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# GENERALIZED JUNCTION CONDITIONS FOR DEGENERATE PARABOLIC EQUATIONS

CYRIL IMBERT AND VINH DUC NGUYEN

ABSTRACT. We are interested in the study of parabolic equations on a *multi-dimensional junction* (Imbert, Monneau (2014)), i.e. the union of a finite number of copies of a half-hyperplane of  $\mathbb{R}^{d+1}$  whose boundaries are identified. The common boundary is referred to as the *junction hyperplane*. The parabolic equations on the half-hyperplanes are in non-divergence form, fully non-linear and possibly degenerate, and they do degenerate along the junction hyperplane, i.e. along the junction hyperplane the nonlinearities do not depend on second order derivatives. The parabolic equations are supplemented with a *generalized junction condition* (or boundary condition of Neumann type), which is compatible with the maximum principle. Our main result states that, in the case where the non-linearities at the junction have convex sublevel sets with respect to the gradient variable, then these general junction conditions can be classified: they are equivalent to junction conditions of control type. This classification extends the one obtained by Imbert and Monneau for Hamilton-Jacobi equations on networks and multi-dimensional junctions. We give two applications of this classification result. On the one hand, we give the first complete answer to an open question about these equations: we prove in the two-domain case that the vanishing viscosity limit associated with quasi-convex Hamilton-Jacobi equations coincides with the maximal Ishii solution identified by Barles, Briani and Chasseigne (2012). On the other hand, we give a short and simple PDE proof of a large deviation results of Boué, Dupuis and Ellis (2000).

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## 1. INTRODUCTION

**1.1. Degenerate parabolic equations on junctions.** *Multi-dimensional junctions* are introduced in [14]:

$$J = \bigcup_{i=1}^N J_i \quad \text{with} \quad \begin{cases} J_i = \{x = (x', x_i) : x' \in \mathbb{R}^d, x_i \geq 0\} \simeq \mathbb{R}_+^{d+1} \\ J_i \cap J_j = \Gamma \simeq \mathbb{R}^d \quad \text{for } i \neq j. \end{cases}$$

Given  $T \in [0, +\infty]$ , we consider a general degenerate parabolic equation posed on a junction,

$$(1.1) \quad \begin{cases} u_t + F_i(x, Du, D^2u) = 0 & (t, x) \in (0, T) \times J_i^*, i = 1, \dots, N, \\ L(-u_t, \partial_1 u, \dots, \partial_N u, x', D'u) = 0 & (t, x) \in (0, T) \times \Gamma \end{cases}$$

where  $J_i^*$  denotes  $J_i \setminus \Gamma$ ,  $u_t$  denotes the time derivative,  $Du$  and  $D^2u$  respectively denote the gradient and the Hessian of  $u$  with respect to  $x$ , and for  $x \in \Gamma$ ,  $\partial_i u(x)$  denotes the derivative of  $u_i(x) = u|_{J_i}(x)$  with respect to  $x_i$  (recall  $x = (x', x_i)$ ) and  $D'u$  denotes the derivative with respect to  $x'$ . We make the following assumptions on each  $F_i$ .

**Assumption (F).**

- $F_i$  continuous and degenerate elliptic;
- for all  $R > 0$ , there exists  $C_{i,R} > 0$  such that for all  $y = (y', y_i)$ , all  $p \in \mathbb{R}^{d+1}$ , all  $B \in \mathbb{S}_{d+1}(\mathbb{R})$  and all  $\lambda > 0$

$$\{|y_i| \leq 1, |y'| \leq R, |B| \leq R\} \Rightarrow F_i(y, p, B + \lambda e_{d+1} \otimes e_{d+1}) \geq F_i(y, p, B) - C_{i,R} \lambda |y_i|^2;$$

- for all  $R > 0$ ,

$$\lim_{|p| \rightarrow +\infty} \inf_{(x, B) \in B_R} F_i(x, p, B) = +\infty;$$

- there exists  $H_i : \Gamma \times \mathbb{R}^{d+1} \rightarrow \mathbb{R}$  continuous such that
  - for all  $(x, p, B) \in \Gamma \times \mathbb{R}^{d+1} \times \mathbb{S}_{d+1}(\mathbb{R})$ ,  $F_i(x, p, B) = H_i(x, p)$ ;
  - for all  $x \in \Gamma$ , for all  $\lambda \in \mathbb{R}$ , the set  $\{p \in \mathbb{R}^{d+1} : H_i(x, p) \leq \lambda\}$  is convex.

In the assumption above,  $e_{d+1}$  denotes the unit vector orthogonal to  $\Gamma$  and pointing inside  $J_i$ .

**Example 1.1** (First order case). The first example we give is the one coming from [15, 14]. It reduces to deal with  $F_i(x, p, B) = H_i(x, p)$  for any  $x \in J_i$  (and not only  $x \in \Gamma$ ) and  $p \in \mathbb{R}^d$  with  $H_i$  continuous, coercive in  $p$  uniformly in  $x$ , with convex sublevel sets (in  $p$ ).

**Example 1.2** (The model case). Our results apply to the model case where  $F_i(x, p, B) = H_i(x, p) - \text{Trace}(\sigma_i(x) \sigma_i^T(x) B)$  with  $H_i$  is as in Example 1.1 and where the  $(d+1) \times m$  real matrix  $\sigma_i$  is such that  $\sigma_i \equiv 0$  on  $\Gamma$  and the  $(d+1)$ -th line  $\sigma_i^{d+1}$  of  $\sigma_i$  satisfies  $|\sigma_i^{d+1}(y)| \leq c_i |y_{d+1}|$ . Remark that this latter condition holds true if  $\sigma_i \equiv 0$  on  $\Gamma$  and  $\sigma_i$  is Lipschitz continuous.

We recall that  $F_i(x, p, A)$  is degenerate elliptic if it is non-increasing with respect to  $A$  (where the set of symmetric matrices  $\mathbb{S}_{d+1}(\mathbb{R})$  is equipped with the usual partial order). The function  $H_i$  appearing in (F) is referred to as the *Hamiltonian* from the branch  $J_i$ . As far as the junction function  $L$  is concerned, we assume

**Assumption (L).**

- the function  $L$  is continuous;
- $L(p_0, \dots, p_N, x', p')$  is non-increasing in  $p_i$  for  $i = 0, \dots, N$ ;
- $\exists i_0 : \{\forall i, p_i \leq q_i \text{ and } p_{i_0} \neq q_{i_0}\} \Rightarrow L(p_0, \dots, p_N, x', p') > L(q_0, \dots, q_N, x', p')$
- $\inf_{x', p'} L(p_0, \dots, p_N, x', p') \rightarrow +\infty$  as  $\min_{i=0, \dots, N} p_i \rightarrow -\infty$ ;
- $\sup_{x', p'} L(p_0, \dots, p_N, x', p') \rightarrow -\infty$  as  $\max_{i=0, \dots, N} p_i \rightarrow +\infty$ .

**Example 1.3** (Kirchoff conditions). A model for  $L$  is

$$L(p_0, \dots, p_N) = - \sum_{i=1}^N -\beta_i p_i$$

with  $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{R}^N \setminus \{0\}$ . Such a condition is called a Kirchoff condition.

**Example 1.4** (Flux-limited junction conditions). An second important example of junction functions  $L$  is the one related to flux-limited solutions [15, 14]. Given a *flux-limiter*  $A$ ,

$$(A) \quad \begin{cases} A : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R} \text{ continuous} \\ \text{for all } x' \in \mathbb{R}^d, \lambda \in \mathbb{R}, \{p' \in \mathbb{R}^d : A(x', p') \leq \lambda\} \text{ convex} \end{cases}$$

we consider the associated junction condition  $L_A$  defined by

$$(1.2) \quad L_A(p_0, \dots, p_N, x', p') = -p_0 + \max(A(x', p'), \max_i h_i^-(x', p', p_i))$$

where  $h_i^-(x, p', p_i)$  denotes the non-increasing part of  $p_i \mapsto h_i(x', p', p_i) := H_i((x', 0), (p', p_i))$  [14]: if  $p_i \mapsto h_i(x', p', p_i)$  reaches its minimum at  $\pi_i^0(x', p')$ , which is the minimal minimizer, then

$$h_i^-(x', p', p_i) = \begin{cases} h_i(x', p', p_i) & \text{if } p_i \leq \pi_i^0(x', p') \\ h_i(x', p', \pi_i^0(x', p')) & \text{if } p_i \geq \pi_i^0(x', p') \end{cases}$$

The appropriate notion of weak solutions for Hamilton-Jacobi equations is the one introduced by Crandall and Lions [10, 11], namely viscosity solutions. It is explained in [15, 14] that two notions of viscosity solutions are needed in the study of Hamilton-Jacobi equations on networks, depending on the type of junction conditions we impose. We will see that it is also the case for the degenerate parabolic equations we consider in this work. For general junction functions  $L$  in (1.1), the junction condition has to be understood in the “viscosity sense”: either the junction condition  $L = 0$  or one of the equations  $u_t + F_i = 0$  is satisfied. We refer to such (viscosity) solutions as *relaxed solutions* (see Definition 2.2 below). But for the special junction conditions  $L_A$  (see (1.2)), relaxed solutions satisfy the junction condition in a stronger sense (made precise in Definition 2.8 below): such (viscosity) solutions are referred to as *flux-limited solutions*.

