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# COMPUTING DISTANCES AND GEODESICS BETWEEN MANIFOLD-VALUED CURVES IN THE SRV FRAMEWORK

ALICE LE BRIGANT

ABSTRACT. This paper focuses on the study of open curves in a Riemannian manifold  $M$ , and proposes a reparametrization invariant metric on the space of such paths. We use the square root velocity function (SRVF) introduced by Srivastava et al. in [12] to define a Riemannian metric on the space of immersions  $\mathcal{M} = \text{Imm}([0, 1], M)$  by pullback of a natural metric on the tangent bundle  $T\mathcal{M}$ . This induces a first-order Sobolev metric on  $\mathcal{M}$  and leads to a distance which takes into account the distance between the origins in  $M$  and the  $L^2$ -distance between the SRV representations of the curves. The geodesic equations for this metric are given and exploited to define an exponential map on  $\mathcal{M}$ . Two possibilities are presented to effectively compute the optimal deformation of one curve into another – that is, the geodesic linking two elements of  $\mathcal{M}$  – geodesic shooting, and path straightening. The particular case of curves lying in the hyperbolic half-plane  $\mathbb{H}$  is considered as an example.

## 1. INTRODUCTION

Computing distances between shapes of open or closed curves is of interest in many applications, from medical imaging to radar detection, as soon as one wants to compare, classify or statistically analyze trajectories or contours of objects. While the shape of an organ or the trajectory of a car on a short distance can be modeled by a plane curve, some applications provide curves in a non-flat space : if we consider a trajectory on a larger scale, such as that of a hurricane or a ship, the curvature of the earth can no longer be ignored and the appropriate model is an  $S^2$ -valued curve. However trajectories are not restricted to the description of movement but can more generally represent the evolution of any given object – such as the covariance matrix of a Gaussian process – evolving in a certain space – the space of hermitian positive definite matrices. This motivates the study of curves lying in a general manifold, and more specifically the choice of a metric on the space of such curves.

Here we consider open oriented curves in a Riemannian manifold  $M$ , more precisely the space of immersions  $c : [0, 1] \rightarrow M$ ,

$$\mathcal{M} = \text{Imm}([0, 1], M).$$

One way to proceed is to equip  $\mathcal{M}$  with a Riemannian structure, that is to locally define a scalar product  $G$  on its tangent space  $T\mathcal{M}$ . A property that is usually required of this metric is reparametrization invariance, that is that the metric be the same at all points of  $\mathcal{M}$  representing curves that are identical modulo reparametrization. Two curves are identical modulo reparametrization when they pass through the same points of  $M$  but at different speeds. Reparametrizations are represented by increasing diffeomorphisms  $\phi : [0, 1] \rightarrow [0, 1]$  (so that they

preserve the end points of the curves), and their set is denoted by  $\text{Diff}^+([0, 1])$ . Elements  $h, k \in T_c\mathcal{M}$  of the tangent space in  $c \in \mathcal{M}$  are infinitesimal deformations of  $c$  and can be seen as vector fields along the curve  $c$  in  $M$ . The Riemannian metric  $G$  is reparametrization invariant if the action of  $\text{Diff}^+([0, 1])$  is isometric for  $G$

$$(1) \quad G_{c \circ \phi}(h \circ \phi, k \circ \phi) = G_c(h, k),$$

for any  $c \in \mathcal{M}$ ,  $h, k \in T_c\mathcal{M}$  and  $\phi \in \text{Diff}^+([0, 1])$ . This is often called the *equivariance property*, and it guarantees that the induced distance between two curves  $c_0$  and  $c_1$  does not change if we reparametrize them by the same diffeomorphism  $\phi$

$$d(c_0 \circ \phi, c_1 \circ \phi) = d(c_0, c_1).$$

What's more, a reparametrization invariant metric on the space of curves induces a Riemannian structure on the shape space, where the space of reparametrizations is quotiented out. A shape can be seen as the equivalence class of all the curves that are identical modulo a change of parameterization, and the shape space as the associated quotient space

$$\mathcal{S} = \text{Imm}([0, 1], M) / \text{Diff}^+([0, 1]).$$

We get a formal –the manifold structure of the space  $\text{Imm}([0, 1], M)$  has not yet been thoroughly studied to our knowledge – principal bundle structure  $\pi : \mathcal{M} \rightarrow \mathcal{S}$ , which induces a decomposition of the tangent bundle  $T\mathcal{M} = V\mathcal{M} \oplus H\mathcal{M}$  into a vertical subspace  $V\mathcal{M} = \ker(T\pi)$  consisting of all vectors tangent to the fibers of  $\mathcal{M}$  over  $\mathcal{S}$ , and a horizontal subspace  $H\mathcal{M} = (V\mathcal{M})^{\perp G}$  defined as the orthogonal complement of  $V\mathcal{M}$  according to the metric  $G$  that we put on  $\mathcal{M}$ . If  $G$  verifies the equivariance property, then it induces a Riemannian metric  $\hat{G}$  on the shape space, for which the geodesics are simply the projected horizontal geodesics of  $\mathcal{M}$  for  $G$ . The geodesic distances  $d$  on  $\mathcal{M}$  and  $\hat{d}$  on  $\mathcal{S}$  are then simply linked by

$$\hat{d}([c_0], [c_1]) = \inf \{ d(c_0, c_1 \circ \phi) \mid \phi \in \text{Diff}^+([0, 1]) \},$$

where  $[c_0]$  and  $[c_1]$  denote the shapes of two given curves  $c_0$  and  $c_1$ , and  $\hat{d}$  verifies the stronger property

$$\hat{d}(c_0 \circ \phi, c_1 \circ \psi) = \hat{d}(c_0, c_1),$$

for any reparametrizations  $\phi, \psi \in \text{Diff}^+([0, 1])$ . This motivates the choice of a reparametrization invariant metric on  $\mathcal{M}$ .

Riemannian metrics on the space of curves lying in a flat space have been widely studied ([8],[9],[14],[1]). The most natural candidate for a reparametrization invariant metric is the  $L^2$ -metric with integration over arc length  $d\ell = \|c'(t)\| dt$

$$G_c(h, k) = \int \langle h, k \rangle d\ell,$$

but Michor and Mumford have shown in [7] that the induced metric on the shape space always vanishes. This has motivated the study of Sobolev metrics ([9],[1],[2]), where higher order derivatives are introduced. One first-order Sobolev metric has proved particularly interesting for the applications ([4],[13])

$$(2) \quad G_c(h, k) = \int \langle D_\ell h^\perp, D_\ell k^\perp \rangle + \frac{1}{4} \langle D_\ell h^\parallel, D_\ell k^\parallel \rangle d\ell,$$

where  $c$  is a curve in a euclidean space  $\mathbb{R}^n$ ,  $\langle \cdot, \cdot \rangle$  denotes the euclidean metric on  $\mathbb{R}^n$ ,  $D_\ell h = \frac{1}{\|c'\|} h'$  is the derivation of  $h$  according to arc length,  $D_\ell h^\parallel = \langle D_\ell h, v \rangle v$

is the projection of  $D_\ell h$  on the unit tangent vector field  $v = \frac{1}{\|c'\|}c'$  to  $c$ , and  $D_s h^\perp = D_s h - D_s h^\parallel$ . Srivastava et al. introduced in [12] a simple framework to study this metric, by showing that it could be obtained by pullback of the  $L^2$ -metric via a simple transformation called the Square Root Velocity Function (SRVF) which associates to each curve its velocity renormalized by the square root of its norm. This framework can be extended to curves in a general manifold by using parallel transport, in a way which allows us to move the computations to the tangent plane to the origin of one of the two curves under comparison, see [5] and [15]. In [5] the transformation used is a generalization of the SRV function introduced by Bauer et al. in [1] as a tool to study a more general form of the Sobolev metric (2). In [15] a Riemannian framework is given, including the associated Riemannian metric and the geodesic equations.

In this paper we propose another generalization of the SRV framework to curves in a manifold. Instead of moving the computations to one tangent plane as in [5] and [15], the distance is computed in the manifold itself which enables us to take into account a greater amount of information on the space separating two curves. Therefore, we feel that the distance we introduce here will be more directly dependent on the geometry of the manifold. In the following section, we introduce our metric as the pullback of a quite natural metric on the tangent bundle  $T\mathcal{M}$ , and show that it induces a formal fiber bundle structure over the manifold  $M$  seen as the set of starting points of the curves of  $\mathcal{M}$ . In section 3, we give the induced geodesic distance and highlight the difference with respect to the distance introduced in [15]. In section 4, we give the geodesic equations associated to our metric and exploit them to build the exponential map. Geodesics of the space of curves can then be computed using either that exponential map and geodesic shooting, or by path straightening. The first method requires the characterization of Jacobi fields on  $\mathcal{M}$ . Finally, in section 5, we consider the special case of curves lying in the hyperbolic half-plane, for which we give some simple computational tools needed in the algorithms previously presented, as well as explicit formulas for the path straightening in that simpler setting.

