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## Non-Arbitrage up to Random Horizon for Semimartingale Models

Anna Aksamit · Tahir Choulli · Jun  
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**Abstract** This paper quantifies the impact of stopping at a random time on non-arbitrage, for a class of semimartingale models. We focus on No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) concept, also known in the literature as the arbitrage of the first kind. The first principal result lies in describing the pairs of market model and random times for which the resulting stopped model fulfills the NUPBR condition. The second principal result characterises the random time models that preserve the NUPBR property after stopping for any quasi-left-continuous market model. The analysis that drives these results is based on new stochastic developments in martingale theory with progressive enlargement of filtration. Furthermore, we construct explicit martingale densities (deflators) for a subclass of local martingales when stopped at a random time.

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## 1 Introduction

Our goal in this paper resides in understanding the interplay between the two important financial/economic topics of Arbitrage and Informational Markets. There is no doubt about the rôle of information in the behavior of agents and/or the evolution of markets. We call a market model informational if there are two groups of agents and one group receives extra information over time. In mathematical finance context, this boils down to the case where there are two flows of information (called filtrations).

### 1.1 A Class of Informational Markets for Finance and Insurance

Herein, we focus on the case where the additional information is generated by a random time dynamically in time. In the probabilistic literature, this is called progressive enlargement of the public flow of information by the random time. Our motivations are numerous and most of these reside in financial applications. In fact, the random time can represent, in real life, a death time, a default time of a firm, or any occurrence time of an event that might impact the market in some way. The case of death time is essentially popular in insurance, life insurance and actuarial sciences where both risks of mortality and longevity present serious challenges (see [8] and the references therein). However the situation of default time has its roots in credit risk theory, where the information about the default is added progressively through time.

In contrast to informational market, arbitrage has competing definitions that varies from one context to another, in both areas of mathematical finance and finance. The most popular ones are “classical Non-Arbitrage” (NA hereafter), No-Free-Lunch-with-Vanishing-Risk (NFLVR hereafter), No-Free-Lunch, No-Unbounded-Profit-with-Bounded-Risk (NUPBR hereafter), Cheap thrills, free snacks, and so on. Philosophically, an arbitrage opportunity is a transaction with no cash outlay that results in a sure profit. Most of these concepts coincide in the discrete-time framework with finite and deterministic horizon. In this context, NA is equivalent to the existence of a pricing rule (equivalent martingale measure), as well as to the existence of optimal portfolios for “nice utilities” (market’s viability). These facts make NA a building block in the financial modelling, and a vital property that market models should satisfy for the pricing rules and the hedging tools to remain applicable.

However, in the case of infinite horizon and/or continuous-time settings, most of the above no arbitrage concepts differ tremendously from each others, and even finding their relationships become challenging. NA, NFLVR and NUPBR are the most studied non-arbitrage concepts. The Fundamental Theorem of Asset Pricing –established in full generality by Delbaen and Schachermayer in [19]– asserts that NFLVR holds if and only if there exists an equivalent pricing/martingale measure. Thus, the fulfillment of NFLVR provides a huge advantage for quantifying risks due to the existence of equivalent martingale

measures. However, in this paper, we focus on the NUPBR, which is a weaker condition of non-arbitrage.

## 1.2 Why this Precise Choice of NUPBR?

Our leitmotif reasons are inspired from both finance/economic and mathematical finance. First of all, it is known from Kabanov's work (see [28]) that NFLVR holds if and only if both NUPBR and NA hold. Thus, addressing the NUPBR for informational markets is an important and crucial step towards understanding the NFLVR for these markets on the one hand. On the other hand, in [35], the authors developed many interesting financial models failing NFLVR while fulfilling NUPBR (see also [2] where the authors provides models of informational markets satisfying the NUPBR and violating the NFLVR). In the same line of thinking, Ruf [41] developed delta hedging for a model violating NFLVR and satisfying NUPBR only. More recently, the important rôle of numéraire portfolio was recognized by many researchers (see [30], [36], [38] and [41]) in a setting of pricing, hedging and optimisation problems, where the existence of a solution is obtained under a condition weaker than NFLVR (see [9], [13], [14], [35], [41], and the references therein). In fact, the existence of a growth-optimal portfolio does not require the NFLVR assumption, and the necessary and sufficient condition for the existence of the numéraire portfolio is the NUPBR (see [13], [23] and [30]). Furthermore, the NUPBR property is the non-arbitrage concept that is intimately related to the weakest form of markets' viability (see [13], [32], and [36] for details about this issue). It has been proved recently that for a model violating the NUPBR, the optimal portfolio will not exist even locally, and the pricing rules fail as well. As recognized in [13] and in [43], the NUPBR property is mathematically very attractive and possesses the 'dynamic/localization' feature that the NFLVR and other arbitrage concepts lack to possess. By localization feature, we mean that if the property holds locally (i.e. the property holds for the stopped models with a sequence of stopping times that increases to infinity), then it holds globally. All these facts point to the conclusion that NFLVR might be restrictive, while NUPBR is vital (we can not go lower than that) and sufficient for a market model to be acceptable financially and quantitatively.

## 1.3 The Literature on Arbitrage for Informational Markets and Our Aim

Many papers, in both finance and mathematical finance, are devoted to the investment/consumption problem and/or NFLVR for insider trading, where the insider has a private information at the beginning of the period, which requires the study of an initial enlargement of filtration (see [5], [21], [39]), or in a more general enlargement of filtration setting [34]. The case of a progressive enlargement of filtration, as in credit risk modeling, is less investigated and presents an interest, for pricing derivatives or solving optimization problems.

The NFLVR condition in progressive enlargement setting was studied in [2], and in [20] where it was proved that in a complete market, NFLVR fails before and after honest times. No necessary and sufficient condition is known for the stability of NFLVR under progressive enlargement. The NUPBR condition under a progressive enlargement with a random time was studied for the particular case of complete market with continuous filtration, when  $\tau$  avoids  $\mathbb{F}$ -stopping times in [20].

It is possible to derive some of our results using a new optional decomposition formula, established recently in [4]. We have also to mention that, after that a first version of our results has been available, Acciaio et al. [1] proved some of them, using a different method. See also Song [42], where the author proves very recently that NUPBR holds on  $[0, \tau[$ .

In this paper, we consider a subclass of semimartingale model  $S$ —for the sake of simplicity—, an arbitrary random time  $\tau$ , and we address the following:

$$\text{For which pairs } (S, \tau), \text{ does NUPBR property hold for } S^\tau? \quad (1.1)$$

and

$$\text{For which } \tau, \text{ is the NUPBR preserved for any } S^\tau? \quad (1.2)$$

We emphasize that our necessary and sufficient conditions, in the characterisations aimed above, are given in terms of the behavior of  $\mathbb{F}$ -adapted processes. This sounds remarkable as it conveys that one can check non-arbitrage for the stopped model at the random time using public information only. Furthermore, these characterisations contain important details about how/where arbitrages might occur in the informational market. This will be discussed below.

#### 1.4 Our Financial and Mathematical Achievements

The analysis of these questions led to innovative results with mathematical and financial interests. When stopping a market model with a random time, arbitrages will occur only if the asset price process jumps, and when the signal process—that the uninformed agent receives about the possibility that the random time is still ahead, and called by public signal process for simplicity—brutally disappear (i.e. it jumps and vanishes). Notice that as soon as the signal process vanishes, the uninformed agent knows that  $\tau$  has already occurred. As a result, models with continuous asset price processes fulfill the NUPBR when stopped at any random time, while the continuity of asset price processes at the random time have no effect on the occurrence of arbitrages in any sense. We conclude that *some correlation* between the jumps of the initial market and the jumps of the public signal process is the main source of arbitrages. At the quantitative finance level, we quantify—with deep precision—the jumpy part of the public signal process that play key role in arbitrage, and specify the continuous-time correlation between this part and the jumps of the initial market as well. The importance of these achievements lies in providing method to evaluate whether an informational market model (of this type

addressed here) is worthy financially and quantitatively or not. On the mathematical side, our paper presents many innovative stochastic results that, we believe, are useful in engineering and other mathematical fields such as signal processing and filtering. We explain when and how many stochastic structures (compensating, projections, and localisation) can be recovered in the smallest filtration from the biggest and vice versa.

This paper is organized as follows. The next section (Section 2) presents our principal results in different contexts, and discusses their meaning and/or their economical interpretations and their consequences as well. It contains three subsections, where we develop preliminary results on the NUPBR, analysis for practical examples, and the statement of the main results respectively. Section 3 develops new stochastic results, that are the key mathematical ideas behind the answers to (1.1)-(1.2). Section 4 gives an explicit form for the deflator in the case where  $S$  is a local martingale whose jumps do not occur simultaneously as the jumps of the public signal process about  $\tau$ . Section 5 contains the proofs of the main theorems announced, without proofs, in Section 2. The paper concludes with an Appendix, where some classical results on the predictable characteristics of a semimartingale and other related results are recalled. Some technical proofs and other innovative results are also postponed to the Appendix, for the ease of the reader.

## 2 Main Results and their Interpretations

This section is devoted to the presentation of our main results and their immediate consequences. To this end, we start specifying our mathematical setting and the economical concepts that we will address.

### 2.1 Notations and Preliminary Results on NUPBR

We consider a stochastic basis  $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P)$ , where  $\mathbb{F}$  is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness), and  $\mathcal{F}_\infty \subseteq \mathcal{G}$ . Financially speaking, the filtration  $\mathbb{F}$  represents the flow of public information through time. On this basis, we consider an arbitrary but fixed  $d$ -dimensional càdlàg semimartingale  $S$ . This represents the discounted price processes of  $d$ -stocks, while the riskless asset's price is assumed to be constant. Beside the initial model  $(\Omega, \mathcal{G}, \mathbb{F}, P, S)$ , we consider a random time  $\tau$ , i.e., a non-negative  $\mathcal{G}$ -measurable random variable. To this random time, we associate the process  $D$  and the filtration  $\mathbb{G}$  given by

$$D := I_{[\tau, +\infty[}, \quad \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \quad \mathcal{G}_t = \bigcap_{s > t} \left( \mathcal{F}_s \vee \sigma(D_u, u \leq s) \right).$$

In other terms, the filtration  $\mathbb{G}$  is the smallest right-continuous filtration which contains  $\mathbb{F}$  and makes  $\tau$  a stopping time. In the probabilistic literature,  $\mathbb{G}$  is

called the progressive enlargement of  $\mathbb{F}$  with  $\tau$ . In addition to  $\mathbb{G}$  and  $D$ , we associate to  $\tau$  two important  $\mathbb{F}$ -supermartingales given by

$$Z_t := P(\tau > t \mid \mathcal{F}_t) \quad \text{and} \quad \tilde{Z}_t := P(\tau \geq t \mid \mathcal{F}_t). \quad (2.1)$$

The supermartingale  $Z$  is right-continuous with left limits and is the  $\mathbb{F}$ -optional projection of  $I_{]0, \tau[}$ , while  $\tilde{Z}$  admits right limits and left limits only and is the  $\mathbb{F}$ -optional projection of  $I_{]0, \tau]}$ . The  $\mathbb{F}$ -martingale  $m$ , given by

$$m := Z + D^{\circ, \mathbb{F}}, \quad (2.2)$$

where  $D^{\circ, \mathbb{F}}$  is the  $\mathbb{F}$ -dual optional projection of  $D$  (See [26] for more details), satisfies  $\Delta m + Z_- = \tilde{Z}$  and will play an important rôle in the paper.

In what follows,  $\mathbb{H}$  is a filtration satisfying the usual hypotheses and  $Q$  a probability measure on the filtered probability space  $(\Omega, \mathbb{H})$ . The set of martingales for the filtration  $\mathbb{H}$  under  $Q$  is denoted by  $\mathcal{M}(\mathbb{H}, Q)$ . When  $Q = P$ , we simply denote  $\mathcal{M}(\mathbb{H})$ . As usual,  $\mathcal{A}^+(\mathbb{H})$  denotes the set of increasing, right-continuous,  $\mathbb{H}$ -adapted and integrable processes.

If  $\mathcal{C}(\mathbb{H})$  is a class of  $\mathbb{H}$  adapted processes, we denote by  $\mathcal{C}_0(\mathbb{H})$  the set of processes  $X \in \mathcal{C}(\mathbb{H})$  with  $X_0 = 0$ , and by  $\mathcal{C}_{loc}(\mathbb{H})$  the set of processes  $X$  such that there exists a sequence  $(T_n)_{n \geq 1}$  of  $\mathbb{H}$ -stopping times that increases to  $+\infty$  and the stopped processes  $X^{T_n}$  belong to  $\mathcal{C}(\mathbb{H})$ . We put  $\mathcal{C}_{0,loc}(\mathbb{H}) = \mathcal{C}_0(\mathbb{H}) \cap \mathcal{C}_{loc}(\mathbb{H})$ . In all the paper, we shall write  $A = \emptyset$  (or  $X = Y$ ) as a shortcut for  $A$  is a negligible set (resp.  $X = Y$  a.s.).

For a process  $K$  with  $\mathbb{H}$ -locally integrable variation, we denote by  $K^{\circ, \mathbb{H}}$  its dual optional projection. The dual predictable projection of  $K$  is denoted  $K^{p, \mathbb{H}}$ . For a process  $X$ , we denote  ${}^{\circ, \mathbb{H}}X$  (resp.  ${}^{p, \mathbb{H}}X$ ) its optional (resp. predictable) projection with respect to  $\mathbb{H}$ .

For an  $\mathbb{H}$ -semi-martingale  $Y$ , the set  $L(Y, \mathbb{H})$  is the set of  $\mathbb{H}$  predictable processes integrable w.r.t.  $Y$  and for  $H \in L(Y, \mathbb{H})$ , we denote  $H \bullet Y_t := \int_0^t H_s dY_s$ . As usual, for a process  $X$  and a random time  $\vartheta$ , we denote by  $X^\vartheta$  the stopped process. To distinguish the effect of filtration, we will denote  $\langle \cdot, \cdot \rangle^{\mathbb{F}}$ , or  $\langle \cdot, \cdot \rangle^{\mathbb{G}}$  the sharp bracket (predictable covariation process) calculated in the filtration  $\mathbb{F}$  or  $\mathbb{G}$ , if confusion may rise. We recall that, for general semi-martingales  $X$  and  $Y$ , the sharp bracket is (if it exists) the dual predictable projection of the covariation process  $[X, Y]$ .

We recall that a set  $A \subset \mathbb{R}^+ \times \Omega$  is evanescent if the process  $I_A$  is indistinguishable from the null process. We shall sometimes write  $A = \emptyset$ .

We introduce the non-arbitrage notion that will be addressed in this paper.

**Definition 2.1** An  $\mathbb{H}$ -semimartingale  $X$  satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under  $(\mathbb{H}, Q)$  (hereafter called  $\text{NUPBR}(\mathbb{H}, Q)$ ) if for any  $T \in (0, +\infty)$  the set

$$\mathcal{K}_T(X, \mathbb{H}) := \left\{ (H \bullet X)_T \mid H \in L(X, \mathbb{H}), \text{ and } H \bullet X \geq -1 \right\}$$

is bounded in probability under  $Q$ . When  $Q \sim P$ , we simply write  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$ .

*Remark 2.2* (a) It is important to notice that this definition for NUPBR condition appeared first in [31] (up to our knowledge), and it differs when the time horizon is infinite from that of the literature given in Karatzas and Kardaras [30] where the definition is taken for any  $T \in (0, +\infty]$ . It is obvious that, when the horizon is deterministic and finite, the current NUPBR condition coincides with that of the literature. We could name the current NUPBR as  $\text{NUPBR}_{loc}$ , but for the sake of simplifying notation, we opted for the usual terminology.

(b) When the horizon is infinite, the NUPBR condition of the literature implies the NUPBR condition defined above. However, the reverse implication may not hold in general. In fact if we consider  $S_t = \exp(W_t + t)$ ,  $t \geq 0$ , then it is clear that  $S$  satisfies our  $\text{NUPBR}(\mathbb{H})$ , while the  $\text{NUPBR}(\mathbb{H})$  of the literature is violated. To see this last claim, it is enough to remark that

$$\lim_{t \rightarrow +\infty} (S_t - 1) = +\infty \quad P - a.s. \quad S^t - 1 = H \cdot S \geq -1 \quad H := I_{\llbracket 0, t \rrbracket}.$$

The following proposition slightly generalizes the results obtained for a finite horizon in [43] (see Theorem 2.6) to our NUPBR context.

**Proposition 2.3** *Let  $X$  be an  $\mathbb{H}$ -semimartingale. Then the following assertions are equivalent.*

(a)  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$ .

(b) There exist a positive  $\mathbb{H}$ -local martingale  $Y$  and an  $\mathbb{H}$ -predictable process  $\theta$  satisfying  $0 < \theta \leq 1$  and  $Y(\theta \cdot X)$  is a local martingale.

*Proof* The proof of the implication (b) $\Rightarrow$ (a) is based on [43] and is omitted. Thus, we focus on proving the reverse implication and suppose that assertion (a) holds. Therefore, a direct application of Theorem 2.6 in [43] to each  $(X_{t \wedge n})_{t \geq 0}$ , we obtain the existence of a positive  $\mathbb{H}$ -local martingale  $Y^{(n)}$  and an  $\mathbb{H}$ -predictable process  $\theta_n$  such that  $0 < \theta_n \leq 1$  and  $Y^{(n)}(\theta_n \cdot X^n)$  is a local martingale. Then, it is obvious that the process

$$N := \sum_{n=1}^{+\infty} I_{\llbracket n-1, n \rrbracket} (Y_-^{(n)})^{-1} \cdot Y^{(n)}$$

is a local martingale and  $Y := \mathcal{E}(N) > 0$ . Moreover, the  $\mathbb{H}$ -predictable process  $\theta := \sum_{n \geq 1} I_{\llbracket n-1, n \rrbracket} \theta_n$  satisfies  $0 < \theta \leq 1$  and  $Y(\theta \cdot X)$  is a local martingale. This ends the proof of the proposition.  $\square$

For any  $\mathbb{H}$ -semimartingale  $X$ , the local martingales fulfilling the assertion (b) of Proposition 2.3 are called  $\sigma$ -martingale densities or deflators for  $X$ . The set of these  $\sigma$ -martingale densities will be denoted throughout the paper by

$$\mathcal{L}(\mathbb{H}, X) := \{Y > 0 \mid \exists \theta \in \mathcal{P}(\mathbb{H}), 0 < \theta \leq 1, (Y(\theta \cdot X), Y) \in \mathcal{M}_{loc}(\mathbb{H})\} \quad (2.3)$$

where, as usual,  $\mathcal{P}(\mathbb{H})$  stands for the predictable  $\sigma$ -field on  $\Omega \times [0, \infty[$  and by abuse of notation  $\theta \in \mathcal{P}(\mathbb{H})$  means that  $\theta$  is  $\mathcal{P}(\mathbb{H})$ -measurable.



