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Abdeljalil Saghe

► To cite this version:

Abdeljalil Saghe. Nonstandard analysis : The field of Omicran-reals and its applications. 2016. <hal-01252675v4>

HAL Id: hal-01252675

<https://hal.science/hal-01252675v4>

Preprint submitted on 25 May 2016

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Nonstandard analysis : The field of Omicran-reals and its applications

Abdeljalil. Saghe*

Cpge de Settat, Lycée Qualifiant Technique PB : 576 - Maroc

May 25, 2016

Keyword. Nonstadard analysis, Omicran-reals, hyppereals, Abraham Robinson, infinitesimal.

Abstract

From the results of my first paper [1] which represents a new approach to nonstandard analysis, we can construct a new extension of the real numbers which contains the infinite and infinitesimal numbers, the new set called the field of Omicran-reals [1] is different from the set of the hyperreals [2], and we can define the new field without using the properties given by the model theory.

Our aims in this paper, is to give the applications of the new method, and we can see that the new approach is very simple in practice compared to other [2] [3].

1 Introduction

This paper is organized as follows, we restart in the first section of preliminary results by some notions for define the field of Omicran-reals [1], in the next section we define the exact limit, and we show that the limit of a sequence (u_n) is an approximation of the exact limit of (u_n) , this important result leads to define the projection of an Omicran-real x onto \mathbb{R} , and we can verify that the limit of (u_n) is the projection of the exact limit of (u_n) onto the set of the real numbers. In the section 3, we present the exact derivate with its applications. In the fifth section, we give the relations which calculates the logarithm, and we verify that $\ln(x) \approx \frac{x^\delta - 1}{\delta}$, which implies that $\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha}$, from the above example, we deduce the possibility to obtain an exact equality by using an approximate relationship. In the sixth section we give a simple method for calculating the length of an arc \widehat{AB} of a curve \mathcal{C}_f .

2 Preliminary results

Definition 1 Let u be a map from $]0, 1]$ to \mathbb{R} , such that :

- There exists a map \tilde{u} is defined on $D'(0, 1)$, and holomorphic in a neighborhood of 0.
- There exists $\varepsilon > 0$, such that $\forall x \in]0, \varepsilon[$ we have a : $\tilde{u}(x) = u(x)$.

The map u is called a *metallic map*, and \tilde{u} is a *metallic extension* of u .

Notations :

We note :

- $\Delta_1 = \{ u, u \text{ is a metallic map} \}$.
- $\mathcal{A}_2 = \{ \frac{1}{u}, u \in \Delta_1 \ \forall x \in]0, 1] \ u(x) \neq 0 \text{ and } \lim u(\frac{1}{n}) = 0 \}$.
- $\Delta_2 = \{ v / v :]0, 1] \longrightarrow \mathbb{R}, \text{ there exists } 1/u \in \mathcal{A}_2 \text{ and } \varepsilon > 0 \text{ such that } v_{/]0, \varepsilon]} = (\frac{1}{u})_{/]0, \varepsilon]} \}$.
- $\Delta = \Delta_1 \cup \Delta_2$, $(\Delta, +, \cdot)$ is a unitary ring.
- Let \mathcal{I}_0 the set of maps defined in $]0, 1]$ and zero on $]0, \varepsilon[$ (for $0 < \varepsilon \leq 1$).
The set \mathcal{I}_0 is a maximal ideal of Δ , then Δ/\mathcal{I}_0 is a field.
- We define $\Delta(\frac{1}{n}) = \{ h(\frac{1}{n})_{n \geq 1}, h \in \Delta \}$.

