there is a compact $K \subset M_{\varepsilon, k_0}$ such that $\mu(K) > 1 - \varepsilon$, $\int_{M \setminus K} \log^+ \|df^{\pm 1}\| d\mu < \varepsilon$ and the maps $x \mapsto E_j(x)$ are continuous over K . For $x \in K$ and $n \geq 0$, define the sets of integers $I_{x, n}$ and $I_{x, n}^c$ as

$$I_{x, n} = \{k_0 \leq i \leq n : f^i x \in K\}$$

and

$$I_{x, n}^c = \{0 \leq i \leq n : f^i x \notin K\}.$$

From the definition of a tangent dynamical ball, we have

$$\mathcal{C}(x, n, r) = \mathcal{C}_K(x, n, r) \cap \mathcal{C}_{K^c}(x, n, r),$$

where

$$\begin{aligned} \mathcal{C}_K(x, n, r) &= \bigcap_{i \in I_{x, n}} (d_{f^i x} f^{-i})(B_{f^i x}(0, r)), \quad \text{and} \\ \mathcal{C}_{K^c}(x, n, r) &= \bigcap_{i \in I_{x, n}^c} (d_{f^i x} f^{-i})(B_{f^i x}(0, r)). \end{aligned}$$

To estimate the volume $\text{vol}_x(\mathcal{C}(x, n, r))$ we will use that

$$(3.4) \quad (d_x f^i)(B_x(0, r)) \subset B_{f^i x}(0, \|d_x f^i\| r) \subset T_{f^i x} M,$$

and

$$(3.5) \quad B_x(0, \|d_x f^i\|^{-1} r) \subset (d_{f^i x} f^{-i})(B_{f^i x}(0, r)).$$

By continuity of the maps $x \mapsto E_j(x)$ on K , we can assume that there exists $\alpha > 0$ such that for every $x \in K$ and every pair (j_1, j_2) , with $j_1 \neq j_2$, we have

$$(3.6) \quad \angle(E_{j_1}(x), E_{j_2}(x)) \geq \alpha.$$

Part I: Upper bound of the volume. Inequality (3.3) follows directly from $\mathcal{C}(x, n, r) \subset \mathcal{C}_K(x, n, r)$. Let $w = (d_{f^i x} f^{-i})v$, where $v \in B_{f^i x}(0, r) \cap E_j(f^i x)$, $1 \leq j \leq s$ and $i \in I_{x, n}$. Using (3.4), we have

$$\begin{aligned} \|w\| &= \|d_{f^i x} f^{-i} v\| \leq \exp(-i(\lambda(f^i x, v) - \varepsilon)) \|v\| \\ &= \exp(-i(\lambda_j - \varepsilon)) \|v\|. \end{aligned}$$

Let $v \in B_z(0, r)$, for $z \in K$, and let $v = \sum_j v_j$ be the Oseledec's decomposition of v . The *law of sines* implies

$$\|v_j\|_z \leq \frac{\|v\|_z}{\sin(\alpha)}.$$

Therefore,

$$\begin{aligned} (d_{f^i x} f^{-i})(B_{f^i x}(0, r)) &\subseteq (d_{f^i x} f^{-i}) \left(\prod_{j=1}^l B_{f^i x}^j(0, r/\sin(\alpha)) \right) \\ &\subseteq \prod_{j=1}^l (d_{f^i x} f^{-i})(B_{f^i x}^j(0, r/\sin(\alpha))) \\ &\subseteq \left(\prod_{\lambda_j \leq 0} B^j(0, r/\sin(\alpha)) \right) \times \\ &\quad \left(\prod_{\lambda_j > 0} B^j(0, \exp(-i(\lambda_j - \varepsilon))r/\sin(\alpha)) \right). \end{aligned}$$

The last term above is a parallelepiped (of dimension d). His volume is comparable to the volume of a rectangular parallelepiped whose sides have the same lengths that the original one. The constant of comparison depends only on the angle of the sides, which is greater than α by (3.6). Therefore, there exists a constant $\tilde{C}' = \tilde{C}'(\alpha) > 0$, such that

$$\text{vol}_x(\mathcal{C}(x, n, r)) \leq \tilde{C}'(r/\sin(\alpha))^d \prod_{\lambda_j > 0} \exp(-n \dim(E_j(x))(\lambda_j - \varepsilon)).$$

In particular, Inequality (3.3) is deduced for $C' = \tilde{C}'(r/\sin(\alpha))^d$.