## 2. EXTENSION OF THE SRV FRAMEWORK TO MANIFOLD-VALUED CURVES

**2.1. Our metric on the space of curves.** Let  $c : [0, 1] \rightarrow M$  be a curve in  $M$  and  $h, k \in T_c\mathcal{M}$  two infinitesimal deformations. We consider the following first-order Sobolev metric on  $\mathcal{M}$

$$G_c(h, k) = \langle h(0), k(0) \rangle + \int \langle \nabla_\ell h^\perp, \nabla_\ell k^\perp \rangle + \frac{1}{4} \langle \nabla_\ell h^\parallel, \nabla_\ell k^\parallel \rangle \, d\ell,$$

where we integrate according to arc length,  $\langle \cdot, \cdot \rangle$  denotes the Riemannian metric of the manifold  $M$ ,  $\nabla_\ell h = \frac{1}{\|c'\|} \nabla_{c'} h$  is the covariant derivative of  $h$  according to arc length, and  $\nabla_\ell h^\parallel = \langle D_\ell h, v \rangle v$  and  $\nabla_\ell h^\perp = \nabla_\ell h - \nabla_\ell h^\parallel$  are its tangential and normal components respectively. If  $M$  is a flat euclidean space, we obtain the metric (2) studied in [12], with an added term involving the origins. Without this extra term, the bilinear form  $G$  is not definite since we are no longer translation invariant as soon as the curves lie in a non-flat manifold. Here we show that  $G$  can be obtained as the pullback of a very natural metric  $\tilde{G}$  on the tangent bundle  $T\mathcal{M}$ . We consider the square root velocity function (SRVF, introduced in [12]) on

the space of curves in  $M$ ,

$$R : \mathcal{M} \rightarrow T\mathcal{M}, \quad c \mapsto \frac{c'}{\sqrt{\|c'\|}},$$

where  $\|\cdot\|$  is the norm associated to the Riemannian metric on  $M$ . In order to define  $\tilde{G}$ , we introduce the following projections from  $TTM$  to  $TM$ . Let  $\xi \in T_{(p,u)}TM$  and  $(x, U)$  be a curve in  $TM$  that passes through  $(p, u)$  at time 0 at speed  $\xi$ . Then we define the vertical and horizontal projections

$$\begin{aligned} \text{vp}_{(p,u)} & : T_{(p,u)}TM \rightarrow T_pM, & \xi & \mapsto \xi_V := \nabla_{x'(0)}U, \\ \text{hp}_{(p,u)} & : T_{(p,u)}TM \rightarrow T_pM, & \xi & \mapsto \xi_H := x'(0). \end{aligned}$$

The horizontal and vertical projections live in the tangent bundle  $TM$  and are not to be confused with the horizontal and vertical parts which live in the double tangent bundle  $TTM$  and will be denoted by  $\xi^H, \xi^V$ . Furthermore, let us point out that the horizontal projection is simply the differential of the natural projection  $TM \rightarrow M$ , and that according to these definitions, the Sasaki metric ([10], [11]) can be written

$$g_{(p,u)}^S(\xi, \eta) = \langle \xi_H, \eta_H \rangle + \langle \xi_V, \eta_V \rangle,$$

where  $\langle \cdot, \cdot \rangle$  denotes the Riemannian metric on  $M$ . Now we can define the metric that we put on  $T\mathcal{M}$ . Let us consider  $h \in T\mathcal{M}$  and  $\xi, \eta \in T_hT\mathcal{M}$ . We define

$$\tilde{G}_h(\xi, \eta) = \langle \xi(0)_H, \eta(0)_H \rangle + \int_0^1 \langle \xi(t)_V, \eta(t)_V \rangle dt,$$

where  $\xi(t)_H = \text{hp}(\xi(t))$  and  $\xi(t)_V = \text{vp}(\xi(t))$  are the horizontal and vertical projections of  $\xi(t) \in TTM$  for all  $t$ . Then we have the following result.

**Proposition 1.** *The metric  $G$  on the space of curves  $\mathcal{M}$  can be obtained as pullback of the metric  $\tilde{G}$  by the square root velocity function  $R$ .*

*Proof.* For any  $c \in \mathcal{M}$ , and  $h, k \in T_c\mathcal{M}$ , the metric  $G$  is defined by

$$G_c(h, k) = \tilde{G}_{R(c)}(T_cR(h), T_cR(k)).$$

For any  $t \in [0, 1]$ , we have  $T_cR(h)(t)_H = h(t)$  and  $T_cR(h)(t)_V = \nabla_h R(c)(t)$ . To prove this proposition, we just need to compute the latter. Let  $s \mapsto c(s, \cdot)$  be a curve in  $\mathcal{M}$  such that  $c(0, \cdot) = c$  and  $c_s(0, \cdot) = h$ . Here and in all the paper we use the notations  $c_s = \partial c / \partial s$  and  $c_t = \partial c / \partial t$ . Then

$$\begin{aligned} \nabla_h R(c)(t) &= \frac{1}{\|c'\|^{1/2}} \nabla_h c' + h \left( \|c'\|^{-1/2} \right) c' \\ &= \frac{1}{\|c_t\|^{1/2}} \nabla_s c_t + \partial_s \langle c_t, c_t \rangle^{-1/4} c_t \\ &= \frac{1}{\|c_t\|^{1/2}} \nabla_t c_s - \frac{1}{2} \langle c_t, c_t \rangle^{-5/4} \langle \nabla_s c_t, c_t \rangle c_t \\ &= \|c'\|^{1/2} \left( (\nabla_\ell h)^\perp + \frac{1}{2} \langle \nabla_\ell h, \frac{c'}{\|c'\|} \rangle \frac{c'}{\|c'\|} \right), \end{aligned}$$

where in the last step we use again the inversion  $\nabla_s c_t = \nabla_t c_s$ .  $\square$

**2.2. Fiber bundle structures.** This choice of metric induces two formal fiber bundle structures.

*Principal bundle over the shape space.* Just as in the planar case, the fact that the square root velocity function  $R$  satisfies

$$R(c \circ \phi) = \sqrt{\phi'} (R(c) \circ \phi),$$

for all  $c \in \mathcal{M}$ ,  $h, k \in T_c \mathcal{M}$  and  $\phi \in \text{Diff}^+([0, 1])$ , guarantees that the integral part of  $G$  is reparametrization invariant. Remembering that the reparametrizations  $\phi \in \text{Diff}^+([0, 1])$  preserve the origins of the curves, we notice that  $G$  is constant along the fibers and verifies the equivariance property (1). As mentioned in the introduction, we then have a formal principal bundle structure over the shape space

$$\pi : \mathcal{M} = \text{Imm}([0, 1], M) \rightarrow \mathcal{S} = \mathcal{M}/\text{Diff}^+([0, 1]).$$

which induces a decomposition  $T\mathcal{M} = V\mathcal{M} \oplus H\mathcal{M}$ . There exists a Riemannian metric  $\hat{G}$  on the shape space  $\mathcal{S}$  such that  $\pi$  is (formally) a Riemannian submersion from  $(\mathcal{M}, G)$  to  $(\mathcal{S}, \hat{G})$

$$G_c(h^H, k^H) = \hat{G}_{\pi(c)}(T_c\pi(h), T_c\pi(k)),$$

where  $h^H$  and  $k^H$  are the horizontal parts of  $h$  and  $k$ , as well as the horizontal lifts of  $T_c\pi(h)$  and  $T_c\pi(k)$ , respectively. This expression does in fact define  $\tilde{G}$  in the sense that it does not depend on the choice of the representatives  $c$ ,  $h$  and  $k$ . For more details, see the theory of Riemannian submersions and G-manifolds in [6].

*Fiber bundle over the starting points.* The special role that plays the starting point in the metric  $G$  induces another formal fiber bundle structure, where the base space is the manifold  $M$ , seen as the set of starting points of the curves, and the fibers are composed of the curves which have the same origin. The projection is then

$$\pi^{(*)} : \mathcal{M} \rightarrow M, \quad c \mapsto c(0).$$

It induces another decomposition of the tangent bundle in vertical and horizontal bundles

$$\begin{aligned} V_c^{(*)}\mathcal{M} &= \ker T\pi^{(*)} = \{h \in T_c\mathcal{M} \mid h(0) = 0\}, \\ H_c^{(*)}\mathcal{M} &= \left(V_c^{(*)}\mathcal{M}\right)^{\perp_G}. \end{aligned}$$

**Proposition 2.** *We have the usual decomposition  $T\mathcal{M} = V^{(*)}\mathcal{M} \oplus H^{(*)}\mathcal{M}$ , the horizontal bundle  $H_c^{(*)}\mathcal{M}$  consists of parallel vector fields along  $c$ , and  $\pi^{(*)}$  is (formally) a Riemannian submersion for  $(\mathcal{M}, G)$  and  $(M, \langle \cdot, \cdot \rangle)$ .*

*Proof.* Let  $h$  be a tangent vector. Consider  $h_0$  the parallel vector field along  $c$  with initial value  $h_0(0) = h(0)$ . It is a horizontal vector, since its vanishing covariant derivative along  $c$  assures that for any vertical vector  $l$  we have  $G_c(h_0, l) = 0$ . The difference  $\tilde{h} = h - h_0$  between those two horizontal vectors has initial value 0 and so it is a vertical vector, which gives a decomposition of  $h$  into a horizontal vector and a vertical vector. The definition of  $H^{(*)}\mathcal{M}$  as the orthogonal complement of  $V^{(*)}\mathcal{M}$  guaranties that their sum is direct. Now if  $k$  is another tangent vector, then the scalar product between their horizontal parts is

$$\begin{aligned} G_c(h^H, k^H) &= \langle h^H(0), k^H(0) \rangle_{c(0)} \\ &= \langle h(0), k(0) \rangle_{c(0)} = \langle T_c\pi^{(*)}(h^H), T_c\pi^{(*)}(k^H) \rangle_{\pi^{(*)}}, \end{aligned}$$

and this completes the proof.  $\square$

## 3. INDUCED DISTANCE ON THE SPACE OF CURVES

Here we give an expression of the geodesic distance induced by the metric  $G$ . We show that it can be written similarly to the product distance given in [5] and [15], with an added curvature term. Let us consider two curves  $c_0, c_1 \in \mathcal{M}$ , and a path of curves  $s \mapsto c(s, \cdot)$  linking them in  $\mathcal{M}$

$$c(0, t) = c_0(t), \quad c(1, t) = c_1(t),$$

for all  $t \in [0, 1]$ . We denote by  $q(s, \cdot) = R(c(s, \cdot))$  the image of this path of curves by the SRVF  $R$ . Note that  $q$  is a vector field along the surface  $c$  in  $M$ . Let now  $\tilde{q}$  be the raising of  $q$  in the tangent plane  $T_{c(0,0)}M$  in the following way

$$\tilde{q}(s, t) = P_{c(\cdot,0)}^{s,0} \circ P_{c(s,\cdot)}^{t,0} (q(s, t)),$$

where we denote by  $P_\gamma^{t_1, t_2} : T_{\gamma(t_1)}M \rightarrow T_{\gamma(t_2)}M$  the parallel transport along a curve  $\gamma$  from  $\gamma(t_1)$  to  $\gamma(t_2)$ . Notice that  $\tilde{q}$  is a surface in a vector space, as illustrated in Figure 1. Lastly, we introduce a vector field  $(a, \tau) \mapsto \omega^{s,t}(a, \tau)$  in  $M$ , which parallel translates  $q(s, t)$  along  $c(s, \cdot)$  to its origin, then along  $c(\cdot, 0)$  and back down again, as shown in Figure 1. More precisely

$$\omega^{s,t}(a, \tau) = P_{c(a,\cdot)}^{0,\tau} \circ P_{c(\cdot,0)}^{s,a} \circ P_{c(s,\cdot)}^{t,0} (q(s, t))$$

for all  $b, s$ . That way the quantity  $\nabla_s \omega^{s,t}$  measures the holonomy along the rectangle of infinitesimal width shown in Figure 1.