*corresponding author: saghe007@gmail.com

- Let \sim the equivalence relation on $\Delta(\frac{1}{n})$ defined by :

$$g(\frac{1}{n})_{n \geq 1} \sim h(\frac{1}{n})_{n \geq 1} \Leftrightarrow \exists n_0 / \forall n \geq n_0, \quad h(\frac{1}{n}) = g(\frac{1}{n}).$$

- The equivalence class of $(f(\frac{1}{n}))_{n \geq 1}$ modulo \sim is noted $\overline{(f(\frac{1}{n}))_{n \geq 1}}$.
- Let $\overline{\Delta(\frac{1}{n})}$ the set of the equivalence classes $\overline{(f(\frac{1}{n}))_{n \geq 1}}$.
- Let $\Delta_0 = \Delta/\mathcal{I}_0$, and \overline{g} the equivalence class of g .
If it has not ambiguity, we replace \overline{g} by g .
- The field of Omicran-reals is defined as follows [1] :

$$\mathcal{O} = \Delta_0(\delta) = \{ g(\delta), g \in \Delta_0 \},$$

where δ is an indeterminate.

Remark 1 • We define on \mathcal{O} the following total order :

$$g(\delta) \leq h(\delta) \Leftrightarrow g(\frac{1}{n}) \leq h(\frac{1}{n}) \text{ from a certain rank } n_0.$$

- The field of Omicran-reals is an extension of \mathbb{R} , in addition, the total order relation defined on \mathcal{O} extends the usual order relation \leq .
- The number δ is infinitesimal [1].

3 The exact limit

Proposition 1 The map φ defined as :

$$\varphi : \begin{array}{l} \overline{(\Delta(\frac{1}{n}), +, \cdot)} \longrightarrow (\mathcal{O}, +, \cdot) \\ \overline{(g(\frac{1}{n}))_{n \geq 1}} \longrightarrow g(\delta) \end{array}$$

is isomorphism.

If we want to define a new concept more precise than the limit that allows to give the value taken by the sequence $(f(\frac{1}{n}))_{n \geq 1}$ on ad infinitum, then this concept (called exact limit) is dependent on the values taken by $(f(\frac{1}{n}))_{n \geq 1}$ from a certain rank n_0 . Intuitively, the equivalence class $\overline{(f(\frac{1}{n}))_{n \geq 1}}$ is a only concept can give these values independently n_0 . On other hand, if f is an element of Δ , then we can identify the equivalence class $\overline{(f(\frac{1}{n}))_{n \geq 1}}$ by $f(\delta)$ from the above proposition, so, we deduce that we can define the new concept as follow :

Definition 2 Let $f \in \Delta$.

The Omicran-real $\varphi(\overline{(f(\frac{1}{n}))_{n \geq 1}}) = f(\delta)$ is called the exact limit of the sequence $(f(\frac{1}{n}))_{n \geq 1}$.

We note :

$$\lim_{exact} f(\frac{1}{n}) = f(\delta).$$

Remark 2 We remark that $\lim_{exact} = \varphi \circ s$, where s is a canonical surjection defined as :

$$s : \begin{array}{l} (\Delta(\frac{1}{n}), +, \cdot) \longrightarrow \overline{(\Delta(\frac{1}{n}), +, \cdot)} \\ (f(\frac{1}{n}))_{n \geq 1} \longrightarrow \overline{(f(\frac{1}{n}))_{n \geq 1}} \end{array}.$$

Example 1 • $\lim_{exact} \frac{1}{n} = \delta$.

- $\lim_{exact} \sin(\frac{1}{n}) = \sin(\delta)$.
- $\lim_{exact} \frac{1}{n+1} = \frac{\delta}{\delta+1}$.
- $\lim_{exact} \sin(2\pi n) = 0$, the writing $\sin(\frac{2\pi}{\delta})$ doesn't make sense, because the function $z \longrightarrow \sin(\frac{2\pi}{z})$ is not element of Δ .

- We can verify that there does not exist an element $f \in \Delta$, such that $f(\frac{1}{n}) = (-1)^n$ from a certain rank, then we can't define the exact limit $\lim_{exact} (-1)^n$.

Definition 3 (The projection)

Let f a metalic function, and $x \in \mathcal{O}$ such that $x = f(\delta)$.

If we find an element $x^* \in \mathbb{R}$ such that $|x - x^*| \leq |x - y|$, $\forall y \in \mathbb{R}$, then the real x^* is called the projection of x onto \mathbb{R} .

