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# Nonshifted calculus of variations on time scales with $\nabla$ -differentiable $\sigma$

Loïc Bourdin\*

## Abstract

In calculus of variations on general time scales, an Euler-Lagrange equation of integral form is usually derived in order to characterize the critical points of nonshifted Lagrangian functionals, see *e.g.* [R.A.C. Ferreira and co-authors, *Optimality conditions for the calculus of variations with higher-order delta derivatives*, Appl. Math. Lett., 2011]. In this paper, we prove that the  $\nabla$ -differentiability of the forward jump operator  $\sigma$  is a sharp assumption on the time scale in order to  $\nabla$ -differentiate this integral Euler-Lagrange equation. This procedure leads to an Euler-Lagrange equation of differential form. Furthermore, from this differential form, we prove a Noether-type theorem providing an explicit constant of motion for Euler-Lagrange equations admitting a symmetry.

**Keywords:** Time scale; calculus of variations; Euler-Lagrange equations; Noether's theorem.

**AMS Classification:** 34N05; 39A12; 39A13; 39A10.

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## 1 Introduction

The time scale theory was introduced by S. Hilger in his PhD thesis [18] in 1988 in order to unify discrete and continuous analyses. The general idea is to extend classical theories on an arbitrary non-empty closed subset  $\mathbb{T}$  of  $\mathbb{R}$ . Such a subset  $\mathbb{T}$  is called a *time scale*. In this paper, it is assumed to be bounded with  $a = \min \mathbb{T}$  and  $b = \max \mathbb{T}$ . Hence, the time scale theory establishes the validity of some results both in the continuous case  $\mathbb{T} = [a, b]$  and in the purely discrete case  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$ . Moreover, it also treats more general models involving

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both continuous and discrete time elements, see *e.g.* [15, 30] for dynamical populations whose generations do not overlap. Another example of application is to consider  $\mathbb{T} = \{a\} \cup \{a + \lambda^{\mathbb{N}}\}$  with  $0 < \lambda < 1$  allowing the time scale theory to cover the quantum calculus [23].

Since S. Hilger defined the  $\Delta$ - and  $\nabla$ -derivatives on time scales, many authors have extended to time scales various results from the continuous or discrete standard calculus theory. We refer to the surveys [1, 2, 9, 10] of M. Bohner and co-authors. In the continuous case  $\mathbb{T} = [a, b]$ , the operators  $\Delta$  and  $\nabla$  coincide with the usual derivative operator  $d/dt$ , *i.e.*  $u^\Delta = u^\nabla = \dot{u}$  where  $\dot{u}$  is the classical derivative of a function  $u$ . In the discrete case  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$ ,  $\Delta$  is the usual forward Euler approximation of  $d/dt$ , *i.e.*  $\Delta u(t_k) = (u(t_{k+1}) - u(t_k))/(t_{k+1} - t_k)$  and similarly,  $\nabla$  is the usual backward Euler approximation of  $d/dt$ , *i.e.*  $\nabla u(t_k) = (u(t_k) - u(t_{k-1}))/t_k - t_{k-1}$ .

**Context in shifted calculus of variations.** The pioneering work on calculus of variations on general time scales is due to M. Bohner in [7]. In particular, he obtains a necessary condition for local optimizers of Lagrangian functionals of type

$$\mathcal{L}(u) = \int_a^b L(u^\sigma(\tau), u^\Delta(\tau), \tau) \Delta\tau, \quad (1)$$

where  $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ ,  $(x, v, t) \mapsto L(x, v, t)$  is a continuous Lagrangian of class  $\mathcal{C}^1$  in its two first variables,  $u$  is a  $C_{\text{rd}}^{1,\Delta}(\mathbb{T})$ -function,  $u^\sigma = u \circ \sigma$ ,  $u^\Delta$  is the  $\Delta$ -differential of  $u$  and  $\int \Delta\tau$  denotes the Cauchy  $\Delta$ -integral defined in [9, p.26]. We refer to Section 2.1 for the precise definitions.

*Notation:* in the whole paper, the function defined by  $t \mapsto \partial L / \partial v(u^\sigma(t), u^\Delta(t), t)$ , where  $u \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$ , is denoted by  $\partial L / \partial v(u^\sigma, u^\Delta, \cdot)$ .

Precisely, M. Bohner characterizes the critical points of  $\mathcal{L}$  as the solutions of the following  $\Delta \circ \Delta$ -differential Euler-Lagrange equation, see [7, Theorem 4.2]:

$$\left[ \frac{\partial L}{\partial v}(u^\sigma, u^\Delta, \cdot) \right]^\Delta(t) = \frac{\partial L}{\partial x}(u^\sigma(t), u^\Delta(t), t). \quad (\text{EL}_{\text{diff}}^{\Delta \circ \Delta})$$

Here, the notation  $\Delta \circ \Delta$  refers to the composition of  $\Delta$  with itself in the left-hand term of  $(\text{EL}_{\text{diff}}^{\Delta \circ \Delta})$ . As it is mentioned in [7], the result of M. Bohner recovers the usual continuous case  $\mathbb{T} = [a, b]$  (where  $\sigma$  is the identity) where the critical points of Lagrangian functionals of type

$$\mathcal{L}(u) = \int_a^b L(u(\tau), \dot{u}(\tau), \tau) d\tau \quad (2)$$

are characterized by the solutions of the well known continuous Euler-Lagrange equation (see *e.g.* [4, p.12]) given by

$$\frac{d}{dt} \left[ \frac{\partial L}{\partial v}(u, \dot{u}, \cdot) \right](t) = \frac{\partial L}{\partial x}(u(t), \dot{u}(t), t). \quad (3)$$

Moreover, the work of M. Bohner in [7] also recovers the following discrete case  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$  where the critical points of discrete Lagrangian functionals of type

$$\mathcal{L}(u) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(u(t_{k+1}), \Delta u(t_k), t_k) \quad (4)$$

are characterized by the solutions of the well known discrete Euler-Lagrange equation (see *e.g.* [3]) given by

$$\Delta \left[ \frac{\partial L}{\partial v}(u^\sigma, \Delta u, \cdot) \right](t_k) = \frac{\partial L}{\partial x}(u(t_{k+1}), \Delta u(t_k), t_k). \quad (5)$$

In what follows, we will speak about  $\mathcal{L}$  (defined in (1)) as a *shifted* Lagrangian functional in reference to the presence of  $u^\sigma$  (instead of  $u$ ) in its definition. Note that this characteristic has no consequence on the continuous case but let us mention the presence of  $u(t_{k+1})$  (instead of  $u(t_k)$ ) in the discrete case. We will see that this difference is important at the discrete level and *a fortiori* at the time scale one too. In particular, this shifted framework does not cover the variational integrator studied in [17, 27]. We refer to the next paragraph for more details.