**Proposition 3.** *With the above notations, the geodesic distance induced by the Riemannian metric  $G$  between two curves  $c_0$  and  $c_1$  on the space  $\mathcal{M} = \text{Imm}([0, 1], M)$  of parameterized curves is given by*

$$(3) \quad \text{dist}(c_0, c_1) = \inf_{c(0,\cdot)=c_0, c(1,\cdot)=c_1} \int_0^1 \sqrt{\|c_s(s, 0)\|^2 + \int_0^1 \|\nabla_s q(s, t)\|^2 dt} ds,$$

where  $q = R(c)$  is the Square Root Velocity representation of the curve  $c$  and the norm is the one associated to the Riemannian metric on  $M$ . It can also be written

$$(4) \quad \text{dist}(c_0, c_1) = \inf_{c(0,\cdot)=c_0, c(1,\cdot)=c_1} \int_0^1 \sqrt{\|c_s(s, 0)\|^2 + \int_0^1 \|\tilde{q}_s(s, t) + \Omega(s, t)\|^2 dt} ds,$$

where  $\tilde{q}$  is the raising of  $q$  in the tangent plane  $T_{c(0,0)}M$  and the curvature term  $\Omega$  is given by

$$\begin{aligned} \Omega(s, t) &= P_{c(\cdot,0)}^{s,0} \circ P_{c(s,\cdot)}^{t,0} (\nabla_s \omega^{s,t}(s, t)) \\ &= P_{c(\cdot,0)}^{s,0} \circ P_{c(s,\cdot)}^{t,0} \left( \int_0^t P_{c(s,\cdot)}^{\tau,t} \left( \mathcal{R}(c_\tau, c_s) P_{c(s,\cdot)}^{t,\tau} q(s, t) \right) d\tau \right), \end{aligned}$$

if  $\mathcal{R}$  denotes the curvature tensor of the manifold  $M$ .

**Remark 1.** *The second expression (4) highlights the difference with respect to the distance given in [5] and [15]. In the first term under the square root we can see the velocity vector of the curve  $c(\cdot, 0)$  linking the two origins, and in the second the velocity vector of the curve  $\tilde{q}$  linking the TSRVF-images of the curves – Transported*

Square Root Velocity Function, as introduced by Su et al. in [13]. If instead we equip the tangent bundle  $T\mathcal{M}$  with the metric

$$\tilde{G}'_h(\xi, \xi) = \|\xi_h(0)\|^2 + \int_0^1 \left\| \xi_v(t) - \int_0^t P_c^{\tau,t} (\mathcal{R}(c', \xi_h) P_c^{t,\tau} h(t)) d\tau \right\|^2 dt,$$

for  $h \in T\mathcal{M}$  and  $\xi, \eta \in T_h T\mathcal{M}$ , then the curvature term  $\Omega$  vanishes and the geodesic distance on  $\mathcal{M}$  becomes

$$(5) \quad \text{dist}'(c_0, c_1) = \inf_{c(0,\cdot)=c_0, c(1,\cdot)=c_1} \int_0^1 \sqrt{\|c_s(s,0)\|^2 + \|\tilde{q}_s(s,\cdot)\|_{L^2}^2} ds,$$

which corresponds exactly to the geodesic distance introduced by Zhang et al. in [15] on the space  $\mathbb{C} = \cup_{p \in M} L^2([0,1], T_p M)$ . The difference between the two distances (3) and (5) resides in the curvature term  $\Omega$ , which, as previously mentioned, measures the holonomy along the rectangle of infinitesimal width shown in Figure 1, and translates the fact that in the first one, we compute the distance in the manifold, whereas in the second, it is computed in the tangent space to one of the origins of the curves. Therefore, the first one takes more directly into account the "relief" of the manifold between the two curves under comparison. For example, if there is a "bump" between two curves in an otherwise relatively flat space, the second distance (5) might not see it, whereas the first one (3) will thanks to the curvature term.

**Remark 2.** Let us briefly consider the flat case : if the manifold  $M$  is flat, the two distances (3) and (5) coincide. If two curves  $c_0$  and  $c_1$  in a flat space have the same starting point  $p$ , the first summand under the square root vanishes and the distance becomes the  $L^2$ -distance between the two SRV representations  $q_0 = R(c_0)$  and  $q_1 = R(c_1)$ . If two curves in a flat space differ only by a translation, then the distance is simply the distance between their origins.

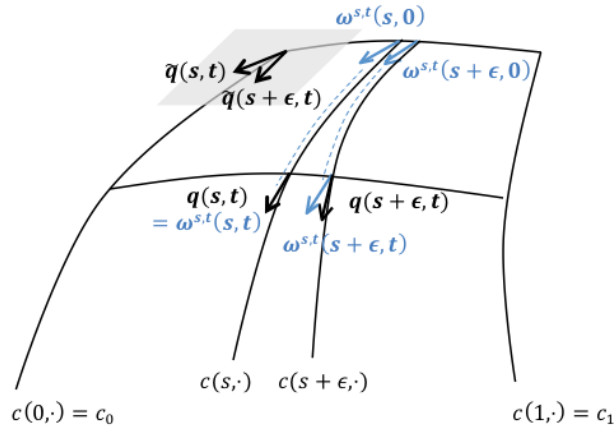


FIGURE 1. Illustration of the distance between two curves  $c_0$  and  $c_1$  in the space of curves  $\mathcal{M}$



*Proof.* Since  $G$  is defined by pullback of  $\tilde{G}$  by the SRVF  $R$ , we know that the lengths of  $c$  in  $\mathcal{M}$  and of  $q = R(c)$  in  $\mathbb{T}\mathcal{M}$  are equal and so that

$$\text{dist}(c_0, c_1) = \inf_{c(0,\cdot)=c_0, c(1,\cdot)=c_1} \int_0^1 \sqrt{\tilde{G}(q_s(s,\cdot), q_s(s,\cdot))} \, ds,$$

with

$$\tilde{G}(q_s(s,\cdot), q_s(s,\cdot)) = \|c_s(s,0)\|^2 + \int_0^1 \|\nabla_s q(s,t)\|^2 \, dt.$$

Now let us fix  $t \in [0, 1]$ . Then  $s \mapsto P_{c(s,\cdot)}^{t,0}(q(s,t))$  is a vector field along  $c(\cdot, 0)$ , and so

$$\nabla_s \left( P_{c(s,\cdot)}^{t,0} q(s,t) \right) = P_{c(\cdot,0)}^{0,s} \left( \frac{\partial}{\partial s} P_{c(\cdot,0)}^{s,0} \circ P_{c(s,\cdot)}^{t,0} (q(s,t)) \right) = P_{c(\cdot,0)}^{0,s} (\tilde{q}_s(s,t)).$$

We consider the vector field  $\nu$  along the surface  $(s,\tau) \mapsto c(s,\tau)$  that is parallel along all curves  $c(s,\cdot)$  and takes value  $\nu(s,t) = q(s,t)$  in  $\tau = t$  for any  $s \in [0, 1]$ , that is

$$\nu(s,\tau) = P_{c(s,\cdot)}^{t,\tau} (q(s,t)),$$

for all  $s \in [0, 1]$  and  $\tau \in [0, 1]$ . That way we know that

$$\begin{aligned} \nabla_s \nu(s,t) &= \nabla_s q(s,t), \\ \nabla_s \nu(s,0) &= P_{c(\cdot,0)}^{0,s} (\tilde{q}_s(s,t)), \\ \nabla_\tau \nu(s,\tau) &= 0, \end{aligned}$$

for all  $s, \tau \in [0, 1]$ . Then we can express its covariant derivative in the following way

$$\begin{aligned} \nabla_s \nu(s,t) &= P_{c(s,\cdot)}^{0,t} (\nabla_s \nu(s,0)) + \int_0^t P_{c(s,\cdot)}^{\tau,t} (\nabla_\tau \nabla_s \nu(s,\tau)) \, d\tau \\ (6) \quad &= P_{c(s,\cdot)}^{0,t} \circ P_{c(\cdot,0)}^{0,s} (\tilde{q}_s(s,t)) + \int_0^t P_{c(s,\cdot)}^{\tau,t} \left( \mathcal{R}(c_\tau, c_s) P_{c(s,\cdot)}^{t,\tau} q(s,t) \right) \, d\tau. \end{aligned}$$

Now let us fix  $s \in [0, 1]$  as well. Notice that the vector field  $\omega^{s,t}$  defined above verifies

$$\begin{aligned} \omega^{s,t}(s,t) &= q(s,t), \\ \nabla_\tau \omega^{s,t}(a,\tau) &= 0, \\ \nabla_a \omega^{s,t}(a,0) &= 0, \end{aligned}$$

for all  $a, \tau \in [0, 1]$ . Note that unlike  $\nu$ , we do *not* have  $\nabla_s \omega^{s,t}(s,t) = \nabla_s q(s,t)$  because  $\omega^{s,t}(a,t) = q(a,t)$  is only true for  $a = s$ . It is easy to verify that the last term of equation (6) is precisely the covariant derivative of the vector field  $\omega^{s,t}$

$$\nabla_s \omega^{s,t}(s,t) = \int_0^t P_{c(s,\cdot)}^{\tau,t} \left( \mathcal{R}(c_\tau, c_s) P_{c(s,\cdot)}^{t,\tau} q(s,t) \right) \, d\tau,$$

since for any  $\tau \in [0, 1]$ ,  $\omega^{s,t}(s,\tau) = P_{c(s,\cdot)}^{t,\tau} q(s,t)$ , and finally by composing by  $P_{c(\cdot,0)}^{s,0} \circ P_{c(s,\cdot)}^{t,0}$ , we obtain the second expression (4), which completes the proof.  $\square$

## 4. COMPUTING GEODESICS

**4.1. Geodesic equations on  $\mathcal{M}$ .** To be able to compute the distance given by (3) between two curves, we first need to compute the optimal deformation  $s \mapsto c(s, \cdot)$  from one to the other. That is, we need to characterize the geodesics of  $\mathcal{M}$  for our metric. In order to do so, taking inspiration from [15], we use the variational principle. In what follows, we use the lighter notation  $u(t_1)^{t_1, t_2} = P_c^{t_1, t_2}(u(t_1))$  to denote the parallel transport of a vector  $u(t_1) \in T_{c(t_1)}M$  along a curve  $c$  from  $c(t_1)$  to  $c(t_2)$ , when there is no ambiguity on the choice of  $c$ .