Remark 3 The distance from x to \mathbb{R} is $d_{\mathbb{R}}(x) = \inf_{y \in \mathbb{R}} |x - y| = |x - x^*|$.

Example 2 • $\delta^* = 0$.

$$\bullet \left(\frac{1}{\delta^2 + 1} \right)^* = 1.$$

Theorem 1 Let f a metalic function, and $x \in \mathcal{O} / x = f(\delta)$.

The projection x^* of x onto \mathbb{R} exist and unique, in addition : $x^* = \lim_{n \rightarrow +\infty} f(\frac{1}{n})$.

Proof.

Let $x_0 = \lim f(\frac{1}{n})$.

Then : $\forall \varepsilon > 0 \exists n \geq n_0, |f(\frac{1}{n}) - x_0| \leq \varepsilon$.

\Leftrightarrow for every $n \geq n_0$, we have : $-\varepsilon \leq f(\frac{1}{n}) - x_0 \leq \varepsilon$.

$\Leftrightarrow \lim_{exact} f(\frac{1}{n}) - x_0 \leq \varepsilon$ and $-\varepsilon \leq \lim_{exact} f(\frac{1}{n}) - x_0$.

$\Leftrightarrow f(\delta) - x_0 \leq \varepsilon$ and $-\varepsilon \leq f(\delta) - x_0$.

$\Leftrightarrow |f(\delta) - x_0| \leq \varepsilon$ (1)

Next, we can show that : $\forall y \in \mathbb{R}$ on a : $|f(\delta) - x_0| \leq |f(\delta) - y|$.

If $\exists y \in \mathbb{R}$ and $|f(\delta) - y| \leq |f(\delta) - x_0| \leq \varepsilon$ ($\forall \varepsilon \in \mathbb{R}^+$)

Then, we deduce that $|y - x_0| \leq 2\varepsilon$ ($\forall \varepsilon \in \mathbb{R}^+$)

Then $y = x_0 = x^*$.

Finally, we deduce the existence and the unicity of $x^* \in \mathbb{R}$ such that :

$$\forall y \in \mathbb{R}, |f(\delta) - x^*| \leq |f(\delta) - y|.$$

In addition $x^* = \lim f(\frac{1}{n})$.

Theorem 2 Let f the metalic map, and $x = f(\delta)$.

Then : $\forall \varepsilon > 0$ real, we have :

$$|x - x^*| \leq \varepsilon$$

x^* is a unique element of \mathbb{R} which verify this property.

Proof.

For every $n \geq n_0$ we have :

$$|f(\frac{1}{n}) - x^*| \leq \varepsilon,$$

then $\forall n \geq n_0$:

$$x^* - \varepsilon \leq f(\frac{1}{n}) \leq x^* + \varepsilon,$$

which implies :

$$x^* - \varepsilon \leq f(\delta) \leq x^* + \varepsilon.$$

We deduce that :

$$|x - x^*| \leq \varepsilon.$$

To show the unicity of x^* , we assume there exists an other element $y \in \mathbb{R}$ such that : $|x - y| \leq \varepsilon$.

Then : $|x^* - y| \leq 2\varepsilon$, finally we get $y = x^*$.

Theorem 3 (of the exact limit)

If $f \in \Delta_1$, then the real $\lim f(\frac{1}{n})$ is the projection of $\lim_{exact} f(\frac{1}{n})$ onto \mathbb{R} , so we get :

$$\left(\lim_{exact} f(\frac{1}{n}) \right)^* = \lim f(\frac{1}{n}).$$

4 The exact derivative

Definition 4 Let f be a function that is differentiable at the point $x_0 \in \mathbb{R}$.

If the function $h \rightarrow \frac{f(x_0+h)-f(x_0)}{h}$ is metallic, then the exact limit of $(\frac{f(x_0+\frac{1}{n})-f(x_0)}{\frac{1}{n}})$ exist.

We note :

$$\widehat{f}(x_0) = \lim_{\text{exact}} \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} = \frac{f(x_0 + \delta) - f(x_0)}{\delta}$$

The Omicran-real $\widehat{f}(x_0)$ is called the exact derivative of the function f at a point x_0 .