*Remark 2.4* Proposition 2.3 implies that for any process  $X$  and any **finite** stopping time  $\sigma$  (i.e.  $\sigma < +\infty$   $P$ -a.s.), the two concepts of  $\text{NUPBR}(\mathbb{H})$  (the current concept and the one of the literature) coincide for  $X^\sigma$ .

Below, we prove that, in the case of infinite horizon, the current NUPBR condition is stable under localization, while this is not the case for the NUPBR condition defined in the literature.

**Proposition 2.5** *Let  $X$  be an  $\mathbb{H}$ -semimartingale. Then, the following assertions are equivalent.*

- (a) *There exists a sequence  $(T_n)_{n \geq 1}$  of  $\mathbb{H}$ -stopping times that increases to  $+\infty$ , such that for each  $n \geq 1$ , there exists a probability  $Q_n$  on  $(\Omega, \mathbb{H}_{T_n})$  such that  $Q_n \sim P$  and  $X^{T_n}$  satisfies  $\text{NUPBR}(\mathbb{H})$  under  $Q_n$ .*
- (b)  *$X$  satisfies  $\text{NUPBR}(\mathbb{H})$ .*
- (c) *There exists an  $\mathbb{H}$ -predictable process  $\phi$ , such that  $0 < \phi \leq 1$  and  $(\phi \cdot X)$  satisfies  $\text{NUPBR}(\mathbb{H})$ .*

*Proof* The proof for (a)  $\iff$  (b) follows from the stability of NUPBR condition for a finite horizon under localization which is due to [43] (see also [11] for further discussion about this issue), and the fact that the NUPBR condition is stable under any equivalent probability change.

The proof of (b)  $\implies$  (c) is trivial and is omitted. To prove the reverse, we assume that (c) holds. Then Proposition 2.3 implies the existence of an  $\mathbb{H}$ -predictable process  $\psi$  such that  $0 < \psi \leq 1$  and a positive  $\mathbb{H}$ -local martingale  $Y$  such that  $Y(\psi\phi \cdot X)$  is a local martingale. Since  $\psi\phi$  is predictable and  $0 < \psi\phi \leq 1$ , we deduce that  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$ . This ends the proof of the proposition.  $\square$

We end this section with a simple, but useful result for predictable processes with finite variation.

**Lemma 2.6** *Let  $X$  be an  $\mathbb{H}$ -predictable process with finite variation. Then  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$  if and only if  $X \equiv X_0$  (i.e. the process  $X$  is constant).*

*Proof* It is obvious that if  $X \equiv X_0$ , then  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$ . Suppose that  $X$  satisfies  $\text{NUPBR}(\mathbb{H})$ . Consider a positive  $\mathbb{H}$ -local martingale  $Y$ , and an  $\mathbb{H}$ -predictable process  $\theta$  such that  $0 < \theta \leq 1$  and  $Y(\theta \cdot X)$  is a local martingale. Let  $(T_n)_{n \geq 1}$  be a sequence of  $\mathbb{H}$ -stopping times that increases to  $+\infty$  such that  $Y^{T_n}$  and  $Y^{T_n}(\theta \cdot X)^{T_n}$  are true martingales. Then, for each  $n \geq 1$ , define  $Q_n := (Y_{T_n}/Y_0) \cdot P$ . Since  $X$  is predictable, then  $(\theta \cdot X)^{T_n}$  is also predictable with finite variation and is a  $Q_n$ -martingale. Thus, we deduce that  $(\theta \cdot X)^{T_n} \equiv 0$  for each  $n \geq 1$ . Therefore, we deduce that  $X$  is constant (since  $X^{T_n} - X_0 = \theta^{-1} \cdot (\theta \cdot X)^{T_n} \equiv 0$ ). This ends the proof of the lemma.  $\square$

## 2.2 Examples and Discussion

In this section, we will discuss some simple cases and examples, for which the  $\text{NUPBR}(\mathbb{G})$  for  $S^\tau$  is either valid or violated. This is based on the following.

**Proposition 2.7** *The following assertions hold.*

(a) *Let  $M$  be an  $\mathbb{F}$ -local martingale. Then, for any random time  $\tau$ , the process*

$$\widehat{M}_t := M_t^\tau - \int_0^{t \wedge \tau} \frac{d\langle M, m \rangle_s^{\mathbb{F}}}{Z_{s-}} \quad (2.4)$$

*is a  $\mathbb{G}$ -local martingale, where  $m$  is defined in (2.2).*

(b) *The  $\mathbb{G}$ -predictable process*

$$H_t := (Z_{t-})^{-1} I_{\llbracket 0, \tau \rrbracket}(t), \quad (2.5)$$

*is  $\mathbb{G}$ -locally bounded.*

*Proof* 1) The proof of the assertion (a) can be found in [18, 26].

2) The process  $X := Z^{-1} I_{\llbracket 0, \tau \rrbracket}$  is a càdlàg  $\mathbb{G}$ -supermartingale (see [44] for details). Thus, its left limit is locally bounded. Then, due to

$$(Z_-)^{-1} I_{\llbracket 0, \tau \rrbracket} = X_-,$$

the local boundedness of  $H$  follows. This ends the proof of the proposition.  $\square$

*Example 2.8* Suppose that  $S = W$  is a Brownian motion and  $\mathbb{F}$  is the natural augmentation of the filtration generated by  $W$ . Then, for any random time  $\tau$ , the process

$$S^\tau \mathcal{E}(Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \widehat{m}) \quad \text{is a } \mathbb{G}\text{-local martingale,} \quad (2.6)$$

where  $\widehat{m} := m^\tau - Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \langle m \rangle^{\mathbb{F}}$ . Indeed, since  $m$  is continuous, we deduce that  $\widehat{m}$  is a continuous  $\mathbb{G}$ -local martingale. Hence  $\mathcal{E}(Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \widehat{m})$  is a positive  $\mathbb{G}$ -local martingale due to Proposition 2.7. Thus, (2.6) follows directly from integration by parts and Proposition 2.7-(a), and  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).

*Remark 2.9* This example was generalized in [20], using a different approach, to the case of any continuous  $S$  defined on the Brownian filtration in addition to the assumption that  $\tau$  avoids  $\mathbb{F}$ -stopping times. As a result of this work, many interesting mathematical and financial questions arose.

(1) Is the continuity of  $S$  the main assumption to avoid arbitrages (i.e. does the avoiding hypothesis plays a rôle)?

(2) Suppose that the continuity of  $S$  played a key rôle in eliminating arbitrages. Can this assumption be weakened to the continuity of  $S$  at the random time  $\tau$  only (instead of global continuity) while preserving non-arbitrage (NUPBR)?

The following answers the first question.

**Theorem 2.10** *If  $S$  is continuous and satisfies the NUPBR( $\mathbb{F}$ ), then  $S^\tau$  satisfies the NUPBR( $\mathbb{G}$ ).*

*Proof* If  $S$  satisfies the NUPBR( $\mathbb{F}$ ), then  $\mathcal{L}(\mathbb{F}, S)$  defined in (2.3) is not empty. Since  $S$  is continuous, then for any element  $Y$  of  $\mathcal{L}(\mathbb{F}, S)$ ,  $YS$  is an  $\mathbb{F}$ -local martingale (see Proposition 3.3 and Corollary 3.5 of [6]). In other terms, there exists a positive  $\mathbb{F}$ -local martingale  $Y$  such that  $YS$  is an  $\mathbb{F}$ -local martingale. Let  $(T_n)_n$  be a sequence of  $\mathbb{F}$ -stopping times that increases to infinity and  $Y^{T_n}$  is a martingale, and put  $Q_n := Y_{T_n}/E(Y_{T_n}) \cdot P$ . Thanks to Proposition 2.5, the proof of the theorem will be achieved if we prove that each  $S^{T_n \wedge \tau}$  satisfies the NUPBR( $\mathbb{G}$ ) under  $Q_n$ . Thus, for the sake of simplicity, there is no loss of generality in assuming –for the rest of the proof– that  $S$  is a continuous  $\mathbb{F}$ -local martingale. In this case, we consider  $m^c$  the continuous local martingale part of  $m$  and  $\widehat{m}^c := I_{\llbracket 0, \tau \rrbracket} \cdot m^c - Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \langle m^c, m \rangle^{\mathbb{F}}$ , and remark that  $[m, S] = \langle m, S \rangle^{\mathbb{F}} = \langle m^c, S \rangle^{\mathbb{F}}$ . Thus, similarly as in the analysis of the previous example, we can easily prove that both  $Y^{\mathbb{G}} := \mathcal{E}(Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \widehat{m}^c)$  and  $S^\tau Y^{\mathbb{G}}$  are  $\mathbb{G}$ -local martingales and  $Y^{\mathbb{G}} > 0$ .  $\square$

*Example 2.11* Suppose that  $S_t := N_t - t$ , where  $N$  is a Poisson process with intensity one, and the filtration  $\mathbb{F}$  is the complete filtration generated by  $N$ . Let  $(T_i)_{i \geq 1}$  be the jumps of the process  $N$ , and

$$\tau := \alpha T_1 + (1 - \alpha) T_2, \quad \alpha \in (0, 1).$$

Note that  $S_{t \wedge \tau} - S_{t \wedge T_1} = t \wedge T_1 - t \wedge \tau$  is nondecreasing. Therefore,  $S^\tau$  does not satisfy the NUPBR( $\mathbb{G}$ ) (see also Lemma 2.6). One can also prove (see Proposition 5.3 in [2]) that  $\{\widetilde{Z} = 0 < Z_-\} = \llbracket T_2 \rrbracket$ .

*Example 2.12* Consider the same initial market model as in Example 2.11, and

$$\tau := (aT_2) \wedge T_1, \quad a \in (0, 1).$$

Then  $m = m_0 + \phi \cdot S^{T_1}$ , where  $\phi_t := \beta t e^{-\beta t}$ . This property follows immediately from the fact that  $m$  is a pure jump local martingale (as any  $\mathbb{F}$  martingale),  $\Delta m = \widetilde{Z} - Z_-$ , and

$$\widetilde{Z}_t = e^{-\beta t}(\beta t + 1)I_{\llbracket 0, T_1 \rrbracket}(t) + e^{-\beta t}I_{\llbracket T_1 \rrbracket}(t), \quad \widetilde{Z}_{t-} = Z_{t-} = e^{-\beta t}(\beta t + 1)I_{\llbracket 0, T_1 \rrbracket}(t),$$

where  $\beta = (1/a) - 1$ . The above calculations can be derived easily using the independent increments of the Poisson process, and the reader can also consult [2] (Subsection 5.2.2, page 108) or [7] for detailed calculations. Thus, by combining Proposition 2.7, Itô's formula, and simple calculation such as  $[S^\tau, S^\tau]_t = N_{t \wedge \tau} = S_{t \wedge \tau} + \tau \wedge t$ , we deduce that  $\widehat{S}_t = S_{t \wedge \tau} - \int_0^{\tau \wedge t} \frac{\beta u}{\beta u + 1} du$  and  $S^\tau \mathcal{E}(\psi \cdot \widehat{S})$  are  $\mathbb{G}$ -local martingales, where  $1 + \psi_u := (2 + (\beta u / (1 + \beta u)))^{-1}$ . Hence, we conclude that  $S^\tau$  satisfies the NUPBR( $\mathbb{G}$ ).

*Remark 2.13* 1) Example 2.11 (respectively 2.12) showed that the set  $\{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket$  can be empty and arbitrages occur (respectively coincides with  $\llbracket \widetilde{T} \rrbracket \neq \emptyset$ , where  $\widetilde{T} := T_1 I_{\{T_1 \leq aT_2\}} + (+\infty) I_{\{T_1 > aT_2\}}$ , and no arbitrages occur). These, clearly, answer the question asked in Remark 2.9–(2). In other words, this simple analysis shows that the assumption  $\{\Delta S \neq 0\} \cap \llbracket \tau \rrbracket = \emptyset$  has, in general, no

impact on arbitrage occurrence in any sense, while the key fact behind eliminating arbitrage is much deeper and one should look at different direction. Thus, the right path for eliminating arbitrage, when stopping with  $\tau$ , will be singled out as follows.

2) By comparing Examples 2.11 and 2.12, we conclude that the main difference between these examples –when viewed under  $\mathbb{F}$ – lies in the set  $\{\tilde{Z} = 0 < Z_-\}$ : For the case of Example 2.12 (and of course Example 2.8), this set is evanescent, while for Example 2.11 it is not. This provides sufficient evidence to support the claim that arbitrage of the first kind would occur only if the set  $\{\tilde{Z} = 0 < Z_-\}$  is not evanescent. However, the questions of how and where these arbitrage might occur are not clear enough from these three examples. The following will provide a simple but important detail that allows us to get closer to our objective of understanding how  $\mathbb{G}$ -arbitrage might occur.

*Example 2.14* Consider two independent Poisson processes  $N = \sum_n I_{[T_n, +\infty[}$  and  $\bar{N}$  with intensity one,  $\mathbb{F}$  is the completed filtration that is generated by  $N$  and  $\bar{N}$ ,  $S_t = \bar{N}_t - t$ , and  $\tau := \alpha T_1 + (1 - \alpha)T_2$ . Then, obviously  $S^\tau$  satisfies the NUPBR( $\mathbb{G}$ ).

Both Examples 2.11 and 2.14 have the same  $\{\tilde{Z} = 0 < Z_-\}$ , while they have different sets for  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$ . In fact,  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$  is equal to  $[[T_2]]$  for Example 2.11 and is empty for Example 2.14. The set  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$  is also empty for Example 2.8 since  $S$  is continuous.

Therefore, from the analysis of these examples, we can conclude that the arbitrage of the first kind for  $S^\tau$  under  $\mathbb{G}$  would occur only if  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$  is not evanescent, (i.e. when  $S$  jumps on the set  $\{\tilde{Z} = 0 < Z_-\}$ ). Hence, the further natural question resulting from the above discussion is

How can we assess precisely the occurrence or not of  $\mathbb{G}$ -arbitrage? (2.7)

### 2.3 The Main Results

In this subsection, we present our principal results on the NUPBR condition under stopping at  $\tau$ . The following is our first main result that consists of characterizing the pairs  $(S, \tau)$  of market represented by quasi-left-continuous processes<sup>1</sup> and random time models, for which  $S^\tau$  fulfills the NUPBR.

**Theorem 2.15** *For any pair  $(S, \tau)$  of  $\mathbb{F}$ -quasi-left-continuous process and random time, there exists an  $\mathbb{F}$ -local martingale, denoted  $m^{(0)}$ , that is pure jumps local martingale (i.e its continuous local martingale part is null), quasi-left-continuous,  $m_0^{(0)} = 0$ ,*

$$\Delta m^{(0)} \in \{0, 1\}, \quad \{\Delta m^{(0)} \neq 0\} \subset \{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}, \quad (2.8)$$

<sup>1</sup> A process  $X$  is quasi-left-continuous if it does not jump at predictable jump times.

and the following three assertions are equivalent.

- (a)  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).
- (b) For any  $\delta > 0$ ,

$$I_{\{Z_- \geq \delta\}} \cdot \mathcal{T}(S) \text{ satisfies NUPBR}(\mathbb{F}), \text{ where } \mathcal{T}(S) := S - [S, m^{(0)}]. \quad (2.9)$$

- (c) For any  $n \geq 1$ , the process  $(\mathcal{T}(S))^{\sigma_n}$  satisfies NUPBR( $\mathbb{F}$ ), where

$$\sigma_n := \inf\{t \geq 0 : Z_t < 1/n\}. \quad (2.10)$$

The proof of this theorem is technical and is delegated to Section 5. Below, we discuss the meaning and the financial/economic as well as the practical importance of this theorem.

*Remark 2.16* 1) The  $\mathbb{F}$ -local martingale  $m^{(0)}$  is given explicitly in (5.8), for any pair  $(S, \tau)$ . This explicit description requires technical notations, and hence we opted for postponing this description together with the proof of the theorem to Section 5. Via the process  $m^{(0)}$ , we quantify the jumpy part of the “public signal process”,  $\tilde{Z}$ , that plays central rôle in generating arbitrages for  $S^\tau$ .

2) The appearance of the sets  $\{Z_- \geq \delta\}$ ,  $\delta > 0$  in the assertion (b) (or equivalently the sequence of  $\mathbb{F}$ -stopping times  $(\sigma_n)_{n \geq 1}$  in assertion (c)) reflects exactly the translation of the  $\mathbb{G}$ -localization in terms of the  $\mathbb{F}$ -localization. It is technical somehow, but very important as it explains with precision how the important property of localisation can be recovered when the filtration varies. This fact and others are detailed in Section 3, and in order to understand the contribution of Theorem 2.15 in arbitrage theory and informational markets, we advice the reader to ignore this fact (temporarily) in the following remark.

3) Two semimartingales  $X$  and  $Y$  are called uncorrelated if  $[X, Y] \equiv 0$ . Theorem 2.15 claims that the NUPBR( $\mathbb{G}$ ) for  $S^\tau$  is *essentially* equivalent to the NUPBR( $\mathbb{F}$ ) of the corresponding *uncorrelated* process  $\mathcal{T}(S)$  to  $m^{(0)}$  (it is easy to check that  $[\mathcal{T}(S), m^{(0)}] \equiv 0$ ). The word *essentially* in the sentence refers to the predictable sets neglected here and mentioned in the second remark above. It is also worthy to mention that, due to the second property of (2.8) and  $\{\tilde{Z} = 0\} \subset ]\tau, +\infty[$ ,  $S$  and  $\mathcal{T}(S)$  coincide after stopping at  $\tau$  (i.e.  $S^\tau = (\mathcal{T}(S))^\tau$ ). Thus, this interpretation of Theorem 2.15 conveys that the theorem quantifies the financial claim –as announced in Subsection 1.4– that the correlation, between the jumps of the initial market and the jumps of the signal process, constitutes the key source for arbitrage after stopping with  $\tau$ .

4) Theorem 2.15 not only provides complete answers to questions (1.1) and (2.7), but it also provides a method to check whether  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ) or not, using the small filtration  $\mathbb{F}$  only, by calculating  $m^{(0)}$  for  $(S, \tau)$  —as instructed in Section 5—, and checking afterwards the NUPBR( $\mathbb{F}$ ) for  $\mathcal{T}(S)$ . We refer the reader to [3] for a similar result when  $S$  is general. In virtue of this interpretation, Theorem 2.15 also furnishes explicit examples of pairs  $(S, \tau)$  for which  $S^\tau$  fulfills the NUPBR( $\mathbb{G}$ ). These consequences will be outlined in the forthcoming Corollary 2.17 and Theorem 2.20.

5) Where  $\mathbb{G}$ -arbitrages would occur? It is clear that, on the set  $\{\tilde{Z} = 0 < Z_-\}$ ,

the uninformed agent (the agent who uses the public flow of information  $\mathbb{F}$  only) loses *brutally* the signal about  $\tau$ , and this optional set occurs after  $\tau$ . However, the informed agent –who uses the flow  $\mathbb{G}$ – would make arbitrages on  $\llbracket 0, \tau \rrbracket$ . Thus, the natural question is where the informed agent would make arbitrages? The answer to this question lies in interpreting  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$  as a support of a random optional measure whose compensator random measure has a predictable support  $\mathcal{S}$  that may intersect  $\llbracket 0, \tau \rrbracket$  (i.e.  $\mathcal{S} \cap \llbracket 0, \tau \rrbracket \neq \emptyset$ ). On this intersection, the informed agent would make arbitrages as she has a substantial advantage regarding the occurrence of  $\tau$ .