Since the publication of [7], the *shifted* calculus of variations is widely investigated in several directions: with double integral [8], with higher-order  $\Delta$ -derivatives [14], with nonfixed boundary conditions and transversality conditions [20], with double integral mixing  $\Delta$ - and  $\nabla$ -derivatives [25], with higher-order  $\nabla$ -derivatives [28], etc. We also refer to [21, 22] for *shifted* optimal control problems. Let us mention that *shifted* variational problems are particularly suitable (in comparison with the *nonshifted* ones) because of the emergence of a *shift* in the integration by parts formula on time scales (see [9, Theorem 1.77 p.28]) given by

$$\int_a^b u(\tau) \cdot v^\Delta(\tau) \Delta\tau = u(b) \cdot v(b) - u(a) \cdot v(a) - \int_a^b u^\Delta(\tau) \cdot v^\sigma(\tau) \Delta\tau. \quad (6)$$

**Context in nonshifted calculus of variations.** As mentioned in the previous paragraph, the *shifted* calculus of variations on general time scales developed in [7] does not cover an important discrete calculus of variations. Precisely, recall that the critical points of (nonshifted) discrete Lagrangian functionals of type

$$\mathcal{L}(u) = \sum_{k=0}^{N-1} (t_{k+1} - t_k) L(u(t_k), \Delta u(t_k), t_k) \quad (7)$$

are characterized by the solutions of the well known discrete Euler-Lagrange equation (see *e.g.* [17]) given by

$$\nabla \left[ \frac{\partial L}{\partial v}(u, \Delta u, \cdot) \right] (t_k) = \frac{t_{k+1} - t_k}{t_k - t_{k-1}} \frac{\partial L}{\partial x}(u(t_k), \Delta u(t_k), t_k). \quad (8)$$

Recall that the above discrete Euler-Lagrange equation (8) corresponds to the variational integrator obtained and well studied in [17, 27]. In particular, it is an efficient numerical scheme for the continuous Euler-Lagrange equation (3) preserving its variational structure and relative properties at the discrete level. In this (nonshifted) discrete case, note the emergence of the composition between the operators  $\nabla$  and  $\Delta$ . In numerical analysis, it is well known that such a composition allows to get a larger order of convergence.

Up to our knowledge, only few studies treat on the *nonshifted* calculus of variations on general time scales, see [12, 13, 19]. In these papers, the critical points of (nonshifted) Lagrangian functionals of type

$$\mathcal{L}(u) = \int_a^b L(u(\tau), u^\Delta(\tau), \tau) \Delta\tau \quad (9)$$

are characterized by the solutions of the following  $\Delta$ -integral Euler-Lagrange equation, see *e.g.* [19, Theorem 4]:

$$\frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) = \int_a^{\sigma(t)} \frac{\partial L}{\partial x}(u(\tau), u^\Delta(\tau), \tau) \Delta\tau + c. \quad (\text{EL}_{\text{int}}^\Delta)$$

The objective of this paper is to  $\nabla$ -differentiate  $(\text{EL}_{\text{int}}^\Delta)$  in order to get a  $\nabla \circ \Delta$ -differential Euler-Lagrange equation of type

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) \right]^\nabla (t) = \omega(t) \frac{\partial L}{\partial x}(u(t), u^\Delta(t), t) \quad (10)$$

that encompasses both the classical continuous case (3) and the (nonshifted) discrete case (8). Towards this goal, a difficulty emerges due to the presence of  $\sigma$  in the upper bound of the  $\Delta$ -integral in  $(\text{EL}_{\text{int}}^\Delta)$ . In fact, we will exhibit a counter-example (see Example 1) showing that we

cannot  $\nabla$ -differentiate  $(\text{EL}_{\text{int}}^\Delta)$  on general time scales with a general Lagrangian  $L$ . It leads us to consider a subclass of time scales. Precisely, we prove that the  $\nabla$ -differentiability of  $\sigma$  is a sharp assumption on the time scale in order to  $\nabla$ -differentiate  $(\text{EL}_{\text{int}}^\Delta)$  and to obtain a  $\nabla \circ \Delta$ -differential Euler-Lagrange equation of type (10). Moreover, in such a case, an interesting phenomena is the direct emergence of this assumption in (10) since we prove that  $\omega = \sigma^\nabla$ .

In this paper, we study the consequences of the  $\nabla$ -differentiability of  $\sigma$  on the structure of  $\mathbb{T}$ . In particular, we will see how this assumption allows us to apply  $\nabla$  on  $(\text{EL}_{\text{int}}^\Delta)$  in order to obtain (10) with  $\omega = \sigma^\nabla$ . Note that the  $\nabla$ -differentiability of  $\sigma$  is not a loss of generality since it is satisfied in the continuous case  $\mathbb{T} = [a, b]$  (with  $\sigma^\nabla = 1$ ) and in the discrete case  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$  (with  $\sigma^\nabla(t_k) = (t_{k+1} - t_k)/(t_k - t_{k-1})$ ). In particular, note that our main result recovers both the usual continuous case (3) and the nonshifted discrete case (8).

**Derivation of Noether-type results.** In *shifted* calculus of variations, we refer to the paper [6] studying the existence of a constant of motion for  $\Delta \circ \Delta$ -differential Euler-Lagrange equations  $(\text{EL}_{\text{diff}}^{\Delta \circ \Delta})$ . We refer to [29] for a similar study with  $\nabla$ -derivatives. The common strategy is to generalize the celebrated Noether's theorem [24, 32] to time scales. Precisely, under invariance assumption on the Lagrangian  $L$ , authors prove that a conservation law can be obtained.

In *nonshifted* calculus of variations, the nondifferential form of  $(\text{EL}_{\text{int}}^\Delta)$  is an obstruction in order to develop the same strategy. A direct application of our main result is then to provide a Noether-type theorem based on the differential form (10). This will be done in Section 4.

**Remark 1.** *For sake of completeness of this introduction, we mention that an Euler-Lagrange equation of differential form in the nonshifted case is obtained in [19, Remark 4]. Precisely, the author characterizes the critical points of  $\mathcal{L}$  (defined in (9)) as the solutions of the following differential Euler-Lagrange equation:*

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) - \mu \frac{\partial L}{\partial x}(u, u^\Delta, \cdot) \right]^\Delta(t) = \frac{\partial L}{\partial x}(u(t), u^\Delta(t), t). \quad (11)$$

*The advantage of this result is to be valid on every time scale. Nevertheless, this differential form does not directly coincide with the usual discrete Euler-Lagrange equation (8) and the obtaining of a Noether-type result from this differential form remains an open problem. These observations both give a particular interest to the  $\nabla$ -differentiation of  $(\text{EL}_{\text{int}}^\Delta)$  and to the  $\nabla \circ \Delta$ -differential formulation (10).*

**Remark 2.** *Finally, it has to be noted that our whole study is made in terms of Lagrangian functionals involving  $\Delta$ -integral and  $\Delta$ -derivative. However, thanks to the duality principle introduced in [11], all results can be analogously derived for Lagrangian functionals involving  $\nabla$ -integral and  $\nabla$ -derivative. This will be done in Section 5.*

**Organization of the paper.** We first give basic recalls on time scale calculus in Section 2.1 and on nonshifted calculus of variations on general time scales in Section 2.2. Section 2.3 is devoted to our main result (Theorem 1). A study on time scales with continuous  $\sigma$  is provided in Section 3.1 and with  $\nabla$ -differentiable  $\sigma$  in Sections 3.2 and 3.3. The results obtained in Section 3 are instrumental in order to prove Theorem 1. We prove a Noether-type theorem (Theorem 2) in Section 4. In Section 5, we conclude this paper with the analogous results for nonshifted calculus of variations defined in terms of  $\nabla$ -integral and  $\nabla$ -derivative.