Example 3 Consider the following function defined as: $f : x \rightarrow x^2$.

The exact derivative of f is defined by : $\widehat{f}(x_0) = 2x_0 + \delta$.

Theorem 4 Let f be a function that is differentiable at the point $x_0 \in \mathbb{R}$.

If the function $h \rightarrow \frac{f(x_0+h)-f(x_0)}{h}$ is metallic, then :

$$(\widehat{f}(x_0))^* = f'(x_0)$$

Proof.

We can apply the theorem of the exact limit.

Example 4 For $f : x \rightarrow x^2$, the exact derivative at x_0 is $\widehat{f}(x_0) = 2x_0 + \delta$, and the derivative at x_0 is $f'(x_0) = 2x_0$.

We verify easily that : $(2x_0 + \delta)^* = 2x_0$.

Lemma 1 Let f be a metallic function such that for every integer $k \in \mathbb{N}$, the function $t \rightarrow f(x_0 + kt)$ is metallic, then :

$$f(x_0 + N\delta) = f(x_0) + \delta(\widehat{f}(x_0) + \widehat{f}(x_0 + \delta) + \widehat{f}(x_0 + 2\delta) + \dots + \widehat{f}(x_0 + (N-1)\delta)).$$

Proof.

From the definition of \widehat{f} , we get :

$$\begin{aligned} f(x_0 + \delta) &= f(x_0) + \delta\widehat{f}(x_0) \\ f(x_0 + 2\delta) &= f(x_0 + \delta) + \delta\widehat{f}(x_0 + \delta) \\ &\dots\dots\dots \\ f(x_0 + N\delta) &= f(x_0 + (N-1)\delta) + \delta\widehat{f}(x_0 + (N-1)\delta). \end{aligned}$$

Summing these equalities, and we find the desired result .

Application 1 (Calculate the sum of Σk^n)

- For $n = 1$, if $f(x) = x^2$, then $\widehat{f}(x) = 2x + \delta$.
From the above proposition in the case of $x_0 = 0$, we find :

$$N^2\delta^2 = (\delta(\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \dots + \widehat{f}((N-1)\delta))).$$

Which implies that : $N^2\delta^2 = \delta.(\sum_{k=0}^{N-1} 2k\delta + \delta)$.

Then : $N^2 = \sum_{k=0}^{N-1} (2k + 1) = 2 \sum_{k=0}^{N-1} k + N \Rightarrow \frac{N^2 - N}{2} = \sum_{k=0}^{N-1} k$.

- In the case of $n = 2$, we choose $f(x) = x^3$, then $\widehat{f}(x) = 3x^2 + 3x\delta + \delta^2$, by using the above proposition for $x_0 = 0$, we find :

$$\begin{aligned} N^3\delta^3 &= \delta.(\widehat{f}(0) + \widehat{f}(\delta) + \widehat{f}(2\delta) + \dots + \widehat{f}((N-1)\delta)) \\ &= \delta. \sum_{k=0}^{N-1} (3k^2\delta^2 + 3k\delta.\delta + \delta^2) = \delta^3.(\sum_{k=0}^{N-1} 3k^2 + 3k + 1). \end{aligned}$$

Then : $N^3 = 3 \sum_{k=0}^{N-1} k^2 + 3 \sum_{k=0}^{N-1} k + N$.

We deduce : $\sum_{k=0}^{N-1} k^2 = \frac{N^3 - N - 3 \sum_{k=0}^{N-1} k}{3}$.

Finally, we get : $\sum_{k=0}^{N-1} k^2 = \frac{N(N-1)(2N-1)}{6}$.