6) It is important to notice that we do not assume the NUPBR for  $S$ , and the NUPBR( $\mathbb{G}$ ) for  $S^\tau$  does not imply in general the NUPBR( $\mathbb{F}$ ) for  $S$ . In fact it is enough to consider  $X := [m^{(0)}, m^{(0)}]$ , and remark that  $X^\tau \equiv 0$ .

The following corollary describes practical cases of  $\mathbb{F}$ -quasi-left-continuous model  $S$  and  $\tau$  such that  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ) when  $S$  does under  $\mathbb{F}$ .

**Corollary 2.17** *Suppose that  $S$  satisfies NUPBR( $\mathbb{F}$ ). Then, the following hold.*

- (a) *If  $S - [S, m^{(0)}]$  satisfies NUPBR( $\mathbb{F}$ ), then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*
- (b) *If  $(S, [S, m^{(0)}])$  satisfies NUPBR( $\mathbb{F}$ ), then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*
- (c) *If  $m^{(0)} \equiv 0$  (or equivalently  $[S, m^{(0)}] \equiv 0$ ), then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*
- (d) *If  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset$ , then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*
- (e) *If  $\tilde{Z} > 0$  (equivalently  $Z > 0$  or  $Z_- > 0$ ), then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*
- (f) *If  $S$  is continuous, then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*

*Remark 2.18* 1) After posting the first version of our result on Arxiv, assertion (d) drew the attention of the authors of [1], where they proved it with a different method. However, we do not see how their approach can be used to prove a more deeper result such as assertion (a) or assertion (b) or Theorem 2.15. In fact, assertion (d) is the only sufficient condition given in [1] for NUPBR( $\mathbb{G}$ ). Note that we shall also obtain this result in Theorem 4.1, which is simpler than Theorem 2.15.

2) Assertion (c) asserts that if  $S$  is *uncorrelated* to the jumpy part of the signal process  $m^{(0)}$ , then  $S^\tau$  is arbitrage free. A fortiori, this conclusion holds when  $S$  is uncorrelated to the left-jumps of  $\tilde{Z}$  (i.e.  $\tilde{Z} - \tilde{Z}_- = \tilde{Z} - Z_-$ ) on the set where this signal vanishes. Assertion (b) claims that  $S^\tau$  is arbitrage free only when  $S$  is *weakly uncorrelated* to  $m^{(0)}$  under an equivalent martingale measure for  $S$ . Herein, the *weak correlation* of  $S$  and  $m^{(0)}$ , under  $Q$ , is defined by  $\langle S, m^{(0)} \rangle^Q$ . Assertion (e) confirms that  $S^\tau$  is arbitrage free in the case where the uninformed agent never loses the signal  $\tilde{Z}$ .

*Proof of Corollary 2.17:* (1) If  $(S, [S, m^{(0)}])$  satisfies NUPBR( $\mathbb{F}$ ), then it is obvious that  $S - [S, m^{(0)}]$  satisfies NUPBR( $\mathbb{F}$ ) also. Hence, assertion (c) follows from assertion (b), assertion (b) follows from assertion (a), and this latter assertion follows immediately from Theorem 2.15.

(2) If  $\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset$ , then (2.8) implies that  $\Delta m^{(0)} \equiv 0$ . Since  $m^{(0)}$  is a pure jumps local martingale and  $m_0^{(0)} = 0$ , then it is null. Thus,

assertion (d) follows from assertion (c).

(3) Both assertions (e) and (f), obviously, follow from assertion (d), and the proof of the corollary is completed.  $\square$

*Remark 2.19* 1) It is worth mentioning that  $X - Y$  may satisfy NUPBR( $\mathbb{H}$ ), while  $(X, Y)$  may not satisfy NUPBR( $\mathbb{H}$ ). For a non trivial example, consider  $X_t = \lambda t$  and  $Y_t = N_t$  where  $N$  is the Poisson process with intensity  $\lambda$ .

2) It is possible that both  $X$  and  $Y$  satisfy NUPBR, while  $(X, Y)$  does not fulfill NUPBR. Indeed, it is enough to put  $X = \sigma W_t + at$  and  $Y_t = \gamma W_t + ct$ , where  $\sigma > 0$ ,  $\nu > 0$  and  $a\gamma \neq c\sigma$ .

In the spirit of the practical side of Theorem 2.15, we state the following.

**Theorem 2.20** *Let  $\mu$  be the optional random measure associated to the jumps of  $S$ , and  $\nu^{\mathbb{F}}$  and  $\nu^{\mathbb{G}}$  be its  $\mathbb{F}$ -compensator and the  $\mathbb{G}$ -compensator of  $I_{\llbracket 0, \tau \rrbracket} \cdot \mu$  respectively. If  $S$  satisfies NUPBR( $\mathbb{F}$ ) and  $P - a.s.$*

$$I_{\llbracket 0, \tau \rrbracket} \cdot \nu^{\mathbb{F}} \text{ is equivalent to } \nu^{\mathbb{G}}, \quad (2.11)$$

then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ).

The proof of this theorem follows immediately from Corollary 2.17 once we prove that the assumption (2.11) implies that the corresponding  $\mathbb{F}$ -local martingale  $m^{(0)}$  is null. Thus, this proof is delegated to Section 5.

*Remark 2.21* 1) It is worth mentioning that, we always have  $P - a.s.$   $\nu^{\mathbb{G}} \ll I_{\llbracket 0, \tau \rrbracket} \cdot \nu^{\mathbb{F}}$  (i.e. absolute continuity). This can be seen from the fact that  $\nu^{\mathbb{G}}$  lives on  $\llbracket 0, \tau \rrbracket$  only and it is absolutely continuous with respect to  $\nu^{\mathbb{F}}$ .

2) **The Lévy Framework:** If  $S$  is a Lévy process and  $F(dx)$  is its Lévy measure under  $\mathbb{F}$ , then  $\nu(\omega, dt, dx) = F(dx)dt$ , while  $\nu^{\mathbb{G}}(\omega, dt, dx) = I_{\llbracket 0, \tau \rrbracket}(\omega, t)F_{(\omega, t)}^{\mathbb{G}}(dx)dt$ , where  $F^{\mathbb{G}}$  is  $\mathbb{G}$ -predictable kernel on  $\mathbb{R}^d$  (a kind of “generalized” Lévy measure), that can be calculated explicitly. Thus, Theorem 2.20 says that if  $S$  satisfies the NUPBR( $\mathbb{F}$ ) and  $P \otimes \lambda$ -almost all  $(\omega, t)$  (where  $\lambda(dt) = dt$ ),  $F^{\mathbb{G}}(\omega, t, dx) \sim F(dx)$ , then  $S^\tau$  satisfies the NUPBR( $\mathbb{G}$ ).

3) **How Examples 2.11-2.12 fit into the context of Theorem 2.20?**

To answer this question, we remark that  $\mu(\omega, dt, dx) = \delta_1(dx)dN_t(\omega)$  and  $\nu^{\mathbb{F}}(\omega, dt, dx) = \delta_1(dx)dt$  (i.e.  $F(dx) = \delta_1(dx)$ ) on the one hand. On the other hand, for both examples, there exists an  $\mathbb{F}$ -predictable process  $\phi$  such that  $m = m_0 + \phi \cdot S$ ,  $Z_- + \phi \geq 0$ , and

$$\nu^{\mathbb{G}}(\omega, dt, dx) = \delta_1(dx) \left(1 + (\phi_t/Z_{t-})\right) I_{\llbracket 0, \tau \rrbracket}(t)dt.$$

Therefore, for the case of Example 2.12, we have  $\phi \equiv -\beta t e^{-\beta t} I_{\llbracket 0, T_1 \rrbracket}$  and  $1 + \phi_t/Z_{t-} = I_{\llbracket T_1, +\infty \rrbracket} + (1 + \beta t)^{-1} I_{\llbracket 0, T_1 \rrbracket} > 0$ . Hence, in this case, the assumption (2.11) is obviously fulfilled. However, for the case of Example 2.11, by using simple calculations (see [2] for details), one can easily derive

$$Z_t = \tilde{Z}_t = I_{\llbracket 0, T_1 \rrbracket}(t) + I_{\llbracket T_1, T_2 \rrbracket}(t) \exp\left(-\frac{\alpha}{1-\alpha}(t - T_1)\right).$$

Thus, these equalities imply that  $\phi = -I_{\llbracket T_1, T_2 \rrbracket}(t) \exp[-(\alpha/(1-\alpha))(t-T_1)]$ , and  $1+(\phi/Z_-) = 0$  on  $\llbracket T_1, \tau \rrbracket$ . As a consequence, for the case of Example 2.11, the condition (2.11) is violated, since  $\nu^{\mathbb{G}}(\llbracket T_1, \tau \rrbracket) \equiv 0 < \nu^{\mathbb{F}}(\llbracket T_1, \tau \rrbracket)$ .

The remaining part of this subsection focuses on determining the models of random times  $\tau$  such that, for any quasi-left-continuous semi-martingale  $S$  enjoying NUPBR( $\mathbb{F}$ ), the stopped process  $S^\tau$  enjoys NUPBR( $\mathbb{G}$ ). This will answer completely the question (1.2).

**Proposition 2.22** *The following assertions are equivalent:*

- (a) *The thin set  $\{\tilde{Z} = 0 < Z_-\}$  is accessible.*
- (b) *For any (bounded)  $X$  that is  $\mathbb{F}$ -quasi-left-continuous and satisfies NUPBR( $\mathbb{F}$ ), the process  $X^\tau$  satisfies NUPBR( $\mathbb{G}$ ).*

*Proof* The implication (a) $\Rightarrow$ (b) follows from Corollary 2.17–(c), since we have

$$\{\Delta X \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset.$$

We now focus on proving the reverse implication. To this end, we suppose that assertion (b) holds, and we consider an  $\mathbb{F}$ -stopping time  $\sigma$  such that  $\llbracket \sigma \rrbracket \subset \{\tilde{Z} = 0 < Z_-\}$ . It is known that  $\sigma$  can be decomposed into a totally inaccessible part  $\sigma^i$  and an accessible part  $\sigma^a$  such that  $\sigma = \sigma^i \wedge \sigma^a$ . Consider the quasi-left-continuous  $\mathbb{F}$ -martingale

$$M = V - \tilde{V} \in \mathcal{M}_{0,loc}(\mathbb{F})$$

where  $V := I_{\llbracket \sigma^i, +\infty \rrbracket}$  and  $\tilde{V} := V^{p,\mathbb{F}}$ . It is known from [18, paragraph 14, Chapter XX], that

$$\{\tilde{Z} = 0\} \text{ and } \{Z_- = 0\} \text{ are disjoint from } \llbracket 0, \tau \rrbracket. \quad (2.12)$$

This implies that  $\tau < \sigma \leq \sigma^i$   $P$ -a.s.. Hence, we get

$$M^\tau = -\tilde{V}^\tau \text{ is } \mathbb{G}\text{-predictable.}$$

By hypothesis (b),  $M$  being a quasi-left continuous martingale,  $M^\tau$  satisfies NUPBR( $\mathbb{G}$ ), then we conclude that this process is null (i.e.  $\tilde{V}^\tau = 0$ ) due to Lemma 2.6. Thus, we get

$$0 = E(\tilde{V}_\tau) = E\left(\int_0^{+\infty} Z_{s-} d\tilde{V}_s\right) = E(Z_{\sigma^i-} I_{\{\sigma^i < +\infty\}}),$$

or equivalently  $Z_{\sigma^i-} I_{\{\sigma^i < +\infty\}} = 0$   $P$ -a.s. This is possible only if  $\sigma^i = +\infty$   $P$ -a.s. since on  $\{\sigma^i < +\infty\} \subset \{\sigma = \sigma^i < +\infty\}$  we have  $Z_{\sigma^i-} = Z_{\sigma-} > 0$ . This proves that  $\sigma$  is an accessible stopping time. Since  $\{\tilde{Z} = 0 < Z_-\}$  is an optional thin set, assertion (a) follows. This ends the proof of the proposition.  $\square$

As a result, when the filtration itself is quasi-left-continuous, the answer to (1.2) is more precise and simpler.



**Theorem 2.23** *Suppose that  $\mathbb{F}$  is a quasi-left-continuous filtration. Then the following assertions are equivalent.*

- (a) *The thin set  $\{\tilde{Z} = 0 < Z_-\}$  is evanescent.*  
 (b) *For any (bounded)  $X$  satisfying  $NUPBR(\mathbb{F})$ , the process  $X^\tau$  satisfies  $NUPBR(\mathbb{G})$ .*

*Proof* The proof of (a)  $\implies$  (b) follows immediately from Proposition 2.22. Thus, the remaining part of the proof will focus on proving the reverse sense, where we will use the fact that, when  $\mathbb{F}$  is a quasi-left-continuous filtration, any accessible  $\mathbb{F}$ -stopping time is predictable (see [15] or [22, Th. 4.26]). Suppose that assertion (b) holds. Since  $\mathbb{F}$  is a quasi-left-continuous filtration, then any  $\mathbb{F}$ -martingale is quasi-left-continuous, and from Proposition 2.22, again, we deduce that the thin set,  $\{\tilde{Z} = 0 < Z_-\}$ , is  $\mathbb{F}$ -predictable. Now take any  $\mathbb{F}$ -predictable stopping time  $T$  such that

$$\llbracket T \rrbracket \subset \{\tilde{Z} = 0 < Z_-\}.$$

This implies that  $\{T < +\infty\} \subset \{\tilde{Z}_T = 0 < Z_{T-}\}$ , and thus

$$E(I_{\{T < +\infty\}} Z_{T-}) = E(I_{\{T < +\infty\}} \tilde{Z}_T) = 0.$$

This is equivalent to  $T = +\infty$   $P$ -a.s due to  $\{T < +\infty\} \subset \{Z_{T-} > 0\}$ . This ends the proof of the Theorem.

*Remark 2.24* 1) The results of Theorem 2.23 remain valid in full generality and without the quasi-left continuity assumption on  $\mathbb{F}$ . In the general framework, the proof of the results requires technical notations and intermediary results that we opted for avoiding herein for the sake of simple exposition of ideas with less technicalities. For these general results, we refer the reader to our earliest versions available online (see [3]).

2) To see how Examples 2.11 and 2.12 can be deduced from Theorem 2.23, one can remark that the set  $\{\tilde{Z} = 0 < Z_-\}$  is a totally inaccessible thin set equal to  $\llbracket T_2 \rrbracket$  for Example 2.11, while it is evanescent for Example 2.12.

### 3 Stochastics from–and–for Informational Non-Arbitrage

In this section, we develop new stochastic results that will play a key rôle in the proofs and/or the statements of the main results outlined in the previous section. The first subsection gives the link between the  $\mathbb{G}$ -compensators and the  $\mathbb{F}$ -compensators, while the second subsection presents a  $\mathbb{G}$ -martingale that is vital in the explicit construction of deflators developed in the next section.

#### 3.1 Exact Relationship between Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

The main results of this subsection are summarized in Lemmas 3.1 and 3.2, where we address the question of how to compute  $\mathbb{G}$ -dual predictable projections in term of  $\mathbb{F}$ -dual predictable projections and vice versa.

To any  $\mathbb{F}$  semi-martingale  $X$ , we associate a sequence of  $\mathbb{F}$ -predictable stopping times  $(T_n^X)_{n \geq 1}$  that exhaust the accessible jump times of  $X$ . We can decompose  $X$  as follows.

$$X = X^{(qc)} + X^{(a)}, \quad X^{(a)} := I_{\Gamma_X} \cdot X, \quad \Gamma_X := \bigcup_{n=1}^{\infty} \llbracket T_n^X \rrbracket. \quad (3.1)$$

The process  $X^{(a)}$  (the accessible part of  $X$ ) is a thin process with predictable jumps only, while  $X^{(qc)}$  is an  $\mathbb{F}$ -quasi-left-continuous process (the quasi-left-continuous part of  $X$ ).

We start by expressing the  $\mathbb{G}$ -dual predictable projection of an  $\mathbb{F}$ -locally integrable variation process in terms of an  $\mathbb{F}$ -dual predictable projection, and  $\mathbb{G}$ -predictable projection in terms of  $\mathbb{F}$ -predictable projection.

**Lemma 3.1** *The following assertions hold.*

(a) *For any  $\mathbb{F}$ -adapted process  $V$  with locally integrable variation, we have*

$$(V^\tau)^{p,\mathbb{G}} = (Z_-)^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot (\tilde{Z} \cdot V)^{p,\mathbb{F}}. \quad (3.2)$$

(b) *For any  $\mathbb{F}$ -local martingale  $M$ , we have, on  $\llbracket 0, \tau \rrbracket$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\tilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}(\Delta M I_{\{\tilde{Z} > 0\}})}{Z_-}, \quad \text{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{\tilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}(I_{\{\tilde{Z} > 0\}})}{Z_-}. \quad (3.3)$$

(c) *For any quasi-left-continuous  $\mathbb{F}$ -local martingale  $M$ , we have, on  $\llbracket 0, \tau \rrbracket$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\tilde{Z}}\right) = 0, \quad \text{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}, \quad (3.4)$$

where  $m^{(qc)}$  is the quasi-left-continuous  $\mathbb{F}$ -martingale defined in (3.1).

*Proof* (a) Using the notation (2.5), the equality (2.4) takes the form

$$M^\tau = \widehat{M} + H I_{\llbracket 0, \tau \rrbracket} \cdot \langle M, m \rangle^{\mathbb{F}}.$$

By taking  $M = V - V^{p,\mathbb{F}}$ , we obtain

$$V^\tau = I_{\llbracket 0, \tau \rrbracket} \cdot V^{p,\mathbb{F}} + \widehat{M} + H I_{\llbracket 0, \tau \rrbracket} \cdot \langle V, m \rangle^{\mathbb{F}} = \widehat{M} + I_{\llbracket 0, \tau \rrbracket} \cdot V^{p,\mathbb{F}} + \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot (\Delta m \cdot V)^{p,\mathbb{F}},$$

which, using the fact that  $Z_- + \Delta m = \tilde{Z}$ , proves assertion (a).