The main result of this article is about the classification of generalized junction conditions. Roughly speaking, we prove that imposing a general junction condition (in a relaxed

sense, see Definition 2.2) amounts to imposing one of the type  $L_A$  presented above for some  $A$  only depending on the junction function  $L$  (see 4.1 for a formula).

**Theorem 1.5** (Classification of generalized junction conditions). *Assume (F), (L). Then there exists a continuous function  $A_L : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

$$\lim_{|p'| \rightarrow +\infty} A_L(x', p') = +\infty$$

*and such that any  $L$ -relaxed sub-solution (resp. super-solution) of (1.1) is a  $A_L$ -flux-limited sub-solution (resp. super-solution) of (1.1). Moreover, if*

$$\{p' \in \mathbb{R}^d : L(p_0, p_1, \dots, p_N, x', p') \leq \lambda\} \text{ is convex}$$

*for all  $p_0, \dots, p_N, \lambda \in \mathbb{R}$  and all  $x' \in \mathbb{R}^d$ , then*

$$\{p' \in \mathbb{R}^d : A(x', p') \leq \lambda\} \text{ is convex}$$

*for all  $\lambda \in \mathbb{R}$  and  $x' \in \mathbb{R}^d$ .*

*Remark 1.6* (The definition of the effective flux limiter). Some notation is needed. Let

$$A_0(x', p') = \max_{i=1, \dots, N} \min_{p_i \in \mathbb{R}} H_i((x', 0), (p', p_i)).$$

Let  $p_i^0 \geq \pi_i^0(x', p')$  be minimal such that

$$h_i(x', p', p_i) := H_i((x', 0), (p', p_i)) = A_0(x', p').$$

For all  $(x', p')$ , the real number  $A_L(x', p')$  is defined as follows: if

$$L(A_0(x', p'), p_1^0, \dots, p_N^0, x', p') < 0,$$

then  $A_L(x', p') = A_0(x', p')$ , else  $A_L(x', p')$  is the only real number  $\lambda \geq A_0(x', p')$  such that there exists  $p_i^+ \geq p_i^0$  with

$$h_i(x', p', p_i^+) = \lambda \quad \text{and} \quad L(\lambda, p_1^+, \dots, p_N^+, x', p') = 0.$$

This definition is equivalent to (4.1) where  $\pi_i^+$  is defined in (1.6) below. We note that even if  $\lambda$  is unique (thanks to (L)), the  $p_i^+$  are not (in general). See the example in the center of Figure 1 in Example 1.8.

*Remark 1.7.* Such a classification can also be obtained for stationary problems. In this case, the proof is even easier.

**1.2. Comments on the main result.** Our main result, Theorem 1.5, deals with the classification of generalized junction conditions: we show that imposing a generalized junction condition  $L$  (in a relaxed sense) reduces to imposing a junction condition  $L_A$  of control-type (in a stronger sense). It extends the results contained in [15, 14] in two directions: first, we can deal with Kirchoff conditions (see Example 1.3), second we can deal with second order terms (but degenerating along the junction).

As in [15, 14], the classification is quite a straightforward consequence of the following important result (Theorem 3.2): in order to check that a function is a flux-limited sub- and super-solution, it is enough to use a reduced set of test functions  $\varphi$  whose normal derivatives  $\partial_i \varphi$  have specific slopes along  $\Gamma$ . For instance, these slopes equal  $\pi_i^+(p', A(p'))$

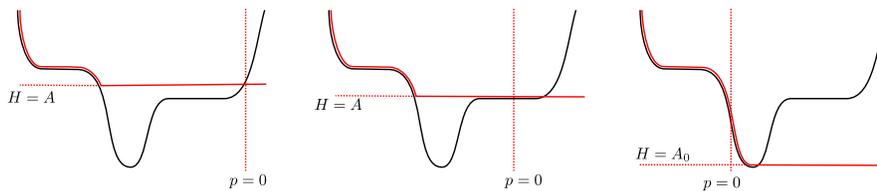


FIGURE 1. In the three cases, the plain black curve represents the Hamiltonian  $H$  and the plain red curve represents the limited-flux function  $F_A$  associated with the Neumann condition. Depending on the position of the graph of  $H$  and the line  $\{p = 0\}$ , the effective flux limiter  $A$  associated with the Neumann condition varies. On the left, the line  $\{p = 0\}$  intersects the graph of  $H$  in its increasing part. On the center, the line  $\{p = 0\}$  intersects the graph of  $H$  in the increasing part, but on a flat part. On the right, the line  $\{p = 0\}$  intersects the graph of  $H$  in its decreasing part, which implies that the flux limiter coincides with  $A_0 = \min H$ .

if the Hamiltonian has no flat parts and does not depend on  $x'$ . We recall that, roughly speaking,  $\pi_i^+$  is the inverse function of the non-decreasing part of  $H_i$ , see (1.6) below.

**Example 1.8** (The Neumann problem in the case  $(d, N) = (0, 1)$ ). We illustrate our result on the simplest example:

$$\begin{cases} u_t + H(u_x) - x^2 u_{xx} = 0, & x > 0, \\ u_x = 0, & x = 0 \end{cases}$$

where  $H$  is a quasi-convex function as illustrated in Figure 1.

A first version of this paper contained a comparison principle for (1.1) (under stronger assumptions on  $F$ ). On the one hand, the proof was quite difficult, relying on the vertex test function introduced in [15, 14], for which  $C^2$  regularity was to be proved in the multi-dimensional setting (following ideas developed in [13]). On the other hand, new and simpler techniques now emerge to attack this problem, see for instance [4, 12, 20, 21]. In particular, it is explained in [4] that the equations considered in the present work can be handled in the two-domain case. For these two reasons, we decided to restrict ourselves to the core of the work, that is to say the classification result.

**1.3. Comments on assumptions.** Before presenting the application to the vanishing viscosity problem we would like now to make a few comments about assumptions (F) and (L). We recall that our goal is to classify generalized junction conditions for degenerate parabolic equations. In particular, we want to understand what are the *effective* junction conditions that are imposed at the junction. From this point of view, it seems interesting to consider degenerate parabolic equations which actually degenerate at  $\Gamma$ . This is exactly the third condition appearing in (F). We also assume that the Hamiltonians have convex sublevel sets, see the fourth condition in (F). This condition can probably be relaxed but until the two very recent contributions [21, 12], the non-convex case was out of reach.

As far as the second condition in (F) is concerned, it implies that the Hamiltonians are coercive. It is used in order to derive the “weak continuity” of subsolutions (see Lemma 2.3 below). The first condition in (F) is used in an essential way when proving that the set of test functions can be reduced (see the proof of Lemma 3.4 below). Remark that this condition is (much) weaker than the one which is needed in order to prove uniqueness, see [9, Condition (3.14)].

**1.4. An application: the vanishing viscosity limit.** Because we are able to deal with Kirchoff conditions, we are in position to adress an open problem about Hamilton-Jacobi equations from “regional control” problem: the identification of the vanishing viscosity limit.

We study the limit as  $\varepsilon \rightarrow 0$  of the equation posed in  $(0, +\infty) \times \mathbb{R}^{d+1}$

$$(1.3) \quad \begin{cases} v_t^\varepsilon + \tilde{H}_1(x, Dv^\varepsilon) = \varepsilon \Delta v^\varepsilon, & x_{d+1} < 0, t > 0 \\ v_t^\varepsilon + \tilde{H}_2(x, Dv^\varepsilon) = \varepsilon \Delta v^\varepsilon, & x_{d+1} > 0, t > 0 \\ v^\varepsilon(0, x) = v_0(x), & x \in \mathbb{R}^{d+1} \end{cases}$$

where  $x = (x', x_{d+1}) \in \mathbb{R}^{d+1}$ . In the previous equation, we do not need to impose any condition since the Laplacian is strong enough to ensure the existence of  $C^1$  functions despite the discontinuity of the first order term. In particular, the following condition holds at  $x_{d+1} = 0$ ,

$$(1.4) \quad \partial_{x_{d+1}} v^\varepsilon(x', 0+) = \partial_{x_{d+1}} v^\varepsilon(x', 0-).$$

In this specific singular perturbation problem, the limit is identified by remarking that (1.4) is a Kirchoff condition and that consequently we can pass to the limit using relaxed solutions; more precisely, the limit of  $v^\varepsilon$  corresponds to a relaxed solution associated with this specific generalized junction condition. But the classification theorem tells us that the limit thus corresponds to a flux-limited solution associated with a flux-limiter  $A$  that is explicitly given by a formula (see Remark 1.6). Looking closely at this formula, we can prove that it corresponds to the maximal Ishii solution of the limit equation recently identified by Barles, Briani and Chasseigne [2, 3].