## 2 Nonshifted calculus of variations on time scales with $\nabla$ -differentiable $\sigma$

In this paper,  $\mathbb{N}$  denotes the set of nonnegative integers,  $\mathbb{N}^*$  denotes the set of positive integers and  $\mathbb{R}^+$  denotes the set of nonnegative reals. We denote by  $\mathbb{T}$  a bounded time scale with  $a = \min(\mathbb{T})$ ,  $b = \max(\mathbb{T})$  and  $\text{card}(\mathbb{T}) \geq 3$ . In Section 2.1, we give basic recalls on time scale calculus.

Section 2.2 is devoted to recalls on nonshifted calculus of variations on general time scales developed in [12, 13, 19]. In particular, the  $\Delta$ -integral Euler-Lagrange equation ( $\text{EL}_{\text{int}}^{\Delta}$ ) is given as a necessary condition for local optimizers of nonshifted Lagrangian functionals, see Proposition 3.

In Section 2.3, under the assumption of  $\nabla$ -differentiability of  $\sigma$ , our main result provides a  $\nabla \circ \Delta$ -differential Euler-Lagrange equation of type (10) as a necessary condition for local optimizers of nonshifted Lagrangian functionals, see ( $\text{EL}_{\text{diff}}^{\nabla \circ \Delta}$ ) in Theorem 1. We also prove that this assumption is sharp, see Example 1.

## 2.1 Basic recalls on time scale calculus

We refer to the surveys [1, 2, 9, 10] for more details on time scale calculus. The backward and forward jump operators  $\rho, \sigma : \mathbb{T} \rightarrow \mathbb{T}$  are respectively defined by

$$\forall t \in \mathbb{T}, \rho(t) = \sup\{s \in \mathbb{T}, s < t\} \text{ and } \sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \quad (12)$$

where we put  $\sup \emptyset = a$  and  $\inf \emptyset = b$ . A point  $t \in \mathbb{T}$  is said to be right-scattered (resp. left-scattered) if  $\sigma(t) > t$  (resp.  $\rho(t) < t$ ). A point  $t \in \mathbb{T}$  is said to be right-dense (resp. left-dense) if  $t \neq b$  and  $\sigma(t) = t$  (resp.  $t \neq a$  and  $\rho(t) = t$ ). Let RS (resp. LS, RD and LD) denote the set of all right-scattered (resp. left-scattered, right-dense and left-dense) points of  $\mathbb{T}$ . Note that  $a \notin \text{LD}$  and  $b \notin \text{RD}$ . The graininess (resp. backward graininess) function  $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$  (resp.  $\nu : \mathbb{T} \rightarrow \mathbb{R}^+$ ) is defined by  $\mu(t) = \sigma(t) - t$  (resp.  $\nu(t) = t - \rho(t)$ ) for any  $t \in \mathbb{T}$ .

We set  $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \rho(b), b]$ ,  $\mathbb{T}_{\kappa} = \mathbb{T} \setminus [a, \sigma(a)[$  and  $\mathbb{T}_{\kappa}^{\kappa} = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$ . Note that  $\mathbb{T}_{\kappa}^{\kappa} \neq \emptyset$  since  $\text{card}(\mathbb{T}) \geq 3$ . Let us recall the usual definitions of  $\Delta$ - and  $\nabla$ -differentiability. A function  $u : \mathbb{T} \rightarrow \mathbb{R}^n$ , where  $n \in \mathbb{N}^*$ , is said to be  $\Delta$ -differentiable at  $t \in \mathbb{T}^{\kappa}$  (resp.  $\nabla$ -differentiable at  $t \in \mathbb{T}_{\kappa}$ ) if the following limit exists in  $\mathbb{R}^n$ :

$$\lim_{\substack{s \rightarrow t \\ s \neq \sigma(t)}} \frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} \left( \text{resp. } \lim_{\substack{s \rightarrow t \\ s \neq \rho(t)}} \frac{u(s) - u(\rho(t))}{s - \rho(t)} \right). \quad (13)$$

In such a case, this limit is denoted by  $u^{\Delta}(t)$  (resp.  $u^{\nabla}(t)$ ). Let us recall the following results on  $\Delta$ -differentiability, see [9, Theorem 1.16 p.5] and [9, Corollary 1.68 p.25]. The analogous results for  $\nabla$ -differentiability are also valid.

**Proposition 1.** *Let  $u : \mathbb{T} \rightarrow \mathbb{R}^n$  and  $t \in \mathbb{T}^{\kappa}$ . The following properties hold:*

1. *if  $u$  is  $\Delta$ -differentiable at  $t$ , then  $u$  is continuous at  $t$ .*
2. *if  $t \in \text{RS}$  and if  $u$  is continuous at  $t$ , then  $u$  is  $\Delta$ -differentiable at  $t$  with*

$$u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)}. \quad (14)$$

3. *if  $\sigma(t) = t$ , then  $u$  is  $\Delta$ -differentiable at  $t$  if and only if the following limit exists in  $\mathbb{R}^n$ :*

$$\lim_{\substack{s \rightarrow t \\ s \neq t}} \frac{u(t) - u(s)}{t - s}. \quad (15)$$

*In such a case, this limit is equal to  $u^{\Delta}(t)$ .*

**Proposition 2.** *Let  $u : \mathbb{T} \rightarrow \mathbb{R}^n$ . Then,  $u$  is  $\Delta$ -differentiable on  $\mathbb{T}^{\kappa}$  with  $u^{\Delta} = 0$  if and only if there exists  $c \in \mathbb{R}^n$  such that  $u(t) = c$  for every  $t \in \mathbb{T}$ .*

From Proposition 1 and for every  $t \in \text{RS}$ , note that a function  $u$  is  $\Delta$ -differentiable at  $t$  if and only if  $u$  is continuous at  $t$ . Still from Proposition 1, note that every  $\Delta$ -differentiable function on  $\mathbb{T}^{\kappa}$  is continuous on  $\mathbb{T}$ .

Recall that a function  $u$  is said to be rd-continuous on  $\mathbb{T}$  if it is continuous at every  $t \in \text{RD}$  and if it admits a left-sided limit at every  $t \in \text{LD}$ , see [9, Definition 1.58 p.22]. We respectively denote by  $C_{\text{rd}}^0(\mathbb{T})$  and  $C_{\text{rd}}^{1,\Delta}(\mathbb{T})$  the functional spaces of rd-continuous functions on  $\mathbb{T}$  and of  $\Delta$ -differentiable functions on  $\mathbb{T}^\kappa$  with rd-continuous  $\Delta$ -derivative. Recall the following results, see [9, Theorem 1.60 p.22]:

- $\sigma$  is rd-continuous.
- if  $u \in C_{\text{rd}}^0(\mathbb{T})$ , the composition  $u^\sigma = u \circ \sigma$  is rd-continuous.
- if  $u \in C_{\text{rd}}^0(\mathbb{T})$ , the composition  $f \circ u$  with any continuous function  $f$  is rd-continuous.