Similarly, we can calculate $\sum_{k=0}^{N-1} k^3, \sum_{k=0}^{N-1} k^4, \dots$

Application 2 (The Riemann sum)

Let f, g be two metallic functions, such that : $f(\delta) = g(\delta)$, then $f(\frac{1}{n}) = g(\frac{1}{n})$ from a certain rank. Consider the function defined as follows :

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$$

From the above lemma, we deduce that there exists a natural number n_0 such that $\forall n \geq n_0$, we have :

$$f(x_0 + \frac{N}{n}) = f(x_0) + \frac{1}{n}(f_n(x_0) + f_n(x_0 + \frac{1}{n}) + f_n(x_0 + \frac{2}{n}) + \dots + f_n(x_0 + \frac{N-1}{n})).$$

If f is differentiable on \mathbb{R} , then $f_n(a + \frac{k}{n}) = f'(a + \frac{k}{n}) + o(\frac{1}{n})$.
Then

$$f(a + \frac{N}{n}) = f(a) + \frac{1}{n} \cdot \sum_{k=0}^{N-1} f_n(a + \frac{k}{n}) = f(a) + \frac{1}{n} \cdot \sum_{k=0}^{N-1} (f'(a + \frac{k}{n}) + o(\frac{1}{n})).$$

We assume that $b > a$, we can choose $N = E[(b-a)n]$, then we get :

$$\begin{aligned} f(a + \frac{N}{n}) - f(a) &= \frac{1}{n} \cdot \sum_{k=0}^{N-1} f'(a + \frac{k}{n}) + \frac{1}{n} \sum_{k=0}^{N-1} o(\frac{1}{n}) \\ &= \frac{1}{n} \cdot \sum_{k=0}^{N-1} f'(a + \frac{k}{n}) + \frac{N}{n} o(\frac{1}{n}). \end{aligned}$$

Since $N = E[(b-a)n]$, then $b-a - \frac{1}{n} < \frac{N}{n} \leq b-a$.

We deduce $\lim f(a + \frac{N}{n}) = f(b)$, we pass to the limit and we find :

$$f(b) - f(a) = \lim \frac{1}{n} \cdot \sum_{k=0}^{E[(b-a)n]-1} f'(a + \frac{k}{n}).$$

For $b = 1$, and $a = 0$, we get :

$$f(1) - f(0) = \int_0^1 f'(t) dt = \lim \frac{1}{n} \sum_{k=0}^{n-1} f'(\frac{k}{n}).$$

5 The logarithmic function

We know that : $\forall x \in \mathbb{R}, \lim_{n \rightarrow +\infty} (1 + \frac{x}{n})^n = e^x$.

Let x be a real number, the function $f : z \rightarrow (1 + xz)^{\frac{1}{z}} = e^{\frac{1}{z} \ln(1+xz)}$ is a holomorphic function on $D(0, \varepsilon) - \{0\}$.

In addition, $\ln(1+xz) = zx - \frac{z^2 x^2}{2} + o(z^2 x^2)$ (for $|z| < 1$), we deduce that : $\lim_{z \rightarrow 0} \frac{\ln(1+xz)}{z} = x$, and $\lim_{z \rightarrow 0} f(z) = e^x$,

then the function f can be extended to a holomorphic function on a neighborhood of 0, which implies that $\lim_{exact} (1 + \frac{x}{n})^n$

exist and we have : $\lim_{exact} (1 + \frac{x}{n})^n = (1 + x\delta)^{\frac{1}{\delta}}$, and $((1 + x\delta)^{\frac{1}{\delta}})^* = e^x$.

From the above results we deduce that e^x is an infinitesimal approximate value of $(1 + \delta x)^{\frac{1}{\delta}}$.

Consider the function $\xi(x) = (1 + \delta x)^{\frac{1}{\delta}}$, if this function $x \rightarrow \xi(x)$ has an inverse function, then the calcul gives us the function ξ^{-1} which is defined as : $\xi^{-1}(x) = \frac{x^\delta - 1}{\delta}$ (if it exists !!!).

We can justify the existence of this function, for that, we consider the real number $x \in \mathbb{R}^{*+}$.

The map defined as $g : z \rightarrow \frac{x^z - 1}{z} = \frac{e^{z \ln(x)} - 1}{z}$ is a holomorphic function on $D(0, \varepsilon) - \{0\}$, and $\lim_{z \rightarrow 0} g(z) = \ln(x)$, then 0 is an artificial singularity of g , we deduce that there exists the exact limit of the sequence $(n(x^{\frac{1}{n}} - 1))_{n \geq 1}$, and his value

is equal to $\lim_{exact} n(x^{\frac{1}{n}} - 1) = \frac{x^\delta - 1}{\delta}$.