**Theorem 1.9** (The vanishing viscosity limit selects the maximal Ishii solution). *Assume*

$$\begin{cases} \tilde{H}_i \text{ continuous} \\ \forall x = (x', 0) \in \mathbb{R}^{d+1}, \forall \lambda \in \mathbb{R}, \{p \in \mathbb{R}^{d+1} : \tilde{H}_i(x, p) \leq \lambda\} \text{ convex} \\ \lim_{|p| \rightarrow +\infty} \inf_{x=(x', 0) \in \mathbb{R}^{d+1}} \tilde{H}_i(x, p) = +\infty \end{cases}$$

and  $v_0$  is uniformly continuous. Then any solution  $v^\varepsilon$  of (1.3) converges towards the maximal Ishii solution  $v$  of

$$(1.5) \quad \begin{cases} v_t + \tilde{H}_1(x, Dv) = 0, & x_{d+1} < 0, t > 0 \\ v_t + \tilde{H}_2(x, Dv) = 0, & x_{d+1} > 0, t > 0 \end{cases}$$

subject to the initial condition

$$v(0, x) = v_0(x), x \in \mathbb{R}^{d+1}.$$

*Remark 1.10.* The function  $v$  is associated with the unique  $A_I^-$ -flux-limited solution  $u$  of the previous Hamilton-Jacobi equation for some flux limiter  $A_I^-(x', p')$ . Precisely,  $v(t, x', x_{d+1}) = u(t, x', |x_{d+1}|)$ , see Theorem 5.7 in Section 5.

**1.5. Review of literature.** Semi-linear uniformly parabolic equations on compact networks were studied in [25, 28, 18, 22] where uniqueness, existence, strong maximum principle among other results were proved to be true.

The first results for Hamilton-Jacobi equations on networks were obtained in [23] for eikonal equations. Some years later, the results were extended in [24, 1, 16]. Many new results were obtained since then, see for instance [15, 14] and references therein.

In [2, 3], the authors study regional control, i.e. control with dynamics and costs which are regular on either side of a hyperplane but with no compatibility or continuity assumption along the hyperplane. They identify the maximal and minimal Ishii solutions as value functions of two different optimal control problems. They also use the vanishing viscosity limit on a 1D example in order to prove that the two Ishii solutions can be different. Moreover, the authors ask about the vanishing viscosity limit in the general case.

In [8], the authors study the vanishing viscosity limit associated with Hamilton-Jacobi equations posed on a junction (the simplest network, see above). The main difference with our results is that the authors impose some compatibility conditions on Hamiltonians. In particular, this allows them to construct viscosity solutions which satisfy Kirchoff conditions in a strong sense. We proceed in a different setting and in a different way: no compatibility conditions on Hamiltonians are imposed, and Kirchoff conditions are understood in a relaxed sense, which is stable under local uniform convergence (and even relaxed semi-limits). We then use the general classification theorem 1.5 to prove that imposing Kirchoff conditions reduce to the study of a flux-limited problem (for which uniqueness holds true).

In his lectures at Collège de France [19], Lions also treats problems related to Hamilton-Jacobi equations with discontinuities. After posting a first version of this paper, Lions and Souganidis [20] wrote a note about a new approach for Hamilton-Jacobi equations posed on junctions with coercive Hamiltonians that are possibly not convex.

Also after the first post of this work, two articles [12, 21] about classification in the non-convex cases were written independently. In [12], the author studies the case  $N = 1$  in the 1D setting, which amounts to studying first order Hamilton-Jacobi equations with nonlinear boundary conditions of Neumann type. In [21], the author can deal with the multi-dimensional setting and several branches.

To finish with, the link between the theory developed in [2, 3] and flux-limited solutions from [15, 14] is explored in [4]. In particular, [4] contains alternative proofs in the two-domain case of the comparison principle from [14] and of the vanishing viscosity limit obtained in the present work.

**1.6. Organization of the paper.** In Section 2, two notions of viscosity solutions are presented and their properties studied. In Section 3, it is proved that in order to check that a function is a flux-limited solution, the set of test functions can be reduced. In Section 4, we prove the main result of this paper, Theorem 1.5. Section 5 is devoted to the study of the vanishing viscosity limit. The last section is devoted to the proof of a known result about large deviations using the main result of this work.

**1.7. Notation.** The Hamiltonian  $H_i(x, p)$  is defined for  $x, p \in \mathbb{R}^{d+1}$ . For  $x' \in \mathbb{R}^d, p' \in \mathbb{R}^d$ , the function  $h_i(x', p', p_i)$  is defined as

$$h_i(x', p', p_i) = H_i((x', 0), (p', p_i)).$$

The minimal minimizer of  $h_i$  is denoted by  $\pi_i^0(x', p')$ . The functions  $h_i^-$  and  $h_i^+$  are defined as follows

$$h_i^-(x', p', p_i) = \begin{cases} h_i(x', p', p_i) & \text{if } p_i \leq \pi_i^0(x', p') \\ h_i(x', p', \pi_i^0(x', p')) & \text{if } p_i \geq \pi_i^0(x', p') \end{cases}$$

$$h_i^+(x', p', p_i) = \begin{cases} h_i(x', p', p_i) & \text{if } p_i \geq \pi_i^0(x', p') \\ h_i(x', p', \pi_i^0(x', p')) & \text{if } p_i \leq \pi_i^0(x', p'). \end{cases}$$

For  $\lambda \geq \min_{p_i \in \mathbb{R}} h_i(x', p', p_i)$ , the function  $\pi_i^+$  is defined by

$$(1.6) \quad \pi_i^+(x', p', \lambda) = \inf\{p_i : h_i(x', p', p_i) = h_i^+(p', p_i) = \lambda\}.$$

The function  $A_0$  is defined for  $x', p' \in \mathbb{R}^d$  as

$$A_0(x', p') = \max_{i \in \{1, \dots, N\}} \left( \min_{p_i} h_i(x', p', p_i) \right).$$

The functions  $p_i^0(x', p')$  are defined as

$$p_i^0(x', p') = \pi_i^+(x', p', A_0(x', p')).$$

## 2. RELAXED AND FLUX-LIMITED SOLUTIONS

**2.1. Test functions.** In order to define viscosity solutions, the set of test functions has to be made precise.

**Definition 2.1** (Test functions). A function  $\phi : (0, T) \times J \rightarrow \mathbb{R}$  is a *test function* for (1.1) if it is continuous in  $(0, T) \times J$ ,  $\phi|_{(0, T) \times J_i}$  is  $C_{t, x}^{1,1}$  and  $\phi|_{(0, T) \times J_i^*}$  is  $C_x^2$ .

We classically say that a function  $\phi$  touches another function  $u$  at a point  $(t, x)$  from below (respectively from above) if  $u \leq \phi$  (respectively  $u \geq \phi$ ) in a neighbourhood of  $(t, x)$  with equality at  $(t, x)$ .

## 2.2. Relaxed solutions.

**Definition 2.2** (*L*-relaxed solutions). A function  $u: (0, T) \times J \rightarrow \mathbb{R}$  is an *L*-relaxed subsolution (resp. *L*-relaxed super-solution) of (1.1) if it is upper semi-continuous (resp. lower semi-continuous) and for all test function  $\phi$  touching  $u$  from above (resp. from below) at  $(t, x) \in (0, T) \times J_i$ , we have

$$\begin{aligned} \phi_t + F_i(x, D\phi, D^2\phi) &\leq 0 \text{ at } (t, x) \\ \text{(resp. } \phi_t + F_i(x, D\phi, D^2\phi) &\geq 0 \text{ at } (t, x)) \end{aligned}$$

if  $x \notin \Gamma$ , and

$$\left( \text{resp. } \begin{cases} \text{either } \phi_t + H_i(x, D\phi) \leq 0 \text{ at } (t, x) \text{ for some } i \in \{1, \dots, N\}, \\ \text{or } L(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, D'\phi) \leq 0 \text{ at } (t, x) \end{cases} \right)$$

if  $x \in \Gamma$ .

The following observation is important for stability and the reduction of the set of test functions. The proof contained in [15] can be easily extended to generalized junction conditions. We give a sketch of proof for the reader's convenience.

**Lemma 2.3** (“Weak continuity” of relaxed subsolutions). *Assume (F) and (L). Let  $u: (0, T) \times J \rightarrow \mathbb{R}$  be a *L*-relaxed subsolution of (1.1). Then for all  $i \in \{1, \dots, N\}$ , and  $x \in \Gamma$ ,*

$$u(t, x) = \limsup_{(s, y) \rightarrow (t, x), y \in J_i^*} u(s, y).$$

*Proof.* Since  $u$  is upper semi-continuous, we have

$$u(t, x) \geq \limsup_{(s, y) \rightarrow (t, x), y \in J_i^*} u(s, y).$$

In order to prove the reverse inequality, we assume that there exists  $(t_*, x_*) \in (0, T) \times \Gamma$  such that

$$u(t_*, x_*) > \limsup_{(s, y) \rightarrow (t_*, x_*), y \in J_i^*} u(s, y).$$

Since  $u$  is upper continuous, there exists a  $C^1$  function  $\Psi$  defined in  $(0, T) \times \Gamma$  and  $(t_0, x_0) \in (0, T) \times \Gamma$  such that

$$u(t_0, x_0) > \limsup_{(s, y) \rightarrow (t_0, x_0), y \in J_i^*} u(s, y)$$

and  $\Psi$  strictly touches  $u$  from above at  $(t, x) \in B_r(t_0, x_0) \subset (0, T) \times \Gamma$ . In particular  $\Psi - u \geq \delta_1 > 0$  in a neighbourhood (respectively to  $(0, T) \times J$ ) of  $\partial B_r(t_0, x_0) \subset (0, T) \times \Gamma$ . We now consider the test function  $\Phi(t, x) = \Psi(t, x') + p_i x_i$  for  $x \in J_i$ . We now use the coercivity of the  $F_i$  (see (F)) to show that for  $p_i > 0$  large enough,  $\Phi$  touches  $u$  from above at  $(t_0, x_0)$ . But this implies that

$$L(-\partial_t \Phi(t_0, x_0), p_1, \dots, p_N, x'_0, D' \Phi(t_0, x_0)) \leq 0$$

which contradicts the assumption (L) since the  $p_i$  are arbitrarily large. The proof is now complete.  $\square$

**2.3. Stability and existence.** The following results related to stability of relaxed sub- and super-solutions are expected. We recall that the lower semi-continuous envelope of a function  $f$  is the largest semi-continuous function lying below  $f$ . We also recall that for a family of functions  $(u_\varepsilon)_\varepsilon$ , we can define an upper and lower semi-limit. They were first introduced by Barles and Perthame [5, 6], see [9] for their definition and their classical use in the viscosity solution theory.