(b) Let  $M$  be an  $\mathbb{F}$ -local martingale, then, for any positive integers  $(n, k)$  the process  $V^{(n,k)} := \sum \frac{\Delta M}{\tilde{Z}} I_{\{|\Delta M| \geq k^{-1}, \tilde{Z} \geq n^{-1}\}}$  has a locally integrable variation. Then, by using the known equality  ${}^{p,\mathbb{G}}(\Delta V) = \Delta(V^{p,\mathbb{G}})$  (see Theorem 76 in pages 149–150 of [17] or Theorem 5.27 in page 150 of [22]), and applying assertion (a) to the process  $V^{(n,k)}$ , we get, on  $\llbracket 0, \tau \rrbracket$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\tilde{Z}} I_{\{|\Delta M| \geq k^{-1}, \tilde{Z} \geq n^{-1}\}}\right) = \frac{1}{Z_-} {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M| \geq k^{-1}, \tilde{Z} \geq n^{-1}\}}\right).$$

Since  $M$  is a local martingale, by stopping we can exchange limits with projections in both sides. Then by letting  $n$  and  $k$  go to infinity, and using the fact that  $\tilde{Z} > 0$  on  $\llbracket 0, \tau \rrbracket$  (see [27]), we deduce that

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\tilde{Z}}\right) = \frac{1}{Z_-} {}^{p,\mathbb{F}}\left(\Delta M I_{\{\tilde{Z} > 0\}}\right).$$

This proves the first equality in (3.3), while the second equality follows from  $\tilde{Z} = \Delta m + Z_-$ :

$$\begin{aligned} Z_- {}^{p,\mathbb{G}}(\tilde{Z}^{-1}) &= {}^{p,\mathbb{G}}\left(\frac{\tilde{Z} - \Delta m}{\tilde{Z}}\right) = 1 - {}^{p,\mathbb{G}}\left(\frac{\Delta m}{\tilde{Z}}\right) \\ &= 1 - (Z_-)^{-1} {}^{p,\mathbb{F}}\left(\Delta m I_{\{\tilde{Z} > 0\}}\right) = 1 - {}^{p,\mathbb{F}}\left(I_{\{\tilde{Z} = 0\}}\right) = {}^{p,\mathbb{F}}\left(I_{\{\tilde{Z} > 0\}}\right). \end{aligned}$$

In the above string of equalities, the third equality follows from the first equality in (3.3), while the fourth equality is due to  ${}^{p,\mathbb{F}}(\Delta m) = 0$  and  $\Delta m I_{\{\tilde{Z} = 0\}} = -Z_- I_{\{\tilde{Z} = 0\}}$ . This ends the proof of assertion (b).

(c) If  $M$  is a quasi-left-continuous  $\mathbb{F}$ -local martingale, then  ${}^{p,\mathbb{F}}\left(\Delta M I_{\{\tilde{Z} > 0\}}\right) = 0$ , and the first property of the assertion (c) follows. Applying the first property to  $M = m^{(qc)}$  and using that, on  $\llbracket 0, \tau \rrbracket$ , one has  $\Delta m^{(qc)} (Z_- + \Delta m)^{-1} = \Delta m^{(qc)} (Z_- + \Delta m^{(qc)})^{-1}$ , we obtain

$$\frac{1}{Z_-} {}^{p,\mathbb{G}}\left(\frac{Z_-}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-} \left(1 - {}^{p,\mathbb{G}}\left(\frac{\Delta m^{(qc)}}{Z_- + \Delta m^{(qc)}}\right)\right) = \frac{1}{Z_-}.$$

This proves assertion (c), and the proof of the lemma is achieved.  $\square$

The next lemma proves that  $\tilde{Z}^{-1} I_{\llbracket 0, \tau \rrbracket}$  is Lebesgue-Stieljes-integrable with respect to any process that is  $\mathbb{F}$ -adapted with  $\mathbb{F}$ -locally integrable variation. Using this fact, the lemma addresses the question of how an  $\mathbb{F}$ -compensator stopped at  $\tau$  can be written in terms of a  $\mathbb{G}$ -compensator, and constitutes a *sort of* converse result to Lemma 3.1-(a).

**Lemma 3.2** *Let  $V$  be an  $\mathbb{F}$ -adapted càdlàg process. Then the following properties hold.*

(a) *If  $V$  belongs to  $\mathcal{A}_{loc}^+(\mathbb{F})$  (respectively  $V \in \mathcal{A}^+(\mathbb{F})$ ), then the process*

$$U := \tilde{Z}^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot V, \quad (3.5)$$

*belongs to  $\mathcal{A}_{loc}^+(\mathbb{G})$  (respectively to  $\mathcal{A}^+(\mathbb{G})$ ).*

(b) *If  $V$  has  $\mathbb{F}$ -locally integrable variation, then the process  $U$  is well defined, its variation is  $\mathbb{G}$ -locally integrable, and its  $\mathbb{G}$ -dual predictable projection is given by*

$$U^{p,\mathbb{G}} = \left(\frac{1}{\tilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot V\right)^{p,\mathbb{G}} = \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \left(I_{\{\tilde{Z} > 0\}} \cdot V\right)^{p,\mathbb{F}}. \quad (3.6)$$

*In particular, if  $\text{supp} V \subset \{\tilde{Z} > 0\}$ , then, on  $\llbracket 0, \tau \rrbracket$ , one has  $V^{p,\mathbb{F}} = Z_- \cdot U^{p,\mathbb{G}}$ .*

*Proof* (a) Suppose that  $V \in \mathcal{A}_{loc}^+(\mathbb{F})$ . First, remark that, due to the fact that  $\tilde{Z}$  is positive on  $\llbracket 0, \tau \rrbracket$ ,  $U$  is well defined. Let  $(\vartheta_n)_{n \geq 1}$  be a sequence of  $\mathbb{F}$ -stopping times that increases to  $+\infty$  such that  $E(V_{\vartheta_n}) < +\infty$ . Then, if  $E(U_{\vartheta_n}) \leq E(V_{\vartheta_n})$ , assertion (a) follows. Thus, we calculate

$$\begin{aligned} E(U_{\vartheta_n}) &= E\left(\int_0^{\vartheta_n} I_{\{0 < t \leq \tau\}} \frac{1}{\tilde{Z}_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau \geq t | \mathcal{F}_t)}{\tilde{Z}_t} I_{\{\tilde{Z}_t > 0\}} dV_t\right) \\ &\leq E(V_{\vartheta_n}). \end{aligned}$$

The last inequality is obtained due to  $\tilde{Z}_t := P(\tau \geq t | \mathcal{F}_t)$ . This ends the proof of assertion (a) of the lemma.

(b) Suppose that  $V \in \mathcal{A}_{loc}(\mathbb{F})$ , and denote by  $W := V^+ + V^-$  its variation. Then  $W \in \mathcal{A}_{loc}^+(\mathbb{F})$ , and a direct application of the first assertion implies that

$$\left(\tilde{Z}\right)^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot W \in \mathcal{A}_{loc}^+(\mathbb{G}).$$

As a result, we deduce that  $U$  given by (3.5) for the case of  $V = V^+ - V^-$  is well defined and has variation equal to  $\left(\tilde{Z}\right)^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot W$  which is  $\mathbb{G}$ -locally integrable. By setting  $U_n := I_{\llbracket 0, \tau \rrbracket} \cdot \left(\tilde{Z}^{-1} I_{\{\tilde{Z} \geq 1/n\}} \cdot V\right)$ , we derive, due to (3.2),

$$(U_n)^{p, \mathbb{G}} = \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \left(I_{\{\tilde{Z} \geq 1/n\}} \cdot V\right)^{p, \mathbb{F}}.$$

Hence, since  $U^{p, \mathbb{G}} = \lim_{n \rightarrow +\infty} (U_n)^{p, \mathbb{G}}$ , by taking the limit in the above equality, (3.6) follows immediately, and the lemma is proved.  $\square$

### 3.2 An Important $\mathbb{G}$ -local martingale

In this subsection, we will introduce a  $\mathbb{G}$ -local martingale that will be crucial for the construction of the deflator. The following lemma provides an important element for this construction.

**Lemma 3.3** *The nondecreasing process*

$$V_t^{\mathbb{G}} := \sum_{0 \leq u \leq t} {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z}=0\}} \right)_u I_{\{u \leq \tau\}} \quad (3.7)$$

is  $\mathbb{G}$ -predictable, càdlàg, locally bounded, and  $(1 - \Delta V^{\mathbb{G}})^{-1}$  is  $\mathbb{G}$ -locally bounded.

*Proof* The  $\mathbb{G}$ -predictability of  $V^{\mathbb{G}}$  being obvious, it remains to prove that this process is  $\mathbb{G}$ -locally bounded. Since  $Z_-^{-1} I_{\llbracket 0, \tau \rrbracket}$  is  $\mathbb{G}$ -locally bounded, then there exists a sequence of  $\mathbb{G}$ -stopping times  $(\tau_n^{\mathbb{G}})_{n \geq 1}$  increasing to infinity such that

$$\left(\frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket}\right)^{\tau_n^{\mathbb{G}}} \leq n + 1.$$

Consider a sequence of  $\mathbb{F}$ -stopping times  $(\sigma_n)_{n \geq 1}$  that increases to infinity such that  $\langle m, m \rangle_{\sigma_n} \leq n + 1$ . Then, for any nonnegative  $\mathbb{F}$ -predictable process  $\phi$  which is bounded by  $C > 0$ , we calculate

$$\begin{aligned} (\phi \cdot V^{\mathbb{G}})_{\sigma_n \wedge \tau_n^{\mathbb{G}}} &= \sum_{0 \leq u \leq \sigma_n \wedge \tau_n^{\mathbb{G}}} \phi_u \, {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z}=0\}} \right)_u I_{\{u \leq \tau\}} I_{\{Z_{u-} \geq \frac{1}{n+1}\}} \\ &\leq \sum_{0 \leq u \leq \sigma_n} \phi_u \, {}^{p, \mathbb{F}} \left( I_{\{\Delta m \leq -\frac{1}{n+1}\}} \right)_u \\ &\leq (n+1)^2 \phi \cdot \langle m, m \rangle_{\sigma_n} \leq C(n+1)^3. \end{aligned}$$

Furthermore, due to  $I_{\{\tilde{Z} > 0\}} \geq \tilde{Z}$ , we obtain that

$$\begin{aligned} 1 - \Delta V^{\mathbb{G}} &= 1 - {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z}=0\}} \right) I_{[0, \tau]} = I_{\tau, +\infty[} + {}^{p, \mathbb{F}} \left( I_{\{\tilde{Z} > 0\}} \right) I_{[0, \tau]} \\ &\geq I_{\tau, +\infty[} + Z_- I_{[0, \tau]}. \end{aligned}$$

Hence, thanks to Proposition 2.7-(b), this ends the proof of the proposition.  $\square$

This important  $\mathbb{G}$ -local martingale will result from an optional integral. For the notion of compensated stochastic integral (or optional stochastic integral), we refer the reader to [24] (Chapter III.4.b p. 106-109) and [17] (Chapter VIII.2 sections 32-35 p. 356-361). Below, for the sake of completeness, we give the definition of this integration.

**Definition 3.4** (see [24], Definition (3.80)) Let  $N$  be an  $\mathbb{H}$ -local martingale with continuous martingale part  $N^c$ , and let  $K$  be an  $\mathbb{H}$ -optional process.

i) The process  $K$  is said to be integrable with respect to  $N$  if  ${}^{p, \mathbb{H}} K$  is  $N^c$  integrable,  ${}^{p, \mathbb{H}}(|K \Delta N|) < +\infty$  and the process

$$\left( \sum_{s \leq t} (K_s \Delta N_s - {}^{p, \mathbb{H}}(K \Delta N)_s)^2 \right)^{1/2}$$

is locally integrable. The set of integrable processes with respect to  $N$  is denoted by  ${}^o L_{loc}^1(N, \mathbb{H})$ .

ii) For  $K \in {}^o L_{loc}^1(N, \mathbb{H})$ , the compensated stochastic integral of  $K$  with respect to  $N$ , denoted by  $K \odot N$ , is the unique local martingale  $M$  which satisfies

$$M^c = {}^{p, \mathbb{H}} K \cdot N^c \quad \text{and} \quad \Delta M = K \Delta N - {}^{p, \mathbb{H}}(K \Delta N).$$

Among the most useful results of the literature involving this integral is the following

**Proposition 3.5** (see [17]) (a) *The compensated stochastic integral  $M = K \odot N$  is the unique  $\mathbb{H}$ -local martingale such that, for any  $\mathbb{H}$ -local martingale  $Y$ , the process  $[M, Y] - K \cdot [N, Y]$  is an  $\mathbb{H}$ -local martingale.*

(b) *For any  $\mathbb{H}$ -local martingale  $Y$ , one has  $[M, Y] \in \mathcal{A}_{loc}(\mathbb{H})$  if and only if  $K \cdot [N, Y] \in \mathcal{A}_{loc}(\mathbb{H})$  and in this case we have*

$$\langle M, Y \rangle^{\mathbb{H}} = (K \cdot [N, Y])^{p, \mathbb{H}}.$$

Now, we are in the stage of defining the  $\mathbb{G}$ -local martingale which will play the rôle of deflator for a class of processes.

**Proposition 3.6** *Consider the following  $\mathbb{G}$ -local martingale*

$$\widehat{m} := I_{\llbracket 0, \tau \rrbracket} \cdot m - \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle m \rangle^{\mathbb{F}},$$

and the process

$$K := \frac{Z_-^2}{Z_-^2 + \Delta \langle m \rangle^{\mathbb{F}}} \frac{1}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket}. \quad (3.8)$$

Then,  $K$  belongs to the space  ${}^{\circ}L_{loc}^1(\widehat{m}, \mathbb{G})$  defined in Definition 3.4. Furthermore, the  $\mathbb{G}$ -local martingale

$$L := -K \odot \widehat{m}, \quad (3.9)$$

satisfies the following

(a)  $\mathcal{E}(L) > 0$  (or equivalently  $1 + \Delta L > 0$ ).

(b) For any  $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ , setting  $\widehat{M} := M^\tau - Z_-^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot \langle M, m \rangle^{\mathbb{F}}$ , we have

$$[L, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad (\text{i.e. } \langle L, \widehat{M} \rangle^{\mathbb{G}} \text{ exists}). \quad (3.10)$$

*Proof* We shall prove that  $K \in {}^{\circ}L_{loc}^1(\widehat{m}, \mathbb{G})$  in the appendix B. For the sake of simplicity in notations, throughout this proof, we will use  $\kappa := Z_-^2 + \Delta \langle m \rangle^{\mathbb{F}}$ . We now prove assertions (a) and (b). Due to (B.1), we have, on  $\llbracket 0, \tau \rrbracket$ ,

$$-\Delta L = K \Delta \widehat{m} - {}^p, \mathbb{G}(K \Delta \widehat{m}) = 1 - Z_- \left( \widetilde{Z} \right)^{-1} - {}^p, \mathbb{F}(I_{\{\widetilde{Z}=0\}}).$$

Thus, we deduce that  $1 + \Delta L > 0$ , and assertion (a) is proved. In the rest of this proof, we will prove (3.10). To this end, let  $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ . Thanks to Proposition 3.5, (3.10) is equivalent to

$$K \cdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad (\text{or equivalently } \frac{1}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G})),$$

for any  $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ . Then, it is easy to check that

$$\begin{aligned} \frac{1}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [\widehat{m}, \widehat{M}] &= \frac{1}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [m, \widehat{M}] - \frac{1}{Z_- \widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [\langle m \rangle^{\mathbb{F}}, \widehat{M}] \\ &= \frac{1}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [m, M] - \frac{1}{Z_- \widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [m, \langle M, m \rangle^{\mathbb{F}}] \\ &\quad - \frac{1}{Z_- \widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [\langle m \rangle^{\mathbb{F}}, M] + \frac{1}{Z_-^2 \widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} \cdot [\langle m \rangle^{\mathbb{F}}, \langle M, m \rangle^{\mathbb{F}}]. \end{aligned}$$

Since  $m$  is an  $\mathbb{F}$ -locally bounded local martingale, all the processes

$$[m, M], [m, \langle M, m \rangle^{\mathbb{F}}], [\langle m \rangle^{\mathbb{F}}, M], \text{ and } [\langle m \rangle^{\mathbb{F}}, \langle M, m \rangle^{\mathbb{F}}]$$

belong to  $\mathcal{A}_{loc}(\mathbb{F})$ . Thus, by combining this fact with Lemma 3.2 and the  $\mathbb{G}$ -local boundedness of  $Z_-^{-p} I_{\llbracket 0, \tau \rrbracket}$  for any  $p > 0$ , the result follows. This ends the proof of the proposition.  $\square$

#### 4 Explicit Deflators for a Class of $\mathbb{F}$ -Local Martingales

This section describes some classes of  $\mathbb{F}$ -quasi-left-continuous local martingales for which the NUPBR is preserved after stopping at  $\tau$ . For these stopped processes, we describe explicitly their local martingale densities in Theorems 4.1–4.2 with an increasing degree of generality. We recall that  $m^{(qc)}$  is defined in (3.1) and  $L$  is defined in Proposition 3.6.

**Theorem 4.1** *Suppose  $S$  is quasi-left-continuous.*

(1) *If  $S$  is an  $\mathbb{F}$ -local martingale such that  $(S, \tau)$  satisfies*

$$\{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\} = \emptyset, \quad (4.1)$$

*then  $\mathcal{E}(L)S^\tau$  is a  $\mathbb{G}$ -local martingale.*

(2) *If  $S$  satisfies NUPBR( $\mathbb{F}$ ) and (4.1) holds, then  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ )*

*Proof* We start by giving some useful observations. Since  $S$  is  $\mathbb{F}$ -quasi-left-continuous, on the one hand we deduce that  $(\Gamma_m)$  is defined in (3.1))

$$\langle S, m \rangle^{\mathbb{F}} = \langle S, m^{(qc)} \rangle^{\mathbb{F}} = \langle S, I_{\Gamma_m^c} \cdot m \rangle^{\mathbb{F}}. \quad (4.2)$$

On the other hand, we note that assertion (a) is equivalent to  $\mathcal{E}(L^{(qc)})S^\tau$  is a  $\mathbb{G}$ -local martingale, where  $L^{(qc)}$  is the quasi-left-continuous local martingale part of  $L$  given by  $L^{(qc)} := I_{\Gamma_m^c} \cdot L = -K \odot \widehat{m}^{(qc)}$ . Here  $K$  is given in Proposition 3.6 and

$$\widehat{m}^{(qc)} := I_{\llbracket 0, \tau \rrbracket} \cdot m^{(qc)} - (Z_-)I_{\llbracket 0, \tau \rrbracket} \cdot \langle m^{(qc)} \rangle^{\mathbb{F}}.$$

It is easy to check that (4.1) is equivalent to

$$I_{\{\tilde{Z}=0 < Z_-\}} \cdot [S, m] = 0. \quad (4.3)$$

We now compute  $-\langle L^{(qc)}, \widehat{S} \rangle^{\mathbb{G}}$ , where  $\widehat{S}$  is the  $\mathbb{G}$ -local martingale given by

$$\widehat{S} := S^\tau - (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, m \rangle^{\mathbb{F}}.$$

Due to the quasi-left continuity of  $S$  and that of  $m^{(qc)}$ , the two processes  $\langle S, m \rangle^{\mathbb{F}}$  and  $\langle m^{(qc)} \rangle^{\mathbb{F}}$  are continuous and  $[S, m^{(qc)}] = [S, m]$ . Hence, we obtain

$$\begin{aligned} K \cdot [\widehat{S}, \widehat{m}^{(qc)}] &= K \cdot [S, \widehat{m}^{(qc)}] - K \Delta \widehat{m}^{(qc)} (Z_-)^{-1} \cdot \langle S, m \rangle^{\mathbb{F}} \\ &= (\tilde{Z})^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot [S, m^{(qc)}] = (\tilde{Z})^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot [S, m]. \end{aligned}$$

It follows that

$$\begin{aligned} -\langle L^{(qc)}, \widehat{S} \rangle^{\mathbb{G}} &= \left( K \cdot [\widehat{S}, \widehat{m}^{(qc)}] \right)^{p, \mathbb{G}} = \left( (\tilde{Z})^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot [S, m] \right)^{p, \mathbb{G}} \\ &= (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \left( I_{\{\tilde{Z} > 0\}} \cdot [S, m] \right)^{p, \mathbb{F}} \\ &= (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, m \rangle^{\mathbb{F}} - (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \left( I_{\{\tilde{Z}=0 < Z_-\}} \cdot [S, m] \right)^{p, \mathbb{F}} \\ &= (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, m \rangle^{\mathbb{F}} + (Z_-)^{-1}I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, -I_{\{\tilde{Z}=0 < Z_-\}} \odot m^{(qc)} \rangle^{\mathbb{F}}. \end{aligned} \quad (4.4)$$

The first and the last equality follow from Proposition 3.5 applied to  $L^{(qc)}$  and  $-I_{\{\tilde{Z}=0 < Z_-\}} \odot m^{(qc)}$  respectively. The second and the third equalities are due to (4.2) and (3.6) respectively.