Part II: Lower bound of the volume. As we have no control on the behavior of the differential on $M \setminus K$, we need to reduce the estimation problem of the volume of $\mathcal{C}(x, n, r)$ to a problem of estimation of the volume of $\mathcal{C}_K(x, n, r)$. For $i \in I_{x,n}$, define $j(i)$ as the number of consecutive indices greater than i belonging to $I_{x,n}^c$, that is,

$$j(i) = \begin{cases} \max\{j \geq 1 : i + m \in I_{x,n}^c, \forall 1 \leq m \leq j\}, & \text{if } i + 1 \in I_{x,n}^c \\ 0, & \text{otherwise.} \end{cases}$$

Let $i \in I_{x,n}$ and suppose $j(i) \geq 1$. By definition, we have $i + m \in I_{x,n}^c$ for every $1 \leq m \leq j(i)$ and $i + j(i) + 1 \in I_{x,n}$. Using inclusion (3.5), it follows

$$\begin{aligned} d_{f^{i+m} x} f^{-(i+m)}(B_{f^{i+m} x}(0, r)) &= d_{f^i x} f^{-i} d_{f^{i+m} x} f^{-m}(B_{f^{i+m} x}(0, r)) \\ &\supseteq d_{f^i x} f^{-i} B_{f^i x}(0, \|d_{f^i x} f^m\|^{-1} r) \\ &\supseteq d_{f^i x} f^{-i} B_{f^i x} \left(0, \left(\prod_{m'=0}^{m-1} \min\{1, \|d_{f^{i+m'} x} f\|^{-1}\} \right) r \right) \\ &\supseteq d_{f^i x} f^{-i} B_{f^i x} \left(0, \left(\prod_{k \in I_{x,n}^c} \min\{1, \|d_{f^k x} f\|^{-1}\} \right) r \right). \end{aligned}$$

The above sequence of inclusions implies that every set of the form $d_{f^k x} f^{-k}(B_{f^k x}(0, r))$, for $k \in I_{x,n}^c$, contains a set of the form $(d_{f^i x} f^{-i})(B_{f^i x}(0, t_n r))$, where $i \in I_{x,n} \cup \{0\}$ and $t_n = \prod_{k \in I_{x,n}^c} \min\{1, \|d_{f^{k-1} x} f\|^{-1}\}$. Thus,

$$(3.7) \quad \mathcal{C}(x, n, r) \supset \mathcal{C}_K(x, n, t_n r).$$

Claim (2) follows from Birkhoff's Ergodic Theorem. More precisely,

$$\begin{aligned} -\frac{1}{n} \log t_n &= -\frac{1}{n} \sum_{k \in I_{x,n}^c} \log \min\{1, \|d_{f^k x} f\|^{-1}\} \\ &\leq -\frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}_{K^c}(f^k x) \log \min\{1, \|d_{f^k x} f\|^{-1}\} \\ &\leq \int_{M \setminus K} \log^+ \|df\| d\mu + \varepsilon \\ &\leq 2\varepsilon \end{aligned}$$

for every n large enough.

Let $i \in I_{x,n}$. For $w = (d_{f^i x} f^{-i})v$, where $v \in B_{f^i x}(0, r) \cap E_j(f^i x)$ and $1 \leq j \leq l$, we have $w \in E_j(x)$. In particular, by definition of M_{ε, k_0} , we obtain for $i \geq k_0$ the following

$$(3.8) \quad \begin{aligned} \|w\| = \|d_{f^i x} f^{-i} v\| &\geq \exp(-i(\lambda(f^i x, v) + \varepsilon)) \|v\| \\ &= \exp(-i(\lambda_j + \varepsilon)) \|v\|. \end{aligned}$$

Consider now $B_x^j(0, r)$ the r -ball centered at 0 on $E_j(x)$, for the norm $g_x|_{E_j(x)}$, for all $1 \leq j \leq l$. Thus, using (3.8), it follows

$$\begin{aligned} (d_{f^i x} f^{-i})(B_{f^i x}(0, r)) &\supseteq (d_{f^i x} f^{-i}) \left(\prod_{j=1}^l B_{f^i x}^j(0, r/l) \right) \\ &= \prod_{j=1}^l (d_{f^i x} f^{-i})(B_{f^i x}^j(0, r/l)) \\ &\supset \prod_{j=1}^l B_x^j(0, \exp(-i(\lambda_j + \varepsilon))r/l) \\ &\supset \left(\prod_{\lambda_j \leq 0} B_x^j(0, r/l) \right) \times \left(\prod_{\lambda_j > 0} B_x^j(0, \exp(-i(\lambda_j + \varepsilon))r/l) \right). \end{aligned}$$