**Proposition 2.4** (Stability of relaxed solutions - I). *Assume (F) and (L). If  $(u_\alpha)_\alpha$  is a family of relaxed sub-solutions (resp. relaxed super-solutions) of (1.1) which is locally uniformly bounded from above (resp. from below), then the upper semi-continuous (resp. lower semi-continuous) envelope of  $\sup_\alpha u_\alpha$  (resp.  $\inf_\alpha u_\alpha$ ) is a relaxed sub-solution (resp. relaxed super-solution) of (1.1).*

*Proof.* We only treat the sub-solution case since the super-solution one is similar. Let  $u$  denote the upper semi-continuous envelope of  $\sup_\alpha u_\alpha$ . Consider a test function  $\phi$  strictly touching  $u$  from above at  $(t, x)$ . There then exist a sequence  $(t_n, x_n) \rightarrow (t, x)$  and  $\alpha_n$  such that  $\phi$  touches  $u_{\alpha_n}$  from above at  $(t_n, x_n)$ . Writing the viscosity inequalities and passing to the limit yields the desired result.  $\square$

**Proposition 2.5** (Stability of relaxed solutions - II). *Assume (F) and (L). If  $\{u_n\}$  is a sequence of relaxed sub-solutions (resp. relaxed super-solutions) of (1.1) which is locally uniformly bounded from above (resp. from below), then the relaxed upper limit (resp. relaxed lower limit) of  $\{u_n\}$  is a relaxed sub-solution (resp. relaxed super-solution) of (1.1).*

*Proof.* We only treat the sub-solution case since the super-solution one is similar. Let  $u$  denote the relaxed upper limit of the sequence  $\{u_n\}$ . Consider a test function  $\phi$  strictly touching  $u$  from above at  $(t, x)$ . There then exist a sequence  $(t_k, x_k) \rightarrow (t, x)$  and  $n_k$  such that  $\phi$  touches  $u_{n_k}$  from above at  $(t_k, x_k)$ . Writing the viscosity inequalities and passing to the limit yields the desired result.  $\square$

We now turn to an existence result.

**Theorem 2.6** (Existence of discontinuous relaxed solutions). *Assume (F) and (L) and consider  $u_0$  uniformly continuous. There exists  $u$  such that its upper semi-continuous (resp. lower semi-continuous) envelope is an relaxed sub-solution (resp. relaxed super-solution) of (1.1) such that*

$$u(0, x) = u_0(x) \text{ for } x \in J.$$

*Remark 2.7.* This theorem states the existence of discontinuous solutions in the sense of Ishii [17].

*Proof.* In view of the stability results, it is enough to construct a solution for some initial datum  $u_0 \in C^{1,1}(J_i) \cap \text{Lip}(J)$ . For such  $u_0$ 's, we can construct barriers in the classical way:  $u^\pm(t, x) = u_0(x) \pm Ct$ . For  $C$  large enough,  $u^+$  is a relaxed supersolution while  $u^-$  is

a relaxed subsolution. We then consider  $W$  the set of all functions lying below  $u^+$  whose upper semi-continuous envelope is a relaxed subsolution. Then the supremum of  $w \in W$  is in  $W$  and it is maximal. Let  $w$  denote this maximal element. If the lower semi-continuous envelope is not a relaxed supersolution, there exists a test function  $\phi$  and a point  $(t, x)$  such that  $\phi$  touches  $w_*$  from below at  $(t, x)$  without satisfying the corresponding viscosity inequality. This implies  $\phi < (u_+)_*$  in a neighbourhood of  $(t, x)$  and we can prove that  $\phi$  is a relaxed subsolution in the same neighbourhood. Then we can construct a relaxed subsolution  $w_\delta$  which is not below  $w$ , contradicting its maximality.  $\square$

**2.4. Flux-limited solutions.** It is proved in [15] that, in the special case where  $L = L_A$  defined in (1.2) and for first order Hamilton-Jacobi equations, relaxed solutions satisfy the junction condition in a stronger sense, which is made precise in the following definition. Let  $A_0$  be defined by

$$(2.1) \quad A_0(x', p') = \max_{i \in \{1, \dots, N\}} \left( \min_{p_i} h_i(x', p', p_i) \right).$$

**Definition 2.8** (Flux-limited solutions). Given a function  $A : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $A \geq A_0$ , a function  $u : (0, T) \times J \rightarrow \mathbb{R}$  is a *A-flux-limited sub-solution* (resp. *A-flux-limited super-solution*) of (1.1) if it is upper semi-continuous (resp. lower semi-continuous) and for any test function  $\phi$  in the sense of Definition 2.1 touching  $u$  from above (resp. from below) at  $(t, x) \in (0, T) \times J_i$ , we have

$$\begin{aligned} & \phi_t + F_i(x, D\phi, D^2\phi) \leq 0 \text{ at } (t, x) \\ (\text{resp. } & \phi_t + F_i(x, D\phi, D^2\phi) \geq 0 \text{ at } (t, x)) \end{aligned}$$

if  $x \notin \Gamma$ , and

$$\begin{aligned} & L_A(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, x', D'\phi) \leq 0 \text{ at } (t, x) \\ (\text{resp. } & L_A(-\phi_t, \partial_1\phi, \dots, \partial_N\phi, x', D'\phi) \geq 0 \text{ at } (t, x)) \end{aligned}$$

if  $x \in \Gamma$ .

*Remark 2.9.* If a flux limiter  $A$  is replaced with  $\max(A, A_0)$ , the corresponding flux function  $L_A$  is unchanged.

The following proposition asserts that for the special junction functions  $L_A$ ,  $L_A$ -relaxed solutions coincide with  $A$ -flux-limited solutions. It was proved in [15, 14] in the case of first order equations. We point out that the multidimensional proof of [14] applies *without any change* to degenerate parabolic equations satisfying (F).

**Proposition 2.10** ( $L_A$ -relaxed solutions are  $A$ -flux-limited solutions – [14]). *Assume (F) and (L). Then any  $L_A$ -relaxed sub-solution (resp. super-solution) of (1.1) is an  $A$ -flux-limited sub-solution (resp. super-solution) of (1.1).*

### 3. REDUCED SET OF TEST FUNCTIONS FOR FLUX-LIMITED SOLUTIONS

In this section, we explain why it is sufficient to consider a reduced set of test functions in order to check that a function is a flux-limited (sub/super)solutions of (1.1). Such a result is used in an essential way when proving the first main result of the paper, namely the classification theorem 1.5.

This result generalizes the one contained in [15]. In order to state it, we need to consider the equation away from the junction hyperplane  $\Gamma$ , and more precisely on each (open) branch  $i$ :

$$(3.1) \quad u_t + F_i(x, Du, D^2u) = 0, \quad (t, x) \in (0, T) \times J_i^*.$$

We recall that  $A_0$  is defined in (2.1).

In order to introduce the reduced set of test functions (or reduced test functions for short), we need the following definition

$$h_i^+(x', p', p_i) = \begin{cases} h_i(x', p', p_i) & \text{if } p_i \geq \pi_i^0(x', p') \\ h_i(x', p', \pi_i^0(x', p')) & \text{if } p_i \leq \pi_i^0(x', p'). \end{cases}$$

We recall that  $h_i(x', p', p_i) = H_i((x', 0), (p', p_i))$  and  $p_i \mapsto h_i(x', p', p_i)$  reaches its minimum at  $\pi_i^0(x', p')$ , which is minimal.

**Definition 3.1** (Reduced test functions). Consider a flux limiter  $A \geq A_0$  (see Remark 2.9). A function  $\varphi : (0, T) \times J \rightarrow \mathbb{R}$  is a *reduced test function* for (1.1) if there exists a function  $\phi \in C^1((0, T) \times \mathbb{R}^d)$  and  $\phi_0 \in C^1(\mathbb{R})$  such that for  $x = (x', x_i) \in J_i$ ,

$$\varphi(t, x) = \phi(t, x') + \phi_0(x_i)$$

and for all  $t \in (0, T)$ ,  $(x', 0) \in \Gamma$ ,  $p_i = \phi_0'(t, x')$  and  $p' = D'\phi(t, x')$  satisfy

$$h_i(x', p', p_i) = h_i^+(x', p', p_i) = A(x', p').$$

We can now state and prove the following theorem.