Let us denote by  $\int \Delta\tau$  the Cauchy  $\Delta$ -integral defined in [9, p.26]. For every  $u \in C_{\text{rd}}^0(\mathbb{T}^\kappa)$ , recall that the function  $U$ , defined by  $U(t) = \int_a^t u(\tau)\Delta\tau$  for every  $t \in \mathbb{T}$ , is the unique  $\Delta$ -antiderivative of  $u$  (in the sense that  $U^\Delta = u$  on  $\mathbb{T}^\kappa$ ) vanishing at  $t = a$ , see [9, Theorem 1.74 p.27]. In particular, we have  $U \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$ .

## 2.2 Recalls on nonshifted calculus of variations on general time scales

In this section, we recall some results on nonshifted calculus of variations on general time scales provided in [12, 13, 19]. Let  $L$  be a Lagrangian *i.e.* a continuous map of class  $\mathcal{C}^1$  in its two first variables

$$\begin{aligned} L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}^\kappa &\longrightarrow \mathbb{R} \\ (x, v, t) &\longmapsto L(x, v, t) \end{aligned} \quad (16)$$

and let  $\mathcal{L}$  be the following (nonshifted) Lagrangian functional:

$$\begin{aligned} \mathcal{L} : C_{\text{rd}}^{1,\Delta}(\mathbb{T}) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_a^b L(u(\tau), u^\Delta(\tau), \tau) \Delta\tau. \end{aligned} \quad (17)$$

In this section, our aim is to give a necessary condition for local optimizers of  $\mathcal{L}$  (with or without boundary conditions at  $t = a$  and  $t = b$ ). In this way, we introduce the following notions and notations:

- $C_{\text{rd},0}^{1,\Delta}(\mathbb{T}) = \{w \in C_{\text{rd}}^{1,\Delta}(\mathbb{T}), w(a) = w(b) = 0\}$  is called the set of *variations* of  $\mathcal{L}$ .
- $u \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$  is said to be a *critical point* of  $\mathcal{L}$  if  $D\mathcal{L}(u)(w) = 0$  for every  $w \in C_{\text{rd},0}^{1,\Delta}(\mathbb{T})$ . Let us precise that  $D\mathcal{L}(u)(w)$  denotes the Gâteaux-differential of  $\mathcal{L}$  at  $u$  in direction  $w$ .

In particular, if  $u$  is a local optimizer of  $\mathcal{L}$ , then  $u$  is a critical point of  $\mathcal{L}$ . Finally, let us recall the following characterization of the critical points of  $\mathcal{L}$ , see [12, Theorem 11], [13, Corollary 1] or [19, Theorem 4].

**Proposition 3.** *Let  $u \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$ . Then,  $u$  is a critical point of  $\mathcal{L}$  if and only if there exists  $c \in \mathbb{R}^n$  such that*

$$\frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) = \int_a^{\sigma(t)} \frac{\partial L}{\partial x}(u(\tau), u^\Delta(\tau), \tau) \Delta\tau + c, \quad (\text{EL}_{\text{int}}^\Delta)$$

for every  $t \in \mathbb{T}^\kappa$ .

Hence, Proposition 3 provides a necessary condition for local optimizers of  $\mathcal{L}$ . Precisely, if  $u$  is a local optimizer of  $\mathcal{L}$ , then there exists  $c \in \mathbb{R}^n$  such that  $u$  satisfies the  $\Delta$ -integral Euler-Lagrange equation  $(\text{EL}_{\text{int}}^\Delta)$ . We refer to Example 1 in Section 2.3 for an application of Proposition 3.

### 2.3 Main result

In this paper, we aim to  $\nabla$ -differentiate the  $\Delta$ -integral Euler-Lagrange equation ( $\text{EL}_{\text{int}}^{\Delta}$ ) in order to get a  $\nabla \circ \Delta$ -differential one of type (10). Precisely, we prove the following result under the assumption of  $\nabla$ -differentiability of  $\sigma$ .

**Theorem 1** (Main result). *Let us assume that  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_{\kappa}$  and let  $u \in \text{C}_{\text{rd}}^{1,\Delta}(\mathbb{T})$ . Then,  $u$  is a critical point of  $\mathcal{L}$  if and only if  $u$  is a solution of the following  $\nabla \circ \Delta$ -differential Euler-Lagrange equation:*

$$\left[ \frac{\partial L}{\partial v}(u, u^{\Delta}, \cdot) \right]^{\nabla}(t) = \sigma^{\nabla}(t) \frac{\partial L}{\partial x}(u(t), u^{\Delta}(t), t), \quad (\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$$

for every  $t \in \mathbb{T}_{\kappa}^{\kappa}$ .

*Proof.* We refer to Propositions 2 and 3 and Corollary 1 in Section 3.3. □

Note that this result encompasses both the usual continuous and discrete Euler-Lagrange equations given by (3) and (8) in Introduction. Indeed, as it is mentioned in Example 4 in Section 3.2, the following properties are satisfied:

- if  $\mathbb{T} = [a, b]$ ,  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_{\kappa}$  with  $\sigma^{\nabla} = 1$ .
- if  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$ ,  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_{\kappa}$  with  $\sigma^{\nabla} = \mu/\nu$ .

Let us prove, from the following simple example, that the assumption of  $\nabla$ -differentiability of  $\sigma$  is sharp for the validity of Theorem 1 when the Lagrangian  $L$  is general (*i.e.* not specified).

**Example 1.** *Let us consider  $n = 1$ ,  $L(x, v, t) = x + v^2/2$  and  $u \in \text{C}_{\text{rd}}^{1,\Delta}(\mathbb{T})$  defined by  $u(t) = \int_a^t \sigma(\tau) \Delta \tau$  for every  $t \in \mathbb{T}$ . Since  $u$  satisfies ( $\text{EL}_{\text{int}}^{\Delta}$ ) with  $c = a$ , we conclude that  $u$  is a critical point of  $\mathcal{L}$ , see Proposition 3. However, note that  $\partial L / \partial v(u, u^{\Delta}, \cdot) = u^{\Delta} = \sigma$  and consequently, the  $\nabla \circ \Delta$ -differential Euler-Lagrange equation ( $\text{EL}_{\text{diff}}^{\nabla \circ \Delta}$ ) has no sense if  $\sigma$  is not  $\nabla$ -differentiable.*

Nevertheless, the following example proves that the  $\nabla$ -differentiability of  $\sigma$  is not necessary for some Lagrangian (*e.g.* independent of the variable  $x$ ).

**Example 2.** *Let us consider  $n = 1$ ,  $L(x, v, t) = v^2/2$  and  $u \in \text{C}_{\text{rd}}^{1,\Delta}(\mathbb{T})$  defined by  $u(t) = t$  for every  $t \in \mathbb{T}$ . Since  $u$  satisfies ( $\text{EL}_{\text{int}}^{\Delta}$ ) with  $c = 1$ , we conclude that  $u$  is a critical point of  $\mathcal{L}$ , see Proposition 3. Note that  $u$  also satisfies ( $\text{EL}_{\text{diff}}^{\nabla \circ \Delta}$ ) even if  $\sigma$  is not  $\nabla$ -differentiable. This phenomena comes from  $\partial L / \partial x = 0$  and consequently ( $\text{EL}_{\text{int}}^{\Delta}$ ) is independent of  $\sigma$  and it can be  $\nabla$ -differentiated.*

However, it has to be noted that the independence of the Lagrangian  $L$  with respect to the variable  $x$  is a very restrictive assumption while the  $\nabla$ -differentiability of  $\sigma$  is not. We refer to Example 3 for time scales with continuous and noncontinuous  $\sigma$ . We respectively refer to Examples 4 and 5 for time scales with  $\nabla$ -differentiable and non- $\nabla$ -differentiable  $\sigma$ .