The same arguments of Part I for the upper bound of the volume allow to show the existence of a constant $\tilde{C} = \tilde{C}(\alpha) > 0$ such that

$$\text{vol}_x \left(\prod_{\lambda_j \leq 0} B_x^j(0, r/l) \times \prod_{\lambda_j > 0} B_x^j(0, \exp(-i(\lambda_j + \varepsilon))r/l) \right)$$

is greater than

$$\tilde{C}(r/l)^d \prod_{\lambda_j > 0} \exp(-n \dim(E_j(x))(\lambda_j + \varepsilon)).$$

The inequality above together (3.7) implies (3.2) for $C = \tilde{C}(r/l)^d$. It concludes the proof of Lemma 3.8. \square

We will use Lemma 3.8 to prove Theorem 3.7 for ergodic measures. This is sufficient because of the ergodic decomposition of a measure.

Proof of Theorem 3.7. Suppose that μ is an ergodic measure and let $\varepsilon > 0$ be small enough as in Lemma 3.8. Let $K = K(\varepsilon, k_0)$ be the compact given by Lemma 3.8. Then, for every $x \in K$, we have

$$\begin{aligned} \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \text{vol}_x(\mathcal{C}(x, n, r)) &\leq \limsup_{n \rightarrow \infty} -\frac{1}{n} \log t_n + \sum_{\lambda_j > 0} \dim(E_j(x))(\lambda_j + \varepsilon) \\ &\leq 2d\varepsilon + \sum_{\lambda_j > 0} \dim(E_j(x))(\lambda_j + \varepsilon) \end{aligned}$$

and

$$\lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \text{vol}_x(\mathcal{C}(x, n, r)) \geq \sum_{\lambda_j > 0} \dim(E_j(x))(\lambda_j - \varepsilon).$$

Since we can choose $K = K(\varepsilon)$ such that $\mu(K) \rightarrow 1$ when $\varepsilon \rightarrow 0$, the conclusion of Theorem 3.7 follows. \square

Let $x \in M$. Denote by $r_{inj}(x)$ the injectivity radius at x . For $0 < r \leq r_{inj}(x)$ we define the *linearized* (n, r) -dynamical ball as $\mathcal{C}_n(x, r) = \exp_x(\mathcal{C}(x, n, r))$. Observe that $(\exp_x)_* \text{vol}_x$ is locally comparable with the Riemannian measure \mathcal{L} , therefore Theorem 3.7 implies

Corollary 3.9. *Let M be a Riemannian manifold and let $f : M \rightarrow M$ be a C^1 -diffeomorphism. Suppose that μ is an f -invariant probability measure such that $\log \|df^{\pm 1}\| \in L^1(\mu)$. Then, for μ -a.e. $x \in M$, we have*

$$\begin{aligned} \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathcal{L}(\mathcal{C}_n(x, r)) &= \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mathcal{L}(\mathcal{C}_n(x, r)) \\ &= \chi^+(x). \end{aligned}$$

Theorem 1.1 and Corollary 3.9 are a first step in order to prove Ruelle's inequality for "nice" diffeomorphisms of noncompact manifolds. Heuristically speaking, if a dynamical ball $B_n(x, r)$ is comparable with the linearized dynamical ball $\mathcal{C}_n(x, r)$, then Ruelle's inequality should arise in a natural way since the limits in Corollary 3.9 look like the Riemannian local entropy.

4. GEODESIC FLOW IN NEGATIVE CURVATURE

Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Let T^1X his unit tangent bundle. Recall that the Liouville measure \mathcal{L} on T^1X is the Riemannian volume of the Sasaki metric on T^1X (see for instance [Bal95] for details). It is invariant under the action of the geodesic flow (g^t) on T^1X . Let $v \in T^1X$ and $t \in \mathbb{R}$. Denote by $E^{su}(v)$ the tangent

space of the strong unstable manifold at v . Denote by $J^{su}(v, t)$ the Jacobian of the linear map $d_v g^t|_{E^{su}(v)}$. The *geometric potential* $F^{su} : T^1X \rightarrow \mathbb{R}$ is then defined by

$$F^{su}(v) = -\frac{d}{dt}\Big|_{t=0} \log J^{su}(v, t).$$

Theorem 4.1 (Paulin-Pollicott-Schapira). *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then F^{su} is Hölder-continuous and bounded.*