**Theorem 3.2** (Reducing the set of test functions). *Assume (F) and consider a function  $A : \Gamma \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $A \geq A_0$ . Given a function  $u : (0, T) \times J \rightarrow \mathbb{R}$ , the following properties hold true.*

i) *If  $u$  is a sub-solution of (3.1) and for all  $i$  and  $(t, x) \in (0, T) \times \Gamma$ ,*

$$(3.2) \quad u(t, x) = \limsup_{s \rightarrow t, y \rightarrow x, y \in J_i^*} u(s, y),$$

*then  $u$  is a  $A_0$ -flux limited sub-solution of (1.1).*

ii) *If  $u$  is a sub-solution of (3.1) satisfying (3.2) and if for any reduced test function  $\varphi$  in the sense of Definition 3.1 touching  $u$  from above at  $(t, x) \in (0, T) \times \Gamma$ , we have*

$$\varphi_t(t, x) + A(x', D'\varphi(t, x)) \leq 0,$$

*then  $u$  is a  $A$ -flux-limited sub-solution at  $(t_0, 0)$ .*

iii) If  $u$  is a super-solution of (3.1) and if for any reduced test function  $\varphi$  in the sense of Definition 3.1 touching  $u$  from below at  $(t, x)$  we have

$$\varphi_t(t, x) + A(x, D'\varphi(t, x)) \geq 0,$$

then  $u$  is a  $A$ -flux-limited super-solution at  $(t_0, 0)$ .

*Proof.* The proof of the corresponding result in [15] applies here *without any change* after proving the two lemmas 3.3 and 3.4 about so-called *critical slopes*. Indeed, with such technical results in hands, the proof focuses on what happens on  $\Gamma$  and second derivatives do not appear any more.  $\square$

**Lemma 3.3** (Super-solution property for the critical slope on each branch). *Let  $i \in \{1, \dots, N\}$  be fixed. Let  $u : (0, T) \times J_i \rightarrow \mathbb{R}$  be a lower semi-continuous super-solution of (3.1). Let  $\phi$  be a test function touching  $u$  from below at some point  $(t_0, x_0) \in (0, T) \times \Gamma$ . We consider*

$$\bar{p}_i = \sup\{\bar{p} \in \mathbb{R} : \exists r > 0, \phi(t, x) + \bar{p}x_i \leq u(t, x) \text{ for } (t, x) \in B((t_0, x_0), r) \cap (0, T) \times J_i\}.$$

If  $\bar{p}_i < +\infty$ , then we have

$$\phi_t + H_i(D'\phi, \partial_i\phi + \bar{p}_i) \geq 0 \quad \text{at } (t_0, x_0) \quad \text{with } \bar{p}_i \geq 0.$$

**Lemma 3.4** (Sub-solution property for the critical slope on each branch). *Let  $u : (0, T) \times J_i \rightarrow \mathbb{R}$  be a sub-solution of (3.1). Let  $\phi$  be a test function touching  $u$  from above at some point  $(t_0, x_0) \in (0, T) \times \Gamma$ . For each  $i = 1, \dots, N$ , let us consider*

$$\underline{p}_i = \inf\{\bar{p} \in \mathbb{R} : \exists r > 0, \phi(t, x) + \bar{p}x_i \geq u(t, x) \text{ for } (t, x) \in B((t_0, x_0), r) \cap (0, T) \times J_i\}.$$

If  $u$  satisfies

$$u(t_0, x_0) = \limsup_{s \rightarrow t_0, y \rightarrow x_0, y \in J_i^*} u(s, y),$$

then  $\underline{p}_i > -\infty$ ; moreover, we have in this case

$$\phi_t + H_i(D'\phi, \phi_x + \underline{p}_i) \leq 0 \quad \text{at } (t_0, 0) \quad \text{with } \bar{p}_i \leq 0.$$

We first prove Lemma 3.3.

*Proof of Lemma 3.3.* The proof follows the same lines of [Lemma 2.8, [15]].

From the definition of  $\bar{p}_i$ , for all  $\varepsilon > 0$  small enough, there exists  $\delta = \delta(\varepsilon) \in (0, \varepsilon)$  such that

$$u(s, y) \geq \phi(s, y) + (\bar{p}_i - \varepsilon)y \quad \text{for all } (s, y) \in B((t_0, x_0), \delta) \subset \mathbb{R} \times J_i$$

and there exists  $(t_\varepsilon, x_\varepsilon) \in B_{\delta/2}(t_0, 0)$  such that

$$u(t_\varepsilon, x_\varepsilon) < \phi(t_\varepsilon, x_\varepsilon) + (\bar{p}_i + \varepsilon)x_\varepsilon.$$

We choose a smooth function  $\Psi : \mathbb{R}^{d+2} \rightarrow [-1, 0]$  such that

$$\Psi = \begin{cases} 0 & \text{in } B_{1/2}(t_0, x_0) \\ -1 & \text{outside } B_1(t_0, x_0). \end{cases}$$

We define

$$\Phi(s, y) = \phi(s, y) + 2\varepsilon\Psi_\delta(s, y) + \begin{cases} (\bar{p}_i + \varepsilon)y & \text{if } y \in J_i \\ 0 & \text{if not} \end{cases}$$

with  $\Psi_\delta(Y) = \delta\Psi\left(\frac{Y}{\delta}\right)$ . Now, we have

$$\Phi(s, y) \leq \phi(s, y) \leq u(s, y) \text{ for } (s, y) \in B_\delta(t_0, x_0) \text{ and } y \notin J_i$$

and

$$\begin{cases} \Phi(s, y) = \phi(s, y) - 2\varepsilon\delta + (\bar{p}_i + \varepsilon)y \leq u(s, y) & \text{for } (s, y) \in (\partial B_\delta(t_0, x_0)) \cap (\mathbb{R} \times J_i), \\ \Phi(s, x) \leq \phi(s, x) \leq u(s, x) & \text{for } (s, x) \in (t_0 - \delta, t_0 + \delta) \times \Gamma \end{cases}$$

and

$$\Phi(t_\varepsilon, x_\varepsilon) = \phi(t_\varepsilon, x_\varepsilon) + (\bar{p}_i + \varepsilon)x_\varepsilon > u(t_\varepsilon, x_\varepsilon).$$

We conclude that there exists a point  $(\bar{t}_\varepsilon, \bar{x}_\varepsilon) \in B_\delta(t_0, x_0) \cap (\mathbb{R} \times J_i^*)$  such that  $u - \Phi$  reaches a minimum in  $\overline{B_\delta(t_0, 0)} \cap (\mathbb{R} \times J_{i_0})$ . We thus can write the viscosity inequality

$$\Phi_t + F_i(x, D\Phi, D^2\Phi) \geq 0 \quad \text{at } (\bar{t}_\varepsilon, \bar{x}_\varepsilon)$$

which means that

$$\begin{aligned} & \phi_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon(\Psi_\delta)_t(\bar{t}_\varepsilon, \bar{x}_\varepsilon) \\ & + F_i(\bar{x}_\varepsilon, (D'\phi + \varepsilon D'\Psi_\delta)(\bar{t}_\varepsilon, \bar{x}_\varepsilon), \partial_i\phi(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + 2\varepsilon\partial_i\Psi_\delta(\bar{t}_\varepsilon, \bar{x}_\varepsilon) + \bar{p}_i + \varepsilon, D^2\phi + 2\varepsilon D^2\Psi_\delta(\bar{t}_\varepsilon, \bar{x}_\varepsilon)) \geq 0. \end{aligned}$$

So we can send  $\varepsilon \rightarrow 0$  in the above inequality to obtain the desired inequality. The proof is now complete.  $\square$

We now turn to the proof of Lemma 3.4

*Proof of Lemma 3.4.* The main difference with the previous lemma is the claim that the critical slope is finite. This is the reason why we only explain this point. Here again, we follow closely [15].

Assume that  $\bar{p}_i = -\infty$ . This implies that there exists  $p_n \rightarrow -\infty$  and  $r_n > 0$  such that  $\phi + p_n x \geq u$  in  $B_n = B_{r_n}(t_0, x_0) \cap \mathbb{R} \times J_i$ . Remark first that, replacing  $\phi$  with  $\phi + (t - t_0)^2 + |x - x_0|^2$  if necessary, we can assume that

$$(3.3) \quad u(t, x) < \phi(t, x) + p_n x_i \text{ if } (t, x) \neq (t_0, x_0).$$

In particular, there exists  $\delta_n > 0$  such that  $\phi + p_n x \geq u + \delta_n$  on  $\partial B_n \setminus \Gamma$ .

Since  $u$  satisfies (3.2), there exists  $(t_\varepsilon, x_\varepsilon) \rightarrow (t_0, x_0)$  such that  $x_\varepsilon \in J_i^*$  and  $u(t_0, x_0) = \lim_{\varepsilon \rightarrow 0} u(t_\varepsilon, x_\varepsilon)$ .