We define the the original logarithm by :

$$\ln_o : x \rightarrow \frac{x^\delta - 1}{\delta}.$$

The function ξ is called the function of the the original exponential, we note :

$$\xi(x) = exp_o(x) = (1 + \delta x)^{\frac{1}{\delta}},$$

and we deduce that :

$$(\ln_o(x))^* = \ln(x).$$

Then :

$$\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha}.$$

- **Application 1.** From the above results, we can show the following equality :

$$\ln(x) = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - x^{-\alpha}}{2\alpha}.$$

Remark 4

$$\frac{\ln(x)}{\ln(y)} = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{y^\alpha - 1} = \lim_{\alpha \rightarrow 0} \frac{x^\alpha - x^{-\alpha}}{y^\alpha - y^{-\alpha}}.$$

- **Application 2.** By using the above results we can show the following theorem.

Theorem 5 For every $0 < x$ and $x \neq 1$, we have :

$$\frac{x-1}{\ln(x)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

Proof. We have :

$$(x^{\frac{1}{n}} - 1) \left(\sum_{k=0}^{n-1} x^{\frac{k}{n}} \right) = x - 1.$$

Then :

$$n(x^{\frac{1}{n}} - 1) \left(\frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}} \right) = x - 1.$$

Since $\lim_{\alpha \rightarrow 0} \frac{x^\alpha - 1}{\alpha} = \lim_{n \rightarrow +\infty} n(x^{\frac{1}{n}} - 1) = \ln(x)$, then :

$$\frac{x-1}{\ln(x)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} x^{\frac{k}{n}}.$$

- **Application 3.** Consider \mathcal{P} the set of the prime numbers.
We define the prime-counting function at real values of x by :

$$\pi(x) = \#\{p \leq x : p \in \mathcal{P}\}.$$

Theorem 6 (Hadamard and de la Vallée Poussin). As $x \rightarrow +\infty$,

$$\pi(x) \sim \frac{x}{\ln(x)}.$$

.

By using the above theorem, and the formula $\ln(x) \approx \frac{x^\delta - 1}{\delta}$, we can show the following theorem :

Theorem 7 As $x \rightarrow +\infty$,

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

.

Proof. We approach $\ln(x)$ by $\frac{x^\delta - 1}{\delta}$, and δ by $\frac{1}{x}$, and we deduce that an approximate value of $\frac{x}{\ln(x)}$ is $\frac{1}{x^{\frac{1}{x}} - 1}$.

Next, we can verify that $\frac{x}{\ln(x)} \sim \frac{1}{x^{\frac{1}{x}} - 1}$, in fact : As $x \rightarrow +\infty$, we have :

$$\frac{1}{x^{\frac{1}{x}} - 1} \sim \frac{1}{e^{\frac{\ln(x)}{x}} - 1} = \frac{1}{\frac{e^{\frac{\ln(x)}{x}} - 1}{\frac{\ln(x)}{x}}} \frac{x}{\ln(x)}.$$

Since, $\lim_{x \rightarrow +\infty} \frac{e^{\frac{\ln(x)}{x}} - 1}{\frac{\ln(x)}{x}} = 1$, then : $\frac{1}{x^{\frac{1}{x}} - 1} \sim \frac{x}{\ln(x)}$.

Finally, we obtain :

$$\pi(x) \sim \frac{1}{x^{\frac{1}{x}} - 1}.$$

- **Application 4.** Let (p_n) the sequence of a prime numbers, we have :

Theorem 8

$$p_n \sim n \ln(n), \text{ while } n \rightarrow +\infty$$

By using the above theorem and the approximations $\ln(n) \approx \frac{n^\delta - 1}{\delta}$ and $\delta \simeq \frac{1}{n}$, we get the following theorem :

Theorem 9

$$p_n \sim n^2 (\sqrt[n]{n} - 1), \text{ while } n \rightarrow +\infty$$

Proof. We can verify that $\lim_{n \rightarrow \infty} \frac{n(\sqrt[n]{n} - 1)}{\ln(n)} = 1$.