The potential F^{su} is intimately related to the Lyapunov exponents. Let μ be a probability measure on T^1X invariant under the geodesic flow (g^t). Since the sectional curvature is pinched, the norm $\|dg^{\pm 1}\|$ is bounded. Hence, $\log \|dg^{\pm 1}\|$ is μ -integrable. Oseledec's Theorem implies that μ -almost every $v \in T^1X$ is regular. In particular, for μ -almost every $v \in T^1X$, the tangent space of the strong unstable manifold at v coincides with $\bigoplus_{\lambda_j(v) > 0} E_j(v)$. This fact justifies the notation $E^{su}(v)$ for the direct sum of the spaces $E_j(v)$ associated to $\lambda_j(v) > 0$. Moreover, we have

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \int_0^n F^{su}(g^t v) dt = -\lim_{n \rightarrow +\infty} \frac{1}{n} \log J^{su}(v, n) = -\chi^+(v).$$

The key fact that will allow us to prove Ruelle's inequality for the geodesic flow is the Gibbs property of the Liouville measure for the potential F^{su} ([PPS12, Proposition 7.9]). Recall that a (g^t) -invariant measure m on T^1X satisfies the Gibbs property for the potential $F : T^1X \rightarrow \mathbb{R}$ with constant $c(F)$ if and only if for every compact subset K of T^1X , for every $r > 0$, there exists $C = C(K, r) \geq 1$, such that for every $T \geq 0$, for every $v \in K \cap g^{-T}K$, we have

$$C^{-1} \leq \frac{m(B_T(v, r))}{\exp(\int_0^T (F(g^t v) - c(F)) dt)} \leq C.$$

Proposition 4.2 (Paulin-Pollicott-Schapira). *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then the Liouville measure on T^1X satisfies the Gibbs property for the potential F^{su} and the constant $c(F^{su}) = 0$.*

4.1. Ruelle's inequality and Pesin's formula. Let X be a complete Riemannian manifold satisfying the assumptions of Theorems 1.3 and 1.4. By simplicity we will always consider in the proofs an ergodic (g^t) -invariant probability measure μ . The proofs of the theorems in the non-ergodic case are consequence of the ergodic decomposition of such a measure. We can also assume that $g = g^1$ is an ergodic transformation with respect to μ . If it is not the case, then we can choose an ergodic-time τ for μ (see [LS79, Theorem 3.2]) and prove Theorem 1.3 and Theorem 1.4 for the map g^τ . The validity of Theorems 1.3 and 1.4 for g^τ implies the validity of both theorems for g since $h_\mu(g^\tau) = \tau h_\mu(g)$ and the Lyapunov exponents of g^τ are τ -multiples of the Lyapunov exponents for g .

Proof of Theorem 1.3. Let K be a subset of T^1X of measure $0 < \mu(K) < 1$. Since F^{su} is μ -integrable, Proposition 4.2 implies

$$\begin{aligned} h_\mu^{\text{vol}}(g, K) &= \operatorname{ess\,sup}_{v \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ g^n v \in K}} -\frac{1}{n} \log \mathcal{L}(B_n(v, r)) \\ &\leq \operatorname{ess\,sup}_{v \in K} \lim_{r \rightarrow 0} \limsup_{\substack{n \rightarrow \infty \\ g^n v \in K}} -\frac{1}{n} \log \left(C^{-1} \exp \left(\int_0^n F^{su}(g^t v) dt \right) \right) \\ &= - \int F^{su} d\mu \\ &= \chi^+. \end{aligned}$$

Last equality is a consequence of Birkhoff's Ergodic Theorem. Thus, $h_\mu^{\text{vol}}(g) \leq \chi^+$ and Theorem 1.1 implies directly that

$$h_\mu(g) \leq \chi^+.$$

□

The proof of Theorem 1.4 is similar to those in [LS82], [Led84a] and [LY85]. We only need to corroborate that all technical hypotheses hold. These technical hypotheses are consequences of the assumption on the derivatives of the sectional curvature since it implies the Hölder regularity of strong unstable and strong stable distributions (see for instance [PPS12, Theorem 7.3]). In [OP04] the authors use the regularity of the strong unstable foliation to prove the existence of nice measurable partitions. They follow the ideas in [LS82] and [Led84a] adapted to the geodesic flow in negative curvature. We remark that in [OP04] the authors use the Hölder regularity of strong unstable and strong stable foliations omitting the hypothesis on the derivatives of the sectional curvatures.