We now introduce the following perturbed test function

$$\Psi(t, x) = \phi(t, x) + p_n x_i + \frac{\eta}{x_i}$$

where  $\eta = \eta(\varepsilon)$  is a small parameter to be chosen later. Let  $(s_\varepsilon, y_\varepsilon)$  realizing the infimum of  $\Psi - u$  in  $B_n$ . In particular,

$$(3.4) \quad (\phi + p_n x_i - u)(s_\varepsilon, y_\varepsilon) \leq \Psi(s_\varepsilon, y_\varepsilon) - u(s_\varepsilon, y_\varepsilon) \leq \Psi(t_\varepsilon, x_\varepsilon) - u(t_\varepsilon, x_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

as soon as  $\eta(\varepsilon) = o(x_\varepsilon^i)$  with  $x_\varepsilon = (x'_\varepsilon, x_\varepsilon^i)$ . In particular, in view of (3.3), this implies that  $(s_\varepsilon, y_\varepsilon) \rightarrow (t_0, x_0)$  as  $\varepsilon \rightarrow 0$ . Since  $u$  is a subsolution of (3.1), we know that

$$\phi_t(s_\varepsilon, y_\varepsilon) + F_i(y_\varepsilon, D'\phi(s_\varepsilon, y_\varepsilon), \partial_i\phi(s_\varepsilon, y_\varepsilon) + p_n - \frac{\eta}{(y_\varepsilon^i)^2}, D^2\phi(s_\varepsilon, y_\varepsilon) + \frac{2\eta}{(y_\varepsilon^i)^3}e_{d+1} \otimes e_{d+1}) \leq 0$$

(where  $(e_1, \dots, e_{d+1})$  is an orthonormal basis of  $\mathbb{R}^{d+1}$  and  $e_{d+1}$  is orthogonal to  $\Gamma$ ). Use now (F) in order to get

$$\phi_t(s_\varepsilon, y_\varepsilon) + F_i(y_\varepsilon, D'\phi(s_\varepsilon, y_\varepsilon), \partial_i\phi(s_\varepsilon, y_\varepsilon) + p_n - \frac{\eta}{(y_\varepsilon^i)^2}, D^2\phi(s_\varepsilon, y_\varepsilon)) \leq 2C_i \frac{\eta}{y_\varepsilon^i}.$$

Remark now that (3.4) together with the choice of  $\eta$  implies

$$\frac{\eta}{y_\varepsilon^i} = o_\varepsilon(1).$$

In particular, the coercivity of  $F_i$  (see (F)) implies that  $p_n - \frac{\eta}{(y_\varepsilon^i)^2}$  is bounded as  $\varepsilon \rightarrow 0$ . Hence we can pass to the limit as  $\varepsilon \rightarrow 0$  in the viscosity inequality and get

$$\phi_t(t_0, x_0) + H_i(x_0, D'\phi(t_0, x_0), \partial_i\phi(t_0, x_0) + p_n^0) \leq 0$$

where  $p_n^0 \in (-\infty, 0]$  is any accumulation point of  $p_n - \frac{\eta}{(y_\varepsilon^i)^2}$  as  $\varepsilon \rightarrow 0$ . The previous inequality implies in particular that  $p_n^0$  is bounded from below by a constant  $C$  which only depends on  $H_i, \phi_t, D\phi$  at  $(t_0, x_0)$ . But this also implies that  $p_n \geq C$  which is the desired contradiction. The proof of the finiteness of  $\bar{p}_i$  is now complete.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of the first main result, Theorem 1.5.

In order to prove it, we recall the following notation

$$\pi_i^+(x', p', \lambda) = \inf\{p_i : h_i(x', p', p_i) = h_i^+(p', p_i) = \lambda\}$$

where we recall that  $h_i(x', p', p_i) = H_i((x', 0), (p', p_i))$  and  $h_i^+$  its non-decreasing part. In the remainder of this section, we do not systematically write the  $(x', p')$  dependence of flux limiters in order to unclutter formulas.

We now prove the main theorem.

*Proof of Theorem 1.5.* Let  $A_L$  be defined as in Remark 1.6. Since  $A_L \geq A_0$ , the coercivity is clear. The proof of the continuity of  $A_L$  and the convexity of sublevel sets is the same as in [14, Proof of Theorem 2.13].

In the remainder of the proof, we do not systematically write the dependance of  $x'$  and  $p'$  for the sake of clarity.

Let us first deal with the sub-solution case.

If  $A_L = A_0$ , then Lemma 2.3 and Theorem 3.2 imply that any  $L$ -relaxed sub-solution is a  $A_0$ -flux limited solution.

We thus now consider the case where  $A_L > A_0$ . We remark that the definition of  $A_L$  given in Remark 1.6 can be put into the following form

$$(4.1) \quad A_L = \max(A_0, \sup\{\lambda : L(\lambda, \pi_1^+(\lambda), \dots, \pi_N^+(\lambda)) > 0\}).$$

In particular if  $L(A_0, p_1^0, \dots, p_N^0) > 0$ , then

$$(4.2) \quad A_L = \sup\{\lambda : L(\lambda, \pi_1^+(\lambda), \dots, \pi_N^+(\lambda)) > 0\}.$$

Remark that the set is not empty since it contains  $A_0$ . It is bounded from above since  $L$  is “semi-coercive” i.e.  $L(p_0, \dots, p_N) \rightarrow +\infty$  as  $\max_i \max(0, -p_i) \rightarrow +\infty$  and  $\pi_i^+(\lambda) \rightarrow +\infty$  as  $\lambda \rightarrow +\infty$ . Consequently,  $A_0 < A_L < +\infty$ . Remark also that  $\lambda \mapsto \pi_i^+(\lambda)$  is non-decreasing and lower semi-continuous. In particular,

$$(4.3) \quad L(A_L, \pi_1^+(A_L), \dots, \pi_N^+(A_L)) \geq 0.$$

Let  $u$  be a  $L$ -relaxed sub-solution and let us prove that it is an  $(A_L - \varepsilon)$ -flux-limited sub-solution for all  $\varepsilon > 0$ . In view of Lemma 2.3 and Theorem 3.2, we only have to consider a reduced test function  $\varphi$  touching  $u$  from above at  $(t_0, x_0) \in (0, T) \times \Gamma$ . We recall that

$$\varphi(t, x) = \phi(t, x') + \phi_i(x_i)$$

with  $\phi'_i(x_i^0) = \pi_i^+(A_L - \varepsilon)$  where  $p'_0 = D'\phi(t_0, x'_0)$ . By definition of relaxed solutions, we have

$$(4.4) \quad \text{either} \quad L(\lambda, \pi_1^+(A_L - \varepsilon), \dots, \pi_N^+(A_L - \varepsilon)) \leq 0$$

$$(4.5) \quad \text{or} \quad -\lambda + (A_L - \varepsilon) \leq 0$$

with  $\lambda = -\partial_t \phi(t_0, x_0)$ .

We claim that (4.5) always holds true. We argue by contradiction by assuming that  $(A_L - \varepsilon) > \lambda$ . In view of (L),  $L$  is strictly decreasing in  $p_{i_0}$  for some  $i_0 \in \{0, \dots, N\}$ .

$$L(A_L, \pi_1^+(A_L), \dots, \pi_N^+(A_L)) < L(\lambda, \pi_1^+(A_L - \varepsilon), \dots, \pi_N^+(A_L - \varepsilon)) \leq 0$$

which contradicts (4.3).

We conclude the proof by remarking that (4.5) yields

$$\begin{aligned} L_{A_L - \varepsilon}(-\partial_t \varphi, \partial_1 \varphi, \dots, \partial_N \varphi, x'_0, D'\varphi) &= \partial_t \phi(t_0, x_0) + \max(A_L(x'_0, p'_0) - \varepsilon, \max_i h_i^-(x'_0, p'_0, \pi_i^+)) \\ &= \partial_t \phi(t_0, x_0) + A_L(x'_0, p'_0) - \varepsilon \leq 0 \end{aligned}$$

where the derivatives of  $\varphi$  in the left hand side are computed at  $(t_0, x_0)$  and the function  $\pi_i^+$  are computed at  $(x'_0, p'_0, A_L(x'_0, p'_0))$  in the right hand side.

As far as the super-solution case is concerned, we define

$$\hat{\pi}_i^+(x', p', \lambda) = \sup\{p_i : h_i(x', p', p_i) = h_i^+(p', p_i) = \lambda\}$$

and we remark that, if  $L(A_0, p_1^0, \dots, p_N^0) > 0$ , then

$$A_L = \inf\{\lambda : L(\lambda, \hat{\pi}_1^+(\lambda), \dots, \hat{\pi}_N^+(\lambda)) < 0\}.$$

In particular,

$$L(A_L, \hat{\pi}_1^+(A_L), \dots, \hat{\pi}_N^+(A_L)) \leq 0.$$

If now  $L(A_0, p_1^0, \dots, p_N^0) \leq 0$  then  $A_L = A_0$  and the previous inequality still holds true.

We can follow the reasoning from the sub-solution case and prove that  $L$ -relaxed super-solutions are  $A_L$ -flux limited super-solutions. The proof is now complete.  $\square$

## 5. THE VANISHING VISCOSITY LIMIT

This section is devoted to the study of the limit (as  $\varepsilon \rightarrow 0$ ) of the solution  $u^\varepsilon$  of the following Hamilton-Jacobi equation posed on a multi-dimensional junction  $J$ ,

$$(5.1) \quad \begin{cases} u_t^\varepsilon + H_i(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon & (t, x) \in (0, T) \times J_i^*, \\ L(-u_t^\varepsilon, \partial_1 u^\varepsilon, \dots, \partial_N u^\varepsilon, x', D'u^\varepsilon) = 0 & (t, x) \in (0, T) \times \Gamma \end{cases}$$

subject to the initial condition

$$(5.2) \quad u(0, x) = u_0(x), \quad x \in J.$$

At least formally, the solutions  $u^\varepsilon$  converge towards the solution of

$$(5.3) \quad \begin{cases} u_t + H_i(x, Du) = 0 & (t, x) \in (0, T) \times J_i^*, \\ L(-u_t, \partial_1 u, \dots, \partial_N u, x', D'u) = 0 & (t, x) \in (0, T) \times \{0\}. \end{cases}$$

The first result applies to general junction functions.