Now, we prove the theorem. Thanks to (4.4), it is obvious that assertion (a) is equivalent to  $\langle S, -I_{\{\tilde{Z}=0 < Z_-\}} \odot m^{(qc)} \rangle^{\mathbb{F}} \equiv 0$  which in turn is equivalent to assertion (b). This ends the proof of the equivalence between (a) and (b).

It is also clear that the condition (4.1) or equivalently (4.3) implies assertion (b), due to  $\langle I_{\{\tilde{Z}=0 < Z_-\}} \odot m^{(qc)}, S \rangle^{\mathbb{F}} = \left( I_{\{\tilde{Z}=0 < Z_-\}} \cdot [m, S] \right)^{p, \mathbb{F}} \equiv 0$ .

This ends the proof of assertion (1).

The proof of assertion (2) follows from combining Proposition 2.5, assertion (1), and the fact that, for any probability measure  $Q$  equivalent to  $P$ , we have

$$\{\tilde{Z} = 0 < Z_-\} = \{\tilde{Z}^Q = 0 < Z_-^Q\}.$$

Here  $Z_t^Q = Q(\tau > t | \mathcal{F}_t)$  and  $\tilde{Z}_t^Q = Q(\tau \geq t | \mathcal{F}_t)$ . This last claim is a direct application of the optional and predictable selection measurable theorems, see Theorems 84 and 85 (or apply Theorem 86 directly) in [17].  $\square$

The following theorem affirms the existence of a deflator for  $S^\tau$ , that we explicitly construct in the proof, when both  $S$  and  $[S, m^{(0)}]$  are  $\mathbb{F}$ -local martingales. This corresponds to the case where  $S$  is local martingale that is weakly uncorrelated to  $m^{(0)}$ , see Remark 2.18–(2) for the definition of weak correlation.

**Theorem 4.2** *Suppose that  $S$  is an  $\mathbb{F}$ -quasi-left-continuous local martingale. Consider the  $\mathbb{F}$ -local martingale  $m^{(0)}$ , described in Theorem 2.15. If  $[S, m^{(0)}]$  is an  $\mathbb{F}$ -local martingale, then there exists a  $\mathbb{G}$ -local martingale  $L^{(1)}$  such that  $1 + \Delta L + \Delta L^{(1)} > 0$  and  $\mathcal{E}(L + L^{(1)}) S^\tau$  is a  $\mathbb{G}$ -local martingale, where  $L$  is defined in (3.9).*

The explicit construction of the  $\mathbb{G}$ -local martingale,  $L^{(1)}$ , requires technical notations like the proof of the theorem itself. Thus, both the proof of the theorem and the explicit description of  $L^{(1)}$  are delegated to Section 5.

## 5 Proofs of Theorems 2.15, 2.20 and 4.2

The proof of these theorems, technically involved, is essentially based on the semimartingale characteristics, and will be achieved in three subsections. The first subsection recalls some notations intimately related to the semimartingale characteristics and provide the explicit form of the key  $\mathbb{F}$ -martingale  $m^{(0)}$  mentioned in Theorem 2.15. Other notations and results related to the semimartingale techniques are delegated in the Appendix.



### 5.1 The Explicit Construction of $m^{(0)}$

We start by recalling some definitions and setting some notations. For any filtration  $\mathbb{H}$ , we denote

$$\tilde{\mathcal{O}}(\mathbb{H}) := \mathcal{O}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d), \quad \tilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d),$$

where  $\mathcal{B}(\mathbb{R}^d)$  is the Borel  $\sigma$ -field on  $\mathbb{R}^d$ . For a càdlàg  $\mathbb{H}$ -adapted process  $X$  we associate the following optional random measure  $\mu_X$  defined by

$$\mu_X(dt, dx) := \sum_{u>0} I_{\{\Delta X_u \neq 0\}} \delta_{(u, \Delta X_u)}(dt, dx). \quad (5.1)$$

For a product-measurable functional  $W \geq 0$  on  $\Omega \times [0, +\infty[ \times \mathbb{R}^d$ , we denote  $W \star \mu_X$  (or sometimes, with abuse of notation  $W(x) \star \mu_X$ ) the process

$$(W \star \mu_X)_t := \int_0^t \int_{\mathbb{R}^d - \{0\}} W(u, x) \mu_X(du, dx) = \sum_{0 < u \leq t} W(u, \Delta X_u) I_{\{\Delta X_u \neq 0\}}. \quad (5.2)$$

**Definition 5.1** For a quasi-left-continuous process  $X$   $\mathbb{H}$ -adapted, and its optional random measure  $\mu_X$ ,  $\mathcal{G}_{loc}^1(\mu_X, \mathbb{H})$  (respectively  $\mathcal{H}_{loc}^1(\mu_X, \mathbb{H})$ ) is the set of all  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functions (respectively all  $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable functions)  $W$  such that

$$\sqrt{W^2 \star \mu_X} \in \mathcal{A}_{loc}^+(\mathbb{H}).$$

Also on  $\Omega \times [0, +\infty[ \times \mathbb{R}^d$ , we define the measure  $M_{\mu_X}^P := P \otimes \mu_X$  by

$$\int W dM_{\mu_X}^P := E[(W \star \mu_X)_\infty],$$

(when the expectation is well defined). The conditional “expectation” given  $\tilde{\mathcal{P}}(\mathbb{H})$  of a product-measurable functional  $W$ , is the unique  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional  $\tilde{W}$  satisfying

$$E[(W I_\Sigma \star \mu_X)_\infty] = E[(\tilde{W} I_\Sigma \star \mu_X)_\infty], \quad \text{for all } \Sigma \in \tilde{\mathcal{P}}(\mathbb{H}).$$

When  $X = S$ , for the sake of simplicity, we denote  $\mu := \mu_S$ . Then, the  $\mathbb{F}$ -canonical decomposition of  $S$  is

$$S = S_0 + S^c + h \star (\mu - \nu) + b \cdot A + (x - h) \star \mu, \quad (5.3)$$

where  $h$ , defined as  $h(x) := x I_{\{|x| \leq 1\}}$ , is the truncation function. We associate to  $\mu$  defined in (5.2) when  $X = S$ , its predictable compensator random measure  $\nu$ . A direct application of Theorem A.1 (in Appendix), to the martingale  $m$ , leads to the existence of a local martingale  $m^\perp$  as well as a  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional  $f_m$ , a process  $\beta_m \in L(S^c, \mathbb{F})$  and an  $\tilde{\mathcal{O}}(\mathbb{F})$ -measurable functional  $g_m$  such that  $f_m \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$ ,  $g_m \in \mathcal{H}_{loc}^1(\mu, \mathbb{F})$  and  $\beta_m \in L(S^c)$  such that

$$m = \beta_m \cdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^\perp. \quad (5.4)$$

The canonical decomposition of  $S^\tau$  under  $\mathbb{G}$  is given by

$$S^\tau = S_0 + \widehat{S}^c + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + \frac{c\beta_m}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot A + h \frac{f_m}{Z_-} I_{\llbracket 0, \tau \rrbracket} \star \nu + b \cdot A^\tau + (x - h) \star \mu^{\mathbb{G}}$$

where  $\mu^{\mathbb{G}} := I_{\llbracket 0, \tau \rrbracket} \star \mu$  and  $\nu^{\mathbb{G}}$  is its  $\mathbb{G}$  compensated measure given by

$$\nu^{\mathbb{G}}(dt, dx) := (1 + f_m(x)/Z_{t-}) I_{\llbracket 0, \tau \rrbracket}(t) \nu(dt, dx), \quad (5.5)$$

and  $\widehat{S}^c := I_{\llbracket 0, \tau \rrbracket} \cdot S^c - \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle m, S^c \rangle$ .

**Lemma 5.2** *Let*

$$\psi := M_\mu^P \left( I_{\{\tilde{Z} > 0\}} \Big| \tilde{\mathcal{P}}(\mathbb{F}) \right). \quad (5.6)$$

*Then, the following assertions hold.*

(a)  $P \otimes \mu$ -a.e. we have  $\{\psi = 0\} = \{f_m = -Z_-\} \subset \{\tilde{Z} = 0\}$  or equivalently

$$\{\psi = 0\} = \{f_m = -Z_-\} \subset \{\tilde{Z} = 0\} \quad \text{on} \quad \{\Delta S \neq 0\}. \quad (5.7)$$

(b) *If*

$$m^{(0)} := I_{\{\psi = 0 < Z_-\}} \star (\mu - \nu), \quad (5.8)$$

*then  $m^{(0)}$  is a quasi-left-continuous, pure jumps local martingale,  $m_0^{(0)} = 0$ , which satisfies (2.8).*

*Proof* 1) Herein, we prove the assertion (a). Since both  $\{\psi = 0\}$  and  $\{Z_- + f_m = 0\}$  are  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable, the equality of the two sets under  $P \otimes \nu$  is equivalent to the one under  $P \otimes \mu = M_\mu^P$ . Due to  $\tilde{Z} \leq I_{\{\tilde{Z} > 0\}}$  and  $f_m = M_\mu^P(\Delta m | \tilde{\mathcal{P}}(\mathbb{F}))$ , we have

$$0 \leq Z_- + f_m = M_\mu^P \left( \tilde{Z} \Big| \tilde{\mathcal{P}}(\mathbb{F}) \right) \leq \psi.$$

Thus, we get  $\{\psi = 0\} \subset \{Z_- + f_m = 0\} \subset \{\tilde{Z} = 0\}$   $M_\mu^P$ -a.e. on the one hand. On the other hand, the reverse inclusion follows from

$$0 = M_\mu^P \left( I_{\{Z_- + f_m = 0\}} I_{\{\tilde{Z} > 0\}} \right) = M_\mu^P \left( I_{\{Z_- + f_m = 0\}} \psi \right).$$

2) Remark that, due to (5.7) and  $\{\tilde{Z} = 0 < Z_-\} \subset \llbracket R \rrbracket$  for some  $\mathbb{F}$ -stopping time  $R$ , we have  $E(I_{\{\psi = 0 < Z_-\}} \star \mu_\infty) \leq 1$  and hence  $m^{(0)}$  is a well defined  $\mathbb{F}$ -local martingale. Since  $S$  is quasi-left-continuous, then it is obvious that  $m^{(0)}$  is a pure jumps quasi-left-continuous local martingale and

$$\Delta m^{(0)} = I_{\{\psi(\Delta S) = 0 < Z_- \ \& \ \Delta S \neq 0\}}.$$

Thus, we deduce that  $\Delta m^{(0)} \in \{0, 1\}$ , and due to (5.7) we get  $\{\Delta m^{(0)} \neq 0\} \subset \{\Delta S \neq 0\} \cap \{\tilde{Z} = 0 < Z_-\}$ . This proves assertion (b), and the proof of the lemma is completed.  $\square$

## 5.2 Proof of Theorem 2.15

The proof of Theorem 2.15 will be completed in four steps. The first step provides an equivalent formulation to assertion (a) using the filtration  $\mathbb{F}$  instead. In the second step, we prove (a) $\Rightarrow$ (b), while the reverse implication is proved in the third step. The proof of (b) $\iff$ (c) is given in the last step.

**Step 1: Formulation of assertion (a):** Thanks to Proposition 2.3,  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ) if and only if there exist a  $\mathbb{G}$ -local martingale  $N^\mathbb{G}$  with  $1 + \Delta N^\mathbb{G} > 0$  and a  $\mathbb{G}$ -predictable process  $\phi^\mathbb{G}$  such that  $0 < \phi^\mathbb{G} \leq 1$  and  $\mathcal{E}(N^\mathbb{G})(\phi^\mathbb{G} \cdot S^\tau)$  is a  $\mathbb{G}$ -local martingale. We can reduce our attention to processes  $N^\mathbb{G}$  such that (see Theorem A.3 in the Appendix)

$$N^\mathbb{G} = \beta^\mathbb{G} \cdot \widehat{S}^c + (f^\mathbb{G} - 1) \star (\mu^\mathbb{G} - \nu^\mathbb{G})$$

where  $\beta^\mathbb{G} \in L(\widehat{S}^c, \mathbb{G})$  and  $f^\mathbb{G}$  is positive and such that  $(f^\mathbb{G} - 1) \in \mathcal{G}_{loc}^1(\mu^\mathbb{G}, \mathbb{G})$ . Then, one notes that  $\mathcal{E}(N^\mathbb{G})(\phi^\mathbb{G} \cdot S^\tau)$  is a  $\mathbb{G}$ -local martingale if and only if  $\phi^\mathbb{G} \cdot S^\tau + [\phi^\mathbb{G} \cdot S^\tau, N^\mathbb{G}]$  is a  $\mathbb{G}$ -local martingale, which in turn, is equivalent to

$$\phi^\mathbb{G} |x f^\mathbb{G}(x) - h(x)| \left(1 + \frac{f_m(x)}{Z_-}\right) I_{\llbracket 0, \tau \rrbracket} \star \nu \in \mathcal{A}_{loc}^+(\mathbb{G}), \quad (5.9)$$

and  $P \otimes A - a.e.$  on  $\llbracket 0, \tau \rrbracket$  (using the kernel  $F$  defined in the Appendix A)

$$b + c \left(\frac{\beta_m}{Z_-} + \beta^\mathbb{G}\right) + \int \left[ (x f^\mathbb{G}(x) - h(x)) \left(1 + \frac{f_m(x)}{Z_-}\right) + h(x) \frac{f_m(x)}{Z_-} \right] F(dx) = 0. \quad (5.10)$$

From Lemma C.1, there exist  $\phi^\mathbb{F}$  and  $\beta^\mathbb{F}$  two  $\mathbb{F}$ -predictable processes and a positive  $\widetilde{\mathcal{P}}(\mathbb{F})$ -measurable functional,  $f^\mathbb{F}$ , such that  $0 < \phi^\mathbb{F} \leq 1$ ,

$$\beta^\mathbb{F} = \beta^\mathbb{G}, \quad \phi^\mathbb{F} = \phi^\mathbb{G}, \quad f^\mathbb{F} = f^\mathbb{G} \text{ on } \llbracket 0, \tau \rrbracket. \quad (5.11)$$

In virtue of these and taking into account integrability conditions given in Proposition C.3, we deduce that (5.9)–(5.10) imply that, on  $\{Z_- \geq \delta\}$ , we have

$$W^\mathbb{F} := \int |(x f^\mathbb{F}(x) - h(x))| \left(1 + \frac{f_m(x)}{Z_-}\right) F(dx) < +\infty \quad P \otimes A - a.e., \quad (5.12)$$

and  $P \otimes A - a.e.$  on  $\{Z_- \geq \delta\}$ , we have

$$b + c \left(\beta^\mathbb{F} + \frac{\beta_m}{Z_-}\right) - \int h(x) I_{\{\psi=0\}} F(dx) + \int \left[ x f^\mathbb{F}(x) \left(1 + \frac{f_m(x)}{Z_-}\right) - h(x) \right] I_{\{\psi>0\}} F(dx) = 0. \quad (5.13)$$

Due to (5.12), this latter equality follows due to  $\{\psi = 0\} = \{Z_- + f_m = 0\}$  (see (5.7)) and after inserting (5.11) in (5.10).

**Step 2: Proof of (a)  $\Rightarrow$  (b).** If we denote

$$S^{(0)} := x I_{\{\psi=0 < Z_-\}} \star \mu = [S, m^{(0)}], \quad (5.14)$$

then it is easy to see that assertion (b) is equivalent to

$$(b') \quad I_{\{Z_{-} \geq \delta\}} \cdot (S - S^{(0)}) \quad \text{satisfies NUPBR}(\mathbb{F}),$$

for any  $\delta > 0$ . Suppose that  $S^\tau$  satisfies NUPBR( $\mathbb{G}$ ), hence (5.12)–(5.13) hold. To prove the assertion (b') above, we consider  $\Sigma_0 := \{\psi > 0 \ \& \ Z_{-} \geq \delta\}$  and

$$\beta := \left( \frac{\beta_m}{Z_{-}} + \beta^{\mathbb{F}} \right) I_{\{Z_{-} \geq \delta\}} \quad \text{and} \quad f = f^{\mathbb{F}} \left( 1 + \frac{f_m}{Z_{-}} \right) I_{\Sigma_0} + I_{\Sigma_0^c}.$$

If  $\beta \in L(S^c, \mathbb{F})$  and  $(f - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$ , we conclude that

$$N := \beta \cdot S^c + (f - 1) \star (\mu - \nu).$$

is a well defined  $\mathbb{F}$ -local martingale. Choosing  $\phi = (1 + W^{\mathbb{F}} I_{\{Z_{-} \geq \delta\}})^{-1}$  where  $W^{\mathbb{F}}$  is defined in (5.12), using (5.13), and applying Itô's formula for  $\mathcal{E}(N)$  ( $\phi I_{\{Z_{-} \geq \delta\}} \cdot (S - S^{(0)})$ ), we deduce that this process is a local martingale. Hence,  $I_{\{Z_{-} \geq \delta\}} \cdot (S - S^{(0)})$  satisfies NUPBR( $\mathbb{F}$ ), and the proof of (a) $\Rightarrow$ (b) is completed.