## 3 Time scales with $\nabla$ -differentiable $\sigma$

In Section 3.1, we study the consequences of the continuity of  $\sigma$ . Then, we study the consequences of the  $\nabla$ -differentiability of  $\sigma$  in Section 3.2. From these preliminaries, we prove the most important results of this section (Proposition 6 and some corollaries) in Section 3.3. In particular, Corollary 1 is instrumental to prove our main result (Theorem 1).



### 3.1 Continuity of $\sigma$

Let us prove the following characterizations of the continuity of  $\sigma$  at a given point  $t \in \mathbb{T}_\kappa$ .

**Lemma 1.** *Let  $t \in \mathbb{T}_\kappa$ . The following properties are equivalent:*

1.  $\sigma$  is continuous at  $t$ ;
2.  $\sigma \circ \rho(t) = t$ ;
3.  $t \notin \text{RS} \cap \text{LD}$ .

*Proof.* Let us prove that 1. implies 2.. If  $t = a \in \mathbb{T}_\kappa$ , then  $a \in \text{RD}$ . Then,  $\sigma \circ \rho(a) = \sigma(a) = a$ . If  $t \neq a$ , let us assume by contradiction that  $\sigma \circ \rho(t) \neq t$ . Necessarily, we have  $t \in \text{RS} \cap \text{LD}$ . Then, let  $(s_k) \subset \mathbb{T}$  be a sequence such that  $s_k < t$  for any  $k \in \mathbb{N}$  and  $s_k \rightarrow t$ . Thus, we have  $\sigma(s_k) < t < \sigma(t)$  for any  $k \in \mathbb{N}$  and consequently,  $(\sigma(s_k))$  does not tend to  $\sigma(t)$ . This is a contradiction with the continuity of  $\sigma$  at  $t$ .

Let us prove that 2. implies 3.. If  $t \in \text{RS} \cap \text{LD}$ , then  $\sigma \circ \rho(t) = \sigma(t) \neq t$ .

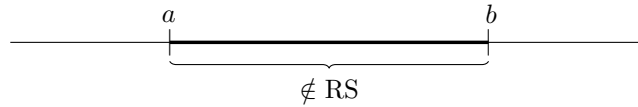
Let us prove that 3. implies 1.. By contradiction, let us assume that  $\sigma$  is not continuous at  $t$ . As a consequence, there exist  $\varepsilon > 0$  and a monotone sequence  $(s_k) \subset \mathbb{T}$  such that  $s_k \rightarrow t$  and  $|\sigma(t) - \sigma(s_k)| \geq \varepsilon$  for every  $k \in \mathbb{N}$ . Firstly, let us assume that  $(s_k)$  is decreasing. Then,  $t \in \text{RD}$  and we have  $t < s_k < s_{k-1}$ . As a consequence,  $t = \sigma(t) < \sigma(s_k) \leq s_{k-1}$  for any  $k \in \mathbb{N}^*$ . It is a contradiction since  $s_{k-1} \rightarrow t$ . Secondly, let us assume that  $(s_k)$  is increasing. As a consequence,  $t \in \text{LD}$  and then  $t \notin \text{RS}$  (see 3.) *i.e.*  $\sigma(t) = t$ . Finally, we have  $s_{k-1} < s_k < t$  and then,  $s_{k-1} < \sigma(s_k) < t = \sigma(t)$  for any  $k \in \mathbb{N}^*$ . It is a contradiction since  $s_{k-1} \rightarrow t$ . In both cases, we have obtained a contradiction.  $\square$

Note that  $\sigma$  is always continuous at  $a$ . Indeed, if  $a \in \text{RD}$ , then  $a \in \mathbb{T}_\kappa$ ,  $a \notin \text{RS} \cap \text{LD}$  and Lemma 1 concludes. If  $a \in \text{RS}$ , then  $a$  is isolated and thus,  $\sigma$  is continuous at  $a$ . This remark and Lemma 1 lead to the following proposition.

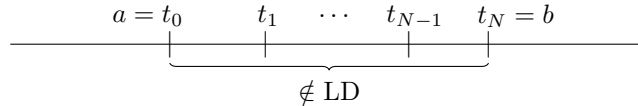
**Proposition 4.**  *$\sigma$  is continuous on  $\mathbb{T}$  if and only if  $\text{RS} \cap \text{LD} = \emptyset$ .*

A similar remark is already done in [9, Example 1.55]. Let us see some examples and counter-examples.

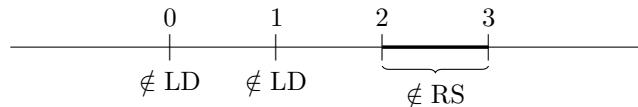
**Example 3.** 1. If  $\mathbb{T} = [a, b]$ ,  $\sigma$  is continuous on  $\mathbb{T}$ .



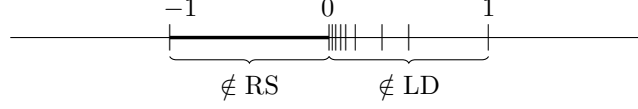
2. If  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$ ,  $\sigma$  is continuous on  $\mathbb{T}$ .



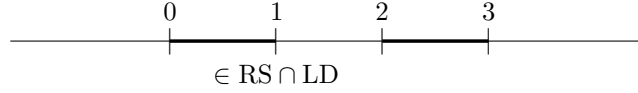
3. If  $\mathbb{T} = \{0, 1\} \cup [2, 3]$ ,  $\sigma$  is continuous on  $\mathbb{T}$ .



4. If  $\mathbb{T} = [-1, 0] \cup \{1/k, k \in \mathbb{N}^*\}$ ,  $\sigma$  is continuous on  $\mathbb{T}$ .



5. If  $\mathbb{T} = [0, 1] \cup [2, 3]$ ,  $\sigma$  is not continuous at  $1 \in \text{RS} \cap \text{LD}$ .



6. If  $\mathbb{T}$  is the usual Cantor set (see [9, Example 1.47 p.18]),  $\sigma$  is not continuous at  $1/3 \in \text{RS} \cap \text{LD}$ .

**Remark 3.** Let us give a short discussion on the notion of regular time scale originally introduced in [16, Definition 9]. We refer to [5, 31] for other applications of this notion. Recall that  $\mathbb{T}$  is said to be regular if for every  $t \in \mathbb{T}$ ,  $\sigma \circ \rho(t) = \rho \circ \sigma(t) = t$ . In particular, for every bounded regular time scale (containing at least two elements),  $a$  is necessarily right-dense and  $b$  is necessarily left-dense, see [16, Proposition 10]. Hence, all finite time scales (containing at least two elements) are not regular. The regularity of a time scale is then an assumption relatively restrictive.

Consequently, we suggest the introduction of the following weakened notion: a time scale is said to be quasi-regular if  $\sigma$  and  $\rho$  are continuous on  $\mathbb{T}$ . Hence, a time scale is quasi-regular if and only if  $\sigma \circ \rho(t) = t$  for every  $t \in \mathbb{T}_\kappa$  and  $\rho \circ \sigma(t) = t$  for every  $t \in \mathbb{T}^\kappa$ . Such a weakened notion allows to cover the finite time scales and to preserve the essence of the initial notion.