**Theorem 5.1** (Vanishing viscosity limit). *Assume (L) and*

$$\begin{cases} H_i \text{ continuous} \\ \forall x \in \Gamma, \lambda \in \mathbb{R}, \{H_i(x, p) \leq \lambda\} \text{ convex} \\ \lim_{|p| \rightarrow +\infty} \inf_{x \in \Gamma} H_i(x, \cdot) = +\infty. \end{cases}$$

*Let  $u_0$  be uniformly continuous in  $J$ . Assume there exists a relaxed solution  $u^\varepsilon$  of (5.1), (5.2). Then  $u^\varepsilon$  converges locally uniformly towards the unique relaxed solution  $u$  of (5.3), (5.2).*

*Remark 5.2.* Even if we will not discuss it, the existence of solutions whose restriction to  $J_i$  are  $C^{1,1}(J_i) \cap C^2(J_i^*)$  is expected in the case of (5.1). Some results are proved in [26, 27] on compact junctions and some others are announced in [20].

*Remark 5.3.* As we previously mentioned it, a special case of the theorem is proved in [8].

*Proof of Theorem 5.1.* By discontinuous stability, the relaxed upper limit  $\bar{u}$  of  $u^\varepsilon$  is a  $L$ -relaxed subsolution of (5.3), i.e. a  $A_L$ -flux-limited subsolution of (5.3). The relaxed lower limit  $\underline{u}$  is a  $L$ -relaxed super-solution of (5.3), i.e. a  $A_L$ -flux-limited super-solution of (5.3). Moreover,  $\bar{u}(0, x) \leq u_0(x) \leq \underline{u}(0, x)$ . By the comparison principle proved in [14], we conclude that  $\bar{u} \leq \underline{u}$  which yields the local uniform convergence towards the unique  $A_L$ -flux-limited solution of (5.3), (5.2) which coincides with  $u$  as defined in the statement.  $\square$

Problem (1.3) can be translated into the junction framework as follows,

$$\begin{cases} u_t^\varepsilon + H_i(x, Du^\varepsilon) = \varepsilon \Delta u^\varepsilon, & x \in J_i^* \simeq (0, +\infty), \quad i = 1, 2 \\ -\partial_1 u^\varepsilon(t, 0) - \partial_2 u^\varepsilon(t, 0) = 0, \\ u^\varepsilon(0, x) = u_0(x), & x \in J \end{cases}$$

with  $H_1(x, p', p_{d+1}) = \tilde{H}_1(x, p', -p_{d+1})$  and  $H_2(x, p', p_{d+1}) = \tilde{H}_2(x, p', p_{d+1})$ . In view of Theorem 5.1,  $u^\varepsilon$  converges towards the relaxed solution

$$(5.4) \quad \begin{cases} u_t + H_i(x, u_x) = 0, & x \in J_i^* \\ u(0, x) = u_0(x), & x \in J \end{cases}$$

associated with the generalized flux function

$$L_\varepsilon(p_0, p_1, p_2, x', p') = -p_1 - p_2.$$

**Corollary 5.4** (The vanishing viscosity limit for the Kirchoff condition). *The function  $u^\varepsilon$  converges towards the  $A_\varepsilon$ -flux-limited solution of (5.4) where  $A_\varepsilon$  is determined as follows: if  $\pi_1^0(x', p') + \pi_2^0(x', p') > 0$  then  $A_\varepsilon = A_0$ ; else  $A_\varepsilon = A_\varepsilon(x', p')$  is the unique  $\lambda \geq A_0$  such that there exists  $p_1^{+,e} \geq \pi_1^0$  and  $p_2^{+,e} \geq \pi_2^0$  such that*

$$h_i(x', p', p_i^{+,e}) = \lambda \text{ for } i = 1, 2, \quad p_1^{+,e} + p_2^{+,e} = 0.$$

*Remark 5.5.* If  $H_1$  and  $H_2$  has no flat parts and  $\pi_1^0(x', p') + \pi_2^0(x', p') \leq 0$ , then  $A_\varepsilon$  is the only  $A$  such that  $\pi_1^+(x', p', A) + \pi_2^+(x', p', A) = 0$ .

We now recall the result about maximal and minimal Ishii solutions from [14].

**Proposition 5.6** (Maximal and minimal Ishii solutions are flux-limited – [14]). *The maximal (respectively the minimal) Ishii solution of (1.5) corresponds to the  $A_I^-$  (respectively  $A_I^+$ ) flux-limited solution of (5.4) with*

$$\begin{cases} A_I^+(x', p') = \max(A_0(x', p'), A^*(x', p')) \\ A_I^-(x', p') = \begin{cases} A_I^+(x', p') & \text{if } \tilde{p}_2^0(x', p') \leq \tilde{p}_1^0(x', p') \\ A_0(x', p') & \text{if } \tilde{p}_2^0(x', p') \geq \tilde{p}_1^0(x', p') \end{cases} \end{cases}$$

where

$$A^*(x', p') = \max_{p_{d+1} \in I(x', p')} (\min(\tilde{H}_2(x', p', p_{d+1}), \tilde{H}_1(x', p', p_{d+1})))$$

and  $I(x', p') = [\min(\tilde{p}_1^0(x', p'), \tilde{p}_2^0(x', p')), \max(\tilde{p}_1^0(x', p'), \tilde{p}_2^0(x', p'))]$  with  $\tilde{p}_2^0$  minimal minimizer of  $\tilde{H}_2$  and  $\tilde{p}_1^0$  maximal minimizer of  $\tilde{H}_1$ .

We now prove the following theorem, which is equivalent to Theorem 1.9.

**Theorem 5.7** (The vanishing viscosity limit selects the maximal Ishii solution). *Assume*

$$\begin{cases} H_i \text{ continuous} \\ H_i(x', 0, \cdot) \text{ quasi-convex} \\ H_i(x, \cdot) \text{ coercive uniformly in } x. \end{cases}$$

*Then the  $u^\varepsilon$  of (5.1), (5.2) converges towards the unique  $A_I^-$ -flux-limited solution of*

$$\begin{cases} u_t + H_i(x, Du) = 0, & x \in J_i^* \\ u(0, x) = u_0(x), & x \in J. \end{cases}$$

*Proof.* In view of Theorem 5.1, we only have to prove that  $A_e = A_I^-$ .

Let  $p_i^0$  denote  $p_i^0(x', p')$ ,  $i = 1, 2$ . We forget about the dependance in  $(x', p')$  in order to simplify the presentation.

If  $p_1^0 + p_2^0 \geq 0$ , then we know from Proposition 5.6 that  $A_I^- = A_0$ . Moreover we have in this case that  $\pi_1^+(A_0) + \pi_2^+(A_0) \geq 0$ ; on the one hand, Lemma 5.4 implies that  $A_e = A_0$ . We thus conclude that  $A_e = A_0 = A_I^-$  in this case.

We now assume that  $\tilde{p}_1^0 := -p_1^0 \geq p_2^0$ . In particular, Proposition 5.6 implies that

$$A_I^- = A_I^+ = \max(A_0, A^*)$$

with

$$A^* = \max_{q \in [p_2^0, \tilde{p}_1^0]} \min(H_1(-q), H_2(q)).$$

We distinguish three cases.

First case: if  $H_2(\tilde{p}_1^0) \geq \tilde{H}_1(\tilde{p}_1^0)$  and  $H_2(p_2^0) \leq \tilde{H}_1(p_2^0)$  then

$$A_0 \leq A_I^+ = A^* = H_2(q_I) = H_1(-q_I).$$

Since  $q_I \geq p_2^0$ ,  $q_I = \pi_2^+(A_I^+)$  and since  $q_I \leq -p_1^0$ , then  $-q_I = \pi_1^+(A_I^+)$ . Consequently,

$$\pi_1^+(A_I^+) + \pi_2^+(A_I^+) = 0$$

which implies that  $A_e = A_I^+ = A_I^-$ .

Second case: if  $H_2(\tilde{p}_1^0) \leq \tilde{H}_1(\tilde{p}_1^0) = A_1$  then  $A_0 = A_1$  and  $A^* = H_2(\tilde{p}_1^0) \leq A_0$ . In turn,

$$A_I^- = A_I^+ = A_0.$$

But in this case,  $\pi_2^+(A_0) = \pi_2^+(A_1) \geq \tilde{p}_1^0 = -\pi_1^+(A_0)$  which implies  $A_e = A_0$  (see Theorem 5.1).

Third case: if  $H_2(p_2^0) \geq \tilde{H}_1(p_2^0)$  then  $A_0 = A_2 > A_1$  and  $A^* \leq A_0$  and  $\pi_1^+(A_0) + \pi_2^+(A_0) \geq 0$ . Hence,  $A_I^+ = A_I^- = A_0 = A_e$ . The proof is now complete.  $\square$

## 6. A LARGE DEVIATION PROBLEM

In [7], the authors study large deviation problems related to diffusion processes whose drift is smooth on either side of a hyperplane. Their proofs rely on probability tools and ideas. Our goal in this section is to propose an analytical/PDE proof. Furthermore, by using the results of previous sections, the rate function is related to the maximal Ishii solution of a Hamilton-Jacobi equation.