6 The geometry of the points

Geometrically, we can represent the infinitesimal numbers as follows :

If f is a metallic function, and $\tilde{f}(\delta)$ is an infinitesimal number, the function f is metallic then the sequence $(f(\frac{1}{n}))$ keeps a constant signe from a certain rank, we can assume that the above sequence is positive from a certain rank n_0 , and we know that : $\tilde{f}(\delta) = \overline{f(\frac{1}{n})}_{n \geq 1}$, consider $I_n =]0, f(\frac{1}{n})]$ then we can represent $\tilde{f}(\delta)$ by the the family of segments $(I_n)_{n \geq n_0}$, we can justify this notation by this equality : $\bigcap_{n \geq n_0} I_n =]0, \tilde{f}(\delta)]$.

Definition 5 Let f and g be two metallic functions.

We assume that the number $\tilde{f}(\delta) - \tilde{g}(\delta)$ is infinitesimal and $\tilde{f}(\delta) < \tilde{g}(\delta)$.

- A geometric quantum of \mathbb{R} is a segment of the type : $[\tilde{f}(\delta), \tilde{g}(\delta)]$, $[\tilde{f}(\delta), \tilde{g}(\delta)[$, $] \tilde{f}(\delta), \tilde{g}(\delta)]$ or $] \tilde{f}(\delta), \tilde{g}(\delta)[$.
- If x_A and x_B are respectively the abscissa of the points A and B , we define the distance between A and B by :

$$\text{dist}(A, B) = |x_B - x_A|.$$

- An elementary Geometric point of \mathbb{R} is a segment of the type : $[x_A, x_A + \delta]$.

7 The length of a curve \mathcal{C}_f

We define the length of an elementary geometric point by :

$$l([x_A, x_A + \delta]) = \delta,$$

where $x_A = g(\delta)$, and g is a metallic function.

Let f be a function such that : $x \rightarrow f(x_A^* + x)$ is metallic, and \tilde{f} the metallic extension of f on a neighborhood of x_A^* , consider $A(x_A, f(x_A))$ and $A'(x_A + \delta, f(x_A + \delta))$ two ordered pairs of E^2 , let ϕ be a function defined as :

$$\phi : z \rightarrow z \sqrt{1 + \left(\frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(\tilde{g}(z))}{z} \right)^2}.$$

On the other hand, $\lim_{z \rightarrow 0} \tilde{g}(z) = x_A^*$, and the map $\theta : z \rightarrow \frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(\tilde{g}(z))}{z}$ is holomorphic on $D(0, \varepsilon) - \{0\}$, and we get :

$$\theta(z) = \frac{\tilde{f}(\tilde{g}(z) + z) - \tilde{f}(x_A^*)}{z} - \frac{\tilde{f}(\tilde{g}(z)) - \tilde{f}(x_A^*)}{z}.$$

Then the $\lim_{z \rightarrow 0} \theta(z)$ exist, and we have :

$$\lim_{z \rightarrow 0} \theta(z) = (g'(0) + 1)f'(x_A^*) - g'(0)f'(x_A^*) = f'(x_A^*) \in \mathbb{R}$$

We deduce that $\lim_{z \rightarrow 0} \phi(z) = 0$, and ϕ is continuously extendable over 0, then the function ϕ is holomorphically extendable over 0, which justify the existence of the exact limit of the sequence $(\phi(\frac{1}{n}))$, and we have

$$\lim_{exact} (\phi(\frac{1}{n})) = \phi(\delta) = \delta \sqrt{1 + \left(\frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2} \in E,$$

which is infinitesimal. We define the length of the segment $[A, A']$ by :

$$l([A, A']) = \delta \sqrt{1 + \left(\frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2}.$$

We note :

$$\psi(x_A) = \delta \sqrt{1 + \left(\frac{f(x_A + \delta) - f(x_A)}{\delta} \right)^2} = \delta \sqrt{1 + \widehat{f}(x_A)^2}.$$