Now, we focus on proving  $\beta \in L(S^c)$  and  $(f - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$ . Since  $\beta_m \in L(S^c)$ , then it is obvious that  $\frac{\beta_m}{Z_{-}} I_{\{Z_{-} \geq \delta\}} \in L(S^c)$  on the one hand. On the other hand,  $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} I_{\{0 \leq Z_{-} < \delta\}} \cdot A \in \mathcal{A}_{loc}^+(\mathbb{F})$  due to  $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} \cdot A^\tau = (\beta^{\mathbb{G}})^T c \beta^{\mathbb{G}} \cdot A^\tau \in \mathcal{A}_{loc}^+(\mathbb{G})$  and Proposition C.3–(c). This completes the proof of  $\beta \in L(S^c)$ .

Now, we prove  $(f - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$ . Thanks to Proposition C.3 and  $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu^{\mathbb{G}}} = \sqrt{(f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}_{loc}^+(\mathbb{G})$ , we deduce that

$$(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \tilde{Z} I_{\{Z_{-} \geq \delta\}} \star \mu \quad \text{and} \quad |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \tilde{Z} I_{\{Z_{-} \geq \delta\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F}). \quad (5.15)$$

By stopping, there is no loss of generality in assuming that these two processes and  $[m, m]$  are integrable. Then we get

$$f - 1 = (f^{\mathbb{F}} - 1) \left( 1 + \frac{f_m}{Z_{-}} \right) I_{\Sigma_0} + \frac{f_m}{Z_{-}} I_{\Sigma_0^c} =: h_1 + h_2.$$

Therefore, we derive that for any  $\alpha > 0$

$$\begin{aligned} E [h_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty] &\leq \delta^{-2} E \left[ (f^{\mathbb{F}} - 1)^2 (Z_{-} + f_m)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_{-} \geq \delta\}} \star \mu_\infty \right] \\ &\leq \delta^{-2} E \left[ (f^{\mathbb{F}} - 1)^2 \tilde{Z} I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_{-} \geq \delta\}} \star \mu_\infty \right] < +\infty, \end{aligned}$$

and

$$\begin{aligned} E [h_2 I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty] &\leq \delta^{-1} E \left[ |f^{\mathbb{F}} - 1| |Z_{-} + f_m| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} I_{\{Z_{-} \geq \delta\}} \star \mu_\infty \right] \\ &= \delta^{-1} E \left[ |f^{\mathbb{F}} - 1| \tilde{Z} I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} I_{\{Z_{-} \geq \delta\}} \star \mu_\infty \right] < +\infty. \end{aligned}$$

By combining the above two inequalities, we conclude that  $(h_1^2 \star \mu)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{F})$ . It is easy to see that  $(h_2^2 \star \mu)^{1/2} \in \mathcal{A}_{loc}^+(\mathbb{F})$  follows from

$$E [h_2^2 \star \mu_\infty] \leq \delta^{-2} E [f_m^2 \star \mu_\infty] \leq \delta^2 E [(\Delta m)^2 \star \mu_\infty] \leq \delta^{-2} E [m, m]_\infty < +\infty.$$

**Step 3: Proof of (b)  $\Rightarrow$  (a).** If for any  $\delta > 0$ , the process  $I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})$  satisfies NUPBR( $\mathbb{F}$ ) where  $S^{(0)}$  is defined in (5.14), then, there exist an  $\mathbb{F}$ -local martingale  $N^{\mathbb{F}}$  and an  $\mathbb{F}$ -predictable process  $\phi$  such that  $0 < \phi \leq 1$  and  $\mathcal{E}(N^{\mathbb{F}}) [\phi I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})]$  is an  $\mathbb{F}$ -local martingale. Again, thanks to Theorem A.3, we can restrict our attention to the case

$$N^{\mathbb{F}} := \beta^{\mathbb{F}} \cdot S^c + (f^{\mathbb{F}} - 1) \star (\mu - \nu),$$

where  $\beta^{\mathbb{F}} \in L(S^c)$  and  $f^{\mathbb{F}}$  is positive such that  $(f^{\mathbb{F}} - 1) \in \mathcal{G}_{loc}^1(\mu, \mathbb{F})$ . Thanks to Itô's formula, the fact that  $\mathcal{E}(N^{\mathbb{F}}) [\phi I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})]$  is an  $\mathbb{F}$ -local martingale implies that on  $\{Z_- \geq \delta\}$

$$k^{\mathbb{F}} := \int |x f^{\mathbb{F}}(x) I_{\{\psi(x) > 0\}} - h(x)| F(dx) < +\infty \quad P \otimes A - a.e.$$

and  $P \otimes A$ -a.e. on  $\{Z_- \geq \delta\}$ , we have

$$b - \int h(x) I_{\{\psi=0\}} F(dx) + c\beta^{\mathbb{F}} + \int [x f^{\mathbb{F}}(x) - h(x)] I_{\{\psi>0\}} F(dx) = 0. \quad (5.16)$$

Recall that  $\{\psi = 0\} = \{Z_- + f_m = 0\}$  (see Lemma 5.2-(a)) and define

$$\beta^{\mathbb{G}} := \left( \beta^{\mathbb{F}} - \frac{\beta_m}{Z_-} \right) I_{\llbracket 0, \tau \rrbracket} \quad \text{and} \quad f^{\mathbb{G}} := \frac{f^{\mathbb{F}}}{1 + f_m/Z_-} I_{\Sigma_0} I_{\llbracket 0, \tau \rrbracket} + I_{\Sigma_0^c \cup \llbracket \tau, +\infty \rrbracket}.$$

If we assume for a while that

$$\beta^{\mathbb{G}} \in L(\widehat{S}^c) \quad \text{and} \quad (f^{\mathbb{G}} - 1) \in \mathcal{G}_{loc}^1(\mu^{\mathbb{G}}), \quad (5.17)$$

then, necessarily  $N^{\mathbb{G}} := \beta^{\mathbb{G}} \cdot \widehat{S}^c + (f^{\mathbb{G}} - 1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$  is a well defined  $\mathbb{G}$ -local martingale satisfying  $\mathcal{E}(N^{\mathbb{G}}) > 0$ . Furthermore, due to (5.16) and to  $\{\psi = 0\} = \{Z_- + f_m = 0\}$ , on  $\llbracket 0, \tau \rrbracket$  we obtain

$$b + c \left( \beta^{\mathbb{G}} + \frac{\beta_m}{Z_-} \right) + \int \left( x f^{\mathbb{G}} \left( 1 + \frac{f_m}{Z_-} \right) - h(x) \right) F(dx) = 0. \quad (5.18)$$

By taking  $\phi^{\mathbb{G}} := (1 + k^{\mathbb{F}} I_{\{Z_- \geq \delta\}})^{-1}$ , and applying Itô to  $(\phi^{\mathbb{G}} I_{\{Z_- \geq \delta\}} \cdot S^{\tau}) \mathcal{E}(N^{\mathbb{G}})$ , we conclude that this process is a  $\mathbb{G}$ -local martingale due to (5.18). Thus,  $I_{\{Z_- \geq \delta\}} \cdot S^{\tau}$  satisfies NUPBR( $\mathbb{G}$ ) as long as (5.17) is fulfilled.

Since  $Z_-^{-1} I_{\llbracket 0, \tau \rrbracket}$  is  $\mathbb{G}$ -locally bounded, then there exists a family of  $\mathbb{G}$ -stopping times  $(\tau_{\delta})_{\delta > 0}$  such that  $\llbracket 0, \tau_{\delta} \rrbracket \subset \{Z_- \geq \delta\}$  (which implies that  $I_{\{Z_- \geq \delta\}} \cdot S^{\tau \wedge \tau_{\delta}} = S^{\tau \wedge \tau_{\delta}}$ ) and  $\tau_{\delta}$  increases to infinity when  $\delta$  goes to zero. Thus, using Proposition 2.5, we deduce that  $S^{\tau}$  satisfies NUPBR( $\mathbb{G}$ ). This achieves the proof of (b) $\Rightarrow$ (a) under (5.17).

To prove that (5.17) holds true, we remark that  $Z_-^{-1} I_{\llbracket 0, \tau \rrbracket}$  is  $\mathbb{G}$ -locally bounded and both  $\beta_m$  and  $\beta^{\mathbb{F}}$  belong to  $L(S^c)$ . This, easily, implies that  $\beta^{\mathbb{G}} \in L(\widehat{S}^c)$ .

Now, we prove that  $\sqrt{(f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}_{loc}^+(\mathbb{G})$ . Since  $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{F})$ , Proposition C.3 allows us again to deduce that for any  $\alpha > 0$

$$(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F}) \text{ and } |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F}).$$

Without loss of generality, we can assume that these two processes and  $[m, m]$  are integrable. Put

$$f^{\mathbb{G}} - 1 = I_{\Sigma_0} I_{\llbracket 0, \tau \rrbracket} \frac{Z_- (f^{\mathbb{F}} - 1)}{f_m + Z_-} - I_{\Sigma_0^c} I_{\llbracket 0, \tau \rrbracket} \frac{f_m}{f_m + Z_-} := f_1 + f_2.$$

Then, setting  $\Sigma_\delta := \{0 < Z_- + f_m \leq \delta/2\} \cap \{Z_- \geq \delta\} = \Sigma_0 \cap \{Z_- + f_m \leq \delta/2\}$ , we calculate

$$E \left( f_1^2 I_{\Sigma_\delta \cap \{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty^{\mathbb{G}} \right) \leq \left(\frac{2}{\delta}\right)^2 E[(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty] < +\infty.$$

and

$$\begin{aligned} E \sqrt{f_1^2 I_{\Sigma_\delta \cap \{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty^{\mathbb{G}}} &\leq \alpha E(I_{\Sigma_\delta} (Z_- + f_m)^{-1} \star \mu_\infty^{\mathbb{G}}) \\ &\leq \alpha E(I_{\{|f_m| \geq \delta/2\}} \star \mu_\infty) \leq \frac{4\alpha}{\delta^2} E[m, m]_\infty < +\infty. \end{aligned}$$

This proves that  $\sqrt{f_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu^{\mathbb{G}}} \in \mathcal{A}_{loc}^+(\mathbb{G})$ . Similarly, we calculate

$$\begin{aligned} E \sqrt{f_1^2 I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty^{\mathbb{G}}} &\leq E(|f_1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty^{\mathbb{G}}) \leq E\left(\frac{|f^{\mathbb{F}} - 1|}{1 + f_m/Z_-} I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty^{\mathbb{G}}\right) \\ &\leq E(|f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \star \mu_\infty) < +\infty. \end{aligned}$$

Thus, by combining all the remarks obtained above, we conclude that  $\sqrt{f_1^2 \star \mu^{\mathbb{G}}}$  is  $\mathbb{G}$ -locally integrable. For the functional  $f_2$ , we proceed by calculating

$$E(f_2^2 I_{\Sigma_\delta} \star \mu_\infty^{\mathbb{G}}) \leq (2/\delta)^2 E(f_m^2 \star \mu_\infty) \leq (2/\delta)^2 E[m, m]_\infty < +\infty,$$

and

$$\begin{aligned} E \sqrt{f_2^2 I_{\Sigma_\delta} \star \mu_\infty^{\mathbb{G}}} &\leq E(|f_m| I_{\{|f_m| \geq \delta/2\}} \star \mu_\infty) \\ &\leq (2/\delta) E(f_m^2 \star \mu_\infty) \leq (2/\delta) E[m, m]_\infty < +\infty. \end{aligned}$$

This proves that  $\sqrt{f_2^2 \star \mu^{\mathbb{G}}}$  is  $\mathbb{G}$ -locally integrable. Therefore, we conclude that (5.17) is valid, and the proof of (b) $\Rightarrow$ (a) is completed.

**Step 4: Proof of (b)  $\iff$  (c).** For any  $\delta > 0$ , we denote

$$\sigma_\infty := \inf\{t \geq 0 : Z_t = 0\}, \quad \tau_\delta := \sup\{t : Z_{t-} \geq \delta\}.$$

Then, due to  $\llbracket \sigma_\infty, +\infty \llbracket \subset \{Z_- = 0\} \subset \{Z_- < \delta\}$ , we deduce

$$\sigma_{1/\delta} \leq \tau_\delta \leq \sigma_\infty \quad \text{and} \quad Z_{\tau_\delta-} \geq \delta > 0 \quad P - a.s. \text{ on } \{\tau_\delta < \infty\}.$$

Here,  $\sigma_{1/\delta}$  is defined by (2.10) ( $k = 1/\delta$ ). Setting  $\Sigma := \bigcap_{n \geq 1} (\sigma_n < \sigma_\infty)$ , we have

$$\text{on } \Sigma \cap \{\sigma_\infty < \infty\}, \quad Z_{\sigma_\infty-} = 0, \quad \text{and} \quad \tau_\delta < \sigma_\infty \quad P - a.s.$$

We introduce the semimartingale  $X := S - S^{(0)}$ . For any  $\delta > 0$ , and any  $H$  predictable such that  $H_\delta := HI_{\{Z_- \geq \delta\}} \in L(X)$  and  $H_\delta \cdot X \geq -1$ , due to Theorem 23 of [17] (page 346 in the French version),

$$(H_\delta \cdot X)_T = (H_\delta \cdot X)_{T \wedge \tau_\delta}, \quad \text{and on } \{\theta \geq \tau_\delta\} \quad (H_\delta \cdot X)_T = (H_\delta \cdot X)_{T \wedge \theta}.$$

Then, for any  $T \in (0, +\infty)$ , we calculate the following

$$\begin{aligned} P((H_\delta \cdot X)_T > c) &= P((H_\delta \cdot X)_T > c \ \& \ \sigma_n \geq \tau_\delta) + P((H_\delta \cdot X)_T > c \ \& \ \sigma_n < \tau_\delta) \\ &\leq 2 \sup_{\phi \in L(X^{\sigma_n}): \phi \cdot X^{\sigma_n} \geq -1} P((\phi \cdot X)_{\sigma_n \wedge T} > c) + P(\sigma_n < \tau_\delta \wedge T). \end{aligned} \tag{5.19}$$

It is easy to prove that  $P(\sigma_n < \tau_\delta \wedge T) \rightarrow 0$  as  $n$  goes to infinity. This can be seen due to the fact that on  $\Sigma$ , we have, on the one hand,  $\tau_\delta \wedge T < \sigma_\infty$  (by differentiating the two cases whether  $\sigma_\infty$  is finite or not). On the other hand, the event  $(\sigma_n < \sigma_\infty)$  increases to  $\Sigma$  with  $n$ . Thus, by combining these, we obtain the following

$$\begin{aligned} P(\sigma_n < \tau_\delta \wedge T) &= P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma) + P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma^c) \\ &\leq P(\sigma_n < \tau_\delta \wedge T < \sigma_\infty) + P((\sigma_n < \sigma_\infty) \cap \Sigma^c) \rightarrow 0. \end{aligned} \tag{5.20}$$

Now suppose that for each  $n \geq 1$ , the process  $(S - S^{(0)})^{\sigma_n}$  satisfies NUPBR( $\mathbb{F}$ ). Then a combination of (5.19) and (5.20) implies that for any  $\delta > 0$ , the process  $I_{\{Z_- \geq \delta\}} \cdot X := I_{\{Z_- \geq \delta\}} \cdot (S - S^{(0)})$  satisfies NUPBR( $\mathbb{F}$ ), and the proof of (c)  $\Rightarrow$  (b) is completed. The proof of the reverse implication is obvious due to the fact that

$$\llbracket 0, \sigma_n \rrbracket \subset \{Z_- \geq 1/n\} \subset \{Z_- \geq \delta\}, \quad \text{for } n \leq \delta^{-1},$$

which implies that  $(I_{\{Z_- \geq \delta\}} \cdot X)^{\sigma_n} = X^{\sigma_n}$ . This ends the proof of (b)  $\iff$  (c), and the proof of the theorem is achieved.  $\square$

### 5.3 Proof of Theorem 2.20

Due to (5.5), we deduce that the condition (2.11) is equivalent to

$$\begin{aligned} 0 &= E \left( I_{\{\psi=0\}} \star \nu_\tau \right) \\ &= E \int_0^\infty Z_{t-} I_{\{\psi(t,x)=0\}} \nu(dt, dx) = E \int_0^\infty Z_{t-} I_{\{\psi(t,x)=0\}} \mu(dt, dx) \end{aligned} \tag{5.21}$$

The second equality is due to the  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurability of  $\psi$  and  $\nu$  that implies that  $I_{\{\psi=0\}} \star \nu$  is  $\mathbb{F}$ -predictable process with integrable variation. It is obvious

that (5.21) implies that both processes  $I_{\{\psi=0<Z_-\}} \star \nu$  and  $I_{\{\psi=0<Z_-\}} \star \mu$  are null. Thus, we conclude that

$$m^{(0)} = I_{\{\psi=0<Z_-\}} \star (\mu - \nu) = I_{\{\psi=0<Z_-\}} \star \mu - I_{\{\psi=0<Z_-\}} \star \nu = 0.$$

As a result, thanks to Corollary 2.17–(c), this ends the proof of the theorem.  $\square$

#### 5.4 Proof of Theorem 4.2

The proof of theorem starts with guessing the form of  $L^{(1)}$  as follows. Put

$$L^{(1)} := g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}), \quad \text{where} \quad g_1 := \frac{1 - \psi}{1 + f_m/Z_-} I_{\{\psi>0\}}, \quad (5.22)$$

$f_m := M_\mu^P(\Delta m | \tilde{\mathcal{P}}(\mathbb{F}))$  and  $\psi$  is given by (5.6). Recall from (5.7) that  $\{\psi = 0\} = \{Z_- + f_m = 0\}$   $M_\mu^P - a.e.$ . Thus the functional  $g_1$  is a well defined non-negative  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional. The proof of the theorem will be completed in two steps. In the first step we prove that the process  $L^{(1)}$ —defined above—is a well defined local martingale, while in the second step we prove that  $\mathcal{E}(L + L^{(1)})$  is a deflator for  $S^\tau$ .

1) Herein, we prove that the integral  $g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$  is well-defined. To this end, it is enough to prove that  $g_1 \star \mu^{\mathbb{G}} \in \mathcal{A}^+(\mathbb{G})$ . Consider the  $\mathbb{F}$  stopping time

$$R := \inf\{t \geq 0 \mid Z_{t-} = 0\}, \quad (5.23)$$

and for  $A = \{\tilde{Z}_R = 0\}$  define  $R_0 := R_A$ . Therefore, remark that

$$(1 - \psi) I_{\{0 < Z_-\}} = M_\mu^P \left( I_{\{\tilde{Z}=0 < Z_-\}} | \tilde{\mathcal{P}}(\mathbb{F}) \right) = M_\mu^P \left( I_{[R_0]} | \tilde{\mathcal{P}}(\mathbb{F}) \right) I_{\{0 < Z_-\}},$$

and calculate

$$\begin{aligned} E(g_1 \star \mu^{\mathbb{G}}(\infty)) &= E(g_1 \tilde{Z} \star \mu_\infty) \\ &\leq E(I_{[R_0]} \star \mu_\infty) = P(\Delta S_{R_0} \neq 0 \ \& \ R_0 < +\infty) \leq 1. \end{aligned}$$

Thus,  $L^{(1)}$  is a  $\mathbb{G}$ -martingale, and  $\mathcal{E}(L + L^{(1)}) > 0$  due to  $\Delta L^{(1)} \geq 0$ .