### 3.2 $\nabla$ -differentiability of $\sigma$

From Lemma 1 and from the nabla version of Proposition 1, we derive the following result.

**Proposition 5.** The following properties are satisfied:

1. if  $\sigma$  is continuous at  $t \in \text{LS}$ , then  $\sigma$  is directly  $\nabla$ -differentiable at  $t$  with  $\sigma^\nabla(t) = \mu(t)/\nu(t)$ .
2. if  $\sigma$  is continuous on  $\mathbb{T}$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$  if and only if for every  $t \in \mathbb{T}_\kappa$  such that  $\rho(t) = t$ , the following limit exists in  $\mathbb{R}$ :

$$\lim_{\substack{s \rightarrow t \\ s \neq t}} \frac{\sigma(s) - t}{s - t}. \quad (18)$$

In such a case, this limit is equal to  $\sigma^\nabla(t)$ .

*Proof.* Let us prove the first point. From the nabla version of Proposition 1 and since  $\sigma$  is continuous at  $t \in \text{LS} \subset \mathbb{T}_\kappa$ ,  $\sigma$  is directly  $\nabla$ -differentiable at  $t$  with

$$\sigma^\nabla(t) = \frac{\sigma(t) - \sigma(\rho(t))}{\nu(t)} = \frac{\sigma(t) - t}{\nu(t)} = \frac{\mu(t)}{\nu(t)}, \quad (19)$$

since  $\sigma \circ \rho(t) = t$  from Lemma 1.

Let us prove the second point. Since  $\sigma$  is continuous on  $\mathbb{T}$ ,  $\sigma$  is directly  $\nabla$ -differentiable at every  $t \in \text{LS}$  from the first point. Consequently,  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$  if and only if  $\sigma$  is  $\nabla$ -differentiable at every  $t \in \mathbb{T}_\kappa$  such that  $\rho(t) = t$  i.e. if and only if for every  $t \in \mathbb{T}_\kappa$  such that  $\rho(t) = t$ , the following limit exists in  $\mathbb{R}$ :

$$\lim_{\substack{s \rightarrow t \\ s \neq \rho(t)}} \frac{\sigma(s) - \sigma(\rho(t))}{s - \rho(t)} = \lim_{\substack{s \rightarrow t \\ s \neq t}} \frac{\sigma(s) - \sigma(t)}{s - t}. \quad (20)$$

To conclude, it is sufficient to note that if  $t = a \in \mathbb{T}_\kappa$ , then  $a \in \text{RD}$  and  $\sigma(a) = a$  and if  $t \neq a$ , then  $t \in \text{LD}$  and the continuity of  $\sigma$  implies that  $t \notin \text{RS}$  (see Proposition 4) i.e.  $\sigma(t) = t$ . The proof is complete.  $\square$

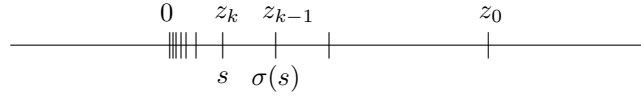
Let us give some examples of time scale with  $\nabla$ -differentiable  $\sigma$ .

**Example 4.** 1. If  $\mathbb{T} = [a, b]$ ,  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$  with  $\sigma^\nabla = 1$ .

2. If  $\mathbb{T} = \{a = t_0 < \dots < t_N = b\}$ ,  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$  with  $\sigma^\nabla = \mu/\nu$ .

3. If  $\mathbb{T} = \{0\} \cup \{z_k, k \in \mathbb{N}\}$  where  $(z_k)$  is a decreasing positive sequence tending to 0 and if  $\lim_{k \rightarrow \infty} z_{k-1}/z_k$  exists (denoted by  $\ell$ ), then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = \ell$ . Indeed, let  $(s_k) \subset \mathbb{T}$  be a positive sequence tending to 0. Then, for every  $k \in \mathbb{N}$ , there exists  $p_k \in \mathbb{N}$  such that  $s_k = z_{p_k}$ . Since  $s_k \rightarrow 0$ , we have  $p_k \rightarrow +\infty$ . Finally, we obtain

$$\lim_{k \rightarrow \infty} \frac{\sigma(s_k) - 0}{s_k - 0} = \lim_{k \rightarrow \infty} \frac{z_{p_k-1}}{z_{p_k}} = \ell. \quad (21)$$



4. Application: if  $\mathbb{T} = \{0\} \cup \{1/r^k, k \in \mathbb{N}\}$  with  $r > 1$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = r$ .

5. Similarly to 3., we can prove that if  $\mathbb{T} = \{0\} \cup \{z_k, k \in \mathbb{N}\}$  where  $(z_k)$  is an increasing negative sequence tending to 0 and if  $\lim_{k \rightarrow \infty} z_{k+1}/z_k$  exists (denoted by  $\ell$ ), then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = \ell$ .



6. Application: if  $\mathbb{T} = \{0\} \cup \{-1/r^k, k \in \mathbb{N}\}$  with  $r > 1$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = 1/r$ .

7. Similarly to 3., we can prove that if  $\mathbb{T} = [-1, 0] \cup \{z_k, k \in \mathbb{N}\}$  where  $(z_k)$  is a decreasing positive sequence tending to 0 and if  $\lim_{k \rightarrow \infty} z_{k-1}/z_k = 1$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = 1$ .

8. Application: if  $\mathbb{T} = [-1, 0] \cup \{1/k^2, k \in \mathbb{N}^*\}$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = 1$ .

9. Similarly to 3., we can prove that if  $\mathbb{T} = \{0\} \cup \{z_k^-, k \in \mathbb{N}\} \cup \{z_k^+, k \in \mathbb{N}\}$  where  $(z_k^-)$  (resp.  $(z_k^+)$ ) is an increasing negative (resp. decreasing positive) sequence tending to 0 and if  $\lim_{k \rightarrow \infty} z_{k+1}^-/z_k^- = \lim_{k \rightarrow \infty} z_{k-1}^+/z_k^+ = \ell$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = \ell$ . Note that, in such a case, we can only have  $\ell = 1$  since  $z_{k+1}^-/z_k^- < 1 < z_{k-1}^+/z_k^+$  for every  $k \in \mathbb{N}^*$ .

10. Application: if  $\mathbb{T} = \{0\} \cup \{z_k^-, k \in \mathbb{N}\} \cup \{z_k^+, k \in \mathbb{N}\}$  with  $z_k^- = -1/k$  and  $z_k^+ = 1/k^2$  for every  $k \in \mathbb{N}$ , then  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ . In particular, we have  $\sigma^\nabla(0) = 1$ .

Let us give some examples of time scale with continuous but non- $\nabla$ -differentiable  $\sigma$ .

**Example 5.** 1. If  $\mathbb{T} = \{0\} \cup \{1/k!, k \in \mathbb{N}\}$ , then  $\sigma$  is not  $\nabla$ -differentiable in 0 since  $k!/(k-1)! = k$  tends to  $+\infty$ .

2. If  $\mathbb{T} = [-1, 0] \cup \{1/2^k, k \in \mathbb{N}\}$ , then  $\sigma$  is not  $\nabla$ -differentiable in 0 since  $2^k/2^{k-1} = 2$  does not tend to 1.