**Proposition 4.** *Let  $[0, 1] \ni s \mapsto c(s, \cdot) \in \mathcal{M}$  be a path of curves. It is a geodesic of  $\mathcal{M}$  if and only if it verifies the following equations*

$$(7a) \quad \nabla_s c_s(s, 0) + r(s, 0) = 0, \quad \forall s$$

$$(7b) \quad \nabla_s \nabla_s q(s, t) + \|q(s, t)\| \left( r(s, t) + r(s, t)^\parallel \right) = 0, \quad \forall t, s$$

where  $q = c_t / \sqrt{\|c_t\|}$  is the SRV representation of  $c$ , the vector field  $r$  is given by

$$r(s, t) = \int_t^1 \mathcal{R}(q, \nabla_s q) c_s(s, \tau)^{\tau, t} d\tau,$$

and  $r^\parallel = \langle r, v \rangle v$  with  $v = c_t / \|c_t\|$ , is the tangential component of  $r$ .

*Proof.* The path  $c$  is a geodesic if and only if it is a critical point of the energy functional  $E : \mathcal{C}^\infty([0, 1], \mathcal{M}) \rightarrow \mathbb{R}_+$ ,

$$E(c) = \int_0^1 G \left( \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} \right) ds.$$

Let  $a \mapsto \hat{c}(a, \cdot, \cdot)$ ,  $a \in (-\epsilon, \epsilon)$ , be a proper variation of the path  $s \mapsto c(s, \cdot)$ , meaning that it coincides with  $c$  in  $a = 0$ , and it preserves its end points

$$\begin{aligned} \hat{c}(0, \cdot, \cdot) &= c, \\ \hat{c}_a(a, 0, t) &= 0 \quad \forall a, t, \\ \hat{c}_a(a, 1, t) &= 0 \quad \forall a, t. \end{aligned}$$

Then  $c$  is a geodesic of  $\mathcal{M}$  if and only if  $\frac{d}{da} \Big|_{a=0} E(\hat{c}(a, \cdot, \cdot)) = 0$  for any proper variation  $\hat{c}$ . If we denote by  $E(a) = E(\hat{c}(a, \cdot, \cdot))$ , for  $a \in (-\epsilon, \epsilon)$ , the energy of a proper variation  $\hat{c}$ , then we have

$$E(a) = \int \langle \hat{c}_s(a, s, 0), \hat{c}_s(a, s, 0) \rangle ds + \int \int \langle \nabla_s \hat{q}(a, s, t), \nabla_s \hat{q}(a, s, t) \rangle dt ds,$$

where  $\hat{q} = \hat{c}_t / \sqrt{\|\hat{c}_t\|}$  is the SRV representation of  $\hat{c}$ . Its derivative is given by

$$\frac{1}{2} E'(a) = \int \langle \nabla_a \hat{c}_s(a, s, 0), \hat{c}_s(a, s, 0) \rangle ds + \int \int \langle \nabla_a \nabla_s \hat{q}(a, s, t), \nabla_s \hat{q}(a, s, t) \rangle dt ds.$$

Considering that the variation preserves the end points, integration by parts gives

$$\begin{aligned} \int \langle \nabla_a \hat{c}_s, \hat{c}_s \rangle ds &= - \int \langle \nabla_s \hat{c}_s, \hat{c}_a \rangle ds \\ \int \langle \nabla_s \nabla_a \hat{q}, \nabla_s \hat{q} \rangle ds &= - \int \langle \nabla_s \nabla_s \hat{q}, \nabla_a \hat{q} \rangle ds, \end{aligned}$$

and so we obtain

$$\begin{aligned} \frac{1}{2}E'(a) &= - \int \langle \nabla_s \hat{c}_s, \hat{c}_a \rangle|_{t=0} ds + \int \int \langle \mathcal{R}(c_a, c_s)q + \nabla_s \nabla_a q, \nabla_s q \rangle dt ds \\ &= - \int \langle \nabla_s \hat{c}_s, \hat{c}_a \rangle|_{t=0} ds - \int \int \langle \mathcal{R}(\hat{q}, \nabla_s \hat{q}) \hat{c}_s, \hat{c}_a \rangle + \langle \nabla_s \nabla_s \hat{q}, \nabla_a \hat{q} \rangle dt ds. \end{aligned}$$

This quantity has to vanish in  $a = 0$  for all proper variations  $\hat{c}$

$$\begin{aligned} &\int \langle \nabla_s c_s|_{t=0}, \hat{c}_a|_{a=0, t=0} \rangle ds \\ &\quad + \int \int \langle \mathcal{R}(q, \nabla_s q) c_s, \hat{c}_a|_{a=0} \rangle + \langle \nabla_s \nabla_s q, \nabla_a \hat{q}|_{a=0} \rangle dt ds = 0. \end{aligned}$$

We cannot yield any conclusions at this point, because  $\hat{c}_a(0, s, t)$  and  $\nabla_a \hat{q}(0, s, t)$  cannot be chosen independently, since  $\hat{q}$  is not any vector field along  $\hat{c}$  but its image via the Square Root Velocity Function. We notice that

$$\nabla_a \hat{c} = \|\hat{q}\| \left( \nabla_a \hat{q} + \nabla_a \hat{q}^\parallel \right),$$

where  $\nabla_a \hat{q}^\parallel = \langle \nabla_a \hat{q}, v \rangle v$  is the projection of  $\nabla_a \hat{q}$  along the unit speed vector field  $v = c_t / \|c_t\|$ . We can then express the variation  $\hat{c}_a$  as follows

$$\hat{c}_a(0, s, t) = \hat{c}_a(0, s, 0)^{0,t} + \int_0^t \|\hat{q}(0, s, \tau)\| \left( \nabla_a \hat{q}(0, s, \tau) + \nabla_a \hat{q}^\parallel(0, s, \tau) \right)^{\tau,t} d\tau,$$

and by inserting this expression in the derivative of the energy we obtain the following, where we omit to write that the variations  $\hat{c}$  and  $\hat{q}$  are always taken in  $a = 0$  for the sake of readability

$$\begin{aligned} &\int_0^1 \langle \nabla_s c_s(s, 0), \hat{c}_a(s, 0) \rangle ds + \int_0^1 \int_0^1 \langle \mathcal{R}(q, \nabla_s q) c_s(s, t), \hat{c}_a(s, 0)^{0,t} \rangle dt ds \\ &\quad + \int_0^1 \int_0^1 \left\langle \mathcal{R}(q, \nabla_s q) c_s(s, t), \int_0^t \|\hat{q}(s, \tau)\| \left( \nabla_a \hat{q}(s, \tau) + \nabla_a \hat{q}^\parallel(s, \tau) \right)^{\tau,t} d\tau \right\rangle dt ds \\ &\quad + \int_0^1 \int_0^1 \langle \nabla_s \nabla_s q(s, t), \nabla_a \hat{q}(s, t) \rangle dt ds \\ &= \int_0^1 \left\langle \nabla_s c_s(s, 0) + \int_0^1 \mathcal{R}(q, \nabla_s q) c_s(s, \tau)^{\tau,0} d\tau, \hat{c}_a(s, 0) \right\rangle ds \\ &\quad + \int_0^1 \int_0^1 \int_t^1 \langle \mathcal{R}(q, \nabla_s q) c_s(s, \tau)^{\tau,t}, \|\hat{q}(s, t)\| (\nabla_a \hat{q}(s, t) + \nabla_a \hat{q}^\parallel(s, t)) \rangle d\tau dt ds \\ &\quad + \int_0^1 \int_0^1 \langle \nabla_s \nabla_s q(s, t), \nabla_a \hat{q}(s, t) \rangle dt ds \\ &= \int_0^1 \left\langle \nabla_s c_s(s, 0) + r(s, 0), \hat{c}_a(s, 0) \right\rangle ds \\ &\quad + \int_0^1 \int_0^1 \langle \nabla_s \nabla_s q(s, t) + \|q(s, t)\| (r(s, t) + r(s, t)^\parallel), \nabla_a \hat{q}(s, t) \rangle dt ds \\ &= 0, \end{aligned}$$

with the previously given definition of  $r$ . Since the variations  $\hat{c}_a(0, s, 0)$  and  $\nabla_a \hat{q}(0, s, t)$  can be chosen independently and take any value for all  $s$  and all  $t$ , it is easy to see that we obtain the desired equations for all  $s$  and  $t$ .  $\square$