If f is a metallic function defined on the interval $[0, 1]$, Let $A(0, f(0))$ and $B(1, f(1))$ two points of the plane which define with f the arc \widetilde{AB} , if the exact limit of $\lim_{exact} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + \widehat{f}(\frac{k}{n})^2}$ exist, we note :

$$\lim_{exact} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + \widehat{f}(\frac{k}{n})^2} = \delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2}.$$

We define the exact lenght of the arc \widetilde{AB} by :

$$l(\widetilde{AB}) = \delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2}.$$

The lenght of the arc \widetilde{AB} is the real denoted by $l^*(\widetilde{AB})$ and defined by :

$$l^*(\widetilde{AB}) = \left(\delta \sum_{k=0}^{\frac{1}{\delta}-1} \sqrt{1 + \widehat{f}(k\delta)^2} \right)^* = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f_n(\frac{k}{n})^2},$$

where $f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}}.$

On other hand, since f is a metallic function, then it can be extended to a function twice differentiable at 0. We assume that the function is twice differentiable on $]0, 1[$.

Then :

$$f_n(\frac{k}{n}) = \frac{f(\frac{k+1}{n}) - f(\frac{k}{n})}{\frac{1}{n}} = f'(\frac{k}{n}) + \frac{1}{2n} f''(\xi_k),$$

where $1 \leq k \leq n-1$, and $\xi_k \in]\frac{k}{n}, \frac{k+1}{n}[$.

Consider $M_1 = \sup_{]0, 1[} (|f'(x)|)$ and $M_2 = \sup_{]0, 1[} (|f''(x)|)$.

We have :

$$f_n^2(\frac{k}{n}) = \left(f'(\frac{k}{n}) + \frac{1}{2n} f''(\xi_k) \right)^2 = f'^2(\frac{k}{n}) + \varepsilon_{n,k},$$

where $\varepsilon_{n,k} = f'(\frac{k}{n}) \frac{1}{n} f''(\xi_k) + \frac{1}{4n^2} f''^2(\xi_k)$.

If $M_1 < +\infty$ and $M_2 < +\infty$, we get :

$$|\varepsilon_{n,k}| \leq \frac{M_1 M_2 + M_2^2}{n}.$$

Next, we deduce that :

$$\begin{aligned} \sqrt{1 + f_n^2(\frac{k}{n})} &= \sqrt{1 + f'^2(\frac{k}{n}) + \varepsilon_{n,k}} \\ &= \sqrt{1 + f'^2(\frac{k}{n})} + \frac{\varepsilon_{n,k}}{2\sqrt{\beta_{n,k}}}, \end{aligned}$$

where $\beta_{n,k} \in]1 + f'^2(\frac{k}{n}), 1 + f'^2(\frac{k}{n}) + \varepsilon_{n,k}|$.

Then :

$$\frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f_n^2(\frac{k}{n})} = \frac{1}{n} \sqrt{1 + f_n^2(0)} + \frac{1}{n} \sum_{k=1}^{n-1} \sqrt{1 + f'^2(\frac{k}{n})} + \frac{1}{2n} \sum_{k=1}^{n-1} \frac{\varepsilon_{n,k}}{\sqrt{\beta_{n,k}}}.$$

We can verify that $\beta_{n,k} > 1$, and we have :

$$\left| \frac{1}{2n} \sum_{k=1}^{n-1} \frac{\varepsilon_{n,k}}{\sqrt{\beta_{n,k}}} \right| \leq \frac{1}{2n} \sum_{k=1}^{n-1} |\varepsilon_{n,k}| \leq \frac{M_1 M_2 + M_2^2}{2n}.$$

Then : $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f_n^2\left(\frac{k}{n}\right)} = \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \sqrt{1 + f'^2\left(\frac{k}{n}\right)}.$

By using the Riemann sum, we deduce the length of the arc \widetilde{AB} :

$$l^*(\widetilde{AB}) = \int_0^1 \sqrt{1 + f'^2(x)} dx.$$

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