Recall that a measurable partition ξ of T^1X is subordinate to the W^{su} -foliation if for μ -a.e. $v \in T^1X$, we have

- (i) the atom $\xi(v)$ is contained in $W^{su}(v)$, and
- (ii) the atom $\xi(v)$ contains a neighborhood of v , open for the submanifold topology on $W^{su}(v)$.

Let vol_v be the volume on $W^{su}(v)$ induced by the Sasaki metric on T^1X restricted to the strong unstable manifold $W^{su}(v)$. The measure μ has absolutely continuous conditional measures on unstable manifolds if for every μ -measurable partition ξ subordinate to W^{su} , the conditional measure $\mu_{\xi(v)}$ of μ on $\xi(v)$ is absolutely continuous with respect to vol_v .

Proposition 4.3. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ an ergodic (g^t) -invariant probability measure and suppose that $g = g^1$ is ergodic. Then, there exists a μ -measurable partition ξ of T^1X , such that*

- (1) the partition ξ is decreasing, i.e. $(g^{-1}\xi)(v) \subset \xi(v)$ for μ -a.e. $v \in T^1X$,
- (2) the partition $\bigvee_{n \geq 0} g^{-n}\xi$ is the partition into points,
- (3) the partition ξ is subordinate to the W^{su} -foliation,
- (4) for μ -a.e. v , we have $\bigcup_{n \in \mathbb{Z}} g^n \xi(g^n v) = W^{su}(v)$,

(5) for all measurable sets $B \subset T^1X$, the map

$$\psi_B(v) = \text{vol}_v(\xi(v) \cap B)$$

is measurable and μ -a.e. finite,

(6) for μ -a.e. $v \in T^1X$, if $w, w' \in \xi(v)$, then the infinite product

$$\Delta(w, w') = \frac{\prod_{n=0}^{\infty} J^{su}(g^{-n}w, 1)}{\prod_{n=0}^{\infty} J^{su}(g^{-n}w', 1)}$$

converges, and

(7) there exist constants $C > 0$ and $0 < \alpha < 1$ such that, if $w \in \xi(v)$, then

$$|\log \Delta(v, w)| \leq C(d(v, w))^\alpha.$$

The existence of μ -measurable partitions satisfying (1) – (4) is proved in [OP04]. Properties (5) – (7) are consequence of the regularity of the strong unstable distribution and the regularity of J^{su} , following the same proof of [Led84a, Proposition 3.1].

The class of μ -measurable partitions satisfying (1) – (4) contains somehow all the complexity of the dynamics of the geodesic flow in the sense that every partition in this class maximises the measure theoretic entropy. This result is proved in [OP04] following the ideas in [Led84a] and [LY85].

Proposition 4.4 (Ledrappier-Young/Otal-Peigné). *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Let μ an ergodic (g^t)-invariant probability measure and suppose that $g = g^1$ is ergodic. If ξ is a partition as in Proposition 4.3, then*

$$h_\mu(g) = h_\mu(g, \xi).$$

Proof of Theorem 1.4. We remark that the computation of the entropy appears in [LS82], but as this fact is not stated explicitly, we give the general idea behind. Suppose that μ has absolutely continuous conditional measures on unstable manifolds. Let ξ be a μ -measurable partition as in Proposition 4.3. We only have to prove that $h_\mu(g, \xi) = \chi^+$. This is equivalent to show that $H_\mu(g^{-1}\xi|\xi) = \int \log J^{su}(v, 1) d\mu(v)$. Define the measure ν on T^1X by

$$\nu(B) = \int \text{vol}_w(\xi(w) \cap B) d\mu(w),$$

for every measurable subset B of T^1X . Property (5) in Proposition 4.3 implies that ν is σ -finite. Since $\mu_{\xi(v)}$ is absolutely continuous with respect to vol_v , the measure μ is absolutely continuous with respect to ν . Moreover, the Radon-Nikodym derivative $\kappa = d\mu/d\nu$ coincide with $d\mu_{\xi(v)}/d\text{vol}_v$, vol_v -almost everywhere on $\xi(v)$, for μ -almost every $v \in T^1X$ (see [LS82, Proposition 4.1]).