**4.2. Exponential map.** Now that we have the geodesic equations, we are able to describe an algorithm which allows us to compute the geodesic  $s \mapsto c(s, \cdot)$  starting from a point  $c \in \mathcal{M}$  at speed  $u \in T_c\mathcal{M}$ . This amounts to finding the optimal deformation of the curve  $c$  in the direction of the vector field  $u$  according to our metric. We initialize this path  $s \mapsto c(s, \cdot)$  by setting  $c(0, \cdot) = c$  and  $c_s(s, 0) = u$ , and we propagate it using iterations of fixed step  $\epsilon > 0$ . The aim is, given  $c(s, \cdot)$  and  $c_s(s, \cdot)$ , to deduce  $c(s + \epsilon, \cdot)$  and  $c_s(s + \epsilon, \cdot)$ . The first is obtained by following the exponential map on the manifold  $M$

$$c(s + \epsilon, \cdot) = \exp_{c(s, \cdot)} \epsilon c_s(s, \cdot),$$

and the second requires the computation of the variation  $\nabla_s c_s(s, \cdot)$

$$c_s(s + \epsilon, \cdot) = [c_s(s, \cdot) + \epsilon \nabla_s c_s(s, \cdot)]^{s, s+\epsilon},$$

where once again, we use the notation  $w(s)^{s, s+\epsilon} = P_c^{s, s+\epsilon}(w(s))$  for the parallel transport of a vector field  $s \mapsto w(s)$  along a curve  $s \mapsto c(s)$  in  $M$ . If we suppose that at time  $s$  we have  $c(s, \cdot)$  and  $c_s(s, \cdot)$  at our disposal, then we also know  $c_t(s, \cdot)$  and  $q(s, \cdot) = c_t(s, \cdot) / \sqrt{\|c_t\|}$ , as well as  $\nabla_t c_s(s, \cdot)$  and

$$(8) \quad \nabla_s q(s, \cdot) = \frac{\nabla_s c_t}{\sqrt{\|c_t\|}}(s, \cdot) - \frac{1}{2} \frac{\langle \nabla_s c_t, c_t \rangle}{\|c_t\|^{5/2}} c_t(s, \cdot),$$

using the fact that  $\nabla_s c_t = \nabla_t c_s$ . The variation  $\nabla_s c_s(s, \cdot)$  can then be computed in the following way

$$\nabla_s c_s(s, t) = \nabla_s c_s(s, 0)^{0, t} + \int_0^t [\nabla_s \nabla_s c_t(s, \tau) - \mathcal{R}(c_t, c_s) c_s(s, \tau)]^{\tau, t} d\tau$$

for all  $t \in [0, 1]$ , where  $\nabla_s c_s(s, 0)$  is given by equation (7a), the second order variation  $\nabla_s \nabla_s c_t(s, \cdot)$  is given by

$$(9) \quad \begin{aligned} \nabla_s \nabla_s c_t &= |c_t|^{1/2} \nabla_s \nabla_s q + \frac{\langle \nabla_t c_s, c_t \rangle}{|c_t|^2} \nabla_t c_s \\ &+ \left( \frac{\langle \nabla_s \nabla_s q, c_t \rangle}{|c_t|^{3/2}} - \frac{3}{2} \frac{\langle \nabla_t c_s, c_t \rangle^2}{|c_t|^4} + \frac{|\nabla_t c_s|^2}{|c_t|^2} \right) c_t, \end{aligned}$$

and  $\nabla_s \nabla_s q$  can be computed via equation (7b).

Now, if in practice we have a series of discrete observations  $p_0, p_1, \dots, p_n \in M$  made at discrete times  $0 = t_0 < t_1 < \dots < t_n = 1$ , then we can be brought back to the continuous case by linking these points by pieces of geodesics. By doing so, we place ourselves in the submanifold of piecewise-geodesic curves. Here we give the exponential map on that submanifold. Let  $c_0$  be the piecewise-geodesic curve going through  $p_0, \dots, p_n$  at times  $t_0, \dots, t_n$ , that is  $c_0(t_i) = p_i$  and  $c|_{[t_i, t_{i+1}]}$  is a geodesic for all  $i$ . Then the optimal deformation of  $c_0$  in the direction of a vector field  $u$  can be computed by the following steps.

**Algorithm 1** (Discrete Exponential Map).

*Input* :  $c_0, u$ .

- *Initialization* : Set  $c(0, t_i) = c_0(t_i)$  and  $c_s(0, t_i) = u(t_i)$  for all  $i = 0, \dots, n$ .
- *Heredity* : If  $c(s, t_i)$  and  $c_s(s, t_i)$  are known for all  $i$ , then

(1) for  $0 \leq i \leq n-1$ , set

$$c_t(s, t_i) = \frac{1}{t_{i+1} - t_i} \log_{c(s, t_i)} c(s, t_{i+1}),$$

$$\nabla_t c_s(s, t_i) = \frac{1}{t_{i+1} - t_i} (c_s(s, t_{i+1})^{t_{i+1}, t_i} - c_s(s, t_i)),$$

where  $\log$  denotes the inverse of the exponential map on  $M$ , and compute  $q(s, t_i) = \frac{1}{\sqrt{\|c_t\|}} c_t(s, t_i)$  and  $\nabla_s q(s, t_i)$  using equation (8).

(2) Compute  $r(s, t_i) = \sum_{k=i}^n \mathcal{R}(q, \nabla_s q) c_s(s, t_k)^{t_k, t_i}$  for all  $0 \leq i \leq n$  and set

$$\nabla_s c_s(s, t_0) = -r(s, t_0),$$

$$\nabla_s \nabla_s q(s, t_i) = -\|q(s, t_i)\| (r(s, t_i) + r(s, t_i)^\parallel) \quad \forall i.$$

(3) For  $i = 0, \dots, n-1$ , compute  $\nabla_s \nabla_s c_t(s, t_i)$  using equation (9), and set

$$\nabla_s c_s(s, t_{i+1}) = [\nabla_s c_s(s, t_i) + (t_{i+1} - t_i) (\nabla_s \nabla_s c_t(s, t_i) - \mathcal{R}(c_s, c_t) c_s(s, t_i))]^{t_i, t_{i+1}}.$$

(4) Finally, for all  $0 \leq i \leq n$ , set

$$c(s + \epsilon, t_i) = \exp_{c(s, t_i)} (\epsilon c_s(s, t_i)),$$

$$c_s(s + \epsilon, t_i) = [c_s(s, t_i) + \epsilon \nabla_s c_s(s, t_i)]^{t_i, i}.$$

where  $\exp$  is the exponential map on the manifold  $M$ .

Output :  $c = \exp_{c_0}^{\mathcal{M}} u$ .

The last step needed to compute the optimal deformation between two curves  $c_0$  and  $c_1$  is to find the appropriate initial speed  $u$ . This can be done by geodesic shooting.

**4.3. Geodesic shooting and Jacobi fields.** The aim of geodesic shooting is to compute the geodesic linking two points  $p_0$  and  $p_1$  of a manifold  $\mathcal{N}$ , knowing the exponential map  $\exp^{\mathcal{N}}$ . An initial speed vector  $u \in T_{p_0} \mathcal{N}$  is chosen, and is iteratively updated after evaluating the gap between the point  $p = \exp_{p_0}^{\mathcal{N}} u$  obtained by taking the exponential map at point  $p_0$  in  $u$  – that is, by shooting from  $p_0$  in the direction  $u$  – and the target point  $p_1$ . The gap between the current point  $p$  and the target point  $p_1$  can be seen as the extremity  $J(1)$  of a Jacobi field  $J : [0, 1] \rightarrow \mathcal{N}$  which has value  $J(0) = 0$  in 0, and the current speed vector can be corrected by  $u \leftarrow u + \dot{J}(0)$ , as shown in Figure 2. In our case, the speed vector  $u$  can be initialized by taking the initial speed vector of the  $L^2$ -geodesic going from  $c_0$  to  $c_1$ . The steps can be summarized as follows.

**Algorithm 2** (Geodesic shooting).

Input :  $c_0, c_1 \in \mathcal{M}$ .

– Initialization : Set  $u = \log_{c_0}^{L^2}(c_1)$ , where  $\log^{L^2}$  is the inverse of the exponential map for the  $L^2$ -metric.

– Fix a threshold  $\delta > 0$ .

(1) Compute  $c = \exp_{c_0}^{\mathcal{M}}(u)$  with Algorithm 1.

(2) Compute the gap  $j = \log_c^{L^2}(c_1)$ .

- (3) If  $j > \delta$ , set  $J(1) = j$  and  $u \leftarrow u + \dot{J}(0)$  where  $\dot{J}(0) = \phi^{-1}(J(1))$ , and go back to the first step.  
 Else, stop.

*Output* :  $c$  approximation of the geodesic linking  $c_0$  and  $c_1$ .

The function  $\phi$  associates the last value  $J(1)$  of a Jacobi field with initial value  $J(0) = 0$  to the initial speed  $\dot{J}(0)$ , and can be deduced from Algorithm 3, which describes the function associating  $J(1)$  to the initial conditions  $J(0)$  and  $\dot{J}(0)$ . To find the inverse of this function, we can consider for example the image of a basis of the tangent vector space in which  $\dot{J}(0)$  lives.

Now, let us characterize the Jacobi fields of  $\mathcal{M}$  to obtain the function  $\phi$ . Consider  $a \mapsto c(a, \cdot, \cdot)$ ,  $a \in (-\epsilon, \epsilon)$ , a family of geodesics in  $\mathcal{M}$ , that is for each  $a \in (-\epsilon, \epsilon)$ ,  $[0, 1] \ni s \mapsto c(a, s, \cdot)$  is a geodesic of  $\mathcal{M}$ . Then for all  $a$ ,  $c(a, \cdot, \cdot)$  verifies the geodesic equations

$$\begin{aligned} \nabla_s c_s(a, s, 0) + r(a, s, 0) &= 0, \quad \forall s \\ \nabla_s \nabla_s q(a, s, t) + \|q(a, s, t)\| \left( r(a, s, t) + r(a, s, t)^\parallel \right) &= 0, \quad \forall t, s \end{aligned}$$

where  $q = c_t / \sqrt{\|c_t\|}$  is the SRV representation of  $c$  and  $r$  is given by

$$r(a, s, t) = \int_t^1 \mathcal{R}(q, \nabla_s q) c_s(a, s, \tau)^{\tau, t} d\tau.$$