3. If  $\mathbb{T} = \{0\} \cup \{\pm 1/2^k, k \in \mathbb{N}\}$ , then  $\sigma$  is not  $\nabla$ -differentiable in 0 since  $2^k/2^{k-1} = 2$ ,  $2^k/2^{k+1} = 1/2$  and  $2 \neq 1/2$ .

Examples 3, 4 and 5 allow to get a better understanding of the restrictions imposed on a time scale by the  $\nabla$ -differentiability of  $\sigma$ . Indeed, we conclude that such a time scale has to satisfy the following properties:

- Due to the continuity of  $\sigma$ , no point can be right-scattered and left-dense.
- Due to the  $\nabla$ -differentiability of  $\sigma$ , the density in a dense point cannot be *too weak*, in contrary to 1. in Example 5. Secondly, in a left- and right-dense point, the left and the right densities have to be *homogeneous* with limit equal to 1, as in 8., 10. of Example 4 and in contrary to 2., 3. of Example 5.

### 3.3 Some results and proof of Theorem 1

The most important result of this section is the following one.

**Proposition 6.** *Let  $u : \mathbb{T} \rightarrow \mathbb{R}^n$  and let  $t \in \mathbb{T}_\kappa^\kappa$ . If the two following properties are satisfied:*

- $\sigma$  is  $\nabla$ -differentiable at  $t$ ;
- $u$  is  $\Delta$ -differentiable at  $t$ ;

*then,  $u^\sigma$  is  $\nabla$ -differentiable at  $t$  with  $(u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)$ .*

*Proof.* Since  $\sigma$  is continuous at  $t$ , recall that  $\sigma \circ \rho(t) = t$  from Lemma 1. We distinguish two cases:  $t \in \text{LS}$  and  $\rho(t) = t$ .

- Firstly, let us consider that  $t \in \text{LS}$ . Since  $\sigma$  is continuous at  $t$ , we have  $\sigma^\nabla(t) = \mu(t)/\nu(t)$ , see Proposition 5. If moreover  $t \in \text{RS}$ , then  $u^\Delta(t) = (u(\sigma(t)) - u(t))/\mu(t)$  and since  $t$  is isolated,  $u^\sigma$  is  $\nabla$ -differentiable at  $t$  with

$$(u^\sigma)^\nabla(t) = \frac{u^\sigma(t) - u^\sigma(\rho(t))}{\nu(t)} = \frac{u(\sigma(t)) - u(t)}{\nu(t)} = \frac{\mu(t)}{\nu(t)}u^\Delta(t) = \sigma^\nabla(t)u^\Delta(t). \quad (22)$$

In the contrary case  $\sigma(t) = t$ , since  $\sigma$  is continuous at  $t$  and since  $u$  is continuous at  $t = \sigma(t)$ , we deduce that  $u^\sigma$  is continuous at  $t \in \text{LS}$ . Then, from the nabla version of Proposition 1,  $u^\sigma$  is  $\nabla$ -differentiable at  $t$  with

$$(u^\sigma)^\nabla(t) = \frac{u^\sigma(t) - u^\sigma(\rho(t))}{\nu(t)} = \frac{u(\sigma(t)) - u(t)}{\nu(t)} = 0, \quad (23)$$

since  $\sigma(t) = t$ . However, in this case, we have  $\sigma^\nabla(t) = \mu(t)/\nu(t) = 0$ . Consequently, we also retrieve  $(u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)$  in this case.

- Secondly, let us consider that  $\rho(t) = t$ . Necessarily,  $t \in \text{RD}$ . Indeed, if  $t = a \in \mathbb{T}_\kappa$ , then  $a \in \text{RD}$ . If  $t \neq a$ , then  $t \in \text{LD}$  and since  $\sigma$  is continuous at  $t$  and  $t \in \mathbb{T}_\kappa$ , we deduce that  $t \in \text{RD}$  from Lemma 1. Finally, since  $u$  is  $\Delta$ -differentiable at  $t$  and since  $\sigma$  is  $\nabla$ -differentiable at  $t$ , we have

$$(u^\sigma)^\nabla(t) = \lim_{\substack{s \rightarrow t \\ s \neq t}} \frac{u^\sigma(s) - u^\sigma(\rho(t))}{s - \rho(t)} = \lim_{\substack{s \rightarrow t \\ s \neq t}} \frac{\sigma(s) - t}{s - t} \frac{u(\sigma(s)) - u(t)}{\sigma(s) - t} = \sigma^\nabla(t)u^\Delta(t). \quad (24)$$

In the previous limit, since  $\sigma$  is continuous at  $t \in \text{LD} \cap \text{RD}$ , we have used that  $s \rightarrow t$ ,  $s \neq t$  implies that  $\sigma(s) \rightarrow \sigma(t) = t$ ,  $\sigma(s) \neq t$ .

The proof is complete. □

From Proposition 6, the following corollary is directly derived.

**Corollary 1.** *Let  $u : \mathbb{T} \rightarrow \mathbb{R}^n$ . If the following properties are satisfied:*

- $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$ ;
- $u$  is  $\Delta$ -differentiable on  $\mathbb{T}_\kappa$ ;

*then,  $u^\sigma$  is  $\nabla$ -differentiable at every  $t \in \mathbb{T}_\kappa^\kappa$  with  $(u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)$ .*

From Propositions 2, 3 and Corollary 1, our main result (Theorem 1) is proved. We conclude this section by introducing the following Leibniz formula useful in order to derive a Noether-type theorem (Theorem 2) in Section 4.

**Proposition 7** (Leibniz formula). *Let  $u, v : \mathbb{T} \rightarrow \mathbb{R}^n$  and  $t \in \mathbb{T}_\kappa^\kappa$ . If the following properties are satisfied:*

- $\sigma$  is  $\nabla$ -differentiable at  $t$ ;
- $u$  is  $\Delta$ -differentiable at  $t$ ;
- $v$  is  $\nabla$ -differentiable at  $t$ ;

*then,  $u^\sigma \cdot v$  is  $\nabla$ -differentiable at  $t$  and the following Leibniz formula holds:*

$$(u^\sigma \cdot v)^\nabla(t) = u(t) \cdot v^\nabla(t) + \sigma^\nabla(t)u^\Delta(t) \cdot v(t). \quad (25)$$

*Proof.* Since  $\sigma$  is continuous at  $t \in \mathbb{T}_\kappa$ , we have  $\sigma \circ \rho(t) = t$  from Lemma 1. From Proposition 6,  $u^\sigma$  is  $\nabla$ -differentiable at  $t$  with  $(u^\sigma)^\nabla(t) = \sigma^\nabla(t)u^\Delta(t)$ . Finally, from the usual Leibniz formula on time scale (see [9, Theorem 1.20 p.8]), we have  $u^\sigma \cdot v$  is  $\nabla$ -differentiable at  $t$  with

$$(u^\sigma \cdot v)^\nabla(t) = u^\sigma(\rho(t)) \cdot v^\nabla(t) + (u^\sigma)^\nabla(t) \cdot v(t) = u(t) \cdot v^\nabla(t) + \sigma^\nabla(t)u^\Delta(t) \cdot v(t). \quad (26)$$

The proof is complete. □

## 4 Application to a Noether-type theorem

We first review the definition of a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$ .