2) In this part, we prove that  $\mathcal{E}(L + L^{(1)}) S^\tau$  is a  $\mathbb{G}$ -local martingale. To this end, it is enough to prove that  $\langle S^\tau, L + L^{(1)} \rangle^{\mathbb{G}}$  exists and

$$S^\tau + \langle S^\tau, L + g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) \rangle^{\mathbb{G}} \quad \text{is a } \mathbb{G}\text{-local martingale.} \quad (5.24)$$

Recall that

$$L = -\frac{Z_-^2}{Z_-^2 + \Delta \langle m \rangle^{\mathbb{F}}} \frac{1}{Z} I_{]0, \tau]} \odot \hat{m},$$



and hence  $\langle S^\tau, L \rangle^{\mathbb{G}}$  exists due to Proposition 3.6–(b). By stopping, there is no loss of generality in assuming that  $S$  is a true martingale. Then, using similar calculation as in the first part, we can easily prove that

$$E [ |x| g_1 \star \mu^{\mathbb{G}}(\infty) ] \leq E ( |\Delta S_{R_0}| I_{\{R_0 < +\infty\}} ) < +\infty.$$

This proves that  $\langle S^\tau, L + L^{(1)} \rangle^{\mathbb{G}}$  exists. Now, we calculate and simplify the expression in (5.24) as follows.

$$\begin{aligned} S^\tau + \langle S^\tau, L + g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) \rangle^{\mathbb{G}} &= \widehat{S} + \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, m \rangle^{\mathbb{F}} + \langle S^\tau, L \rangle^{\mathbb{G}} + x g_1 \star \nu^{\mathbb{G}} \\ &= \widehat{S} + \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \langle S, m \rangle^{\mathbb{F}} - \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \left( I_{\{\tilde{Z} > 0\}} \cdot [S, m] \right)^{p, \mathbb{F}} + x g_1 \star \nu^{\mathbb{G}} \\ &= \widehat{S} + \frac{1}{Z_-} I_{\llbracket 0, \tau \rrbracket} \cdot \left( I_{\{\tilde{Z} = 0\}} \cdot [S, m] \right)^{p, \mathbb{F}} + x M_\mu^P \left( I_{\{\tilde{Z} = 0 < Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) I_{\{Z_- + f_m > 0\}} I_{\llbracket 0, \tau \rrbracket} \star \nu \\ &= \widehat{S} - x M_\mu^P \left( I_{\{\tilde{Z} = 0 < Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) I_{\{\psi = 0\}} I_{\llbracket 0, \tau \rrbracket} \star \nu = \widehat{S} \in \mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

The second equality is due to (4.4), while the last equality follows directly from the fact that  $S^{(0)}$  is an  $\mathbb{F}$ -local martingale (which is equivalent to  $x I_{\{\psi = 0 < Z_-\}} \star \nu \equiv 0$ ) and  $M_\mu^P \left( I_{\{\tilde{Z} = 0 < Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) = I_{\{0 < Z_-\}} (1 - \psi)$ . This ends the proof of the theorem.  $\square$

## APPENDIX

### A Representation of Local Martingales

This section recalls an important result on representation of local martingales. This result relies on the continuous local martingale part and the jump random measure of a given semimartingale. Thus, throughout this section, we suppose given a  $d$ -dimensional semimartingale,  $S = (S_t)_{0 \leq t \leq T}$ . To this semimartingale, we associate its predictable characteristics that we will present below (for more details about these and other related issues, we refer the reader to Section II.2 of [25]). The random measure  $\mu$  associated to the jumps of  $S$  is defined in 5.1. The continuous local martingale part of  $S$  is denoted by  $S^c$ . This leads to the following decomposition, called “*the canonical representation*” (see Theorem 2.34, Section II.2 of [25]), namely the decomposition 5.3. For the matrix  $C$  with entries  $C^{ij} := \langle S^{c,i}, S^{c,j} \rangle$ , the triple  $(b \cdot A, C, \nu)$  is called *predictable characteristics* of  $S$ . Furthermore, we can find a version of the characteristics triple satisfying

$$C = c \cdot A \quad \text{and} \quad \nu(\omega, dt, dx) = dA_t(\omega) F_t(\omega, dx). \quad (\text{A.1})$$

Here  $A$  is an increasing and predictable process which is continuous if and only if  $S$  is quasi-left continuous,  $b$  and  $c$  are predictable processes,  $F_t(\omega, dx)$

is a predictable kernel,  $b_t(\omega)$  is a vector in  $\mathbb{R}^d$  and  $c_t(\omega)$  is a symmetric  $d \times d$ -matrix, for all  $(\omega, t) \in \Omega \times [0, T]$ . In the sequel we will often drop  $\omega$  and  $t$  and write, for instance,  $F(dx)$  as a shorthand for  $F_t(\omega, dx)$ .

The characteristics,  $B = b \cdot A$ ,  $C$ , and  $\nu$ , satisfy

$$F_t(\omega, \{0\}) = 0, \quad \int (|x|^2 \wedge 1) F_t(\omega, dx) \leq 1,$$

$$\Delta B_t = b \Delta A = \int h(x) \nu(\{t\}, dx), \quad \text{and} \quad c = 0 \quad \text{on} \quad \{\Delta A \neq 0\}.$$

We set

$$\nu_t(dx) := \nu(\{t\}, dx), \quad a_t := \nu_t(\mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1.$$

For the following representation theorem, we refer to [24, Theorem 3.75, page 103] and to [25, Lemma 4.24, Chap III].

**Theorem A.1** *Let  $N \in \mathcal{M}_{0,loc}$ . Then, there exist a predictable  $S^c$ -integrable process  $\beta$ ,  $N^\perp \in \mathcal{M}_{0,loc}$  with  $N^\perp$  and  $S$  orthogonal and functionals  $f \in \tilde{\mathcal{P}}$  and  $g \in \tilde{\mathcal{O}}$  such that*

- (a)  $\left( \sum_{s \leq t} f_s(\Delta S_s)^2 I_{\{\Delta S_s \neq 0\}} \right)^{1/2}$  and  $\left( \sum_{s \leq t} g_s(\Delta S_s)^2 I_{\{\Delta S_s \neq 0\}} \right)^{1/2}$  belong to  $\mathcal{A}_{loc}^+$ .
- (b)  $M_\mu^P(g | \tilde{\mathcal{P}}) = 0$ ,  $M_\mu^P - a.e.$ , where  $M_\mu^P := P \otimes \mu$ .
- (c) The process  $N$  satisfies

$$N = \beta \cdot S^c + W \star (\mu - \nu) + g \star \mu + N^\perp, \quad \text{where } W = f + \frac{\hat{f}}{1-a} I_{\{a < 1\}}. \quad (\text{A.2})$$

Here  $\hat{f}_t = \int f_t(x) \nu(\{t\}, dx)$  and  $f$  has a version such that  $\{a = 1\} \subset \{\hat{f} = 0\}$ . Moreover

$$\Delta N_t = \left( f_t(\Delta S_t) + g_t(\Delta S_t) \right) I_{\{\Delta S_t \neq 0\}} - \frac{\hat{f}_t}{1-a_t} I_{\{\Delta S_t = 0\}} + \Delta N_t^\perp. \quad (\text{A.3})$$

The quadruplet  $(\beta, f, g, N^\perp)$  is called the Jacod's parameters of the local martingale  $N$  with respect to  $S$ .

The following lemma is borrowed from Jacod's Theorem 3.75 in [24] (see also Proposition 2.2 in [11]).

**Lemma A.2** *Let  $\mathcal{E}(N)$  be a positive local martingale and  $(\beta, f, g, N^\perp)$  be the Jacod's parameters of  $N$ . Then  $\mathcal{E}(N) > 0$  (or equivalently  $1 + \Delta N > 0$ ) implies that*

$$f > 0, \quad M_\mu^P - a.e.$$

**Theorem A.3** *Let  $S$  be a semi-martingale with predictable characteristic triplet  $(b, c, \nu = A \otimes F)$ ,  $N$  be a local martingale such that  $\mathcal{E}(N) > 0$ , and  $(\beta, f, g, N^\perp)$  be its Jacod's parameters. Then the following assertions hold.*

1)  $\mathcal{E}(N)$  is a  $\sigma$ -martingale density of  $S$  if and only if the following two properties hold:

$$\int |x - h(x) + xf(x)|F(dx) < +\infty, \quad P \otimes A - a.e. \quad (\text{A.4})$$

$$b + c\beta + \int (x - h(x) + xf(x))F(dx) = 0, \quad P \otimes A - a.e. \quad (\text{A.5})$$

2) In particular, we have

$$\int x(1 + f_t(x))\nu(\{t\}, dx) = \int x(1 + f_t(x))F_t(dx)\Delta A_t = 0, \quad P - a.e. \quad (\text{A.6})$$

*Proof* The proof can be found in Choulli et al. [10, Lemma 2.4], and also in Choulli and Schweizer [11].  $\square$

## B Proof of $K \in {}^oL_{loc}^1(\widehat{m}, \mathbb{G})$

We start by calculating on  $\llbracket 0, \tau \rrbracket$ , making use of Lemma 3.1. We recall that  $\kappa := Z_-^2 + \Delta\langle m \rangle^\mathbb{F}$ .

$$\begin{aligned} K\Delta\widehat{m} - {}^{p,\mathbb{G}}(K\Delta\widehat{m}) &= \frac{I_{\llbracket 0, \tau \rrbracket} Z_-^2 \Delta\widehat{m}}{\kappa \widetilde{Z}} - {}^{p,\mathbb{G}}\left(\frac{I_{\llbracket 0, \tau \rrbracket} Z_-^2}{\kappa \widetilde{Z}} \Delta\widehat{m}\right) \\ &= \frac{(Z_-^2 \Delta m - Z_- \Delta\langle m \rangle^\mathbb{F})}{\kappa \widetilde{Z}} + \frac{{}^{p,\mathbb{F}}(I_{\{\widetilde{Z} > 0\}} \Delta\langle m \rangle^\mathbb{F})}{\kappa} - \frac{{}^{p,\mathbb{F}}(\Delta m I_{\{\widetilde{Z} > 0\}}) Z_-}{\kappa} \quad (\text{B.1}) \\ &= \frac{\Delta m}{\widetilde{Z}} I_{\llbracket 0, \tau \rrbracket} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z} = 0\}}\right) I_{\llbracket 0, \tau \rrbracket} =: \Delta V - \Delta V^\mathbb{G}. \end{aligned}$$

Here,  $V^\mathbb{G}$ , defined in (3.7), is nondecreasing, càdlàg and  $\mathbb{G}$ -locally bounded (see Proposition 3.3). Hence, we immediately deduce that  $\sum(\Delta V^\mathbb{G})^2 = \Delta V^\mathbb{G} \cdot V^\mathbb{G}$  is locally bounded, and in the rest of this part we focus on proving  $\sqrt{\sum(\Delta V)^2} \in \mathcal{A}_{loc}^+(\mathbb{G})$ . To this end, we consider  $\delta \in (0, 1)$ , and define  $C := \{\Delta m < -\delta Z_-\}$  and  $C^c$  its complement in  $\Omega \otimes [0, +\infty[$ . Then we obtain

$$\begin{aligned} \sqrt{\sum(\Delta V)^2} &\leq \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_C I_{\llbracket 0, \tau \rrbracket}\right)^{1/2} + \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_{C^c} I_{\llbracket 0, \tau \rrbracket}\right)^{1/2} \\ &\leq \sum \frac{|\Delta m|}{\widetilde{Z}} I_C I_{\llbracket 0, \tau \rrbracket} + \frac{1}{1-\delta} \left(I_{\llbracket 0, \tau \rrbracket} \frac{1}{\widetilde{Z}^2} \cdot [m]\right)^{1/2} =: V_1 + V_2. \end{aligned}$$

The last inequality above is due to  $\sqrt{\sum(\Delta X)^2} \leq \sum |\Delta X|$  and  $\widetilde{Z} \geq Z_-(1-\delta)$  on  $C^c$ . Using the fact that  $(Z_-)^{-1} I_{\llbracket 0, \tau \rrbracket}$  is  $\mathbb{G}$ -locally bounded and that  $m$  is an  $\mathbb{F}$ -locally bounded martingale, it follows that  $V_2$  is  $\mathbb{G}$ -locally bounded. Hence,

we focus on proving the  $\mathbb{G}$ -local integrability of  $V_1$ .

Consider a sequence of  $\mathbb{G}$ -stopping times  $(\vartheta_n)_n$  that increases to  $+\infty$  and

$$\left( (Z_-)^{-1} I_{\llbracket 0, \tau \rrbracket} \right)^{\vartheta_n} \leq n.$$

Also consider an  $\mathbb{F}$ -localizing sequence of stopping times,  $(\tau_n)_n$ , for the process  $V_3 := \sum \frac{(\Delta m)^2}{1+|\Delta m|}$ . Then, it is easy to prove

$$U_n := \sum |\Delta m| I_{\{\Delta m < -\delta/n\}} \leq \frac{n+\delta}{\delta} V_3,$$

and conclude that  $(U_n)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$ . Therefore, due to

$$\begin{aligned} C \cap \llbracket 0, \tau \rrbracket \cap \llbracket 0, \vartheta_n \rrbracket &= \{\Delta m < -\delta Z_-\} \cap \llbracket 0, \vartheta_n \rrbracket \cap \llbracket 0, \tau \rrbracket \\ &\subset \llbracket 0, \tau \rrbracket \cap \llbracket 0, \vartheta_n \rrbracket \cap \{\Delta m < -\frac{\delta}{n}\}, \end{aligned}$$

we derive

$$(V_1)^{\vartheta_n \wedge \tau_n} \leq \left( \tilde{Z} \right)^{-1} I_{\llbracket 0, \tau \rrbracket} \cdot (U_n)^{\tau_n}.$$

Since  $(U_n)^{\tau_n}$  is  $\mathbb{F}$ -adapted, nondecreasing and integrable, then due to Lemma 3.2, we deduce that the process  $V_1^{\vartheta_n \wedge \tau_n}$  is nondecreasing,  $\mathbb{G}$ -adapted and integrable. Since  $\vartheta_n \wedge \tau_n$  increases to  $+\infty$ , we conclude that the process  $V_1$  is  $\mathbb{G}$ -locally integrable. This completes the proof of  $K \in {}^o L_{loc}^1(\hat{m}, \mathbb{G})$ , and the process  $L$  (given via (3.9) and Definition 3.4) is a  $\mathbb{G}$ -local martingale.  $\square$

### C $\mathbb{G}$ -Localization versus $\mathbb{F}$ -Localization

We now present results that are important for the proofs of Subsection 5.2, and are the most innovative results of the appendix.

**Lemma C.1** *The following assertions hold.*

(a) *If  $H^{\mathbb{G}}$  is a  $\tilde{\mathcal{P}}(\mathbb{G})$ -measurable functional, then there exist an  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional  $H^{\mathbb{F}}$  and a  $\mathcal{B}(\mathbb{R}_+) \otimes \tilde{\mathcal{P}}(\mathbb{F})$ -measurable functionals  $K^{\mathbb{F}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

$$H^{\mathbb{G}}(\omega, t, x) = H^{\mathbb{F}}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket} + K^{\mathbb{F}}(\tau(\omega), t, \omega, x) I_{\llbracket \tau, +\infty \rrbracket}. \quad (\text{C.1})$$

(b) *If furthermore  $H^{\mathbb{G}} > 0$  (respectively  $H^{\mathbb{G}} \leq 1$ ), then we can choose  $H^{\mathbb{F}} > 0$  (respectively  $H^{\mathbb{F}} \leq 1$ ) such that*

$$H^{\mathbb{G}}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket} = H^{\mathbb{F}}(\omega, t, x) I_{\llbracket 0, \tau \rrbracket}.$$

(c) *If  $L^{\mathbb{G}}$  is an  $\tilde{\mathcal{O}}(\mathbb{G})$ -measurable functional, then there exist an  $\tilde{\mathcal{O}}(\mathbb{G})$ -measurable functional  $L^{(1)}(t, \omega, x)$ , a  $\tilde{\mathcal{P}}_{prog}(\mathbb{F})$ -measurable functional  $L^{(2)}(t, \omega, x)$ , and an  $\tilde{\mathcal{O}}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functional,  $L^{(3)}(t, \omega, x, v)$ , such that*

$$L^{\mathbb{G}}(t, \omega, x) = L^{(1)}(t, \omega, x) I_{\llbracket 0, \tau \rrbracket} + L^{(2)}(t, \omega, x) I_{\llbracket \tau \rrbracket} + L^{(3)}(t, \omega, x, \tau) I_{\llbracket \tau, +\infty \rrbracket} \quad (\text{C.2})$$

where  $\mathcal{P}_{prog}(\mathbb{F})$  is the  $\mathbb{F}$ -progressive  $\sigma$ -field on  $\Omega \times \mathbb{R}^+$ , and  $\tilde{\mathcal{P}}_{prog}(\mathbb{F}) := \mathcal{P}_{prog}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^d)$ . If furthermore,  $0 < L^{\mathbb{G}}$  (respectively  $L^{\mathbb{G}} \leq 1$ ), then all  $L^{(i)}$  can be chosen such that  $0 < L^{(i)}$  (respectively  $L^{(i)} \leq 1$ ),  $i = 1, 2, 3$ .

(d) For any  $\mathbb{F}$ -stopping time,  $T$ , and any positive  $\mathcal{G}_T$ -measurable random variable  $Y^{\mathbb{G}}$ , there exist two positive  $\mathcal{F}_T$ -measurable random variables,  $Y^{(1)}$  and  $Y^{(2)}$ , satisfying

$$Y^{\mathbb{G}} I_{\{T \leq \tau\}} = Y^{(1)} I_{\{T < \tau\}} + Y^{(2)} I_{\{\tau = T\}}. \quad (\text{C.3})$$

The proof of this lemma can be found in one of our earliest version that is available on Arxiv, see [2]. Below, we state a simple but very useful lemma that generalizes a version elaborated in [12]

**Lemma C.2** *If  $X(t, \omega, x, v)$  is a  $\tilde{\mathcal{O}}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^+)$ -measurable functional, then  $\bar{X}(t, \omega, x) := X(t, \omega, x, t)$  is  $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable.*

*Proof* The proof of this lemma is immediate from a combination of the class monotone theorem, and the proof of the lemma for the generators of  $\tilde{\mathcal{O}}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^+)$  having the form of  $X(t, \omega, x, v) = H(t, \omega, x)k(v)$ . Here  $H$  is  $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable and  $k$  is  $\mathcal{B}(\mathbb{R}^+)$ -measurable. For these generators, we have  $\bar{X}(t, \omega, x) = H(t, \omega, x)k(t)$  which is obviously  $\tilde{\mathcal{O}}(\mathbb{H})$ -measurable

**Proposition C.3** *For any  $\alpha > 0$ , the following assertions hold:*

(a) *Let  $h$  be a  $\tilde{\mathcal{P}}(\mathbb{H})$ -measurable functional. Then,  $\sqrt{(h-1)^2} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{H})$  iff*

$$(h-1)^2 I_{\{|h-1| \leq \alpha\}} \star \mu \text{ and } |h-1| I_{\{|h-1| > \alpha\}} \star \mu \text{ belong to } \mathcal{A}_{loc}^+(\mathbb{H}).$$

(b) *Let  $(\sigma_n^{\mathbb{G}})_n$  be a sequence of  $\mathbb{G}$ -stopping times that increases to infinity. Then, there exists a nondecreasing sequence of  $\mathbb{F}$ -stopping times,  $(\sigma_n^{\mathbb{F}})_{n \geq 1}$ , satisfying the following properties*

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau, \quad \sigma_{\infty} := \sup_n \sigma_n^{\mathbb{F}} \geq R \quad P - a.s., \quad (\text{C.4})$$

$$\text{and} \quad Z_{\sigma_{\infty}^-} = 0 \quad P - a.s. \quad \text{on} \quad \Sigma \cap (\sigma_{\infty} < +\infty), \quad (\text{C.5})$$

where  $R$  is defined in (5.23) and  $\Sigma := \bigcap_{n \geq 1} (\sigma_n^{\mathbb{F}} < \sigma_{\infty})$ .