In what follows, we use the notation  $U^\parallel = \langle U, v(a, s, t) \rangle v(a, s, t)$  with  $v = c_t / \|c_t\|$  to denote the tangential component of any tangent vector  $U \in T_{c(a, s, t)}M$ . Let us consider the Jacobi field  $J : [0, 1] \rightarrow T\mathcal{M}$ ,

$$J(s, \cdot) = \frac{\partial}{\partial a} \Big|_{a=0} c(a, s, \cdot).$$

Computing the derivative of the geodesic equations according to  $a$  leads to a characterization of  $J$

$$(10a) \quad \nabla_a \nabla_s c_s(0, s, 0) + \nabla_a r(0, s, 0) = 0, \quad \forall s$$

$$(10b) \quad \nabla_a \nabla_s \nabla_s q(0, s, t) + \nabla_a \|q(0, s, t)\| \left( r(0, s, t) + r(0, s, t)^\parallel \right) \\ + \|q(0, s, t)\| \left( \nabla_a r(0, s, t) + \nabla_a \left( r(0, s, t)^\parallel \right) \right) = 0, \quad \forall t, s$$

in the sense that we obtain for each  $t \in [0, 1]$  a second order differential equation in  $s$ , the set of which give a unique solution for  $J$ , knowing the initial conditions  $J(0, \cdot)$  and  $\nabla_s J(0, \cdot)$ . Indeed, suppose that we have those two initial conditions and that the curve  $s \mapsto c(0, s, \cdot)$  in  $\mathcal{M}$  is known, as well as all its partial derivatives in  $s$  and  $t$ . We are interested in finding  $J(s, t)$  for all  $s$  and  $t$ . We can write the second order variation of  $J$  as

$$(11) \quad \nabla_s \nabla_s J(s, t) = \left[ (\nabla_a \nabla_s c_s + \mathcal{R}(c_s, J) c_s) \Big|_{a=0, t=0} \right]^{0, t} \\ + \int_0^t \left[ (\nabla_s \nabla_s \nabla_t J + \mathcal{R}(c_t, c_s) \nabla_s J + \nabla_s (\mathcal{R}(c_t, c_s) J)) \Big|_{a=0, t=\tau} \right]^{\tau, t} d\tau,$$

where  $\nabla_s \nabla_s \nabla_t J = W + W^\parallel$  with

$$\begin{aligned}
W &= \langle \nabla_s \nabla_t J, \nabla_s v \rangle v + \langle \nabla_s \nabla_t J, v \rangle \nabla_s v + \langle \nabla_t J, \nabla_s v \rangle \nabla_s v \\
&\quad + \frac{1}{2} \langle \nabla_t J, \nabla_s \nabla_s v \rangle v + \frac{1}{2} \langle \nabla_t J, v \rangle \nabla_s \nabla_s v \\
(12) \quad &\quad + \sqrt{\|c_t\|} \left[ \nabla_a \nabla_s \nabla_s q - 2 \nabla_s \left( \|c_t\|^{-\frac{1}{2}} \right) \left( \nabla_s \nabla_t J - \frac{1}{2} \nabla_s \left( \nabla_t J^\parallel \right) \right) \right. \\
&\quad \left. - \nabla_s \nabla_s \left( \|c_t\|^{-\frac{1}{2}} \right) \left( \nabla_t J - \frac{1}{2} \nabla_t J^\parallel \right) + \mathcal{R}(c_s, J) \nabla_s q + \nabla_s (\mathcal{R}(c_s, J) q) \right],
\end{aligned}$$

and where variations  $\nabla_a \nabla_s c_s(0, s, 0)$  and  $\nabla_a \nabla_s \nabla_s q(0, s, \tau)$  for all  $\tau \in [0, 1]$  are given by equations (10a) and (10b) respectively : for  $t = 0$ , we have

$$(13) \quad \nabla_a \nabla_s c_s = -\nabla_a r,$$

and for all  $t$  we get

$$\begin{aligned}
(14) \quad \nabla_a \nabla_s \nabla_s q &= -\frac{1}{2\sqrt{\|c_t\|}} \langle \nabla_t J, v \rangle (r + r^\parallel) \\
&\quad + \sqrt{\|c_t\|} \left( \nabla_a r + \nabla_a r^\parallel + \langle r, \nabla_s v \rangle v + \langle r, v \rangle \nabla_s v \right).
\end{aligned}$$

The only term left to compute is the variation  $\nabla_a r$ , which can be written

$$\begin{aligned}
\nabla_a r(0, s, t) &= -\int_t^1 (\nabla_J \mathcal{R})(q, \nabla_s q) c_s^{\tau, t} + \mathcal{R}(\nabla_a q, \nabla_s q) c_s^{\tau, t} \\
&\quad + \mathcal{R}(q, \nabla_a \nabla_s q) c_s^{\tau, t} + \mathcal{R}(q, \nabla_s q) \nabla_a (c_s^{\tau, t}) \, d\tau,
\end{aligned}$$

with

$$(15a) \quad \nabla_a q = \frac{1}{\sqrt{\|c_t\|}} \left( \nabla_t J - \frac{1}{2} \nabla_t J^\parallel \right),$$

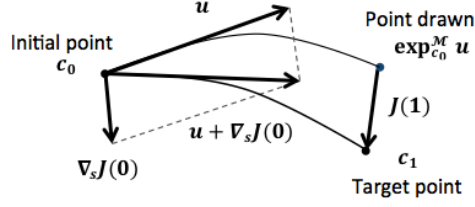
$$\begin{aligned}
(15b) \quad \nabla_a \nabla_s q &= \frac{1}{\sqrt{\|c_t\|}} \left( \nabla_s \nabla_t J - \frac{1}{2} \nabla_s \nabla_t J^\parallel - \frac{1}{2} \langle \nabla_t J, \nabla_s v \rangle v - \frac{1}{2} \langle \nabla_t J, v \rangle \nabla_s v \right) \\
&\quad + \nabla_s \left( \|c_t\|^{-1/2} \right) \left( \nabla_t J - \frac{1}{2} \nabla_t J^\parallel \right) + \mathcal{R}(J, c_s) q,
\end{aligned}$$

and, for all  $\tau \in [t, 1]$ ,

$$\nabla_a (c_s(0, s, \tau)^{\tau, t}) = \nabla_s J(s, t) + \int_t^\tau \mathcal{R}(c_t, J)(c_s(0, s, u)^{u, t}) \, du.$$

We can notice that the obtained equations allow us to iteratively compute  $J(s + \epsilon, \cdot)$  and  $\nabla_s J(s + \epsilon, \cdot)$ , for a fixed step  $\epsilon > 0$ , knowing  $J(s, \cdot)$  and  $\nabla_s J(s, \cdot)$ . Indeed, we then know  $\nabla_t J(s, \cdot)$  since  $J(s, t)$  is known for all  $t$ , as well as  $\nabla_t \nabla_s J(s, \cdot)$  since  $\nabla_s J(s, t)$  is known for all  $t$ , and finally  $\nabla_s \nabla_t J = \nabla_t \nabla_s J + \mathcal{R}(c_s, c_t) J$ . Assuming that we are able to compute the covariant derivative  $\nabla_J \mathcal{R}$  of the curvature tensor, for example if we are in a symmetric space (then it is zero), we obtain an algorithm to compute the Jacobi fields in the space of curves.

As we did for the exponential map, let us consider the discrete case, and place ourselves in the submanifold of piecewise-geodesic curves. We assume that  $M$  is a symmetric space. Consider a discrete path of piecewise-geodesic curves  $c(s_j, t_i)$ , where  $0 = t_0 \leq \dots \leq t_n = 1$  and  $s_j = j/m$  with  $j = 0, \dots, m$ , for  $n, m$  natural integers, and  $c(s_j, \cdot)|_{[t_i, t_{i+1}]}$  is a geodesic of  $M$  for all  $j$  and all  $i$ . We assume that we know the derivatives in  $s$  and  $t$ . Consider two vectors fields  $J_0, W$  along  $c(0, \cdot)$ .


 FIGURE 2. Geodesic shooting in the space of curves  $\mathcal{M}$ 

**Algorithm 3** (Discrete Jacobi fields in the space of curves in a symmetric space).

*Input* :  $c, J_0$  and  $W$ .

- *Initialization* : Set  $J(0, t_i) = J_0(t_i)$  and  $\nabla_s J(0, t_i) = W(t_i)$  for  $i = 0, \dots, n$ .
- *Heredity* : Fix  $j \in \llbracket 1, m \rrbracket$  and set  $s = s_j$ , and  $\epsilon = 1/m$ . We denote by  $a \mapsto c(a, \cdot, \cdot)$  a variation of  $c$ . If  $J(s, t_i)$  and  $\nabla_s J(s, t_i)$  are known for all  $i$ , then

- (1) for all  $0 \leq i \leq n-1$ , set

$$\begin{aligned} \nabla_t J(s, t_i) &= \frac{1}{t_{i+1} - t_i} (J(s, t_{i+1})^{t_{i+1}, t_i} - J(s, t_i)), \\ \nabla_t \nabla_s J(s, t_i) &= \frac{1}{t_{i+1} - t_i} (\nabla_s J(s, t_{i+1})^{t_{i+1}, t_i} - \nabla_s J(s, t_i)), \\ \nabla_s \nabla_t J(s, t_i) &= \nabla_t \nabla_s J(s, t_i) + \mathcal{R}(c_s, c_t) J(s, t_i). \end{aligned}$$

- (2) Compute  $r(0, s, t_i) = \sum_{k=i}^n \mathcal{R}(q, \nabla_s q) c_s(s, t_k)^{t_k, t_i}$  for all  $i = 0, \dots, n$ .
- (3) Compute  $\nabla_a q(0, s, t_i)$  and  $\nabla_s \nabla_a q(0, s, t_i)$  for all  $0 \leq i \leq n$  using (15a) and (15b) and

$$\nabla_a (c_s(0, s, t_k)^{t_k, t_i}) = \nabla_s J(s, t_i) + \sum_{\ell=i}^k \mathcal{R}(c_t, J)(c_s(s, t_\ell)^{t_\ell, t_i}),$$

for all  $i \leq k \leq n$ , and deduce

$$\begin{aligned} \nabla_a r(0, s, t_i) &= - \sum_{k=i}^n [\mathcal{R}(\nabla_a q, \nabla_s q) c_s(0, s, t_k)^{t_k, t_i} \\ &\quad + \mathcal{R}(q, \nabla_a \nabla_s q) c_s(0, s, t_k)^{t_k, t_i} + \mathcal{R}(q, \nabla_s q) \nabla_a (c_s(0, s, t_k)^{t_k, t_i})]. \end{aligned}$$

- (4) Compute  $\nabla_a \nabla_s c_s(0, s, 0)$  and  $\nabla_a \nabla_s \nabla_s q(0, s, t_i)$  for all  $0 \leq i \leq n$  using equations (13) and (14).
- (5) Compute  $W(s, t_i)$  using (12) and  $\nabla_s \nabla_s \nabla_t J(s, t_i) = W(s, t_i) + W^\parallel(s, t_i)$  for all time  $t_i$ , and deduce

$$\begin{aligned} \nabla_s \nabla_s J(s, t_i) &= \left[ (\nabla_a \nabla_s c_s + \mathcal{R}(c_s, J) c_s) \Big|_{a=0, t=0} \right]^{0, t} \\ &\quad + \sum_{k=0}^i \left[ (\nabla_s \nabla_s \nabla_t J + \mathcal{R}(c_t, c_s) \nabla_s J + \nabla_s (\mathcal{R}(c_t, c_s) J)) \Big|_{a=0, t=t_k} \right]^{t_k, t_i}. \end{aligned}$$



(6) Finally, for all  $0 \leq i \leq n$ , set

$$\begin{aligned} J(s + \epsilon, t_i) &= [ J(s, t_i) + \epsilon \nabla_s J(s, t_i) ]^{t_i, t_{i+1}}, \\ \nabla_s J(s + \epsilon, t_i) &= [ \nabla_s J(s, t_i) + \epsilon \nabla_s \nabla_s J(s, t_i) ]^{t_i, t_{i+1}}. \end{aligned}$$

Output :  $J(1)$

The geodesic shooting method presents several drawbacks : we have no convergence result, and stopping before convergence will not be satisfactory since it will yield a path that does not link the curves we want to link. Furthermore, the estimation of the "gap" between the curve obtained by shooting and the target curve is problematic, as it would require shooting as well. In what follows, we present an alternative.