**Definition 1.** *Let  $\eta > 0$ . A map  $\Phi$  is said to be a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$  if  $\Phi$  is a map of class  $\mathcal{C}^2$*

$$\begin{aligned} \Phi : [-\eta, \eta] \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ (\theta, x) &\longmapsto \Phi(\theta, x), \end{aligned} \quad (27)$$

*such that  $\Phi(0, \cdot) = \text{Id}_{\mathbb{R}^n}$ .*

The action of a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$  on a Lagrangian allows us to introduce the notion of symmetry for a  $\nabla \circ \Delta$ -differential Euler-Lagrange equation  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$ .

**Definition 2.** *Let  $\Phi$  be a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$ . A Lagrangian  $L$  is said to be invariant under the action of  $\Phi$  if for every solution  $u$  of  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$  and every  $t \in \mathbb{T}_\kappa^\kappa$ , the map*

$$\theta \longmapsto L(\Phi(\theta, u(t)), \Phi(\theta, u)^\Delta(t), t) \quad (28)$$

*has a null derivative at  $\theta = 0$ . In such a case,  $\Phi$  is said to be a symmetry of the  $\nabla \circ \Delta$ -differential Euler-Lagrange equation  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$  associated.*

The most classical examples of invariance of a Lagrangian under the action of a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$  are given by quadratic Lagrangians and rotations:

**Example 6.** Let us consider  $n = 2$ ,  $L(x, v, t) = \|x\|^2 + \|v\|^2$ ,  $\eta = \pi > 0$  and  $\Phi$  defined by

$$\begin{aligned} \Phi : [-\pi, \pi] \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (\theta, x_1, x_2) &\longmapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \quad (29)$$

Then, for every  $u \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$ , every  $(\theta, t) \in [-\pi, \pi] \times \mathbb{T}_\kappa^\kappa$ , we have  $\Phi(\theta, u)^\Delta(t) = \Phi(\theta, u^\Delta(t))$  from linearity and continuity of  $\Phi$  in its two last variables. Consequently, for every  $u \in C_{\text{rd}}^{1,\Delta}(\mathbb{T})$  and every  $\mathbb{T}_\kappa^\kappa$ , one can easily prove that the map

$$\theta \longmapsto L(\Phi(\theta, u(t)), \Phi(\theta, u)^\Delta(t), t) \quad (30)$$

is independent of  $\theta$  and consequently is constant and has a null derivative at  $\theta = 0$ .

Finally, on time scales with  $\nabla$ -differentiable  $\sigma$ , we prove the following Noether-type theorem providing a constant of motion for  $\nabla \circ \Delta$ -differential Euler-Lagrange equations  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$  admitting a symmetry.

**Theorem 2** (Noether). *Let us assume that  $\sigma$  is  $\nabla$ -differentiable on  $\mathbb{T}_\kappa$  and let  $\Phi$  be a one-parameter family of infinitesimal transformations of  $\mathbb{R}^n$ . If  $L$  is invariant under the action of  $\Phi$ , then for every solution  $u$  of  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$ , there exists  $c \in \mathbb{R}$  such that*

$$\frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\sigma(t)) = c, \quad (31)$$

for every  $t \in \mathbb{T}^\kappa$ .

*Proof.* Let  $u$  be a solution of  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$ . Let us differentiate the map given by (28) at  $\theta = 0$  and let us invert the operators  $\Delta$  and  $\partial/\partial\theta$  from the  $\mathcal{C}^2$ -regularity of  $\Phi$ . We obtain for every  $t \in \mathbb{T}_\kappa^\kappa$ :

$$\frac{\partial L}{\partial x}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u(t)) + \frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u)^\Delta(t) = 0. \quad (32)$$

Finally, multiplying this last equality by  $\sigma^\nabla(t)$  and using that  $u$  is solution of  $(\text{EL}_{\text{diff}}^{\nabla \circ \Delta})$  on  $\mathbb{T}_\kappa^\kappa$ , we obtain

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) \right]^\nabla (t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u(t)) + \sigma^\nabla(t) \frac{\partial L}{\partial v}(u(t), u^\Delta(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u)^\Delta(t) = 0, \quad (33)$$

for every  $t \in \mathbb{T}_\kappa^\kappa$ . Finally, from the Leibniz formula obtained in Proposition 7, it holds

$$\left[ \frac{\partial L}{\partial v}(u, u^\Delta, \cdot) \cdot \frac{\partial \Phi}{\partial \theta}(0, u)^\sigma \right]^\nabla (t) = 0, \quad (34)$$

for every  $t \in \mathbb{T}_\kappa^\kappa$ . From the nabla version of Proposition 2, the proof is complete.  $\square$

Note that this theorem encompasses both the well known Noether's theorems given in the continuous case [4, p.88] and in the (nonshifted) discrete case [17, Theorem 6.4]. For an example of application of Theorem 2, one can consider the framework given in Example 6.

## 5 The $\nabla$ -analogous results

We conclude this paper with the following remark. The whole study made in this paper can be analogously derived for nonshifted calculus of variations with Lagrangian functionals involving a  $\nabla$ -integral dependent on a  $\nabla$ -derivative. We refer to the duality principle introduced in [11] and to [26] for an example of application in calculus of variations on general time scales.

Precisely, let us assume that  $\rho$  is  $\Delta$ -differentiable on  $\mathbb{T}^\kappa$ . Then, the following  $\Delta \circ \nabla$ -differential Euler-Lagrange equation on  $\mathbb{T}^\kappa$ :

$$\left[ \frac{\partial L}{\partial v}(u, u^\nabla, \cdot) \right]^\Delta (t) = \rho^\Delta(t) \frac{\partial L}{\partial x}(u(t), u^\nabla(t), t) \quad (\text{EL}_{\text{diff}}^{\Delta \circ \nabla})$$

characterizes the critical points of the following (nonshifted) Lagrangian functional:

$$\begin{aligned} \mathcal{L} : C_{\text{id}}^{1, \nabla}(\mathbb{T}) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_a^b L(u(\tau), u^\nabla(\tau), \tau) \nabla \tau. \end{aligned} \quad (35)$$

In particular, a necessary condition for local optimizers of  $\mathcal{L}$  is to be a solution of  $(\text{EL}_{\text{diff}}^{\Delta \circ \nabla})$ .

Finally, let us assume that  $L$  is moreover invariant under the action of a one-parameter family  $\Phi$  of infinitesimal transformations of  $\mathbb{R}^n$  in the sense that for every solution  $u$  of  $(\text{EL}_{\text{diff}}^{\Delta \circ \nabla})$  and every  $t \in \mathbb{T}^\kappa$ , the map

$$\theta \longmapsto L(\Phi(\theta, u(t)), \Phi(\theta, u)^\nabla(t), t) \quad (36)$$

has a null derivative at  $\theta = 0$ . Then, for every solution  $u$  of  $(\text{EL}_{\text{diff}}^{\Delta \circ \nabla})$ , there exists  $c \in \mathbb{R}$  such that

$$\frac{\partial L}{\partial v}(u(t), u^\nabla(t), t) \cdot \frac{\partial \Phi}{\partial \theta}(0, u^\rho(t)) = c, \quad (37)$$

for every  $t \in \mathbb{T}^\kappa$ .

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