Consider the stochastic differential equation in  $\mathbb{R}^{d+1}$ ,

$$(6.1) \quad dX^\varepsilon(t) = b(X^\varepsilon(t))dt + \varepsilon^{1/2}\sigma(X^\varepsilon(t))dW(t), \quad X^\varepsilon(0) = x_0, \quad 0 \leq t \leq 1$$

with

$$b(x) = \begin{cases} b_1(x) & \text{if } x_{d+1} < 0 \\ b_2(x) & \text{if } x_{d+1} > 0 \end{cases}$$

and

$$\sigma(x) = \begin{cases} \sigma_1(x) & \text{if } x_{d+1} < 0 \\ \sigma_2(x) & \text{if } x_{d+1} > 0 \end{cases}$$

In order to introduce the rate function, we have to define first Hamiltonians and Lagrangians. Hamiltonians are defined in [7] by

$$\tilde{H}_i(x, p) = \frac{1}{2} \langle a_i(x)p, p \rangle - b_i(x)p, \quad x, p \in \mathbb{R}^{d+1}.$$

with  $a_i = \sigma_i \sigma_i^T$ . Corresponding Lagrangians  $\tilde{L}_1$  and  $\tilde{L}_2$  are related to Hamiltonians  $\tilde{H}_1$  and  $\tilde{H}_2$  by the following formula [7]

$$\tilde{H}_i(x, p) = \sup_{q \in \mathbb{R}^{d+1}} \{-pq - \tilde{L}_i(x, q)\}.$$

Set  $\Omega_1 = \mathbb{R}^d \times (-\infty, 0)$ ,  $\Omega_2 = \mathbb{R}^d \times (0, +\infty)$ ,  $\mathcal{H} = \mathbb{R}^d \times \{0\}$ .

$$(6.2) \quad \tilde{L}(x, p) = \begin{cases} \tilde{L}_1(x, p), & x \in \Omega_1, \\ \tilde{L}_2(x, p), & x \in \Omega_2, \\ \tilde{L}_0(x, p), & x \in \mathcal{H}, \end{cases}$$

where  $\tilde{L}_0$  is defined by

$$\tilde{L}_0(x, p', q) = \inf \left\{ \lambda \tilde{L}_1(x, p', q_1) + (1 - \lambda) \tilde{L}_2(x, p', q_2), \begin{cases} \lambda \in [0, 1], q_1 \geq 0, q_2 \leq 0, \\ \lambda q_1 + (1 - \lambda) q_2 = q \end{cases} \right\}.$$

Call  $\Sigma_{x_0}$  the set of all absolutely continuous function  $\phi \in C([0, 1], \mathbb{R}^{d+1})$  satisfying  $\phi(0) = x_0$ . For any  $\phi \in \Sigma_{x_0}$ , we define the rate function  $I(\phi)$  as follows,

$$(6.3) \quad I_{x_0}(\phi) = \int_0^1 \tilde{L}(\phi(s), \dot{\phi}(s)) ds$$

where  $\tilde{L}$  is defined as in (6.2). We first state the Laplace principle as presented in [7]

**Definition 6.1.** Let  $\{Y^\varepsilon(t), \varepsilon > 0, 0 \leq t \leq 1\}$  with  $Y^\varepsilon(0) = x_0$  be a family of random variables taking values in a Polish space  $\mathcal{Y}$  and let  $I$  be a rate function defined as in (6.3). We say that  $\{Y^\varepsilon\}$  satisfies a Laplace principle with the rate function  $I$  if for every bounded continuous function  $h$  mapping  $\mathcal{Y}$  into  $\mathbb{R}$ , and for any absolutely continuous function  $\phi \in C([0, 1], \mathbb{R}^{d+1})$  satisfying  $\phi(0) = x_0$ , we have

$$(6.4) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \ln \mathbb{E}_{x_0} \left\{ \exp \left[ - \frac{h(Y^\varepsilon)}{\varepsilon} \right] \right\} = - \inf_{\phi \in \Sigma_{x_0}} \{h(\phi(1)) + I_{x_0}(\phi)\}.$$

In [7], the following large deviation result is proved using probabilistic arguments. We will give a PDE proof.

**Theorem 6.2** ([7]). *Assume that*

$$\begin{cases} b_i \text{ is continuous,} \\ \sigma \text{ is continuous and uniformly elliptic, i.e. } \sigma \sigma^T \geq c\mathcal{I} \text{ with } c > 0, \\ (6.1) \text{ has a unique strong solution.} \end{cases}$$

*Then the family  $\{X^\varepsilon, \varepsilon > 0\}$  satisfies the Laplace principle in  $C([0, 1], \mathbb{R}^{d+1})$  with the rate function  $I$  as defined in (6.3).*

*Proof.* Given a function  $h$ , let  $h_\varepsilon$  denote  $\exp(\frac{-h}{\varepsilon})$ . The function  $u_\varepsilon$  given by

$$u_\varepsilon(t, x) = \mathbb{E}_x(h_\varepsilon(X^\varepsilon(t)))$$

is a solution of

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} = \frac{\varepsilon}{2} \text{Trace}(a(x)D^2u_\varepsilon) + b(x)Du_\varepsilon, & t \in (0, 1), x \in \Omega_1 \cup \Omega_2 \\ \frac{1}{2}\partial_{d+1}u_\varepsilon(t, x', 0^+) = \frac{1}{2}\partial_{d+1}u_\varepsilon(t, x', 0^-), & x \in \mathcal{H} \\ u_\varepsilon(0, x) = h_\varepsilon(x), & x \in \Omega_1 \cup \Omega_2 \end{cases}$$

(where  $a = \sigma\sigma^T$ ) The function  $v_\varepsilon = -\varepsilon \ln(u_\varepsilon)$  satisfies

$$\begin{cases} \frac{\partial v_\varepsilon}{\partial t} = \frac{\varepsilon}{2} \text{Trace}(a(x)D^2v_\varepsilon) - \frac{1}{2}\langle a(x)Dv_\varepsilon, Dv_\varepsilon \rangle + b(x)Dv_\varepsilon, & t \in (0, 1), x \in \Omega_1 \cup \Omega_2 \\ \frac{1}{2}\partial_{d+1}v_\varepsilon(t, x', 0^+) = \frac{1}{2}\partial_{d+1}v_\varepsilon(t, x', 0), & x \in \mathcal{H} \\ v_\varepsilon(0, x) = h(x), & x \in \Omega_1 \cup \Omega_2. \end{cases}$$

Moreover, in view of the definition of  $u_\varepsilon$  and  $v_\varepsilon$ , we have

$$v_\varepsilon(t, x) = -\varepsilon \ln \mathbb{E}_x \left\{ \exp \left[ \frac{-h(X^\varepsilon(t))}{\varepsilon} \right] \right\}.$$

Hence, our goal is to prove that

$$\lim_{\varepsilon \rightarrow 0} v_\varepsilon(1, x) = \inf_{\phi \in \Sigma_x} \{h(\phi(1)) + I_x(\phi)\}$$

for any  $\phi \in \Sigma_x$  and  $I$  is defined in (6.3).

We know from Theorem 1.9 that  $v_\varepsilon$  converges locally uniformly towards the maximal Ishii solution  $U^+$  of

$$(6.5) \quad \begin{cases} \frac{\partial U^+}{\partial t} + \tilde{H}_i(x, DU^+) = 0, & x \in \Omega_i, \quad t \in (0, 1) \\ U^+(0, x) = h(x), & x \in \Omega_1 \cup \Omega_2. \end{cases}$$

It thus remains to prove that

$$(6.6) \quad U^+(1, x) = \inf_{\phi \in \Sigma_x} \{h(\phi(1)) + I_x(\phi)\}.$$

. In view of the definition of Lagrangians and Hamiltonians from [7] recalled above, we have

$$\tilde{H}_i(x, p) = \sup_{q \in \mathbb{R}^{d+1}} \{pq - l_i(x, q)\} \quad \text{with} \quad l_i(x, -q) = \tilde{L}_i(x, q),$$

here  $l_i$  corresponds to the running costs considered in [15, Section 6]. In view of the definition of  $\tilde{L}_0$  recalled above, we have

$$\begin{aligned} \tilde{L}_0(x, q', 0) &= \inf \left\{ \lambda \tilde{L}_1(x, q', q_1) + (1 - \lambda) \tilde{L}_2(x, q', q_2), \begin{cases} 0 \leq \lambda \leq 1, \\ q_1 \geq 0, q_2 \leq 0, \lambda q_1 + (1 - \lambda) q_2 = 0 \end{cases} \right\} \\ &= \inf \left\{ \lambda l_1(x, q', v_1) + (1 - \lambda) l_2(x, q', v_2), \begin{cases} 0 \leq \lambda \leq 1, \\ v_1 \leq 0, v_2 \geq 0, \lambda v_1 + (1 - \lambda) v_2 = 0 \end{cases} \right\}. \end{aligned}$$

Hence, the formula of  $U^+$  given in [14, 4] coincides with (6.6). The proof is now complete.  $\square$

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