**4.4. Path straightening.** Another possibility to compute an optimal deformation from one curve to another is the path straightening method, used by Srivastava et al. (see [12]). The advantage of this algorithm is that it provides a path linking our two curves at each step. The idea is to initialize the desired geodesic by a path linking our two curves (for example, the  $L^2$  deformation) and to update it using a gradient descent on the energy functional, while fixing the extremities. Here we will not exactly follow the opposite of the gradient, but the simplest smooth deformation which makes the energy functional decrease. Let  $[0, 1] \ni s \mapsto c(s, \cdot) \in \mathcal{M}$  be a path of curves and  $a \mapsto \hat{c}(a, \cdot, \cdot)$ ,  $a \in (-\epsilon, \epsilon)$  a proper variation of  $c$ . Recall that the differential of the energy  $E : \mathcal{C}^\infty([0, 1], \mathcal{M}) \rightarrow \mathbb{R}_+$ ,

$$E(c) = \int_0^1 G \left( \frac{\partial c}{\partial s}, \frac{\partial c}{\partial s} \right) ds,$$

at point  $c$  and in  $\hat{c}_a(0, \cdot, \cdot)$  is given by

$$\begin{aligned} T_c E(\hat{c}_a(0)) &= - \frac{1}{2} \int_0^1 \left\langle \nabla_s c_s(s, 0) + r(s, 0), \hat{c}_a(0, s, 0) \right\rangle ds \\ &\quad - \frac{1}{2} \int_0^1 \int_0^1 \left\langle \nabla_s \nabla_s q(s, t) + \|q(s, t)\| (r(s, t) + r(s, t)^\parallel), \nabla_a \hat{q}(0, s, t) \right\rangle dt ds. \end{aligned}$$

Then it is easy to verify that the following choice of variation  $\hat{c}$  in  $a = 0$  and  $s \in [0, 1]$  will make the energy functional decrease

$$\begin{aligned} \hat{c}_a(0, s, 0) &= f(s) (\nabla_s c_s(s, 0) + r(s, 0)), \\ \hat{c}_a(0, s, t) &= f(s) \left( \hat{c}_a(0, s, 0)^{0,t} + \int_0^t \|q(s, \tau)\| \left( V(s, \tau) + V^\parallel(s, \tau) \right)^{\tau,t} d\tau \right), \end{aligned}$$

for all  $t \in [0, 1]$ , with

$$V(s, \tau) = \nabla_s \nabla_s q(s, \tau) + \|q(s, \tau)\| \left( r(s, \tau) + r^\parallel(s, \tau) \right) \quad \forall \tau,$$

and where  $f : [0, 1] \rightarrow \mathbb{R}$  satisfies  $f(0) = f(1) = 0$  and  $f \geq 0$ . This multiplicative function guarantees that  $\hat{c}$  is a proper variation of  $c$ , that is preserving the extremities

$$\begin{aligned} \hat{c}_a(0, 0, t) &= 0 \quad \forall t \in [0, 1], \\ \hat{c}_a(0, 1, t) &= 0 \quad \forall t \in [0, 1]. \end{aligned}$$

The steps of the algorithm can be summarized as follows.

**Algorithm 4** (Path straightening).

*Input* :  $c_0, c_1 \in \mathcal{M}$ .

– *Initialization* : Set  $k = 1$ ,  $c = c_{L^2}$  where  $c_{L^2}$  is the  $L^2$ -deformation between  $c_0$  and  $c_1$ , and compute its energy  $E = E(c_{L^2})$ .

– *Fix a threshold*  $\delta > 0$ .

(1) *Compute for all  $s$  and  $t$*

$$V(s, t) = \nabla_s \nabla_s q(s, t) + \|q(s, t)\| \left( r(s, t) + r^\parallel(s, t) \right),$$

*and set*

$$\hat{c}_a(0, s, t) = f(s) \left[ \left( \nabla_s c_s(s, 0) + r(s, 0) \right)^{0,t} + \int_0^t \|q(s, \tau)\| \left( V(s, \tau) + V^\parallel(s, \tau) \right)^{\tau,t} d\tau \right].$$

*with  $f : [0, 1] \rightarrow \mathbb{R}$  satisfying  $f(0) = f(1) = 0$  and  $f \geq 0$ .*

(2) *Update  $c(s, t) \leftarrow \exp_{c(s,t)} \left( \frac{1}{k} \hat{c}_a(0, s, t) \right) \forall s, t$  and compute  $E(c)$ .*

(3) *If  $E - E(c) > \delta$ , then set  $E = E(c)$  and  $k \leftarrow k + 1$  and go back to step (1). Else, stop.*

*Output* :  $c$  approximation of the geodesic linking  $c_0$  to  $c_1$ .

## 5. EXAMPLE : CURVES IN THE HYPERBOLIC HALF-PLANE $\mathbb{H}$

Let us consider, as an example, the particular case of curves in the hyperbolic half-plane  $\mathbb{H}$ , for which we can obtain some explicit formulas. Here we give a few tools in that space which are necessary for the algorithms previously described. We consider the upper half-plane representation of  $\mathbb{H}$ . Recall that in this representation, the metric is given by

$$\langle u, v \rangle = \frac{u^1 v^1 + u^2 v^2}{y^2},$$

for two tangent vectors  $u = u^1 + iu^2$  and  $v = v^1 + iv^2$  in a point  $z = x + iy$ . If  $c(t) = x(t) + iy(t)$  is a curve in  $\mathbb{H}$  and  $v(t) = v^1(t) + v^2(t)$  is a vector field along  $c$ , then the covariant derivative of  $v$  is given by  $\nabla_{\dot{c}(t)} v = X(t) + iY(t)$  where

$$X = \frac{\dot{x}v^2 + \dot{y}v^1}{y}, \quad Y = \frac{\dot{y}v^2 - \dot{x}v^1}{y}.$$

Let us now remind a well-known expression for the Riemann curvature tensor in a manifold of constant sectional curvature. Recall that  $\mathbb{H}$  has constant sectional curvature  $K = -1$ .

**Proposition 5.** *Let  $X, Y, Z$  be three vector fields on a manifold of constant sectional curvature  $K$ . The Riemann curvature tensor can be written*

$$\mathcal{R}(X, Y)Z = K (\langle Y, Z \rangle X - \langle X, Z \rangle Y).$$

We also need to be able to compute the geodesic starting from a point  $p \in \mathbb{H}$  at speed  $u_0 \in T_p \mathbb{H}$  (in other words, the exponential map  $u \mapsto \exp_p(u)$ ), as well as the geodesic linking two points  $p$  and  $q$ , with the associated initial vector speed (which gives the inverse  $q \mapsto \log_p(q)$  of the exponential map). Let us recall that the geodesics of the hyperbolic half-plane are vertical segments and half-circles whose origins are on the x-axis, and that they can be obtained as images of the

vertical geodesic following the y-axis by a Moebius transformation  $z \mapsto \frac{az+b}{cz+d}$ , with  $ad - bc = 1$ .

**Proposition 6** (Geodesics of  $\mathbb{H}$  and logarithm map). *Let  $z_0 = x_0 + iy_0$  and  $z_1 = x_1 + iy_1$  be two elements of  $\mathbb{H}$ . If  $x_0 = x_1$  then the geodesic going from  $z_0$  to  $z_1$  is a segment. Otherwise, it is given by  $\gamma(t) = x(t) + iy(t)$  with*

$$x(t) = \frac{bd + ac\bar{y}(t)^2}{d^2 + c^2\bar{y}(t)^2}, \quad y(t) = \frac{\bar{y}(t)}{d^2 + c^2\bar{y}(t)^2}, \quad t \in [0, 1],$$

where the coefficients of the Moebius transformation can be deduced from the center  $x_\Omega$  and the radius  $R$  of the semi-circle going through  $z_0$  and  $z_1$  :  $a = \frac{1}{2} \left( \frac{x_\Omega}{R} + 1 \right)$ ,  $b = x_\Omega - R$ ,  $c = \frac{1}{2R}$ ,  $d = 1$ , and for all  $t \in [0, 1]$ ,

$$\bar{y}(t) = \bar{y}_0 e^{Kt}, \quad \text{with } K = \ln \frac{\bar{y}_1}{\bar{y}_0}, \quad \bar{y}_0 = -i \frac{az_0 + b}{cz_0 + d} \quad \text{and} \quad \bar{y}_1 = -i \frac{az_1 + b}{cz_1 + d}.$$

The inverse of the exponential map is in turn given by  $\log_{z_0}(z_1) = \dot{x}(0) + i\dot{y}(0)$ , with

$$\dot{x}(0) = \frac{2cdK\bar{y}_0^2}{(d^2 + c^2\bar{y}_0^2)^2}, \quad \dot{y}(0) = \frac{K\bar{y}_0(d^2 - c^2\bar{y}_0^2)}{(d^2 + c^2\bar{y}_0^2)^2}.$$

We now remind the exponential map in  $\mathbb{H}$ .