Recall that

$$H_\mu(g^{-1}\xi|\xi) = \int I_\mu(g^{-1}\xi|\xi) d\mu,$$

where $I_\mu(g^{-1}\xi|\xi)(v) = -\log \mu_{\xi(v)}((g^{-1}\xi)(v))$. Thus,

$$I_\mu(g^{-1}\xi|\xi)(v) = -\log \int_{(g^{-1}\xi)(v)} \kappa(w) d\text{vol}_v(w).$$

Using Change of Variables Theorem, it follows

$$\int_{(g^{-1}\xi)(v)} \kappa(w) d\text{vol}_v(w) = \int_{\xi(gv)} \kappa(g^{-1}w) \frac{1}{J^{su}(g^{-1}w, 1)} d\text{vol}_{g^v}(w).$$

From [LS82, Proposition 4.2], the application $L(w) = \frac{\kappa(w)}{\kappa(g^{-1}w)} J^{su}(g^{-1}w, 1)$ is constant on the atoms of the partition ξ . Therefore,

$$\begin{aligned} \int_{\xi(gv)} \kappa(g^{-1}w) \frac{1}{J^{su}(g^{-1}w, 1)} d\text{vol}_{g^v}(w) &= \int_{\xi(gv)} \frac{\kappa(w)}{L(w)} d\text{vol}_{g^v}(w) \\ &= \frac{1}{L(gv)} \int_{\xi(gv)} \kappa(w) d\text{vol}_{g^v}(w) \\ &= \frac{1}{L(gv)} \int_{\xi(gv)} d\mu_{\xi(gv)}(w) \\ &= \frac{1}{L(gv)}. \end{aligned}$$

Putting all together, we have shown that $I_\mu(g^{-1}\xi|\xi) = \log J^{su}(v, 1) + \log \frac{\kappa(gv)}{\kappa(v)}$. Since $I_\mu(g^{-1}\xi|\xi) \geq 0$ and $\log J^{su}(v, 1)$ is μ -integrable, it follows that the negative part of $\log \frac{\kappa(gv)}{\kappa(v)}$ is μ -integrable. In particular, its μ -integral is equal to zero (see [LS82, Proposition 2.2]), thus

$$h_\mu(g) = \int I_\mu(g^{-1}\xi|\xi) d\mu = \int \log J^{su}(v, 1) d\mu(v) = \chi^+.$$

The converse statement is just the conclusion of [Led84a, Theorem 3.4] under the hypothesis obtained in Proposition 4.4, for a μ -measurable partition ξ as in Proposition 4.3. \square

4.2. Commentaries. We discuss now some consequences of Theorem 1.3 in thermodynamic formalism. The *topological pressure of (g^t) for a potential $F : T^1X \rightarrow \mathbb{R}$* , denoted by $P_{(g^t)}(F)$ (or simply $P(F)$), is defined as

$$P(F) = \sup_{\mu} P(F, \mu),$$

where $P(F, \mu) = h_\mu((g^t)) + \int_{T^1X} F d\mu$ and μ is a (g^t) -invariant probability measure on T^1X . A (g^t) -invariant probability measure m on T^1X is said to be an *equilibrium state for F* , if

$$P(F) = P(F, m).$$

In [PPS12] the authors construct a Gibbs measure for every bounded Hölder-continuous potential F , with constant $c(F)$ equal to the topological pressure $P(F)$. We remark that if a Gibbs measure is finite, its normalization is an equilibrium state for the potential.

As a consequence of Theorem 4.1, there exists a Gibbs measure for F^{su} under the hypotheses of Theorem 1.3, which is denoted by $m_{F^{su}}$. In terms of thermodynamical formalism, Ruelle's inequality can be stated as:

Corollary 4.5. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1. Assume that the derivatives of the sectional curvature are uniformly bounded. Then, for every (g^t) -invariant probability measure μ on T^1X , we have*

$$(4.1) \quad P(F^{su}, \mu) \leq 0.$$

In particular, we can remove inequality (4.1) as a redundant assumption in [PPS12, Theorem 7.2], which gives us

Corollary 4.6. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1 . Assume that the derivatives of the sectional curvature are uniformly bounded. If the geodesic flow on T^1X is conservative with respect to the Liouville measure \mathcal{L} , then \mathcal{L} is proportional to the Gibbs measure $m_{F^{su}}$ associated to the geometric potential F^{su} . Furthermore, the topological pressure $P(F^{su})$ is equal to zero.*

In particular, we also have

Corollary 4.7. *Let X be a complete Riemannian manifold, with dimension at least 2 and pinched sectional curvature at most -1 . Assume that the derivatives of the sectional curvature are uniformly bounded. If X has finite volume, then*

$$\frac{m_{F^{su}}}{m_{F^{su}}(T^1X)} = \frac{\mathcal{L}}{\mathcal{L}(T^1X)}.$$

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