(c) *Let  $V$  be an  $\mathbb{F}$ -predictable and non-decreasing process. Then,  $V^{\tau} \in \mathcal{A}_{loc}^+(\mathbb{G})$  if and only if  $I_{\{Z_- \geq \delta\}} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$  for any  $\delta > 0$ .*

(d) *Let  $h$  be a nonnegative and  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable functional. Then,  $h I_{]0, \tau]} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$  if and only if for all  $\delta > 0$ ,  $h I_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}_{loc}^+(\mathbb{F})$ , where  $\mu^1 := \tilde{Z} \cdot \mu$ .*

(e) *Let  $f$  be positive and  $\tilde{\mathcal{P}}(\mathbb{F})$ -measurable, and  $\mu^1 := \tilde{Z} \cdot \mu$ . Then  $\sqrt{(f-1)^2} I_{]0, \tau]} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$  iff  $\sqrt{(f-1)^2} I_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}_{loc}^+(\mathbb{F})$ , for all  $\delta > 0$ .*

*Proof* (a) Put  $W := (h-1)^2 \star \mu = W_1 + W_2$ , where  $W_1 := (h-1)^2 I_{\{|h-1| \leq \alpha\}} \star \mu$ ,  $W_2 := (h-1)^2 I_{\{|h-1| > \alpha\}} \star \mu$  and  $W'_2 := |h-1| I_{\{|h-1| > \alpha\}} \star \mu$ . Note that

$$\sqrt{W} = \sqrt{W_1 + W_2} \leq \sqrt{W_1} + \sqrt{W_2} \leq \sqrt{W_1} + W'_2.$$

Therefore  $\sqrt{W_1}, W_2' \in \mathcal{A}_{loc}^+$  imply that  $\sqrt{W}$  is locally integrable. Conversely, if  $\sqrt{W} \in \mathcal{A}_{loc}^+$ , then  $\sqrt{W_1}$  and  $\sqrt{W_2}$  are both locally integrable. Since  $W_1$  is locally bounded and has finite variation,  $W_1$  is locally integrable. In the following, we focus on the proof of the local integrability of  $W_2'$ . Denote

$$\tau_n := \inf\{t \geq 0 : V_t > n\}, \quad V := W_2.$$

It is easy to see that  $\tau_n$  increases to infinity and  $V_- \leq n$  on the set  $]0, \tau_n]$ . On the set  $\{\Delta V > 0\}$ , we have  $\Delta V \geq \alpha^2$ . By using the elementary inequality  $\sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}} \leq \sqrt{1+x} - \sqrt{x} \leq 1$ , when  $0 \leq x \leq \frac{n}{\alpha^2}$ , we have

$$\sqrt{V_- + \Delta V} - \sqrt{V_-} \geq \beta_n \sqrt{\Delta V} \quad \text{on } ]0, \tau_n], \quad \text{where } \beta_n := \sqrt{1 + \frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}},$$

and

$$(W_2')^{\tau_n} = \left( \sum \sqrt{\Delta V} \right)^{\tau_n} \leq \frac{1}{\beta_n} \left( \sum \Delta V \right)^{\tau_n} = \frac{1}{\beta_n} \left( \sqrt{W_2} \right)^{\tau_n} \in \mathcal{A}_{loc}^+(\mathbb{H})$$

Therefore  $W_2' \in (\mathcal{A}_{loc}^+(\mathbb{H}))_{loc} = \mathcal{A}_{loc}^+(\mathbb{H})$ .

(b) Due to Jeulin [26], there exists a sequence of  $\mathbb{F}$ -stopping times  $(\sigma_n^{\mathbb{F}})_n$  such that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau. \quad (\text{C.6})$$

By putting  $\sigma_n := \sup_{k \leq n} \sigma_k^{\mathbb{F}}$ , we shall prove that

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n \wedge \tau, \quad (\text{C.7})$$

or equivalently  $\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\}$  is negligible. Due to (C.6) and  $\sigma_n^{\mathbb{G}}$  is nondecreasing, we derive

$$\{\sigma_n^{\mathbb{F}} < \tau\} = \{\sigma_n^{\mathbb{G}} < \tau\} \subset \bigcap_{i=1}^n \{\sigma_i^{\mathbb{G}} = \sigma_i^{\mathbb{F}}\} \subset \{\sigma_n^{\mathbb{F}} = \sigma_n\}.$$

This implies that,

$$\{\sigma_n^{\mathbb{F}} \wedge \tau < \sigma_n \wedge \tau\} = \{\sigma_n^{\mathbb{F}} < \tau, \& \sigma_n^{\mathbb{F}} < \sigma_n\} = \emptyset,$$

and the proof of (C.7) is completed. Without loss of generality we assume that the sequence  $\sigma_n^{\mathbb{F}}$  is nondecreasing. By taking limit in (C.6), we obtain  $\tau = \sigma_\infty \wedge \tau, P$ -a.s. which is equivalent to  $\sigma_\infty \geq \tau, P$ -a.s. Since  $R$  is the smallest  $\mathbb{F}$ -stopping time greater or equal than  $\tau$  almost surely, we obtain,  $\sigma_\infty \geq R \geq \tau, P$ -a.s.. This achieves the proof of (C.4).

On the set  $\Sigma$ , it is easy to show that

$$I_{[0, \sigma_n^{\mathbb{F}}]} \longrightarrow I_{[0, \sigma_\infty^{\mathbb{F}}]}, \quad \text{when } n \text{ goes to } +\infty.$$

Then, thanks again to (C.6) (by taking  $\mathbb{F}$ -predictable projection and let  $n$  go to infinity afterwards), we obtain

$$Z_- = Z_- I_{\llbracket 0, \sigma_\infty^\mathbb{F} \rrbracket}, \quad \text{on } \Sigma. \quad (\text{C.8})$$

Hence, (C.5) follows immediately, and the proof of assertion (b) is completed.

(c) Suppose that  $V^\tau \in \mathcal{A}_{loc}^+(\mathbb{G})$ . Then, there exists a sequence of  $\mathbb{G}$ -stopping times  $(\sigma_n^\mathbb{G})$  increasing to infinity such that  $V^{\tau \wedge \sigma_n^\mathbb{G}} \in \mathcal{A}^+(\mathbb{G})$ . Consider  $(\sigma_n)$  a sequence of  $\mathbb{F}$ -stopping times satisfying (C.4)–(C.5) (its existence is guaranteed by assertion (b)). Therefore, for any fixed  $\delta > 0$

$$W^n := Z_- I_{\{Z_- \geq \delta\}} \cdot V^{\sigma_n} \in \mathcal{A}^+(\mathbb{F}), \quad (\text{C.9})$$

or equivalently, this process is càdlàg predictable with finite values. Thus, it is obvious that proving that the  $\mathbb{F}$ -predictable and nondecreasing process

$$W := I_{\{Z_- \geq \delta\}} \cdot V \quad \text{is càdlàg with finite values} \quad (\text{C.10})$$

is sufficient to prove its  $\mathbb{F}$ -locally integrability. To prove (C.10), we consider the random time  $\tau^\delta$  defined by

$$\tau^\delta := \sup\{t \geq 0 : Z_{t-} \geq \delta\}.$$

Then, it is clear that  $I_{\llbracket \tau^\delta, +\infty \rrbracket} \cdot W \equiv 0$  and

$$\tau^\delta \leq R \leq \sigma_\infty \quad \text{and} \quad Z_{\tau^\delta-} \geq \delta \quad P\text{-a.s.} \quad \text{on } \{\tau^\delta < +\infty\}.$$

The proof of (C.10) will be achieved by considering three sets, namely  $\{\sigma_\infty = \infty\}$ ,  $\Sigma \cap \{\sigma_\infty < +\infty\}$ , and  $\Sigma^c \cap \{\sigma_\infty < +\infty\}$ . It is obvious that (C.10) holds on  $\{\sigma_\infty = \infty\}$ . Due to (C.5), we deduce that  $\tau^\delta < \sigma_\infty$ ,  $P$ -a.s. on  $\Sigma \cap \{\sigma_\infty < +\infty\}$ . Since  $W$  is supported on  $\llbracket 0, \tau^\delta \rrbracket$ , then (C.10) follows immediately on the set  $\Sigma \cap \{\sigma_\infty < +\infty\}$ . Finally, on the set

$$\Sigma^c \cap \{\sigma_\infty < +\infty\} = \left( \bigcup_{n \geq 1} \{\sigma_n = \sigma_\infty\} \right) \cap \{\sigma_\infty < +\infty\},$$

the sequence  $\sigma_n$  increases stationarily to  $\sigma_\infty$ , and thus (C.10) holds on this set. This completes the proof of (C.10), and hence  $I_{\{Z_- \geq \delta\}} Z_- \cdot V$  is locally integrable, for any  $\delta > 0$ .

Conversely, if  $I_{\{Z_- \geq \delta\}} \cdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ , there exists a sequence of  $\mathbb{F}$ -stopping times  $(\tau_n)_{n \geq 1}$  that increases to infinity and  $(I_{\{Z_- \geq \delta\}} \cdot V)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$ . Then, we have

$$E [I_{\{Z_- \geq \delta\}} I_{\llbracket 0, \tau \rrbracket} \cdot V_{\tau_n}] = E [I_{\{Z_- \geq \delta\}} Z_- \cdot V_{\tau_n}] < +\infty. \quad (\text{C.11})$$

This proves that  $I_{\{Z_- \geq \delta\}} I_{[0, \tau]} \cdot V$  is  $\mathbb{G}$ -locally integrable, for any  $\delta > 0$ . Since  $(Z_-)^{-1} I_{[0, \tau]}$  is  $\mathbb{G}$ -locally bounded, then there exists a family of  $\mathbb{G}$ -stopping times  $(\tau_\delta)_{\delta > 0}$  that increases to infinity when  $\delta$  decreases to zero, and

$$[0, \tau \wedge \tau_\delta] \subset \{Z_- \geq \delta\}.$$

This implies that the process  $(I_{[0, \tau]} \cdot V)^{\tau_\delta}$  is  $\mathbb{G}$ -locally integrable, and hence the assertion (c) follows immediately.

(d) The proof of assertion (d) follows from combining the easy fact that  $hI_{[0, \tau]} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$  if and only if  $hI_{[0, \tau]} \star \nu^{\mathbb{G}} = hI_{[0, \tau]}(1 + \frac{f_m}{Z_-}) \star \nu \in \mathcal{A}_{loc}^+(\mathbb{G})$  and assertion (c) using  $V = h(1 + \frac{f_m}{Z_-}) I_{\{Z_- > \delta\}} \star \nu$ .

(e) The proof of assertion (e) follows immediately from combining assertions (d) and (a). This ends the proof of the proposition.  $\square$

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## References

1. Acciaio B., Fontana C., Kardaras C.(2014) Arbitrage of the First Kind and Filtration Enlargements In Semimartingale Financial Models, <http://arxiv.org/pdf/1401.7198.pdf>
2. Aksamit, A., Choulli, T., Deng, J., and Jeanblanc, M. (2014): Arbitrages in a progressive enlargement setting , Arbitrage, Credit and Informational Risks, Peking University Series in Mathematics Vol. 6, 55-88, World Scientific.
3. Aksamit, A., Choulli, T., Deng, J., and Jeanblanc, M. (2014): Non-Arbitrage up to Random Horizon for Semimartingale Models, <http://arxiv.org/abs/1310.1142>.
4. Aksamit, A. (2014): Random times, enlargement of filtration and arbitrages, PhD Thesis, Evry Val d'Essonne University.
5. Amendinger, J., Imkeller, P., Schweizer, M. (1998): Additional logarithmic utility of an insider. Stochastic processes and their applications, 75(2), 263-286.
6. Ansel, J.-P., Stricker, C.: Couverture des actifs contingents. Ann. Inst. Henri Poincaré 30, 303-315 (1994).
7. Assmussen, S. (200): Ruin probability. Scientific World.
8. Barbarin, J.(2008): Valuation, hedging and the risk management of insurance contracts. PhD thesis, Louvain University, <http://hdl.handle.net/2078.1/12686>.
9. Becherer, D. (2001): The numéraire portfolio for unbounded semimartingales. Finance and Stochastics 5(3), 327-341.
10. Choulli, T., Stricker, C., and Li J. (2007): Minimal Hellinger martingale measures of order  $q$ . Finance and Stochastics 11, 3, 399-427.
11. Choulli T. and Schweizer M. (2013): Locally  $\Phi$ -integrable  $\sigma$ -martingale densities for general semimartingales, to appear in Stochastics.
12. Choulli T. and Schweizer M. (2013): On Volterra-type semimartingale and their integrands. Preprint of University of Alberta and ETH-Zurich.



13. Choulli T., Deng, J. and Ma, J. (2014): How non-arbitrage, viability and numéraire portfolio are related. To appear in *Finance and Stochastics*.
14. Christensen, M. M., Larsen, K.: No arbitrage and the growth optimal portfolio. *Stochastic Analysis and Applications* 25(1), 255-280 (2007).
15. Dellacherie, C. (1972): *Capacités et processus stochastiques*, Springer.
16. Dellacherie, C. and Meyer, P-A. (1972): *Probabilités et Potentiel*, chapitres I-VI, Hermann, Paris.
17. Dellacherie, C. and Meyer, P-A. (1980): *Probabilités et Potentiel*, chapitres V-VIII, Hermann, Paris, English translation : *Probabilities and Potentiel*, chapters V-VIII, North-Holland, (1982).
18. Dellacherie, M., Maisonneuve, B. and Meyer, P-A. (1992): *Probabilités et Potentiel*, chapitres XVII-XXIV: *Processus de Markov (fin)*, Compléments de calcul stochastique, Hermann, Paris.
19. Delbaen F. and Schachermayer W. (1994): A general version of the fundamental theorem of asset pricing, *Mathematische Annalen*, 300, 463–520.
20. Fontana, C., Jeanblanc, M. and Song, S. (2013): On arbitrages arising with honest times. To appear in *Finance and Stochastics*
21. Ghorud, A. and Pontier, M. (1998): Insider trading in a continuous time market model, *International Journal of Theoretical and Applied Finance*, 1, 331-347.
22. He, S. W., Wang, C. K., Yan, J. A. (1992): *Semimartingale theory and stochastic calculus*. CRC Press.
23. Hulley, H. and Schweizer, M. (2010): M6 On Minimal Market Models and Minimal Martingale Measures, *Contemporary Quantitative Finance*, pp 35-51.
24. Jacod, J. (1979): *Calcul Stochastique et Problèmes de Martingales*, Lecture Notes in Mathematics, Vol. 714.
25. Jacod, J. and Shiryaev, A.N. (2003): *Limit theorems for stochastic Processes*, Springer Verlag.
26. Jeulin, T. (1980), *Semi-martingales et Grossissement d'une Filtration*, Lecture Notes in Mathematics, vol. 833, Springer, Berlin - Heidelberg - New York.
27. Jeulin, Th. and Yor, M., (1978) *Grossissement d'une filtration et semi-martingales : formules explicites*, Séminaire de Probabilités XII, Dellacherie, C. and Meyer, P-A. and Weil, M., Lecture Notes in Mathematics 649, 78-97, Springer-Verlag.
28. Kabanov, Y. (1997): On the FTAP of Kreps-Delbaen-Schachermayer. In: Kabanov, Y. et al. (eds.): *Statistics and Control of Stochastic Processes: The Liptser Festschrift*, pp. 191–203. World Scientific, Singapore.
29. Kallsen, J. (2004):  $\sigma$ -localization and  $\sigma$ -martingales. *Theory of Probability & Its Applications* 48, 1, 152-163.
30. Karatzas I. and Kardaras C. (2007): The numéraire portfolio in semimartingale financial models, *Finance and Stochastics*, 447-493.
31. Kardaras, K. (2012): A time before which insiders would not undertake [riks.arXiv:2010.1961v2](https://arxiv.org/abs/2010.1961v2).
32. Kardaras, K. (2012): Market viability via absence of arbitrage of the first kind. *Finance and stochastics*, 16(4), 651-667.
33. Kardaras, K. (2014): On the characterisation of honest times that avoid all stopping times. *Stochastic Processes and their Applications* 124, 373–384.
34. Kohatsu-Higa, A., Sulem, A. (2006): Utility maximization in an insider influenced market. *Mathematical Finance*, 16(1), 153-179.
35. Loewenstein, M., Willard, G. A. (2000): Local martingales, arbitrage, and viability Free snacks and cheap thrills. *Economic Theory*, 16(1), 135-161.
36. Larsen, K. and Zitkovic, G. (2013): On Utility Maximization under Convex Portfolio Constraints *The Annals of Applied Probability* Vol. 23, No. 2, 665-692
37. Levental, S., Skorohod, A. V. (1995): A necessary and sufficient condition for absence of arbitrage with tame portfolios. *The Annals of Applied Probability*, 906-925.
38. Platen, E.: A benchmark approach to finance. *Math. Finance* 16, 131-151, 2006
39. Pikovsky, I., Karatzas, I. (1996): Anticipative portfolio optimization. *Advances in Applied Probability*, 1095-1122.
40. Rokhlin, D.B. (2010): On the existence of an equivalent supermartingale density for a fork-convex family of random processes, *Math. Notes* 87, no. 3-4, 556-563.

- 
41. Ruf J. (2013) Hedging under arbitrage, *Mathematical Finance* Volume 23, Issue 2, pages 297-317.
  42. Song, S. (2014): No-arbitrage condition for  $S^{t-}$  in a progressively enlarged filtration arXiv:1405.4474.
  43. Takaoka, K., and Schweizer, M. (2014): A note on the condition of no unbounded profit with bounded risk. *Finance and Stochastics* 18(2), pp: 393-405.
  44. Yor, M. (1978): Grossissement d'une filtration et semi-martingales: théorèmes généraux Séminaire de probabilités, tome XII, *Lecture Notes in Mathematics*, Vol. 649, 61-69.