**Proposition 7** (Exponential map in  $\mathbb{H}$ ). *Let  $z_0 = x_0 + iy_0$  be an element of  $\mathbb{H}$  and  $u_0 = \dot{x}_0 + i\dot{y}_0$  a tangent vector such that  $\dot{x}_0 \neq 0$ . Then the exponential map is given by  $\exp_{z_0}(u_0) = \gamma(1)$ , where  $\gamma(t) = x(t) + iy(t)$  with*

$$x(t) = \frac{bd + ac\bar{y}(t)^2}{d^2 + c^2\bar{y}(t)^2}, \quad y(t) = \frac{\bar{y}(t)}{d^2 + c^2\bar{y}(t)^2}, \quad t \in [0, 1].$$

The coefficients  $a, b, c, d$  of the Moebius transformation can be computed as previously from the center  $x_\Omega = x_0 + y_0 \frac{y_0}{x_0}$  and the radius  $R = \sqrt{(x_0 - x_\Omega)^2 + y_0^2}$  of the semi-circle of the geodesic, and for all  $t \in [0, 1]$ ,

$$\bar{y}(t) = \bar{y}_0 e^{t \frac{\dot{y}_0}{\bar{y}_0}}, \quad \text{with } \bar{y}_0 = -i \frac{az_0 + b}{cz_0 + d} \quad \text{and} \quad \dot{y}_0 = \frac{\dot{x}_0(d^2 + c^2\bar{y}_0^2)^2}{2cd\bar{y}_0}.$$

Finally, we remind how to parallel transport along a geodesic in the hyperbolic plane.

**Proposition 8** (Parallel transport in  $\mathbb{H}$ ). *Let  $t \mapsto \gamma(t)$  be a curve in  $\mathbb{H}$  with coordinates  $x(t), y(t)$ , and  $u_0 \in T_{\gamma(t_0)}\mathbb{H}$  a tangent vector. The parallel transport of  $u_0$  along  $\gamma$  from  $t_0$  to  $t$  is given by*

$$u(t) = \frac{y(t)}{y(t_0)} \begin{pmatrix} \cos \theta(t_0, t) & \sin \theta(t_0, t) \\ -\sin \theta(t_0, t) & \cos \theta(t_0, t) \end{pmatrix} u_0,$$

where  $\theta(t_i, t_f) = \int_{t_i}^{t_f} \frac{\dot{x}(\tau)}{y(\tau)} d\tau$ . If  $\gamma$  is a geodesic, we get

$$\theta(t_i, t_f) = 2(\arg(d + ic\bar{y}(t_f)) - \arg(d + ic\bar{y}(t_i))),$$

where the coefficients  $c$  and  $d$  of the Moebius transformation can be computed as explained previously, and  $\bar{\gamma} = i\bar{y}$  is the pre-image of  $\gamma$  by that transformation.

Using these tools, all computations described in the previous algorithms are explicit. As most of the latter are already detailed, we will only give an explicit and discretized version of the path straightening described in Algorithm 4. Let  $c_0$

and  $c_1$  be piecewise geodesic curves, that is  $c_0|_{[t_i, t_{i+1}]}$  and  $c_1|_{[t_i, t_{i+1}]}$  are portions of geodesic for all  $i = 0, \dots, n-1$ .

**Algorithm 5** (Discrete path straightening in the space of curves in  $\mathbb{H}$ ).

*Input* :  $c_0, c_1$

– *Initialization* : Set  $k = 1$  and for  $i = 0, \dots, n$ ,  $c(\cdot, t_i) = \gamma_i$  where  $\gamma_i$  is the geodesic of  $\mathbb{H}$  linking  $c_0(t_i)$  to  $c_1(t_i)$  and can be computed with Proposition 6. Compute for all  $0 \leq j \leq m-1$  and all  $0 \leq i \leq n$

$$\begin{aligned} c_s(s_j, t_i) &= \frac{1}{\epsilon} \log_{c(s_j, t_i)} c(s_{j+1}, t_i), \\ \nabla_s q(s_j, t_i) &= \frac{1}{\epsilon} (q(s_{j+1}, t_i)^{s_{j+1}, s_j} - q(s_j, t_i)), \end{aligned}$$

where the log is given by Proposition 6 and the parallel transport can be computed using Proposition 8, and set  $E = \sum_{j=0}^{m-1} [\|c_s(s_j, 0)\|^2 + \sum_{i=0}^n \|\nabla_s q(s_j, t_i)\|^2]$ .

– Fix a threshold  $\delta > 0$ .

(1) For all  $j = 0, \dots, m-1$  and all  $i = 0, \dots, n-1$ , set

$$\begin{aligned} c_t(s_j, t_i) &= \frac{1}{t_{i+1} - t_i} \log_{c(s_j, t_i)} c(s_j, t_{i+1}), \\ q(s_j, t_i) &= \frac{c_t(s_j, t_i)}{\sqrt{\|c_t(s_j, t_i)\|}}, \\ r(s_j, t_i) &= \sum_{k=i}^n \left[ \langle q(s_j, t_k), c_s(s_j, t_k) \rangle \nabla_s q(s_j, t_k) \right. \\ &\quad \left. - \langle \nabla_s q(s_j, t_k), c_s(s_j, t_k) \rangle q(s_j, t_k) \right]^{t_k, t_i}, \\ r^\parallel(s_j, t_i) &= \frac{1}{\|c_t(s_j, t_i)\|} \langle r(s_j, t_i), c_t(s_j, t_i) \rangle c_t(s_j, t_i), \end{aligned}$$

(2) For all  $j = 0, \dots, m-2$  and all  $i = 0, \dots, n-1$ , set

$$\begin{aligned} c_s(s_{j+1}, t_i) &= \frac{1}{\epsilon} \log_{c(s_{j+1}, t_i)} c(s_{j+2}, t_i), \\ \nabla_s q(s_{j+1}, t_i) &= \frac{1}{\epsilon} (q(s_{j+2}, t_i)^{s_{j+2}, s_{j+1}} - q(s_{j+1}, t_i)), \\ \nabla_s c_s(s_j, t_i) &= \frac{1}{\epsilon} (c_s(s_{j+1}, t_i)^{s_{j+1}, s_j} - c_s(s_j, t_i)), \\ \nabla_s \nabla_s q(s_j, t_i) &= \frac{1}{\epsilon} (\nabla_s q(s_{j+1}, t_i)^{s_{j+1}, s_j} - \nabla_s q(s_j, t_i)), \\ \text{and } \nabla_s c_s(s_{m-1}, t_i) &= \nabla_s c_s(s_{m-2}, t_i)^{s_{m-2}, s_{m-1}}, \\ \nabla_s \nabla_s q(s_{m-1}, t_i) &= \nabla_s \nabla_s q(s_{m-2}, t_i)^{s_{m-2}, s_{m-1}}. \end{aligned}$$

(3) For all  $j = 0, \dots, m-1$  and all  $i = 0, \dots, n-1$ , compute

$$\begin{aligned} V(s_j, t_i) &= \nabla_s \nabla_s q(s_j, t_i) + \|q(s_j, t_i)\| \left( r(s_j, t_i) + r^\parallel(s_j, t_i) \right), \\ \hat{c}_a(0, s_j, t_i) &= f(s_j) \left[ (\nabla_s c_s(s_j, 0) + r(s_j, 0))^{0, t_i} \right. \\ &\quad \left. + \sum_{k=0}^i \|q(s_j, t_k)\| \left( V(s, t_k) + V^\parallel(s, t_k) \right)^{t_k, t_i} \right]. \end{aligned}$$

with  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $x \mapsto \sqrt{0.25 - (x - 0.5)^2}$ .

(4) Update  $c(s_j, t_i) \leftarrow \exp_{c(s_j, t_i)} \left( \frac{1}{k} \hat{c}_a(0, s_j, t_i) \right)$ ,  $\forall i, j$  and compute

$$\begin{aligned} c_s(s_j, t_i) &= \frac{1}{\epsilon} \log_{c(s_j, t_i)} c(s_{j+1}, t_i), \\ \nabla_s q(s_j, t_i) &= \frac{1}{\epsilon} (q(s_{j+1}, t_i) - q(s_j, t_i)), \\ E(c) &= \sum_{j=0}^{m-1} \left[ \|c_s(s_j, 0)\|^2 + \sum_{i=0}^n \|\nabla_s q(s_j, t_i)\|^2 \right]. \end{aligned}$$

(5) If  $E - E(c) > \delta$ , then set  $E = E(c)$  and  $k \leftarrow k + 1$  and go back to step (1).  
Else, stop.

Output :  $c$  approximation of the geodesic linking  $c_0$  to  $c_1$ .

## 6. CONCLUSION

In the same way that the first-order Sobolev metric (2) on the space of plane curves can be obtained as the pullback of the  $L^2$ -metric by the square root velocity function ([12]), our metric  $G$  on the space of manifold-valued curves can be obtained as the pullback of a natural metric on the tangent bundle  $\mathcal{TM}$  by the same SRVF. As such it is reparametrization invariant, and induces a Riemannian metric on the shape space  $\mathcal{S}$  for which the fiber bundle projection is formally a Riemannian submersion. On the other hand, the special role that  $G$  gives to the starting points of the curves induces another formal fiber bundle structure, this time over the manifold  $M$  seen as the set of starting points of the curves, for which the projection is formally also a Riemannian submersion. The geodesic distance induced by  $G$  takes into account the distance between the origins of the curve in  $M$  and the  $L^2$ -distance between the  $SRV$  representations, without parallel transporting the computations to a unique tangent plane as in [5] and [15]. This should allow us to take into account a greater amount of information on the geometry of the manifold  $M$ . Explicit equations can be obtained for the geodesics, as well as for Jacobi fields. These allow us to iteratively compute the optimal deformation between two curves, either by geodesic shooting using the exponential map, or by path straightening using Jacobi fields. These algorithms are given in the submanifold of piecewise-geodesic curves to reflect the reality of applications, where instead of curves we deal with series of observations. The particular case where the manifold  $M$  is the hyperbolic half-plane illustrates that the algorithms are effectively computable in a simple symmetric space. Future work will include computing means and medians of curves using our metric.

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